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AUTUMN COURSE ON GEOMAGNETISM, THE IONOSPHERE AND MAGNETOSPHERE

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MATHEMATICS (cont.)

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II - Harmonic Analysis.

II-1 Curvilinear Coordinates.

Let (x,y,z) denote Cartesian coordinates, and

$$u_1(x,y,z)\ ,\ u_2(x,y,z)\ ,\ u_3(x,y,z)$$

three continuous functions of coordinates. We shall suppose that the three equations

$$u_1 = u_1(x,y,z)\ ,\ u_2 = u_2(x,y,z)\ ,\ u_3 = u_3(x,y,z) \qquad (1)$$

establish a 1-1 correspondence between the two tuples (x,y,z) and (u_1,u_2,u_3), or, equivalently, that these equations may be solved for x,y,z :

$$x = x(u_1,u_2,u_3)\ ,\ y = y(u_1,u_2,u_3)\ ,\ z = z(u_1,u_2,u_3) \qquad (2)$$

uniquely. The 1-1 correspondence may not hold only at a finite number of points of space.

Equations (1) define a system of coordinates (u_1,u_2,u_3) in space, called curvilinear coordinates.

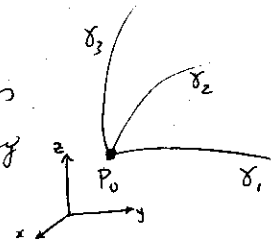
The equation

$$u_i(x,y,z) = \text{Const}\ ,\ i=1,2,3 \qquad (3)$$

defines a family of surfaces (the const. assuming different values). These are the coordinate surfaces in the induced system of curvilinear coordinates.

Each point P of space may now be characterized by its coordinates (x,y,z) or (u_1,u_2,u_3).

Through each point of space, say $P_0 \equiv (u_{10},u_{20},u_{30})$, there pass three coordinate lines $\gamma_1,\gamma_2,\gamma_3$ along each of which only one of the coordinates u_1,u_2,u_3 vary, the other two remaining constant. The equations (parametric) of these coordinate lines through P_0 may be written as:



$$\gamma_1 :\ x = x(u_1,u_{20},u_{30})\ ,\ y = y(u_1,u_{20},u_{30})\ ,\ z = z(u_1,u_{20},u_{30})$$

$$\gamma_2 :\ x = x(u_{10},u_2,u_{30})\ ,\ y = y(u_{10},u_2,u_{30})\ ,\ z = z(u_{10},u_2,u_{30})$$

$$\gamma_3 :\ x = x(u_{10},u_{20},u_3)\ ,\ y = y(u_{10},u_{20},u_3)\ ,\ z = z(u_{10},u_{20},u_3)$$

If, at each point of space, the coordinate lines are mutually orthogonal, the system of curvilinear coordinates is said to be a ~~orthogonal~~ system of orthogonal curvilinear coordinates.

From the definition, it follows that any two coordinate surfaces cut in a coordinate line.

Consider now a point $P \equiv (u_1, u_2, u_3)$ and let $A_1 \equiv (u_1+du_1, u_2, u_3)$, $A_2 \equiv (u_1, u_2+du_2, u_3)$ & $A_3 \equiv (u_1, u_2, u_3+du_3)$ be three neighbouring points lying on the coordinate lines through P, in the directions of increase of u_1, u_2, u_3.

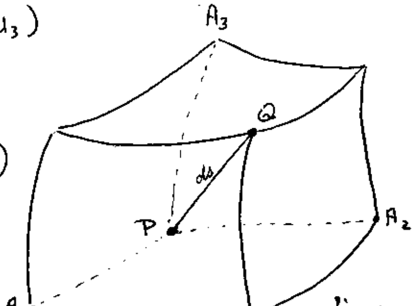


fig. 1

Complete the curvilinear parallellopiped by drawing the coordinate surfaces through P, A_1, A_2 & A_3 and let Q be the pt. diagonally opposite to P. Clearly, $Q \equiv (u_1+du_1, u_2+du_2, u_3+du_3)$

The elements of length PA_1, PA_2 and PA_3 are not simply du_1, du_2 & du_3 respectively, since some of the u_i may have dimension different from length. In general, we suppose that

$$PA_1 = h_1\, du_1 \quad , PA_2 = h_2\, du_2 \quad , PA_3 = h_3\, du_3 \quad ,$$

where h_1, h_2 and h_3 are functions of the coordinates.

For orthogonal curvilinear coordinates, the element of length $ds \equiv PQ$ is given by

$$ds^2 = h_1^2\, du_1^2 + h_2^2\, du_2^2 + h_3^2\, du_3^2$$
$$= dx^2 + dy^2 + dz^2 \qquad (4)$$

(since $h_1 = h_2 = h_3 = 1$ for cartesian coordinates)

For systems of non-orthogonal coordinates, the expression for ds^2 may include cross-terms $du_i\, du_j$ $(i \neq j)$.
We assume ~~Clearly~~, h_1, h_2 & h_3 to be ~~are~~ positive quantities.

II-2 The gradient, divergence and curl operators.

These are differential operators acting on scalar or vector functions of position. It is therefore normal to assume some differentiability properties for these functions. As a rule, we shall suppose that the function is as many times differentiable as needed by the operator applied to it.

The gradient. The gradient of a function of position φ at a certain point P of space is defined to be the vector whose direction is that of maximum rate of increase of φ (at P) and whose magnitude is equal to this rate of increase.

The gradient of φ will be denoted $\text{grad}_P\, \varphi$ (or $\nabla_P \varphi$), where subscript P refers to the point of space where $\nabla\varphi$ is to be calculated. This subscript will sometimes be omitted for simplicity.

From this definition of the gradient it follows that:

(i). The vector $\nabla_P \varphi$ is perpendicular to the surface $\varphi = \text{const}$ passing through P. ~~and is directed so~~ ~~as to~~

(ii) If $\underline{u}$ denotes the unit vector along any direction through P, then the rate of change of φ along that direction is given by $(\text{grad}_P \varphi) \cdot \underline{u}$, where the "dot" denotes scalar product.

The divergence. The divergence of a vector function $\underline{A}$ at point P is defined as follows:

$$\text{div}_P \underline{A} \equiv \lim_{\substack{\Delta V \to 0 \\ \text{at } P}} \frac{1}{\Delta V} \iiint_{\Delta S} \underline{A} \cdot d\underline{\sigma} \qquad (5)$$

where ΔV is the volume of the parallellopiped in fig. 1, ΔS its bounding surface, and the limit is performed so that P be always ~~at the corner~~ a vertex. The integral in the r.h.s. of (5) is the scalar surface integral of vector $\underline{A}$ over surface ΔS.

The curl. The curl of a vector function $\underline{A}$ at point P is defined as follows:

$$\text{curl}_P \underline{A} \equiv \lim_{\substack{\Delta V \to 0 \\ \text{at } P}} \frac{1}{\Delta V} \iint_{\Delta S} \underline{A} \wedge d\underline{\sigma} \qquad (6)$$

where "$\wedge$" denotes the vector product and the integral in the r.h.s. of (6) is the vector surface integral of vector $\underline{A}$ over ΔS.

II-3 Laplace's equation.

Before considering Laplace's equation, we define the following two differential operators:

(i) Laplace's operator (acting on a scalar function):

$$\nabla^2 \varphi \equiv \text{div}(\text{grad}\, \varphi) \qquad (7)$$

(ii) Laplace's operator (acting on a vector function):

$$\nabla^2 \underline{A} \equiv \text{grad}(\text{div}\, \underline{A}) - \text{curl}(\text{curl}\, \underline{A}) \qquad (8)$$

In cartesian coordinates (and only in these coordinates) the following relation holds.

$$\nabla^2 \underline{A} = \nabla^2 (A_x \underline{i} + A_y \underline{j} + A_z \underline{k}) =$$
$$= (\nabla^2 A_x)\,\underline{i} + (\nabla^2 A_y)\,\underline{j} + (\nabla^2 A_z)\,\underline{k} .$$

In what follows, we give the expressions of grad, div, curl & ∇^2 in orthogonal curvilinear coordinates. (The components of a vector $\underline{A}$ will be denoted by A_1, A_2, A_3):

$$\text{grad}\,\varphi \equiv \left(\frac{1}{h_1}\frac{\partial \varphi}{\partial u_1} , \frac{1}{h_2}\frac{\partial \varphi}{\partial u_2} , \frac{1}{h_3}\frac{\partial \varphi}{\partial u_3} \right) \qquad (9)$$

$$\text{div}\,\underline{A} \equiv \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_3 h_1 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)\right] \qquad (10)$$

$$\text{curl}\,\underline{A} \equiv \left\{ \frac{1}{h_2 h_3}\left[\frac{\partial (h_3 A_3)}{\partial u_2} - \frac{\partial (h_2 A_2)}{\partial u_3}\right] , \frac{1}{h_3 h_1}\left[\frac{\partial (h_1 A_1)}{\partial u_3} - \frac{\partial (h_3 A_3)}{\partial u_1}\right] , \right.$$
$$\left. \frac{1}{h_1 h_2}\left[\frac{\partial (h_2 A_2)}{\partial u_1} - \frac{\partial}{\partial u_2}(h_1 A_1)\right] \right\} \qquad (11)$$

$$\nabla^2 \varphi \equiv \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \varphi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial \varphi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \varphi}{\partial u_3}\right)\right] \qquad (12)$$

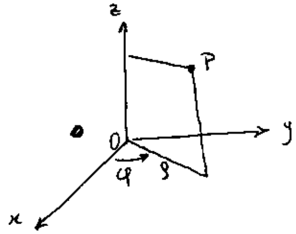
Operator $\nabla^2 \underline{A}$ may easily be obtained from the definition and (9) – (11).

Special cases.

(i). Cartesian coordinates (x, y, z).

$$ds^2 = dx^2 + dy^2 + dz^2 .$$
$$h_1 = h_2 = h_3 = 1.$$

(ii) Cylindrical coordinates (ρ, φ, z).

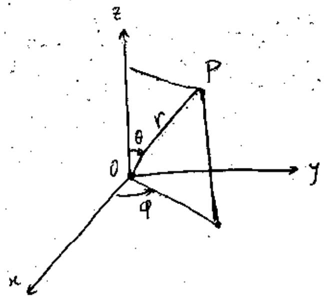


$$ds^2 = d\rho^2 + \rho^2 d\varphi^2 + dz^2 .$$

Here, $h_1 = 1$, $h_2 = \rho$, $h_3 = 1$.

$$\nabla^2 \Psi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Psi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Psi}{\partial \varphi^2} + \frac{\partial^2 \Psi}{\partial z^2}$$

(iii) Spherical polar coordinates (r, θ, φ).



$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$$

$$h_1 = 1, \quad h_2 = r, \quad h_3 = r\sin\theta$$

$$\nabla^2 \Psi = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Psi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Psi}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 \Psi}{\partial \varphi^2}\right]$$

<u>Laplace's equation</u> $\nabla^2 V = 0$ (or $\nabla^2 \underline{A} = 0$) plays an important role in the theory of potential in general, and in Geomagnetism in particular. It is satisfied by the electrostatic ~~field~~ potential in a uniform uncharged dielectric, by the magnetic scalar and vector potentials in non-magnetic media and in media where no electric current flows, by the velocity potential of ideal, incompressible fluid flow, etc.

<u>Laplace's equation in two-dimensions</u> If the function satisfying Laplace's equation is independent of the cartesian coordinate z (say,) the problem is said to be two-dimensional. In such a case, Laplace's equation takes the form:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad \text{in cartesian coords.}$$

or

$$\rho \frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{\partial^2 V}{\partial \varphi^2} = 0 \quad \text{in cylindrical coords.}$$

More generally, let $u(x,y)$ and $v(x,y)$ be given

such that

$$u + iv = f(x+iy), \tag{13}$$

where f is an analytic function of its argument. Consider the curvilinear coordinates u, v, z defined in this way.

$$du + i\,dv = f'(x+iy)\,(dx + i\,dy)$$

Hence

$$ds^2 = dx^2 + dy^2 + dz^2 = \frac{1}{|f'(x+iy)|^2}(du^2 + dv^2) + dz^2$$

Thus, for this system of curvilinear coordinates

$$h_1 = h_2 = \frac{1}{|f'(z)|}, \quad h_3 = 1$$

Hence

$$\nabla^2 V = \frac{1}{h^2}\left(\frac{\partial^2 V}{\partial u^2} + \frac{\partial^2 V}{\partial v^2}\right) + \frac{\partial^2 V}{\partial z^2} =$$

$$= |f'|^2 \left(\frac{\partial^2 V}{\partial u^2} + \frac{\partial^2 V}{\partial v^2}\right) + \frac{\partial^2 V}{\partial z^2}.$$

If, moreover, V is independent of z, ~~and satisfies~~ then

$$\frac{\partial^2 V}{\partial u^2} + \frac{\partial^2 V}{\partial v^2} = 0 \tag{14}$$

We have thus arrived at the following important conclusion: The two-dimensional Laplace's equation retains its form in every system of curvilinear coordinates u, v, z deduced from x, y, z by means of the transformation (13).

The general solution of (14) may be obtained by the method of separation of variables. We find

$$V(u,v) = (A_0 u + B_0)(\alpha_0 v + \beta_0) + \\ + \sum (A_n \cosh nu + B_n \sinh nu).(\alpha_n \cos nv + \beta_n \sin nv) \tag{15}$$

In formula (15), the summation spreads over all possible values of $n (\neq 0)$. Moreover, we might have interchanged u & v in the r.h.s. of (15), since these two variables enter in Laplace's equation on equal foot.

If, however, we search for a solution which is periodic in v with period 2π, then we should set $\alpha_0 \equiv 0$ and the summation is now over positive integer values of n:

$$V(u,v) = A_1 u + B_1 + \sum_{n=1}^{\infty} (A_n \cosh nu + B_n \sinh nu)(\alpha_n \cos nv + \beta_n \sin nv) \tag{16}$$