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Winter College on Quantum Optics: Novel Radiation Sources

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Light localization and band edge lasing in a photonic band gap

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LIGHT LOCALIZATION AND QUANTUM OPTICS IN A PHOTONIC BAND GAP

Tran Quang, University of Toronto

Lecture 1: *Fundamental Phenomena*

- Concepts of a photonic band gap
- Localization of spontaneous emission and photon-atom bound state
- Coherence control of spontaneous emission: a single atom optical memory device
- Band edge superradiance

Lecture 2: *Nonlinear Effects*

- Optical switching and optical transistor
- Inversion without fluctuation
- Quantum optical "spin-glass" state
- Anomalous nonlinear atomic resonant response

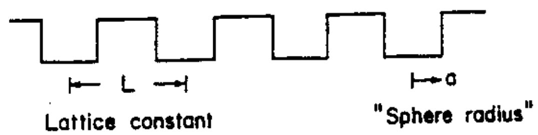
The Photonic Bandgap

Photonic band gap concept
E. Yablonovitch, P.R.L. **58**, 2059 (1987)
S. John, P.R.L. **58**, 2486 (1987)

Illustrative example: one-dimensional PBG (d=1)

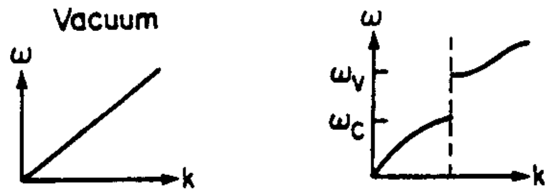
$$-\nabla^2 E - \frac{\omega^2}{c^2}\epsilon_{fluc}(x)E = \frac{\omega^2}{c^2}\epsilon_0 E$$

$\epsilon_{fluc}(x)$

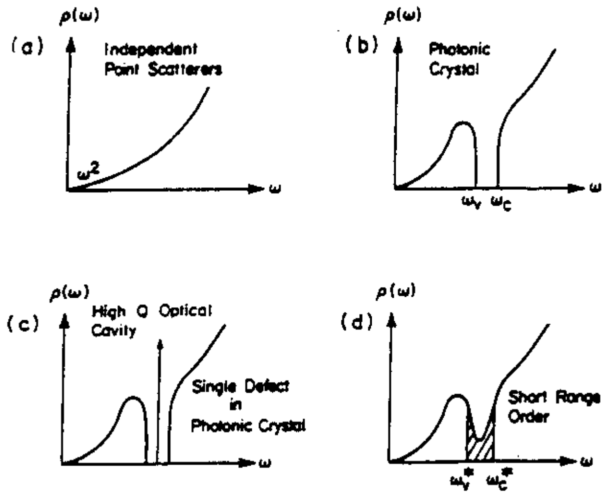


Largest gap occurs when single scattering (microscopic) resonance and Bragg resonance conditions coincide.
Bragg: $\omega/c = \pi/L$.
"Mie": $\lambda/4 = 2a \Rightarrow \omega/c = \pi/2n(2a)$,
n – refractive index.
The largest gap condition: $f = a/L = 1/2n$.

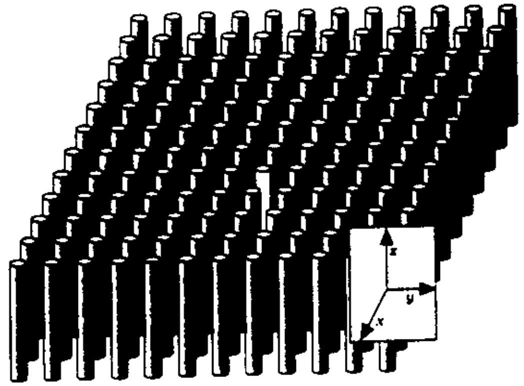
The Photonic Bandgap

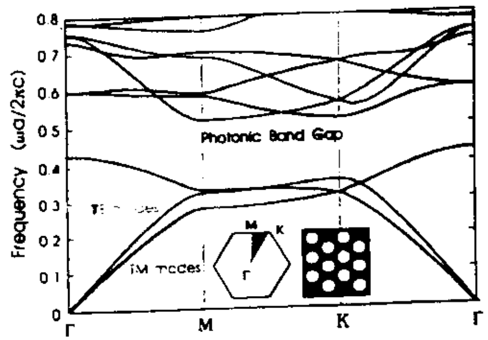


Photon Density of States



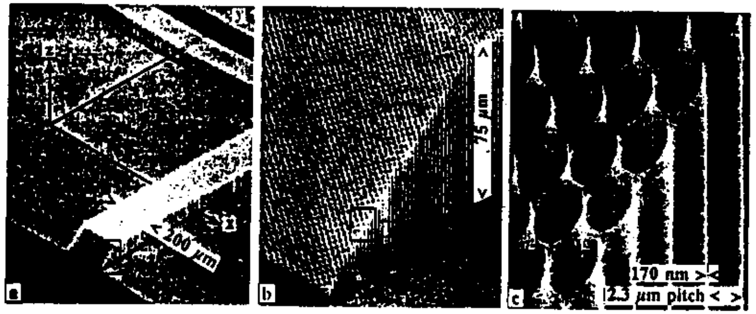
Two-Dimensional Photonic Crystal

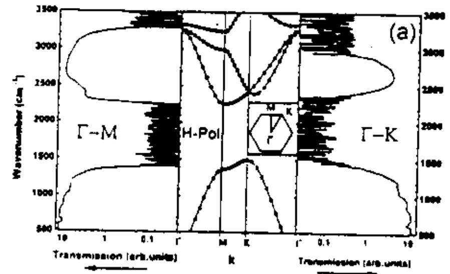


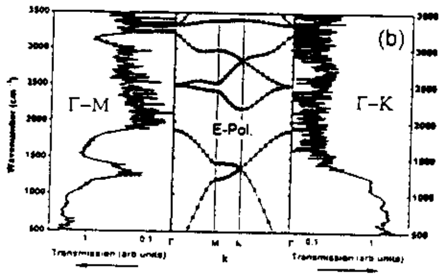


Two-Dimensional Photonic Crystal

(Gruning et.al. Appl. Phys. Lett., 1996)

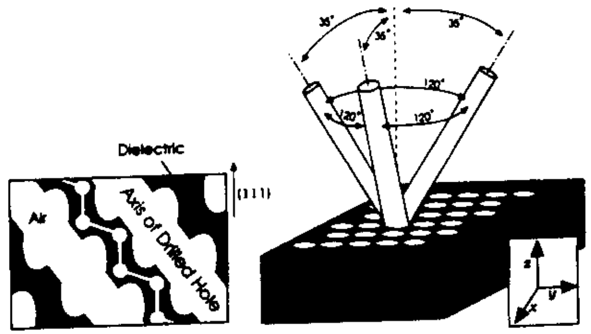


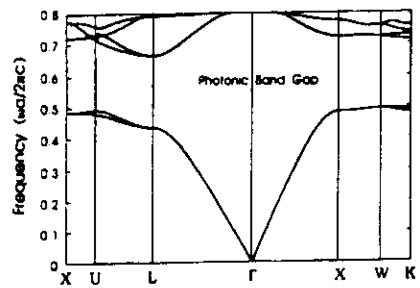




Three-Dimensional Photonic Crystal

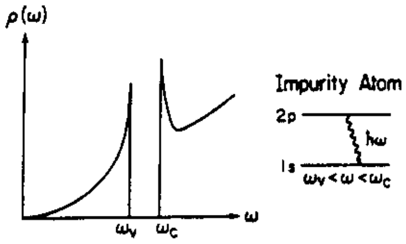
(Yablonovich et.al. P.R.L. 1991)





Localization of Spontaneous Emission and Photon-Atom Bound State

S. John and T. Quang, P.R.A. **50**, *1764, (1994)*



Two-level atom formalism:

$$H = \sum_\lambda \hbar\Delta_\lambda a_\lambda^\dagger a_\lambda + i\hbar \sum_\lambda g_\lambda (a_\lambda^\dagger \sigma_{12} - \sigma_{21} a_\lambda),$$

where

$\Delta_\lambda = \omega_\lambda - \omega_a$ – detuning of radiation mode from atomic resonance;

$\sigma_{ij} = |i\rangle\langle j|$, $i, j = 1, 2$ – atomic operators.

Wave function:

$$|\psi(t)\rangle = b_2(t)|2\rangle + \sum_\lambda b_{1,\lambda}(t)|1,\lambda\rangle e^{-i\Delta_\lambda t}.$$

Initial condition: $b_2(0) = 1$.

Spontaneous Emission

The time-dependent Schrödinger equation:

$$\frac{d}{dt} b_2(t) = -\sum_\lambda g_\lambda b_{1,\lambda}(t) e^{-i\Delta_\lambda t},$$

$$\frac{d}{dt} b_{1,\lambda}(t) = g_\lambda b_2(t) e^{i\Delta_\lambda t}.$$

The solution of the second equation

$$b_{1,\lambda}(t) = g_\lambda \int_0^t b_2(t') e^{i\Delta_\lambda t'} dt'.$$

Using this solution we find

$$\frac{d}{dt} b_2(t) = -\int_0^t G(t-t') b_2(t'),$$

where

$$G(t-t') = -\sum_\lambda g_\lambda^2 e^{-i\Delta_\lambda (t-t') dt'}.$$

In free space, the dispersion relation has the form

$$\omega_k = ck.$$

It yields:

$$G(t-t') \sim \frac{\gamma}{2}\delta(t-t')$$

and the excited atom displays an exponential decay to its ground state.

$$b_2(t) = e^{-\frac{\gamma}{2}t}.$$

Photon-Atom Bound State

Dispersion relation in an *isotropic PBG:*

$$\omega_k = \frac{c}{4na} \arccos\left[\frac{4n\cos(kL) + (1-n)^2}{(1+n)^2}\right].$$

Near the edge $k \cong k_0 \equiv \frac{\pi}{L}$:

$$\omega_k = \omega_c + A(k-k_0)^2,$$

where $A \sim \omega_c/k_0^2$ and ω_c is the band edge frequency.

The Green's function $G(t-t')$ can be found as

$$G(t-t') = \beta^{\frac{3}{2}} e^{-i\frac{\pi}{4}} e^{i\delta_c(t-t')}/\sqrt{\pi(t-t')},$$

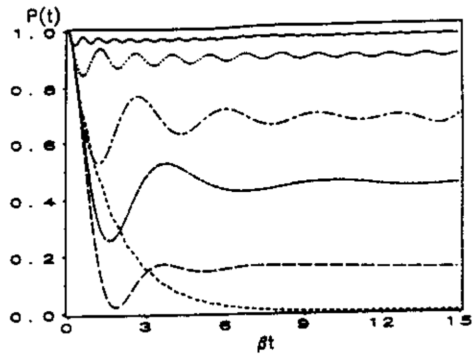
where $\beta^{\frac{3}{2}} = \omega_{21}^{7/2} d_{21}^2/(6\pi\epsilon_0\hbar c^3)$ and $\delta_c = \omega_a - \omega_c$.
$b_2(t)$ can be found using Laplace transform method:

$$b_2(t) = \frac{2}{3}e^{i\beta t} + \frac{2}{3}e^{(-i-\sqrt{3})\beta t/2} + (branch\ cut),$$

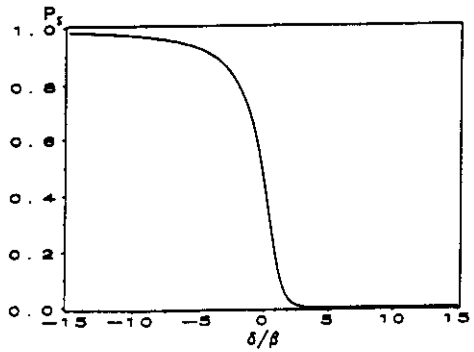
where the branch cut term has the form of an error function and tends to zero at the long time limit.

- Vacuum Rabi splitting into two dressed states: inside and outside of the gap
- Localization of spontaneous emission forms a photon-atom bound state
- Oscillatory behavior of spontaneous emission dynamics

Dynamics of Spontaneous Emission
($\delta_c/\beta = -10, -4, -1, 0, 1, 10$)

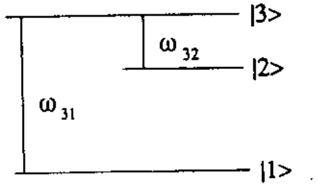


Steady-State Atomic Population



Coherence Control of Spontaneous Emission

(Quang, Woldeyohannes, John, and Agarwal, PRL 1997)



The Hamiltonian of the system for $\omega_L = \omega_{32}$:

$$H = i\hbar\Omega(e^{i\phi_c}\sigma_{23} - e^{-i\phi_c}\sigma_{32}) + \sum_\lambda \hbar\Delta_\lambda a_\lambda^\dagger a_\lambda$$

$$+i\hbar \sum_\lambda g_\lambda(a_\lambda^\dagger \sigma_{13} - \sigma_{31} a_\lambda),$$

where $\sigma_{ij} = |i\rangle\langle j|$ $(i,j = 1,2,3)$; $\Delta_\lambda = \omega_\lambda - \omega_{31}$; Ω and ϕ_c are the amplitude and phase of the control laser field.

The wave function:

$$|\psi(t)\rangle = b_3(t)|3\rangle + b_2(t)|2\rangle + \sum_\lambda c_{1,\lambda}(t)|1,\lambda\rangle e^{-i\Delta_\lambda t}.$$

Initial condition:
$b_3(0) = \cos\theta$, $b_2(0) = sin\theta e^{i\phi_p}$, $b_{1\lambda}(0) = 0$.

A Single-Atom Optical Memory Device

Analytical solution:

$$b_3(t) = a_1 e^{iv_1^2 t} + a_2 e^{iv_2^2 t} + (branch\ cut),$$

where

$$a_1 = 2v_1(i\Omega e^{i\phi}\sin\theta + v_1^2\cos\theta)/[(v_1 - v_2)(v_1 - v_3)(v_1 - v_4)],$$

$$a_2 = 2v_2(i\Omega e^{i\phi}\sin\theta + v_2^2\cos\theta)/[(v_2 - v_1)(v_2 - v_3)(v_2 - v_4)],$$

$$v_{1,3} = \sqrt{u}/2 \pm \left[(u^2/4 + \Omega^2)^{1/2} - u/4\right]^{1/2},$$

$$v_2 = v_4^* = -\sqrt{u}/2 - i\left[(u^2/4 + \Omega^2)^{1/2} + u/4\right]^{1/2}.$$

Here $\phi = \phi_p - \phi_c$ is the relative phase and $u > 0$.

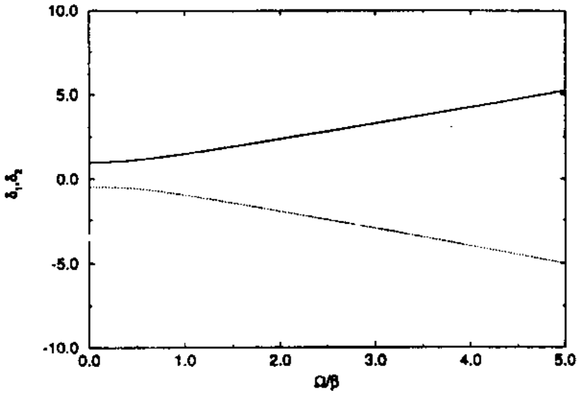
- Photon-atom bound state without decay at $\omega_c - v_1^2$
- Steady state is dependent on initial conditions

For a strong field:

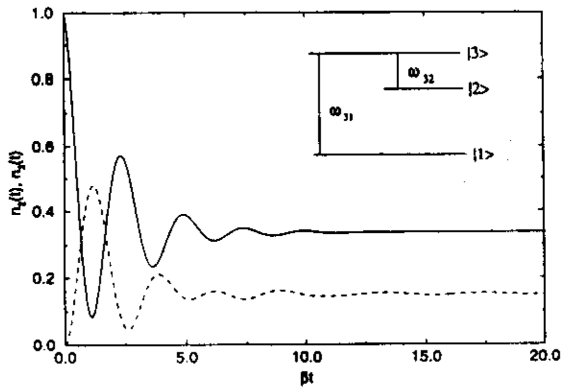
$$n_{3s} = n_{2s} = [1 - \sin(2\theta)\sin\phi]/4.$$

n_{2s} and n_{2s} are dependent on the relative phase ϕ. It may be relevant for a single-atom (phase-sensitive) optical memory device.

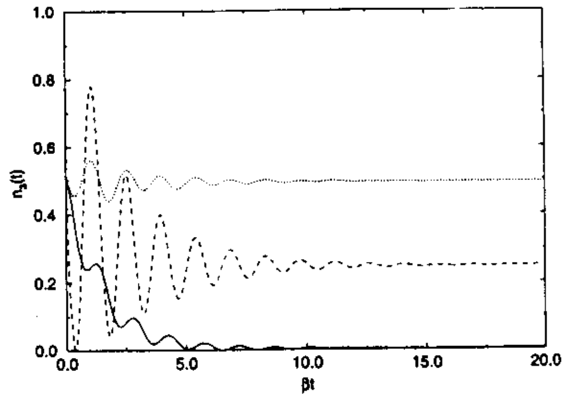
Level Splitting



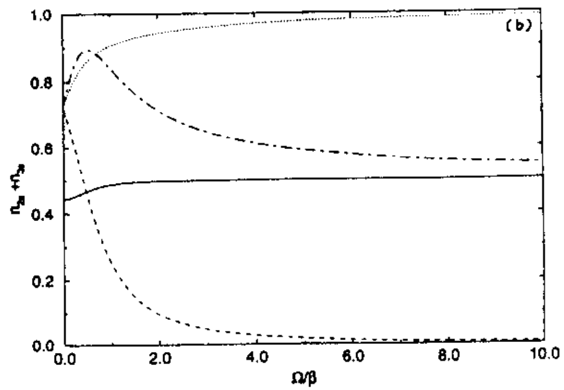
Atomic Population as a Function of Time
$(\theta = 0)$



Atomic Population as a Function of Time
$(\theta = \pi/4;\ \ \phi = 0,\ \pi/2,\ -\pi/2)$



Steady-State Atomic Population
$(\theta = \pi/4;\ \ \phi = 0,\ \pi/2,\ -\pi/2)$



Anomalous Collective Scale Factor

S. John and T. Quang, P.R.A. **50**, *1764, (1994)*

$$H = \sum_\lambda \hbar \Delta_\lambda a_\lambda^\dagger a_\lambda + i\hbar \sum_\lambda g_\lambda (a_\lambda^\dagger J_{12} - J_{21} a_\lambda),$$

where

$$J_{ij} = \sum_{k=1}^{N} |i\rangle_k {}_k\langle j|, \ (i,j = 1,2).$$

Initial state: Single atomic excitation in Dicke state

$$|\psi(0)\rangle = |J, M = 1 - J\rangle$$

where $|J, M\rangle$ is the eigenstate of J_3 and J^2.

In *ordinary vacuum*:

$$b_2(t) \sim e^{-N\gamma t/2}.$$

Near the edge of an isotropic PBG:

$$b_2(t) = \frac{2}{3} e^{iN^{2/3}\beta t} + \frac{2}{3} e^{(-i-\sqrt{3})N^{2/3}\beta t/2} + (branch\ cut).$$

In an *anisotropic PBG*, the collective time scale factor $\tau^{-1} = N^\phi$. Anomalous exponent ϕ is determined by the band edge singularity:

$\phi = 2/3$ for $d = 1$ or isotropic PBG,

$\phi = 1$ or 2 for $d = 2$ or $d = 3$, respectively.

Application: optical devices with fast modulation.

Band Edge Superradiance

John and Quang, P.R.L., **74**, *3419 (1995)*

System Hamiltonian:

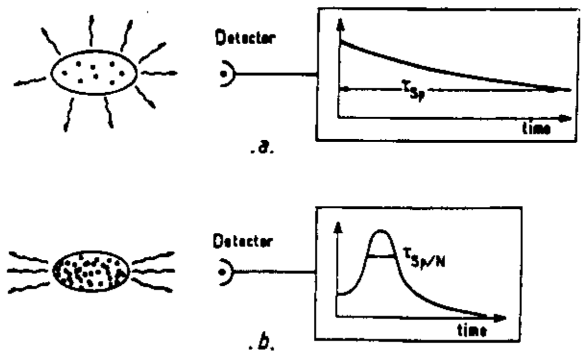
$$H = \sum_\lambda \hbar \Delta_\lambda a_\lambda^\dagger a_\lambda + i\hbar \sum_\lambda g_\lambda (a_\lambda^\dagger J_{12} - J_{21} a_\lambda),$$

where

$$\Delta_\lambda = \omega_\lambda - \omega_{21},$$

$$J_{ij} = \sum_{k=1}^{N} |i\rangle_k {}_k\langle j|, \ (i,j = 1,2).$$

Superradiance in ordinary vacuum



Superradiance: Analytical Solution

The equations of motion:

$$\frac{d}{dt}\langle J_{12}(t)\rangle = \int_0^t G(t-t')\langle J_3(t)J_{12}(t')\rangle dt',$$

$$\frac{d}{dt}\langle J_3(t)\rangle = -2\int_0^t G(t-t')\langle J_{21}(t)J_{12}(t')\rangle dt' + c.c.$$

Here

$$\langle J_3(t)\rangle = \langle J_{22}(t)\rangle - \langle J_{11}(t)\rangle,$$
$$G(t-t') = \sum_\lambda g_\lambda^2 e^{-i\Delta_\lambda(t-t')}$$

is the delay Green's function. The analytical solution in the *Markovian approximation*:

$$\langle J_3(t)\rangle = -N tanh\left\{B[(t/\tau)^{2-d/2} - 1]\right\},$$

where B is a constant; $\tau^{-1} \sim N^\phi$ and $\phi = 2/(4-d)$.
It yields:

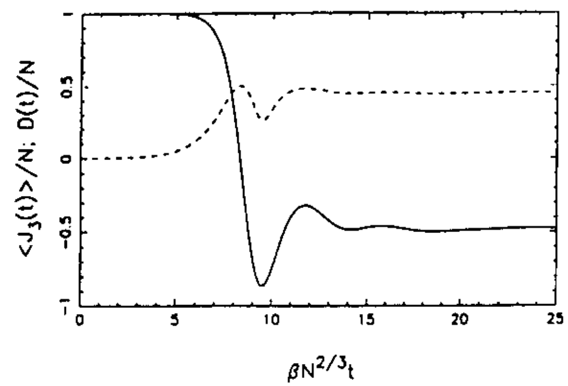
- Collective decay rate $\sim N^\phi$
- Intensity $\sim -\frac{dJ_3}{dt} \sim N^{1+\phi}$

For $d = 3$ we have $I \sim N^3$. It means a band edge superradiant laser would be stronger and with a faster speed of modulation than it is in the ordinary vacuum.

Localization of Superradiance

Markovian approximation cancels out memory effects and localization of superradiance. Numerical simulation (non-Markovian mean field approach) gives:

- Steady-state population inversion $\langle J_3(t)\rangle/N$ is not equal to -1: a fraction of superradiance remains localized in the vicinity of the atoms
- The atomic polarization evolves from its infinitesimal initial value to a steady-state macroscopic value: evidence of spontaneous breaking of symmetry
- The evolution of $\langle J_3(t)\rangle$, $|\langle J_{12}\rangle|$ displays collective self-induced oscillation instead of a simple decay as it is in free space.



Collective Switching in Confined Photonic Systems

(John and Quang, P.R.L. in press)

Master equation in free space:

$$\frac{\partial\rho}{\partial t} = -i[H_{coh}, \rho] + \frac{\partial\rho}{\partial t}|_{\text{loss}},$$

where $\frac{\partial\rho}{\partial t}|_{\text{loss}}$ is taken for the case when $H_{coh} \equiv 0$. It does not work for the confined photonic systems where the photonic mode density exhibits *rapid variation* with frequency.

New approach: Take into account of the differences in the density of modes at different dressed-state transition frequencies.

Collective resonance fluorescence: $H = H_0 + H_1 + H_{dephase}$

$$H_0 = \frac{1}{2}\hbar\Delta J_3 + \hbar\varepsilon(J_{12} + J_{21}) + \sum_\lambda \hbar\delta_\lambda a_\lambda^\dagger a_\lambda,$$

$$H_1 = i\hbar\sum_\lambda g_\lambda(a_\lambda^\dagger J_{12} - J_{21}a_\lambda),$$

$$\left(\frac{\partial\rho}{\partial t}\right)_{dephase} = (\gamma_p/2)(2J_3\rho J_3 - J_3^2\rho - \rho J_3^2),$$

where $J_{ij} = \Sigma_{k=1}^{N} |i\rangle_k\langle j|_k$ (i,j =1,2); $\Delta = \omega_a - \omega_L$; $\delta_\lambda = \omega_\lambda - \omega_L$.

Dressed-State Master Equation

Schwinger (boson) representation:
$J_{ij} = a_i^\dagger a_j$ (i,j =1,2), where $[a_i, a_j^\dagger] = \delta_{ij}$
Canonical transformation:

$$a_1 = \cos\phi q_1 + \sin\phi q_2,$$

$$a_2 = -\sin\phi q_1 + \cos\phi q_2.$$

$$H_0 = \hbar\Omega R_3 + \sum_\lambda \hbar\delta_\lambda a_\lambda^\dagger a_\lambda.$$

Dressed state master equation:

$$\frac{\partial\rho}{\partial t} = \frac{1}{2}A_0[R_3\rho R_3 - R_3^2\rho] + \frac{1}{2}A_-[R_{21}\rho R_{12} - R_{12}R_{21}\rho] + \frac{1}{2}A_+[R_{12}\rho R_{21} - R_{21}R_{12}\rho] + h.c.$$

$$A_0 = \gamma_0\sin^2\phi\cos^2\phi + \gamma_p\cos^2(2\phi),$$

$$A_- = \gamma_-\sin^4\phi + \gamma_p\sin^2(2\phi),$$

$$A_+ = \gamma_+\cos^4\phi + \gamma_p\sin^2(2\phi).$$

$\gamma_0 = 2\pi\,\Sigma_\lambda g_\lambda^2\delta(\omega_a - \omega_L)$;
$\gamma_- = 2\pi\,\Sigma_\lambda g_\lambda^2\delta(\omega_a - \omega_L + 2\Omega)$;
and $\gamma_+ = 2\pi\,\Sigma_\lambda g_\lambda^2\delta(\omega_a - \omega_L - 2\Omega)$.
In vacuum: $\gamma_0 = \gamma_- = \gamma_+ = \gamma$.

Optical Switching and Optical Transistor

Steady-state solution of the master equation:

$$\rho = P_0 \sum_n \xi^n |n\rangle\langle n|,$$

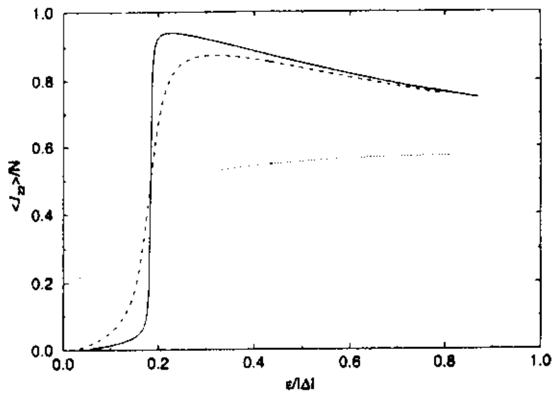
where $\xi = A_-/A_+$; $P_0 = (\xi - 1)/(\xi^{N+1} - 1)$; and $|n\rangle \equiv |N-n, n\rangle$ stands for a N-atom state in which n atoms are in the upper dressed state $|\tilde{2}\rangle$. In the limit of $N >> 1$:

$$\langle R_{22}\rangle/N \cong \begin{cases} 1, & if\ \xi > 1 \\ 1/2, & if\ \xi = 1 \\ 0, & if\ \xi < 1 \end{cases} \quad (1)$$

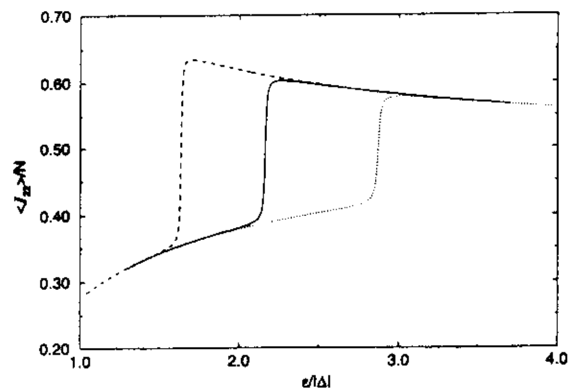
$$\langle J_{22}\rangle/N \cong \begin{cases} \cos^2\phi, & if\ \xi > 1 \\ 1/2, & if\ \xi = 1 \\ \sin^2\phi, & if\ \xi < 1 \end{cases} \quad (2)$$

Photonic material switches from an absorptive medium to a gain medium as a function of the control laser field. It may be relevant for an optical transistor. The optical switching is almost robust against to the dephasing caused by phonons in PBG materials.

Steady-State Atomic Population
($N = 10,\ 500,\ 5000$)



Steady-State Atomic Population
($N = 5000$; $\gamma_-/\gamma_+ = 0.3,\ 0.4,\ 0.5$)



Inversion without Fluctuations

Mandel q-parameters

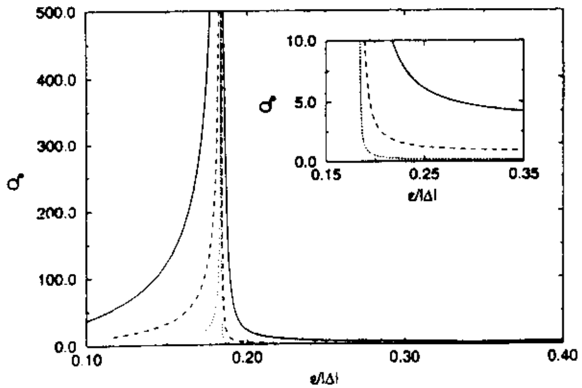
$$Q_d = (\langle R_{22}^2 \rangle - \langle R_{22} \rangle^2)/\langle R_{22} \rangle,$$

$$Q_b = (\langle J_{22}^2 \rangle - \langle J_{22} \rangle^2)/\langle J_{22} \rangle.$$

$Q_d < 1$ means sub-Poissonian distribution of atoms on the upper dressed state.
$Q_b < 1$ means sub-Poissonian distribution of atoms on the upper bared state.
In the limit of $N >> 1$:

$$Q_d \cong \begin{cases} 1/N, & if \;\; \xi > 1 \\ N/12, & if \;\; \xi = 1 \\ 1/(1-\xi), & if \;\; \xi < 1 \end{cases} \quad (3)$$

$$Q_b \cong \begin{cases} \sin^2\phi(\xi+1)/(\xi-1), & if \;\; \xi > 1 \\ (N+2)/6, & if \;\; \xi = 1 \\ \cos^2\phi(\xi+1)/(1-\xi), & if \;\; \xi < 1 \end{cases} \quad (4)$$



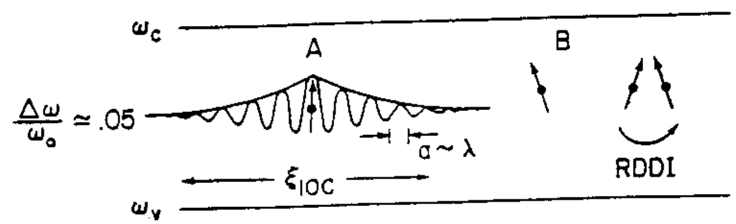
Optical Spin-glass state

John and Quang, P.R.L **76**, *1320 (1996)*
and P.R.A. **52**, *4083 (1995)*

Far inside the gap of a PBG:

- Suppression of spontaneous emission: Spontaneous emission remains localized in the vicinity of the atom
- Resonant dipole-dipole interaction (RDDI) remains strong: New regime of strong coupling limit between atoms
- New coherent propagation effects without damping
- Excited atom i can transfer localized photon to atom j

Wave zone: $\lambda < r_{ij} < l_{loc}$ by tunneling
Near zone: $r_{ij} << \lambda$ by high energy virtual photons



Photon Hopping Conduction

John and Quang, P.R.A. **52**, *4083 (1995)*

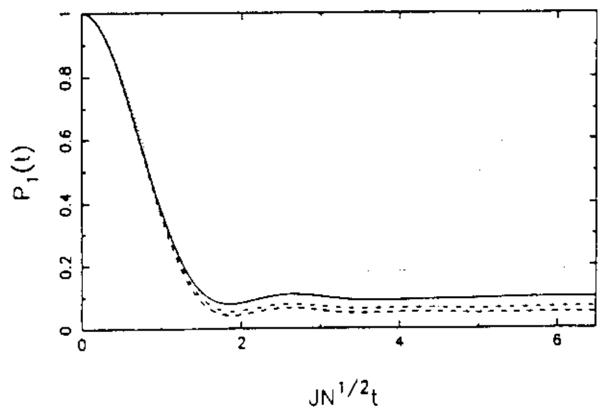
$$H = \sum_j \frac{\hbar\delta_j}{2}\sigma_j^z + \sum_{i\neq j} J_{ij}\sigma_i^\dagger\sigma_j + g\sum_j(\sigma_j^\dagger a + a^\dagger\sigma_j),$$

where $\delta_j = \omega_j - \omega_c$ - detuning;
J_{ij} - resonance dipole-dipole interaction between atoms i^{th} and j^{th}. Single excitation:

$$|\psi(t)\rangle = \sum_{i=1}^{N} c_i(t)|i\rangle,$$

$$|i\rangle = |-,-,...,+,-,....,-\rangle.$$

Initial condition: $c_1(0) = 1$, $c_i(0) = 0$ for $i \neq 1$.
J_{ij} and δ_j are Gaussian random numbers with zero mean value and with variance J and δ.
Applications: Lossless energy transfer device.



Spin Glass

- Collection of spins with random interactions
- Frozen disordered state with zero macroscopic magnetization while the local spontaneous magnetization at a given site i^{th} is non-zero
- Motivation: neural network as a programmable spin glass (Hopfield, 1982; Anderson, 1983)

Sherrington-Kirkpatrick (SK) model for spin glass

$$H = \sum_{i\neq j} J_{ij}S_iS_j,$$

where J_{ij} is Gaussian random number with zero mean value and with variance J:

$$P(J_{ij}) = (2\pi J^2)^{-1/2}e^{-J_{ij}^2/2J^2}.$$

Here S_i is classical Ising spin.

Quantum Optical Spin Glass

(John and Quang P.R.L. **76**, *1320 (1996))*

$$H = \sum_j \frac{\hbar\delta_j}{2}\sigma_j^z + \sum_{i\neq j} J_{ij}\sigma_i^\dagger\sigma_j + g\sum_j(\sigma_j^\dagger a + a^\dagger\sigma_j).$$

Differences between two models:

- Localized defect mode
- Pauli spin operator
- $\hbar\omega_{21} \gg kT \Rightarrow$ Spin glass at room temperature
- Optical neural network

Edwards-Anderson order parameters:

$$m = \frac{1}{N}\sum_{i=1}^{N}[\langle\sigma_i\rangle]_c; \;\; m_c = [\langle a\rangle]_c;$$

$$q = \frac{1}{N}\sum_{i=1}^{N}[\langle\sigma_i^\dagger\rangle\langle\sigma_i\rangle]_c; \;\; q_c = [\langle a^\dagger\rangle\langle a\rangle]_c.$$

$m = 0$, $q = 0$ - paramagnetic state.
$m \neq 0$, $q \neq 0$ - ferromagnetic state.
$m = 0$, $q \neq 0$ - *spin-glass state.*
$m_c = 0$, $q_c = 0$ - incoherent state.
$m_c \neq 0$, $q_c \neq 0$ - coherent state.
$m_c = 0$, $q_c \neq 0$ - *Bose-glass state.*

Optical Spin Glass: Analytical Solution

Low excitation: Holstein-Primakoff approximation:

$$\sigma_i^\dagger \cong b_i^\dagger, \;\; \sigma_i \cong b_i,$$

where b_i satisfies the boson commutation relation. Using the canonical transformation

$$q_\lambda = \sum_i \langle\lambda|i\rangle b_i, \;\; b_i = \sum_\lambda \langle i|\lambda\rangle q_\lambda,$$

we find (for $g = 0$):

$$H = \sum_\lambda J_\lambda q_\lambda^\dagger q_\lambda.$$

The semicircular law for the case of $N \gg 1$:

$$\rho(J_\lambda) = (2\pi\tilde{J}^2)^{-1}\sqrt{4\tilde{J}^2 - J_\lambda^2}.$$

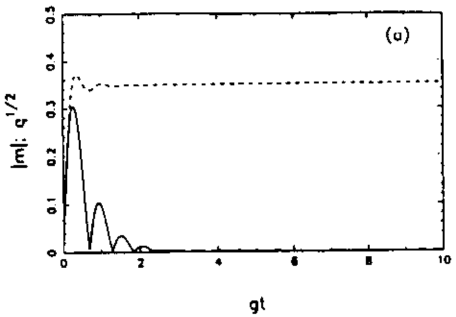
The Edwards-Anderson order parameter is found as:

$$m(t) = \frac{\langle b(0)\rangle}{\tilde{J}t}J_1(2\tilde{J}t),$$

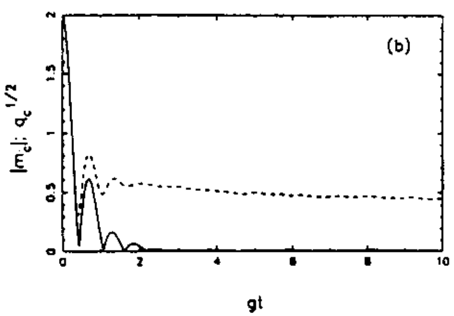
$$q(t) = q(0).$$

- $m(t)$ exhibits oscillatory behavior and tends to zero at $t \to \infty$
- New collective state: an optical analogue to quantum spin-glass state

Spin-Glass State of Atoms



Bose-Glass State of a Defect Mode



Anomalous Resonant Nonlinear Dielectric Response

(*John and Quang, P.R.L.* **76**, *2487 (1996)*)

Model: Two-level atoms +imperfect PBG + applied field

$$H = -\frac{\delta}{2}\sum_i^N \sigma_i^z - \Omega\sum_i^N(\sigma_i^\dagger + \sigma_i) + \sum_{i\neq j}^N J_{ij}\sigma_i^\dagger\sigma_j,$$

where Ω is the resonant Rabi frequency; $\delta = \omega_L - \omega_{21}$; $J_{ij} = J_{ji}$ stands for the RDDI between atoms i and j. Optical Bloch equations:

$$\frac{d}{dt}\langle\sigma_j^-\rangle = (-1/T_2 + i\delta)\langle\sigma_j^-\rangle - i(\Omega - F_j)\langle\sigma_j^z\rangle,$$

$$\frac{d}{dt}\langle\sigma_j^z\rangle = -(\langle\sigma_j^z\rangle + 1)/T_1 + (2i\Omega\langle\sigma_j^\dagger\rangle - 2i\langle\sigma_j^\dagger\rangle F_j + c.c.).$$

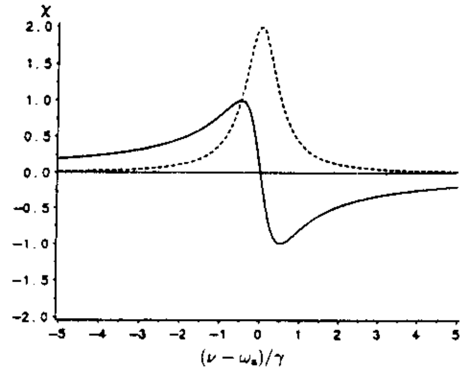
Here, $F_j \equiv \Sigma_{i(\neq j)} J_{ji}\langle\sigma_i^-\rangle$; T_1 and T_2 are the relaxation times of $\langle\sigma^z\rangle$ and $\langle\sigma^-\rangle$, respectively. Atomic susceptibility:

$$\chi = (1/2\tilde{J}^2 w)\{\delta + i/T_2 - [(\delta + i/T_2)^2 - 4w^2\tilde{J}^2]^{1/2}\},$$

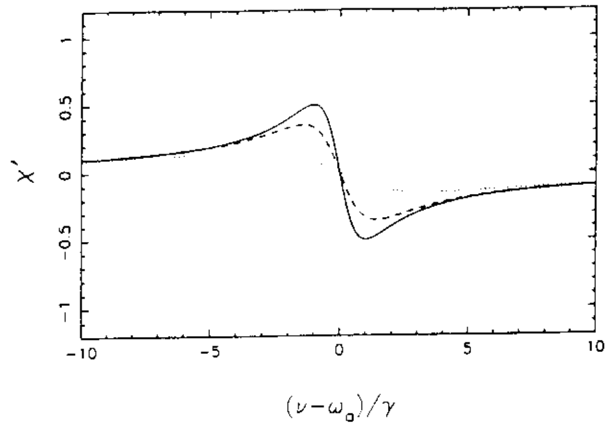
where $w = 4T_1\Omega^2\chi'' - 1$, $\tilde{J} = \sqrt{N}J$.
In the weak coupling case of $\tilde{J} \ll 1/T_2$:

$$\chi = -(\delta - i/T_2)/[\delta^2 + 1/T_2^2 + 4T_1\Omega^2/T_2].$$

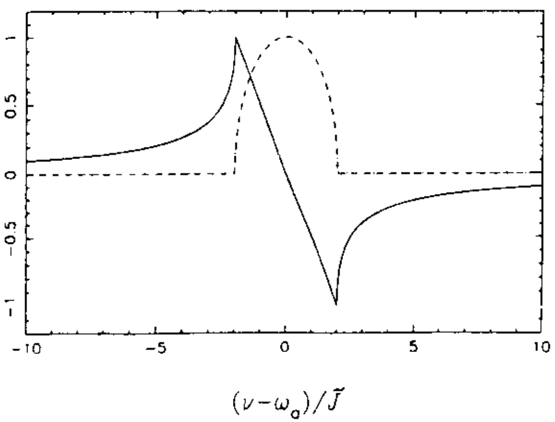
Linear Susceptibility in Free Space



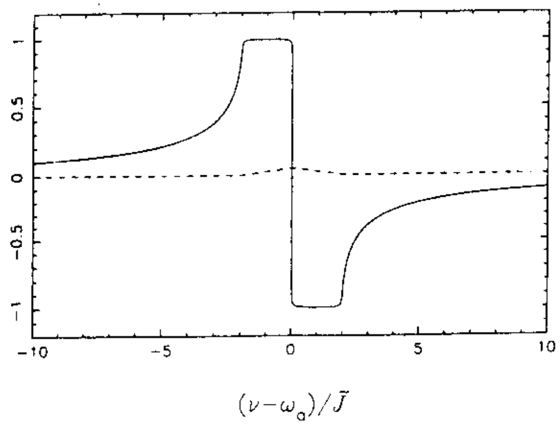
Nonlinear Susceptibility in Free Space



Linear Susceptibility in a PBG

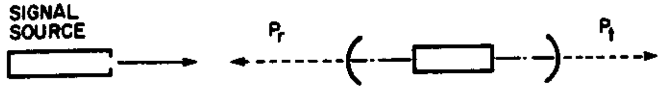


Nonlinear Susceptibility in a PBG



Optical Bistability

(John and Quang, P.R.A. **54**, *4479 (1996))*



Optical bistable system: two output states for the same value of input.

Applications of optical bistability:

- Optical logic elements
- Optical transistors and optical switching

Optical Bloch equations:

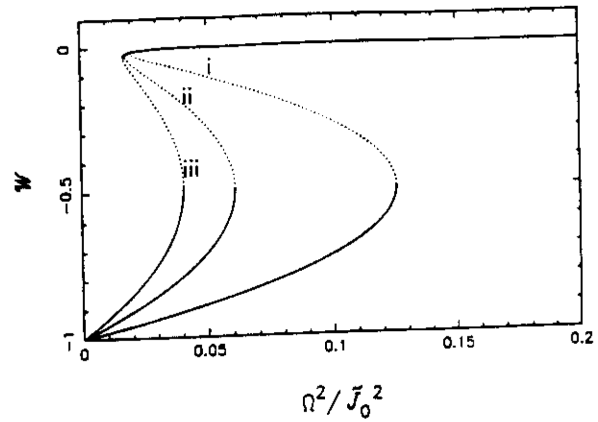
$$\frac{d}{dt}\langle\sigma_j^-\rangle = (-1/T_2+i\delta)\langle\sigma_j^-\rangle + i\tilde{J}_0\langle\sigma_i^z\rangle\langle\sigma_i\rangle - i(\Omega - F_j)\langle\sigma_j^z\rangle,$$

$$\frac{d}{dt}\langle\sigma_j^z\rangle = -(\langle\sigma_j^z\rangle+1)/T_1 + (2i\Omega\langle\sigma_j^\dagger\rangle - 2i\langle\sigma_j^\dagger\rangle F_j + c.c.),$$

where $\tilde{J}_0 = J_0 N$ and J_0 is the mean value of J_{ij}

- Intrinsic optical bistability (without a cavity)
- The threshold can be very low

Atomic Population Inversion
$(\tilde{J}/\tilde{J}_0 = 0,\ 0.4,\ 0.45)$



Switching Threshold