UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL ATOMIC ENERGY AGENCY
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

H4.SMR/984-12

Winter College on Quantum Optics: Novel Radiation Sources

3-21 March 1997

Lasing without inversion

PART I

M.O. Scully

Department of Physics, Texas A&M University, College Station, TX, USA

{Scully Lect I: Lasing with and Without Inversion} 1

Paul M. Intro. to Rad. (Class.) Matter (Q. Mech.) Revisited

= polarization

index of ref.

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial}{\partial t^2}\right)E = +\mu_0 \frac{\partial^2 P}{\partial t^2}$$

$$P = \epsilon_0 \chi E$$

$$\left\{\left(\frac{\partial^2}{\partial x^2} - \underbrace{\frac{1}{c^2}(1+\chi)}\frac{\partial^2}{\partial t}\right)E\right.$$

$$\frac{1}{v} \equiv \frac{n}{c} \qquad n = \sqrt{1+\chi}$$

Index of.

SVAφ $\left(\frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right)E = -\,\omega^2\mu_0\, P$

$$E_0(tz)\, e^{i(kx-\nu t)} + c.c.$$

$$P_0(tz)\, e^{i(kx-\nu t)} + c.c.$$

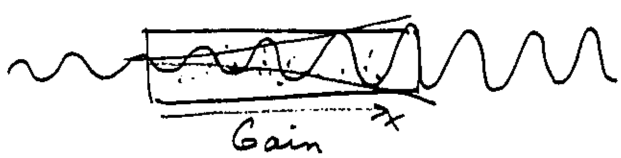
$$\cancel{2ik}\left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right)\underbrace{E_0(tz)}_{\mathcal{E}(tz)\, e^{i\varphi(tz)}} = +i\,\frac{\omega^2\mu_0}{2k}\,P$$

$(\mathrm{Re}\,P + i\,\mathrm{Im}\,P)$

$$\left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right)\mathcal{E}(tz) = -\frac{kc^2\mu_0}{2}\,\mathrm{Im}\,P \qquad \mathrm{Im}\,P = \epsilon_0\chi'' E$$

$$\left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right)\varphi(tz) = \frac{kc^2\mu_0}{2}\,\mathrm{Re}\,P \qquad \mathrm{Re}\,P = \epsilon_0\chi' E$$

Suppose have S.S. cond. (no time dep.)



$$\frac{\partial}{\partial x}\mathcal{E} = \alpha\,\mathrm{Im}\,P$$

const.

$$= \alpha\,\epsilon_0\chi'' E$$

$$\boxed{\text{Gain} = \alpha\,\mathrm{Im}\,P/\mathcal{E}}$$

III Polarization

$P = e\vec{r}\,|\psi|^2 d\vec{r}$

$$P = N e \langle\psi|\vec{r}|\psi\rangle$$

$$= N e \,\mathrm{Tr}(r\rho) \quad *$$

$$= N(\underbrace{e r_{ab}}_{\wp}\,\rho_{ba} + c.c.)$$

$$\equiv N\wp(\rho_{ab} + \rho_{ba}) \qquad \text{taking } \wp \text{ to be real}$$

see (c.f. last page)

$$\underbrace{P_0(x,t)}_{\text{Macro}}\, e^{+i(kx-\nu t)} = N\wp\,\underbrace{\rho_{ab}}_{\text{Micro}}(x,t) = \epsilon_0(\chi' + i\chi'')E$$

$$\boxed{\text{Gain} = \underset{\text{const.}}{\alpha'}\, \mathrm{Im}\,\rho_{ab}/E}$$

---

* density matrix concept

$$\langle\hat{\Phi}\rangle_{q.m.} = \langle\psi|\hat{\Phi}|\psi\rangle$$

$$\langle\langle\hat{\Phi}\rangle_{q.m.}\rangle_{\text{ensemble}} = \sum_{\psi} P_\psi \langle\psi|\hat{\Phi}|\psi\rangle$$

$$= \sum_n \sum_\psi \underbrace{P_\psi \langle n|\hat{\Phi}|\psi\rangle\langle\psi|n\rangle}^{\rho}$$

$$= \sum_n \langle n|\hat{\Phi}\rho|n\rangle = \mathrm{Tr}(\hat{\Phi}\rho)$$

③

IV Int. Rad. w. Matter

$$\dot{\rho} = -i\,[H_0 + V, \rho] \qquad H_0 = \begin{pmatrix} \epsilon_a & 0 \\ 0 & \epsilon_b \end{pmatrix}$$

$$V = \wp E \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & V_{ab} \\ V_{ba} & 0 \end{pmatrix}$$

int. pic.

$$e^{-iH_0 t}\tilde{\rho}\, e^{iH_0 t} = \rho$$

$$\dot{\rho} = -i\Big[H_0 \underbrace{\left(e^{-iH_0 t}\tilde{\rho}\, e^{iH_0 t}\right)}_{\rho} - \left(e^{-iH_0 t}\tilde{\rho}\, e^{iH_0 t}\right) H_0\Big] + e^{-iH_0 t}\dot{\tilde{\rho}}\, e^{iH_0 t}$$

$$= -i[H_0, \rho] \quad -i \quad [V, \rho]$$

$$\dot{\tilde{\rho}} = -i\Big[\underbrace{e^{iH_0 t}\, V\, e^{-iH_0 t}}_{V_I(t)}\,\tilde{\rho} \;-\; \tilde{\rho}\,\underbrace{e^{iH_0 t}\, V\, e^{-iH_0 t}}_{V(t)}\Big]$$

$$\langle b \,|\, V(t) \,|\, a\rangle = e\, r_{ba}\, e^{i\overbrace{(\epsilon_b - \epsilon_a)}^{}t}_{\;-\omega}\left(E_0(x\,t)\, e^{-i(\nu t - kx)} + c.c.\right)$$

$$= e\, r_{ba}\Big(E_0(xt)\,\underbrace{e^{-i[(\nu+\omega)t - kx]}}_{\text{"0"}} + E_0^*(xt)\,\underbrace{e^{+i[(\nu-\omega)t - kx]}}_{\text{keep}}$$

RWA

when $\omega = \nu$

$V(t)_{ab} = e\, r_{ab}\, E_0$ time indep ☺

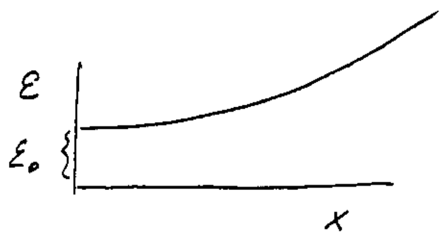
V Laser Gain

$$\dot{\rho}_{ab} = -\Gamma_{ab}\,\rho_{ab} - i\,V_{ab}\,(\rho_{bb}-\rho_{aa})$$

s.s. $$\rho_{ab} = +i\,\frac{\overset{\wp\mathcal{E}}{V_{ab}}}{\Gamma_{ab}}\,(\rho_{aa}-\rho_{bb})$$

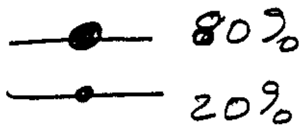
$$\boxed{\text{Gain} = \alpha\,\frac{\mathrm{Im}\,\rho_{ab}}{|\mathcal{E}|} = \frac{\alpha\wp}{\Gamma_{ab}}\,(\rho^{0}_{aa}-\rho^{0}_{bb})}$$

Hence we have Gain



$$\mathcal{E}(x) = \mathcal{E}_0 \exp\Big\{\underbrace{\frac{\alpha\wp}{\Gamma_{ab}}\,(\rho^{(0)}_{aa}-\rho^{(0)}_{bb})}_{+ \text{ if } \rho^{(0)}_{aa} > \rho^{(0)}_{bb}}\, x\Big\}$$

i.e. inversion

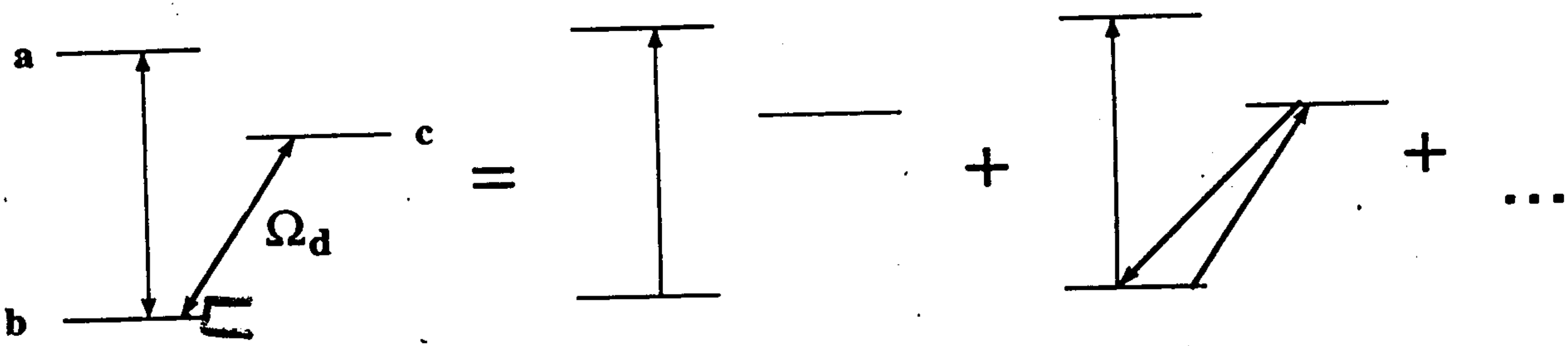


Lasing with Inversion = LwI

5

2.Quantum Interference

interference of the amplitudes of the following absorption processes:



→ reduces absorption and increases gain if $\rho_{bb} > \rho_{cc}$

→ also can be viewed as Fano-type interference between dressed states

Gain without inversion in V-scheme
is not due to coherent population trapping
is not due to the dynamical Stark effect

but is the result of quantum interference

(6)

THEORY OF V-SCHEME

Driving and probe fields interact via upper level coherence ρ_{ac}

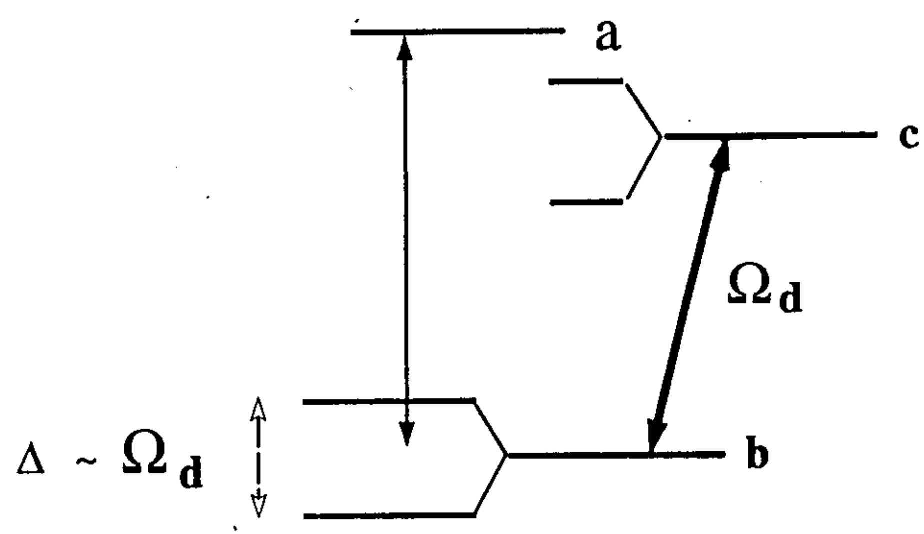
Gain coefficient for a resonant weak probe field

$$gain \sim \frac{(\rho_{aa}-\rho_{bb}) + |\Omega_d|^2/(\gamma_{ac}\gamma_{bd})(\rho_{bb}-\rho_{cc})}{1+|\Omega_d|^2/(\gamma_{ac}\gamma_{ab})}$$

TWO EFFECTS OF UPPER LEVEL COHERENCE

1.Dynamical Stark Splitting

→ Reduces absorption or gain in the center of resonance line



VII Lasing Without Inversion = LWI

1) $$\dot{\rho}_{ab} = -\Gamma_{ab}\rho_{ab} - i\left[V_{ab}\rho_{bb} - \rho_{aa}V_{ab} - \rho_{ac}V_{cb}\right]$$

2) $$\dot{\rho}_{ac} = -\Gamma_{ac}\rho_{ac} - i\left[V_{ab}\rho_{bc} - \rho_{ab}V_{bc}\right]$$

3) $$\dot{\rho}_{bc} = -\Gamma_{bc}\rho_{bc} - i\left[V_{ba}\rho_{ac} + V_{bc}\rho_{cc} - \rho_{bb}V_{bc}\right]$$

3′) $$\rho^{0}_{bc} = \frac{-i}{\Gamma_{bc}} V_{bc}\left(\rho^{0}_{cc} - \rho^{0}_{bb}\right)$$

2′) $$\rho^{0}_{ac} = \frac{-i}{\Gamma_{ac}}\left[-i\frac{V_{ab}V_{bc}}{\Gamma_{bc}}\left(\rho^{0}_{cc} - \rho^{0}_{bb}\right) - \rho_{ab}V_{bc}\right]$$

inserting 2′ into (1) yields, in the $\dot{\rho}_{ab} = 0$ limit

4) $$\rho^{(1)}_{ab} = +\frac{i}{\Gamma_{ab}} V_{ab}\left(\rho^{0}_{aa} - \rho^{0}_{bb}\right)$$

$$+ i\frac{V_{cb}}{\Gamma_{ab}} \cdot \frac{-i}{\Gamma_{ac}} \cdot \left[-i\frac{V_{ab}V_{bc}}{\Gamma_{bc}}\left(\rho^{0}_{cc} - \rho^{0}_{bb}\right) - \rho^{(1)}_{ab}V_{bc}\right]$$

$$\boxed{\rho^{(1)}_{ab} = \frac{i\dfrac{V_{ab}}{\Gamma_{ab}}\left[\left(\rho^{0}_{aa} - \rho^{0}_{bb}\right) - \dfrac{|V_{bc}|^2}{\Gamma_{ac}\Gamma_{bc}}\left(\rho^{0}_{cc} - \rho^{0}_{bb}\right)\right]}{1 + \dfrac{|V_{bc}|^2}{\Gamma_{ab}\Gamma_{ac}}}}$$

Starting point – equations set

$$\begin{cases} \dot{A}^{(1)} = \sum_k g_k^{(1)} e^{i(\omega_1-\omega_k)t} B_k \\ \dot{A}^{(2)} = \sum_k g_k^{(2)} e^{i(\omega_2-\omega_k)t} B_k \\ \dot{B}_k = -g_k^{(1)} A^{(1)} e^{-i(\omega_1-\omega_k)t} - g_k^{(2)} A^{(2)} e^{-i(\omega_2-\omega_k)t} \end{cases}$$

with initial condition $B_k(0)=0$

Integrating $\dot{B}_k$ equation (with initial condition)

$$B_k(t) = -g_k^{(1)} \int_0^t d\tau\, A^{(1)}(\tau)\, e^{+i(\omega_1-\omega_k)\tau}$$
$$- g_k^{(2)} \int_0^t d\tau\, A^{(2)}(\tau)\, e^{+i(\omega_2-\omega_k)\tau}$$

Substitute into $\dot{A}^{(1)}$-equation

$$\dot{A}^{(1)} = -\sum_k g_k^{(1)} e^{i(\omega_1-\omega_k)t} \Big[g_k^{(1)} \int_0^t d\tau\, A^{(1)}(\tau)\, e^{+i(\omega_1-\omega_k)\tau}$$
$$+ g_k^{(2)} \int_0^t d\tau\, A^{(2)}(\tau)\, e^{+i(\omega_2-\omega_k)\tau}$$

$$= -\sum_k g_k^{(1)} g_k^{(1)} \int_0^t d\tau\, A^{(1)}(\tau)\, e^{i(\omega_1-\omega_k)(t-\tau)}$$

$$- \sum_k g_k^{(1)} g_k^{(2)} \int_0^t d\tau\, A^{(2)}(\tau)\, e^{i\omega_1 t - i\omega_2 \tau}\, e^{i\omega_k(t-\tau_k)}$$

But $\omega_1 = \omega_{a_1 b}$ and $\omega_2 = \omega_{a_2 b}$. We write

$$\omega_1 t - \omega_2 \tau = (\omega_{a_1} - \omega_b)\, t - (\omega_{a_2} - \omega_b)\tau$$
$$= \omega_{a_1} t - \omega_{a_2} \tau - \omega_b (t-\tau)$$
$$= \omega_{a_1} t + \omega_{a_2} t - \omega_{a_2} t - \omega_{a_2} \tau - \omega_b (t-\tau)$$
$$= (\omega_{a_1} - \omega_{a_2})\, t + (\omega_{a_2} - \omega_b)(t-\tau)$$

$$\omega_1 t - \omega_2 \tau = \omega_{12} t + \omega_2 (t-\tau)$$

Thus for $\dot{A}^{(1)}$

$$\dot{A}^{(1)} = -\sum_k g_k^{(1)} g_k^{(1)} \int_0^t d\tau\, A^{(1)}(\tau)\, e^{i(\omega_1-\omega_k)(t-\tau)}$$

$$- e^{i\omega_{12}t} \sum_k g_k^{(1)} g_k^{(2)} \int_0^t d\tau\, A^{(2)}(\tau)\, e^{i(\omega_2-\omega_k)(t-\tau)}$$

Now - time for approximations.

1) $\sum_k \to \int_{-\infty}^{+\infty} d\omega_k\, D(\omega_k)$ whatever $D(\omega)$ is - mode density. ω_k - becomes integration variable

2) Assuming that $g_k^{(i)}$ changes slowly around resonance $\omega_i - \omega_k \approx 0$ can be taken out of integrals. So also for $D(\omega)$

Thus first term in $\dot{A}^{(1)}$ equation, and second

$$\dot{A}^{(1)} = -\left[g_k^{(1)2} D\right]_{\omega=\omega_1} \int_0^t d\tau\, A^{(1)}(\tau) \int_{-\infty}^{+\infty} d\omega_k\, e^{i(\omega_1-\omega_k)(t-\tau)}$$

$$- e^{i\omega_{12}t} \left[g_k^{(1)} g_k^{(2)} D\right]_{\omega=\omega_2} \int_0^t d\tau\, A^{(2)}(\tau) \int_{-\infty}^{+\infty} d\omega_k\, e^{i(\omega_2-\omega_k)(t-\tau)}$$

Integrals over ω_k give ~~delta~~ $\delta(t-\tau)$

$$\dot{A}^{(1)} = -\left[g_k^{(1)2} D\right]_{\omega=\omega_1} \int_0^t d\tau\, A^{(1)}(\tau)\, \delta(t-\tau)$$

$$- e^{i\omega_{12}t} \left[g_k^{(1)} g_k^{(2)} D\right]_{\omega=\omega_2} \int_0^t d\tau\, A^{(2)}(\tau)\, \delta(t-\tau)$$

Due to the property of delta function

$$\int_{t_0}^{t} d\tau\, A(\tau)\, \delta(t-\tau) = \frac{1}{2} A(t)$$

we obtain

$$\dot{A}^{(1)} = -\left[g_k^{(1)2} D\right]_{\omega=\omega_1} \frac{1}{2} A^{(1)}(t)$$

$$+ e^{i\omega_{12}t}\left[g_k^{(1)} g_k^{(2)} D\right]_{\omega=\omega_2} \frac{1}{2} A^{(2)}(t)$$

Notation

$$\left[(g_k^{(i)})^2 D\right]_{\omega=\omega_i} = \gamma_i \qquad (i = 1,2)$$

The first term in $\dot{A}^{(1)}$ equation is obvious. The second we write as

$$\left[g_k^{(1)} g_k^{(2)} D\right]_{\omega=\omega_2} \approx \left(g_k^{(1)}\big|_{\omega=\omega_1}\right)\left(g_k^{(2)}\big|_{\omega=\omega_2}\right) D(\omega_2)$$

$$\approx \sqrt{g_k^{(1)2} D(\omega_1)\big|_{\omega=\omega_1}}\, \sqrt{g_k^{(2)} D\big|_{\omega=\omega_2}}\, \sqrt{\frac{D(\omega_2)}{D(\omega_1)}}$$

and we approximate the last quotient by unity since $\omega_1 = \omega_{a_1 b}$ and $\omega_2 = \omega_{a_2 b}$ are pretty close, ~~while~~ while D is slowly varying. Hence finally

$$\dot{A}^{(1)} = -\frac{1}{2}\gamma_1 A^{(1)}(t) - \frac{1}{2}\sqrt{\gamma_1\gamma_2}\, A^{(2)}(t)\, e^{i\omega_{12}t}$$

So the equation for $\dot{A}^{(1)}$ is recovered.

Equation for $\dot{A}^{(2)}$ is obtained in an exatly the same manner, only superscripts (or indices) "1" and "2" must be interchanged. Hence

$$\dot{A}^{(2)} = -\frac{1}{2}\gamma_2 A^{(2)}(t) - \frac{1}{2}\sqrt{\gamma_2\gamma_1}\, A^{(1)}(t)e^{i\omega_{21}t}$$

Since $\omega_{21} = \omega_{a_2} - \omega_{a_1} = -\omega_{12}$ we get

$$\dot{A}^{(2)} = -\frac{1}{2}\gamma_2 A^{(2)}(t) - \frac{1}{2}\sqrt{\gamma_1\gamma_2}\, A^{(1)}(t)e^{-i\omega_{12}t}$$

and this completes the homework