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AUTUMN COURSE ON GEOMAGNETISM, THE IONOSPHERE
AND MAGNETOSPHERE

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IONOSPHERE

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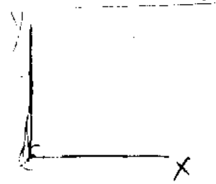
Derivation of the magneto-ionic equation (1)



We consider an e.m. wave propagating with velocity V along the z-axis i.e. with the wave normal along OZ. We suppose that all components of the wave field $\underset{\sim}{E}$ (components $E_{x,y,z}$) $\underset{\sim}{H}$ (components $H_{x,y,z}$) vary with time & distance along the z axis as $e^{i\frac{2\pi}{\lambda}(Vt - z)}$ or $e^{i(\omega t - nkz)}$

$\beta = \frac{\omega}{c}$ $n = \frac{c}{V}$

λ = wavelength

If ω is the angular frequency of the wave $\omega = 2\pi f = \frac{2\pi V}{\lambda}$

For such a wave, the ratio (R) of any two components such as $\frac{E_x}{E_y}$, $\frac{E_y}{E_z}$, $\frac{H_x}{H_y}$ or $\frac{H_x}{H_z}$ (ratios which determine the state of 'polarization' of the wave) — these ratios will be constant for any given value of z. [E_x & E_y will of course each change with z but they change in the same way so that the ratio remains constant] This means that such waves travel through the medium with its polarization unchanged and they are termed 'characteristic waves'. In an isotropic medium the ratio R can have any value but in an anisotropic medium (such as the ionosphere with a superposed magnetic field) only certain discrete values are possible.

We first examine some of the properties of these 'characteristic waves' and determine an expression for the refractive index (or refractive indices) of them.

We note that if t and z vary as stated above then

$$\frac{\partial}{\partial t} = i\frac{2\pi V}{\lambda} = i\omega \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0 \ \& \ \frac{\partial}{\partial z} = -\frac{i\omega}{V}$$

Maxwell's equations in vector form are

$$\text{curl } \underset{\sim}{H} = \dot{\underset{\sim}{D}} = \epsilon_0 \dot{\underset{\sim}{E}} + \dot{\underset{\sim}{P}}$$

$$\text{curl } \underset{\sim}{E} = -\mu_0 \dot{\underset{\sim}{H}} = -\frac{\partial \underset{\sim}{B}}{\partial t} \text{ or } \dot{\underset{\sim}{B}}$$

Here D is the electric displacement of the wave ϵ_0, μ_0 permittivity & permeability of free space

P " " " polarization " " medium

B " " magnetic induction " $\underset{\sim}{B} = \mu_0 \underset{\sim}{H}$

If we write down the x, y & z components of the curls and apply the conditions $\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$ we then Maxwell's equations simplify (for the characteristic wave) to

$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}$ $\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}$ $\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}$

$$\left.\begin{aligned} &\text{(i)} & \omega[\epsilon_0 E_x + P_x] &= \frac{\omega}{V} H_y \\ &\text{(ii)} & \omega[\epsilon_0 E_y + P_y] &= -\frac{\omega}{V} H_x \\ &\text{(iii)} & \omega[\epsilon_0 E_z + P_z] &= 0 \quad \text{i.e. } D_z = 0 \end{aligned}\right\} \text{ from the curl H equation}$$

$$\left.\begin{aligned} &\text{(iv)} & \omega\mu_0 H_x &= -\frac{\omega}{V} E_y \\ &\text{(v)} & \omega\mu_0 H_y &= \frac{\omega}{V} E_x \\ &\text{(vi)} & \omega\mu_0 H_z &= 0. \end{aligned}\right\} \text{ from the curl E equation}$$

From these six equations we conclude

(a) $H_z = 0$ means that the magnetic field of the wave is entirely in the wave front.

(b) E_z is ~~not~~ necessarily zero i.e. there may be* a component of the electric field in the direction of the wave-normal

(c) $D_z = 0$ means that D lies entirely in the wavefront.

*we now or ... fundamental ... what ... property ... the ... paper

If we combine (i) with (v) and (ii) with (iv) we get

$$\left(1 + \frac{P_x}{\epsilon_0 E_x}\right) = \frac{H_y}{V\epsilon_0 E_x} = \frac{1}{\epsilon_0\mu_0 V^2} = \left(1 + \frac{P_y}{\epsilon_0 E_y}\right)$$

$$\text{i.e. } \frac{P_x}{E_x} = \frac{P_y}{E_y} \qquad \text{(vii)}$$

$$\& \text{ from (iv) \& (v) } \frac{H_x}{H_y} = -\frac{E_y}{E_x} \qquad \text{(viii)}$$

By definition refractive index $n = c/V$ where c is the free-space velocity & $\left(= \frac{1}{\sqrt{\epsilon_0\mu_0}}\right)$ so that $\frac{1}{\epsilon_0\mu_0 V^2} = \frac{c^2}{V^2} = n^2$.

$$\text{Hence } \quad n^2 = 1 + \frac{P_x}{\epsilon_0 E_x} \quad \text{or} \quad 1 + \frac{P_y}{\epsilon_0 E_y} \qquad \text{(ix)}$$

In the case of an ionised medium without any external magnetic field we find

$$\underline{P} = Ne\underline{r} = \frac{-Ne^2}{m\omega^2}\underline{E}$$

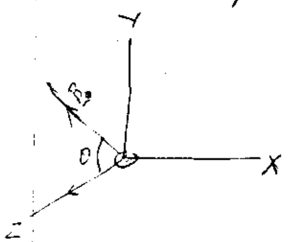
$\overline{P}$ (the electric polarisation of the medium) and $\underline{E}$ (the electric intensity of the w.) are vectors each with components $P_{x,y,z}$ $E_{x,y,z}$

Let us now see what the corresponding relation between $\underline{P}$ and $\underline{E}$ is in the case where we do have an external applied magnetic field.

Let $\underset{\sim}{B_0}$ be the induction of the constant superimposed field

$\underset{\sim}{B_0} = \mu_0 \underset{\sim}{H_0}$ where $\underset{\sim}{H_0}$ is the intensity of the field.

We assume that this external field B_0 (the earth's magnetic field) is in the plane OYZ and makes an angle θ with the Z axis (which is the direction of propagation of the w.)



Let the components of $\underset{\sim}{B_0}$ along the y and z directions be B_0^T & B_0^L respectively so that $B_0^T = B_0 \sin\theta$ $\quad B_0^L = B_0 \cos\theta$

When there is no external magnetic field the effect of a passing e.m. wave is to move the electrons in the direction of the electric vector of the wave. However when there is a superimposed magnetic field the motion is more complicated.

In general a charge e moving with velocity $\underline{v}$ in a magnetic field of induction B_0 is subject to a force $B_0 e v$ which is directed orthogonally to the directions of B_0 & of v i.e. we write the force as $e\,\underset{\sim}{B_0} \wedge \underset{\sim}{v}$ where the sign $\wedge$ stands for the vector product of $\underset{\sim}{B_0}$ and $\underline{v}$.

Consider now what forces operate on a particle with charge e in the three coordinate directions as a result of this magnetic field in the OYZ plane with a component B_0^T in the Y direction and B_0^L in the Z direction.

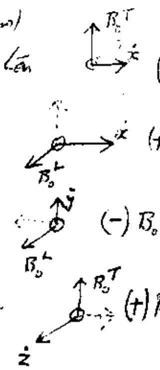
(shown by dotted arrows)

$\dot{x}$ and B_0^T give a force in Z direction $\quad (-)B_0^T e\dot{x}$

$\dot{x}$ and B_0^L . . . . y . . . $\quad (+)B_0^L e\dot{x}$

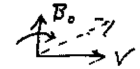
$\dot{y}$ and B_0^L . . . . x . . . $\quad (-)B_0^L e\dot{y}$

$\dot{z}$ and B_0^T . . . . x . . . $\quad (+)B_0^T e\dot{z}$



i.e. the external magnetic field in the plane OYZ (i.e. with components in the Y & Z directions only) gives rise to one force term in the y & z direction and two force terms in the x direction.

If we adopt the sign convention that this arrangement



is called a positive force (shown dotted) i.e. (a right-hand screw of $\underset{\sim}{B_0}$ on $\underline{V}$) is positive in the direction of the dotted arrows then the four forces shown above will have the signs indicated.

The general equation of motion of a charged particle (we are usually concerned with electrons)

$$m\ddot{\underset{\sim}{r}} + m\nu\dot{\underset{\sim}{r}} = e\underset{\sim}{E} + e\dot{\underset{\sim}{r}} \wedge \underset{\sim}{B_0} \qquad (x)$$

e, m charge & mass of electron

The first term is the mass × acceleration term

— second — the term introduced to represent loss of energy due to collisions of electrons (at frequency ν) with heavy particles

$e\underset{\sim}{E}$ is the force term due to the electric vector of the wave and the last term is the one we have just been discussing due to the external magnetic field B_0

In the 3 coordinate directions (x) takes the form

$$eE_x = m\ddot{x} + m\nu\dot{x} + B_0^L e\dot{y} - B_0^T e\dot{z} \qquad (xi)$$
$$eE_y = m\ddot{y} + m\nu\dot{y} - B_0^L e\dot{x} \qquad (xii)$$
$$eE_z = m\ddot{z} + m\nu\dot{z} + B_0^T e\dot{x} \qquad (xiii)$$

Now $\underline{P} = Ne\underline{r}$ and the electric polarisation of the medium will also vary with time as $P = P_0 e^{i\omega t}$
$P_x = P_0^x e^{i\omega t}$ etc.

Hence $\ddot{x} = \frac{\ddot{P}_x}{Ne} = -\frac{\omega^2 P_x}{Ne}$ with similar expressions for $\ddot{y}$ and $\ddot{z}$

$\dot{x} = \frac{\dot{P}_x}{Ne} = \frac{i\omega P_x}{Ne}$ " " " " $\dot{y}$ and $\dot{z}$

Substituting for $\ddot{x}, \dot{x}, \ddot{y}, \dot{y}$ etc in the above gives.

$$eE_x = -\frac{m\omega^2}{Ne}P_x + \frac{i\omega m\nu}{Ne}P_x + eB_0^L\frac{i\omega P_y}{Ne} - eB_0^T\frac{i\omega P_z}{Ne} \qquad (xiv)$$

or $$eE_x = -\frac{m\omega^2 P_x}{Ne}\left[1 - iZ\right] + eB_0^L\frac{i\omega P_y}{Ne} - eB_0^T\frac{i\omega P_z}{Ne} \qquad (xv)$$

where we have set $Z = \frac{\nu}{\omega}$.

~~ie $eE_x = -m\omega^2 P_x U$~~ If we set $1 - iZ = U$ the equations (xi)–(xiii) above become

$$eE_x = -\frac{m\omega^2}{Ne}UP_x + i\omega\frac{eB_0^L}{Ne}P_y - i\omega\frac{eB_0^T}{Ne}P_z \qquad (xvi)$$
$$eE_y = -\frac{m\omega^2}{Ne}UP_y - i\omega\frac{eB_0^L}{Ne}P_x \qquad (xvii)$$
$$eE_z = -\frac{m\omega^2}{Ne}UP_z + i\omega\frac{eB_0^T}{Ne}P_x \qquad (xviii)$$

If we write $X = \frac{Ne^2}{\epsilon_0 m\omega^2}$ $\quad \underset{\sim}{Y} = \frac{\omega_B}{\omega}$ $\quad Y_L = \frac{\omega_B\cos\theta}{\omega}$ $\quad Y_T = \frac{\omega_B\sin\theta}{\omega}$

where $\omega_B = \frac{eB_0}{m}$ and is the so-called 'gyro frequency' then

after a bit of rearrangement the three expressions (xvi–xviii) become

$$\epsilon_0 X E_x = -UP_x - iY_T P_z + iY_L P_y \qquad (xix)$$
$$\epsilon_0 X E_y = -UP_y - iY_L P_x \qquad (xx)$$
$$\epsilon_0 X E_z = -UP_z + iY_T P_x \qquad (xxi)$$

These 3 expressions relate the components of the polarisation $\underline{P}$ and the components of electric field $\underset{\sim}{E}$ when a magnetic field is present.

[If there is no superimposed magnetic field then $Y_L = Y_T = 0$ and we get simply $\epsilon_0 X E = -UP$ and if there are no collisions $U = 1$ and $\epsilon_0 X E = -P$]

We now consider the derivation of the refractive indices of the characteristic waves propagated in this magneto ionic medium.

We have seen that the characteristic waves must satisfy the condition $\frac{P_x}{P_y} = \frac{E_x}{E_y}$

and we now do a little algebraic manipulation on the expressions we have derived to get a quadratic expression for $\frac{P_x}{P_y}$

Equation (iii) gives $E_z = \frac{-P_z}{\epsilon_0}$ & we substitute this expression for E_z in (xxi) above to give

$$(U - X)P_z = iY_T P_x$$
or $$P_z = \frac{iY_T P_x}{U - X}$$

Next Substitute this expression for P_z in (xix) above to give

$$\epsilon_0 X\frac{E_x}{P_x} = -U + \frac{\cancel{i}Y_T^2}{U - X} + iY_L\left(\frac{P_y}{P_x}\right)$$

We also have equation (xx) above
$$\epsilon_0 X\frac{E_y}{P_y} = -U - iY_L\left(\frac{P_x}{P_y}\right)$$

But for the characteristic waves $\frac{E_x}{P_x} = \frac{E_y}{P_y}$ and the LHS of the two expressions immediately above are equal

Let $R = \frac{P_x}{P_y}$

Then $$\frac{Y_T^2}{U-X} + \frac{iY_L}{R} = -iY_L R.$$

or $$R^2 - iR\left[\frac{Y_T^2}{Y_L(U-X)}\right] + 1 = 0 \qquad \text{(xxii)}$$

The R determines the polarisation of the characteristic waves and this quadratic equation in R is called the "wave polarisation equation" and it shows that R can have one of only two possible values.

The solutions of (xxii) are

$$R\left(= \frac{P_x}{P_y}\right) = \frac{iY_T^2}{2Y_L(U-X)} \pm i\left\{1 + \frac{Y_T^4}{4Y_L^2(U-X)^2}\right\}^{1/2} \qquad \text{(xxiii)}$$

We have seen earlier on that for these waves in a magnetoionic medium the magnetic vector is entirely in the wavefront (whereas the electric vector is not) and the 'polarisation of the wave' is usually defined in terms of the magnetic force rather than the electric force. Hence we define the polarisation of the wave

$$R \text{ as } \left(\frac{P_x}{P_y}\right) = -\frac{H_y}{H_x} \quad \text{(see equations vii + viii)}$$

If R is real H_y and H_x have the same phases and we have the straightforward numerical ratio of H_y to H_x & the waves will be linearly polarised waves

If R is complex then we have elliptically polarised waves

If $R = \pm i$ — as is the case for longitudinal propagation ($Y_T = 0$) then H_y + H_x will be in phase quadrature and we have the circularly polarised waves with opposite senses of rotation

If $R = 0$ then $H_y = 0$ and the wave is plane polarised with its magnetic vector entirely in the x direction i.e. perpendicular to the superimposed magnetic field.

If $R = \infty$ then $H_x = 0$ and the wave is plane polarised with its magnetic vector entirely in the plane of the magnetic field [plane OYZ]

R=0 and R=∞ occur for transverse propagation ($Y_L = 0$)

If we denote the two roots of the quadratic (xxii) by R_O and R_X then we note that $R_O \cdot R_X = 1$

Note that the expressions for R involve X (which is proportional to electron density N). Hence the state of polarisation changes if the electron density changes as the wave progresses through the plasma. [Earlier on when we called characteristic waves with the polarisation not changing we were then thinking in terms of a fixed value of X].

Expression for the refractive index n.

We had $n^2 = 1 + \frac{P_x}{\epsilon_0 E_x}$ $\left(\text{or } 1 + \frac{P_y}{\epsilon_0 E_y}\right)$ (eq ix)

also $R = \frac{P_x}{P_y}$ and $\frac{\epsilon_0 X E_y}{P_y} = -U - \frac{iY_L P_x}{P_y}$ (eq xx)

$= -U - iY_L R$

Hence $n^2 = 1 - \frac{X}{U + iY_L R}$ (xxiv) which, when we substitute for R gives

$$n^2 = 1 - \frac{X(U-X)}{U(U-X) - \frac{1}{2}Y_T^2 \pm \left\{Y_L^2(U-X)^2 + \frac{1}{4}Y_T^4\right\}^{1/2}} \qquad \text{(xxv)}$$

— the Appleton-Hartree Magnetoionic formula.

If collisional effects are omitted (U=1) the equation takes the form

$$n^2 = 1 - \frac{1-X}{1-X-\frac{1}{2}Y_T^2 \pm \left\{Y_L^2(1-X)^2 + \frac{1}{4}Y_T^4\right\}^{1/2}} \qquad \text{(xxvi)}$$

If magnetic field effects are also omitted then $Y_T = Y_L = 0$ & the expression becomes $n^2 = 1 - X$ – the Eccles expression derived earlier

The refractive index is now a complex quantity and we may write it as $\mu - i\chi$ where μ is the real part and χ the imaginary part (which is related to the absorption of energy from wave by the medium.) We can relate χ to this absorption as follows:

~~Let us~~ We represented our wave by $e^{i\frac{2\pi}{\lambda}(vt-z)}$

Now $\frac{2\pi}{\lambda} = \frac{\omega}{v}$ and the refractive index $n = \frac{c}{v}$

Hence $$e^{i\frac{2\pi}{\lambda}(vt-z)} = e^{i\frac{\omega}{v}(vt-z)} = e^{i\omega\left(t-\frac{nz}{c}\right)} = e^{i\omega\left(t-\frac{(\mu-i\chi)z}{c}\right)}$$

$$= e^{i\omega\left(t-\frac{\mu z}{c}\right)} e^{-\frac{\omega\chi}{c}.z}$$

See that we now have a 'loss' term here which involves χ

If now we define an absorption coefficient, K (kappa) by assuming that ~~Suppose~~ the amplitude of the wave diminishes by a factor of e^{-K} (K-kappa) in unit distance [or in $e^{-K.s}$ over a distance s for which K may be assumed constant] $K = \frac{\omega\chi}{c}$

$\frac{\omega}{c} = \frac{2\pi f}{f\lambda} = \frac{2\pi}{\lambda}$ Hence $\chi = \frac{\lambda}{2\pi}.K$

and may be said to be a measure of the absorption suffered in a path length $\lambda/2\pi$. This gives physical meaning to χ

Hence $n = \mu - i\chi = \mu - i\frac{cK}{\omega}$

Equation (xxv) is the so-called Appleton-Hartree formula for the refractive index of an ionised medium — a 'plasma' — in the presence of a superimposed magnetic field. It gives the variation of n with frequency f for any ~~given value of~~ direction of propagation relative to the applied magnetic field for given electron density (N) and collisional frequency ν. Remember $X = \frac{Ne^2}{\epsilon_0 m\omega^2}$ & involves electron density N & frequency $X \propto N/f^2$

" $Z = \frac{\nu}{\omega}$ & involves collisional frequency ν & frequency

" $\left\{\begin{array}{l} Y_L = \frac{eB_0\cos\theta}{m\omega} \\ Y_T = \frac{eB_0\sin\theta}{m\omega} \end{array}\right\}$ ~~involve~~ and involve the magnetic field, the direction of propagation and the frequency

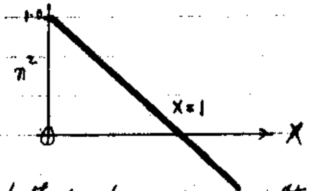
It is instructive to ~~examine~~ consider the variation of n^2 with X & I will do this for one or two simple cases.

I Consider first the simplest possible case — in which we neglect the magnetic field and the effect of collisions i.e. we set $Y_L = Y_T = 0$ & $Z = 0$.

Then $U (= 1 - iZ) = 1$ and (xxv) becomes

$$n^2 (= \mu^2) = 1 - X$$ [the Eccles formula we deduced earlier]

Plotting n^2 against X we get



$X = \frac{Ne^2}{\epsilon_0 m\omega^2} = \frac{Ne^2}{4\pi^2\epsilon_0 m f^2}$

V.G. 7

When X = 1 n = 0 — and the significance of n = 0 is that it is a condition for reflection of the signal.