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INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
34100 TRIESTE (ITALY) - P.O.B. 586 - MIRAMARE - STRADA COSTIERA 11 - TELEPHONES: 224281/2/3/4/5/6
CABLE: CENTRATOM - TELEX 460392 - I

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AUTUMN COURSE ON GEOMAGNETISM, THE IONOSPHERE
AND MAGNETOSPHERE

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MATHEMATICS

A.A. ASHOUR
Department of Mathematics
University of Cairo
Giza, Cairo
Egypt

Some
References:

1. Geomagnetism (vols 1 & 2) — Chapman & Bartels.
2. Static & Dynamic Electricity — Smythe
3. Electromagnetic Theory — Stratton
4. Methods of theoretical Physics — Morse & Feshbach
5. Integral Transforms — Tranter
6. Conduction of Heat in Solids — Carslaw & Jaeger

I. Harmonic Analysis

I-1 Introduction. In Geomagnetism, we often deal with "periodic phenomena", which are represented by periodic functions.

Definition. A function $f(t)$ of the real argument t is said to be periodic in t with period T if it satisfies the relation

$$f(t+kT) = f(t) \qquad (1)$$

for any integer k.

Geomagnetic time changes can be represented in many cases as a combination of a) non-periodic part and b) several periodic components with different periods. As examples of periods, we mention the solar and lunar days, the solar rotation period, the month, the year, the 11-year sun spot cycle and multiples of these times.

Usually, the non-periodic part (or its average) is removed by certain methods which depend on the conditions of the problem. And so, we are left with periodic components, to which we devote the following sections.

I-2 Fourier Series.

~~Consider~~ The differential equation

$$\frac{d^2y}{dt^2} = -n^2 y \quad (n\text{-real}) \qquad (2)$$

gives rise to a solution of the form

$$y = A\cos nt + B\sin nt \qquad (3)$$

where A & B are constants. Such solutions are ~~extremely~~ often met ~~often met~~ in Geomagnetism and they represent periodic motions.

The periodic functions $\cos nt$ & $\sin nt$ play a fundamental role in the theory of expansion of functions, as may be seen from the following:

Fourier Theorem:

Fourier Theorem states that under certain conditions(*), a function $f(t)$ defined on the interval $0 \leq t \leq 2\pi$ can be expressed as an infinite trigonometric series of the form

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nt + b_n \sin nt)$$

$$= a_0 + \sum_{n=1}^{\infty} c_n \sin(nt + \varepsilon_n) \qquad (4)$$

$$\left(c_n = \sqrt{a_n^2+b_n^2}\ , \quad \tan\varepsilon_n = a_n/b_n \right).$$

The equality sign holds over the whole interval $[0, 2\pi]$ except at points of discontinuity of the function $f(t)$.

Coefficients a_0, a_n, b_n $(n=1,2,\dots)$ ~~may be easily calculated in terms of f(t)~~ are called the Fourier coefficients of function $f(t)$. They may be easily calculated with the help of the following

~~(*) It is sufficient that f(t)~~

(*) For our purposes, it is sufficient to assume that $f(t)$ is a piecewise continuous function, i.e. that it is continuous on the interval $[0, 2\pi]$ except, maybe, at a finite number of points where is suffers finite jumps.

orthogonality properties :

$$\left.\begin{aligned}
&\int_0^{2\pi} \cos nt \cos mt \, dt = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \neq 0 \\ 2\pi & \text{if } n = m = 0 \end{cases} \\
&\int_0^{2\pi} \sin nt \sin mt \, dt = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \neq 0 \end{cases} \\
&\int_0^{2\pi} \sin nt \cos mt \, dt = 0
\end{aligned}\right\} \quad (5)$$

A formal multiplication of (4) by 1 or $\cos mt$ or $\sin mt$ followed by an integration over the interval $[0, 2\pi]$ and use of (5) yields the results :

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(t)\, dt \quad , \quad a_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos nt \, dt ,$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin nt \, dt \qquad (6)$$

<u>Note</u> : the theorem remains true when the interval of definition of the function is $\alpha \leq t \leq \alpha + 2\pi$, in which case the integrations in (6) should be performed over this interval instead of $[0, 2\pi]$.

If $\alpha = -\pi$, then $f(t)$ is defined over the interval $[-\pi, +\pi]$. In this case,

i) If $f(t)$ is even, $b_n \equiv 0$ $(n = 1, 2, \cdots)$

ii) If $f(t)$ is odd, $a_n \equiv 0$ $(n = 0, 1, 2, \cdots)$

(The notion of even & odd function may be introduced only if its domain of definition is symmetric about the origin).

Expansion (4) means that we are expanding function $f(t)$ in terms of the set of functions :

$$\{ 1 , \cos nt , \sin nt \} \quad , n = 1, 2, \ldots$$

Sometimes, instead of the aforementioned set we use a set of "orthonormal"(*) functions :

$$\left\{ \frac{1}{\sqrt{2\pi}} , \frac{1}{\sqrt{\pi}} \cos nt , \frac{1}{\sqrt{\pi}} \sin nt \right\}_{n = 1, 2, \ldots}$$

(*) A set of real functions $\{ f_n(t) \}_{n \in N}$ defined on an interval $a \leq t \leq b$ is said to be orthonormal (in the sense of L_2) on that interval if $\int_a^b f_n(t) f_m(t)\, dt = \delta_{mn} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$

I-3 Fourier series with period T.

If function $f(t)$ is defined on the interval $\alpha \le t \le \beta$, where $T = \beta - \alpha$ is not necessarily equal to 2π, $f(t)$ can still be represented as a Fourier series with period T. For this, let

$$\tau = \frac{2\pi t}{T} \tag{7}$$

Clearly, $\tau\big|_{t=\alpha} = \alpha' = \frac{2\pi\alpha}{T}$, $\tau\big|_{t=\beta} = \beta' = \frac{2\pi\beta}{T}$

and $$\beta' - \alpha' = \frac{2\pi(\beta-\alpha)}{T} = 2\pi$$

so that we may apply the ~~aforementioned~~ previous case:

$$f(t) \equiv f\left(\frac{T}{2\pi}\tau\right) \equiv F(\tau) = a_0 + \sum_{n=1}^{\infty}(a_n\cos n\tau + b_n \sin n\tau) \tag{8}$$

with

$$a_0 = \frac{1}{2\pi}\int_{\alpha'}^{\beta'} F(\tau)\,d\tau , \quad a_n = \frac{1}{\pi}\int_{\alpha'}^{\beta'} F(\tau)\cos n\tau\,d\tau$$

and $$b_n = \frac{1}{\pi}\int_{\alpha'}^{\beta'} F(\tau)\sin n\tau\,d\tau \tag{9}$$

or

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2\pi n}{T}t + b_n \sin\frac{2\pi n}{T}t\right) \tag{10}$$

with

$$a_0 = \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} f(t)\,dt , \quad a_n = \frac{2}{\beta-\alpha}\int_{\alpha}^{\beta} f(t)\cos\frac{2\pi n}{T}t\,dt,$$

$$b_n = \frac{2}{\beta-\alpha}\int_{\alpha}^{\beta} f(t)\sin\frac{2\pi n}{T}t\,dt \tag{11}$$

I-4 Approximation to a function by a finite trigonometric series.

The practical problems in geomagnetism require the approximation of a given function by a finite trigonometric series rather than ~~to~~ the exact representation by the corresponding infinite Fourier series.

Let $f(t)$ be a function which satisfies the conditions of Fourier expansion in the interval $[0, 2\pi]$, and let

$$\psi_k(t) = a_0 + \sum_{n=1}^{k} (a_n \cos nt + b_n \sin nt) \qquad (12)$$

where k is a given integer. It is required to determine the coefficients a_0; a_n & b_n in (12) so that $\psi_k(t)$ be the best mean square fit to $f(t)$.

Let
$$u^2 = \frac{1}{2\pi}\int_0^{2\pi}[f(t)-\psi_k(t)]^2\, dt \qquad (13)$$

From previous results, we have ~~we shall evaluate the different terms in the r.h.s. of (13)~~

$$\frac{1}{2\pi}\int_0^{2\pi}\psi_k^2(t)\, dt = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

~~(from previous results)~~

Therefore,

$$u^2 = \frac{1}{2\pi}\int_0^{2\pi} f^2(t)\,dt - \frac{a_0}{\pi}\int_0 f(t)\,dt - \frac{1}{\pi}\sum_{n=1}^{k}\int_0^{2\pi}(a_n\cos nt + b_n \sin nt)f(t)\, dt + a_0^2 + \frac{1}{2}\sum_{n=1}^{k}(a_n^2+b_n^2) \qquad (14)$$

The conditions for minimum of u^2 as a function of a_0, a_n & b_n are:

$$\frac{\partial u^2}{\partial a_0} = \frac{\partial u^2}{\partial a_n} = \frac{\partial u^2}{\partial b_n} = 0 \quad , n=1,2,\dots k \qquad (15)$$

The condition $\frac{\partial u^2}{\partial a_0}=0$ yields

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)\, dt .$$

Similarly, the other conditions for minimum give

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos nt\, dt , \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin nt\, dt. \quad n=1,2,\dots,k.$$

These results mean that the coefficients a_0, a_n & b_n realizing the best mean square fit to $f(t)$ are just the Fourier coefficients of function $f(t)$.

Hence if k & k' are two given integers ~~the~~ and $k' > k$, the first k terms in the function $\psi_{k'}(t)$ realizing the best mean square fit to $f(t)$ are just those forming $\psi_k(t)$.

This is a special case of approximation of functions by a set of orthogonal functions.

I-5 Harmonic Analysis of a set of equidistant values of a function.

When the function $f(t)$ defined on the interval $a \le t \le b$ is not completely specified but is only given as a set of values at equidistant points in the range of definition, then the integrations giving the values of Fourier coefficients may be evaluated numerically by using known methods.

~~The following consideration leads~~

Equivalent results may also be obtained by defining u in the following way:

$$u^2 = \frac{1}{r} \sum_{s=1}^{r} \{ y_s - \Psi_k(t_s) \}^2 , \quad (16)$$

where the range $[0, 2\pi]$ is divided into r equal intervals by the points $t_s = 2\pi s / r$, $s = 0, 1, 2, \ldots, r$, and y_s ~~is~~ the values of $f(t)$ at points t_s.

The result may now be obtained by writing down the conditions for the minimum of u^2 as a function of the coefficients a_0, a_n, b_n $(n = 1, 2, \ldots, k)$.

I-6 Fourier Integral theorem.

Integral transforms are very useful tools in solving problems of Mathematical Physics. These include the Laplace transform, the Fourier transform, the Hankel transform and many others.

All these transforms are based on the relations between the function on which the transform is to be applied and its expansion as series of special forms.

We shall concern ourselves in what follows to the Fourier transform.

Suppose $f(x)$ is a periodic function of period $2\pi\lambda$ defined in the range $-\pi\lambda \le x \le \pi\lambda$. Subsequently, we shall take the limit as $\lambda \to \infty$.

Fourier Series for $f(x)$ is:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{nx}{\lambda} + b_n \sin \frac{nx}{\lambda} \right), \quad (17)$$

where coefficients a_0, a_n & b_n are given by the formulae:

$$2\pi\lambda a_0 = \int_{-\pi\lambda}^{+\pi\lambda} f(x')\,dx' \,, \qquad \pi\lambda a_n = \int_{-\pi\lambda}^{+\pi\lambda} f(x') \cos\frac{\pi x'}{\lambda}\,dx' ,$$

$$\pi\lambda b_n = \int_{-\pi\lambda}^{+\pi\lambda} f(x') \sin\frac{\pi x'}{\lambda}\,dx' .$$

Substituting these expressions in (17) and using elementary trigonometric identities we get:

$$f(x) = \frac{1}{2\pi\lambda} \int_{-\pi\lambda}^{+\pi\lambda} f(x')\,dx' + \frac{1}{\pi\lambda} \sum_{n=1}^{\infty} \int_{-\pi\lambda}^{+\pi\lambda} f(x') \cos\frac{n(\pi - x')}{\lambda}\,dx' . \qquad (18)$$

If we write $\alpha = \frac{n}{\lambda}$ and

$$\Delta\alpha = \frac{n+1}{\lambda} - \frac{n}{\lambda} = \frac{1}{\lambda}$$

equation (18) yields

$$\pi f(x) = \sum_{n=1}^{\infty} \left\{ \int_{-\pi\lambda}^{+\pi\lambda} f(x') \cos\alpha(\pi - x')\,dx' \right\} \Delta\alpha$$

On going to the limit $\lambda \to \infty$, the summation is replaced by an integration and we get:

$$\pi f(x) = \int_0^{\infty} d\alpha \int_{-\infty}^{+\infty} f(x') \cos\alpha(\pi - x')\,dx' \qquad (19)$$

This is the Fourier integral theorem. (Clearly, $f(x)$ should satisfy certain conditions for the integral in (19) to exist).

Formula (19) may also be written as follows:

$$\pi f(x) = \int_0^{\infty} \left[\int_{-\infty}^{+\infty} f(x') \cos\alpha x'\,dx' \right] \cos\alpha x\,d\alpha + \int_0^{\infty} \left[\int_{-\infty}^{+\infty} f(x') \sin\alpha x'\,dx' \right] \sin\alpha x\,d\alpha \qquad (20)$$

Special cases:

(i) Let $f(x)$ be an even function of x. Then the second integral in (20) vanishes. We get:

$$\left.\begin{aligned} \frac{\pi}{2} f(x) &= \int_0^{\infty} \overline{f(\alpha)} \cos\alpha x\,d\alpha , \\ \text{where} \quad \bar{f}(\alpha) &= \int_0^{\infty} f(x') \cos\alpha x'\,dx' \end{aligned}\right\} \qquad (21)$$

$\bar{f}(\alpha)$ is called Fourier transform of $f(x)$. If $\bar{f}(\alpha)$ is known, then $f(x)$ may be calculated from the first of equations (21).

(ii) Let $f(x)$ be an odd function of x. Then

$$\left.\begin{aligned} \frac{\pi}{2} f(x) &= \int_0^{\infty} \overline{f(\alpha)} \sin\alpha x\,d\alpha , \\ \text{where} \quad \bar{f}(\alpha) &= \int_0^{\infty} f(x') \sin\alpha x'\,dx' . \end{aligned}\right\} \qquad (22)$$

<u>Example</u> : Let $f(x) = e^{-x}$

then

$$\bar{F}(t) = \int_0^\infty e^{-x} \begin{Bmatrix} \cos xt \\ \sin xt \end{Bmatrix} dx = \begin{Bmatrix} 1/(1+t^2 \\ t/(1+t^2) \end{Bmatrix}$$

and the inverse transformation yields

$$e^{-x} = \frac{2}{\pi} \int_0^\infty \frac{t}{1+t^2} \sin xt \, dt$$

$$= \frac{2}{\pi} \int_0^\infty \frac{1}{1+t^2} \cos xt \, dt$$