UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL ATOMIC ENERGY AGENCY
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

H4.SMR/984-21

Winter College on Quantum Optics: Novel Radiation Sources

3-21 March 1997

Superradiant laser

M. Kolobov

Department of Physics, University of Essen, Germany

Superradiant Laser

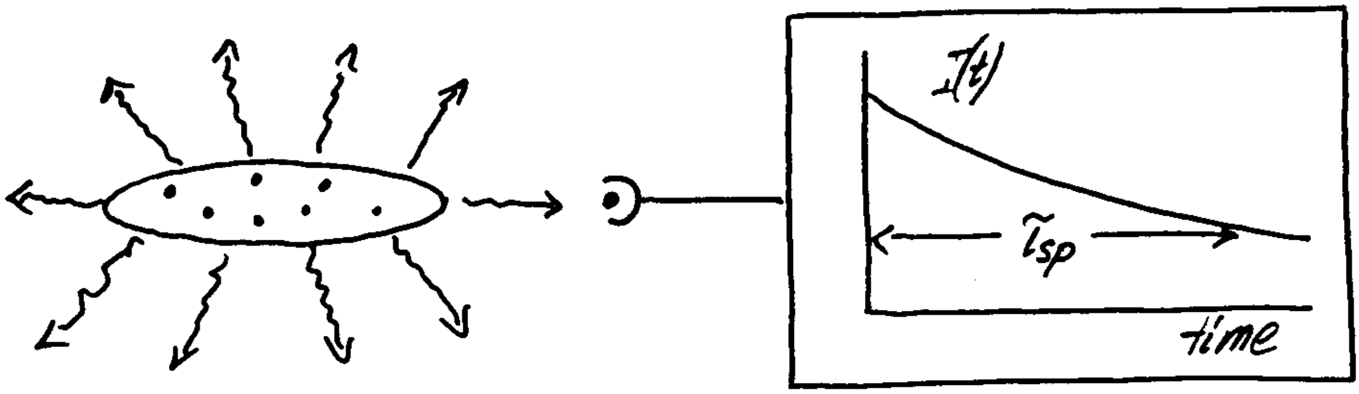
Fritz Haake
Carsten Seeger } Essen
Claude Fabre
Elisabeth Giacobino } Paris
Serge Reynaud
Marek Kuś Warsaw
MK

Program

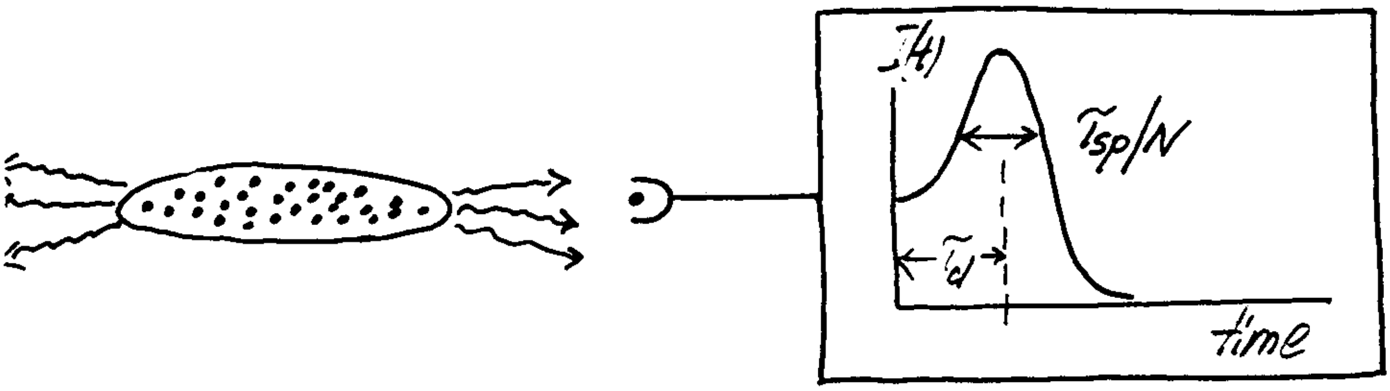
1. Introduction
2. Description of the model of the superradiant laser (SRL)
3. Equations of motion of the SRL
4. Symmetry properties and constants of the motion
5. Admissible cooperativity parameters
6. Full cooperativity: quantum fluctuations
7. Partial cooperativity:
   a) quantum fluctuations
   b) analysis of the atomic Hilbert space
8. Spontaneous emission
9. Shifted frequency solutions
10. Conclusions

Superradiance

Superradiance = collective spontaneous emission [Dicke, 1954; Gross, Haroche, 1982]; interaction of N two-level atoms (initially excited) with a single (damped) mode of electromagnetic field



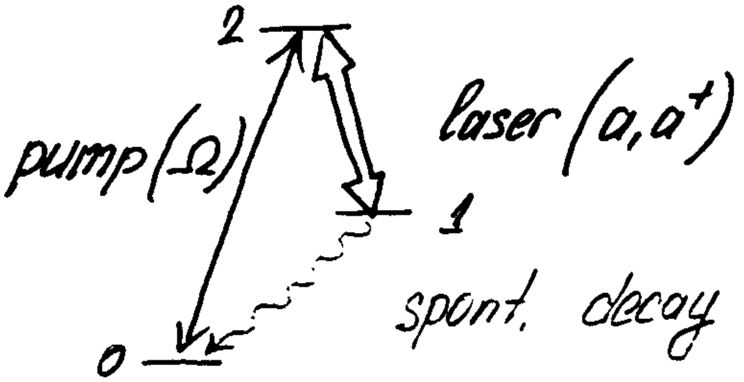
$$I(t) \sim N \exp(-\Gamma t)$$



$I(t) \sim N^2$; line width $\Delta\nu \sim \Gamma N$;
photon statistics close to coherent state

Roman laser

Exemplifies collective and noncollective processes in lasers; N three-level atoms in a cavity + coherent pump



[Ritsch, Marte, Zoller, 1992]

Collective atomic operators

$$S_{ij} = \sum_{\mu=1}^{N} S_{ij}^{\mu} = \sum_{\mu=1}^{N} \left(|i\rangle\langle j|\right)^{\mu}$$

Interactions $0 \leftrightarrow 2$ and $2 \leftrightarrow 1$ are collective

$$H = i\hbar\Omega\left(S_{20} - S_{02}\right) + i\hbar g\left(aS_{21} - a^{+}S_{12}\right)$$

Spont. emission $1 \rightarrow 0$ noncollective

$$H_{sp} = i\hbar \sum_{\mu=1}^{N} g_{\mu}\left(\eta_{\mu} S_{21}^{\mu} - \eta_{\mu}^{+} S_{12}^{\mu}\right)$$

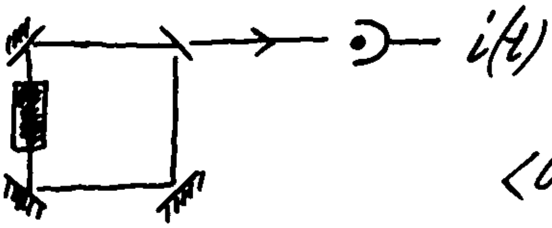
η_{μ} - operators of individual reservoirs

Results of noncollective relaxation: radiation of the Raman laser doesn't have the features of superradiance

1) mean intensity $I \sim N$

2) linewidth $\Delta\nu \sim const$ (independent of N)

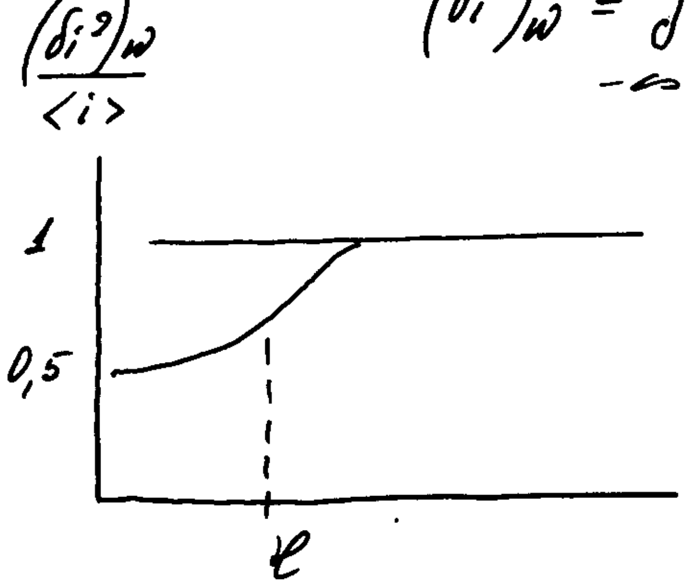
3) photon statistics → photocurrent statistics



$$i(t) = \langle i \rangle + \delta i(t)$$

$$\langle \delta i(t) \delta i(t') \rangle = \langle i \rangle \delta(t-t') + \\ + const \left(\langle a^+(t) a^+(t') a(t') a(t) \rangle \theta(t'-t) + \\ + (t \leftrightarrow t') - \langle a^+ a \rangle \langle a^+ a \rangle \right)$$

$$(\delta i^2)_\omega = \int_{-\infty}^{\infty} dt \, e^{i\omega t} \langle \delta i(0) \delta i(t) \rangle$$ – noise spectrum



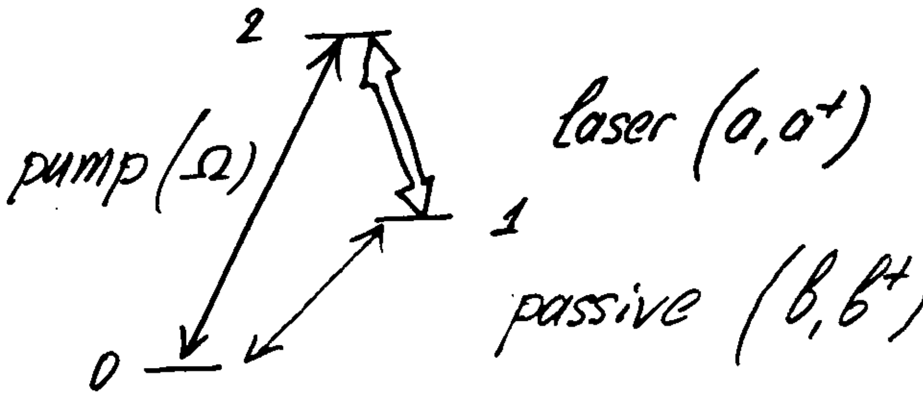
up to 50% of shot-noise reduction

Superradiant laser

What shall happen with the features of the laser radiation if all the processes in laser are collective ??

N three-level atoms in doubly resonant cavity :

active mode → laser
passive mode → collective decay



The Hamiltonian in the interaction picture

$$H = i\hbar\Omega\left(S_{20} - S_{02}\right) + i\hbar g_{12}\left(aS_{21} - a^{+}S_{12}\right) + $$
$$+ \; i\hbar g_{01}\left(bS_{10} - b^{+}S_{01}\right)$$

Ω - external pump field ,

g_{a1}, g_{12} - coupling constants,

a, a^+ and b, b^+ - annihilation and creation operators for the laser and passive mode,

$$[a, a^+] = [b, b^+] = 1 .$$

$$S_{ij} = \sum_{\mu=1}^{N} S_{ij}^{\mu} = \sum_{\mu=1}^{N} (|i\rangle\langle j|)^{\mu}$$ - collective atomic operators

S_{ij}, $i \neq j$ - polarizations

S_{ii} - collective populations of i-th level

$$S_{ij}^{+} = S_{ji} \quad , \quad [S_{ij}, S_{kl}] = \delta_{jk} S_{il} - \delta_{il} S_{kj}$$

Heisenberg equations for S_{ij},

$$\dot{S}_{ij} = (i/\hbar)[H, S_{ij}] ,$$

and Heisenberg-Langevin equations for a, b,

$$\dot{a} = (i/\hbar)[H, a] + (\partial a/\partial t)_{irr}$$
$$\dot{b} = (i/\hbar)[H, b] + (\partial b/\partial t)_{irr}$$

$$\left(\frac{\partial a}{\partial t}\right)_{irr} = -\varkappa_a a + \sqrt{2\varkappa_a}\,\eta_a(t) ,$$

$$\left(\frac{\partial b}{\partial t}\right)_{irr} = -\varkappa_b b + \sqrt{2\varkappa_b}\,\eta_b(t) \quad -$$

relaxation terms with corresponding quantum Langevin forces $\eta_a(t)$ and $\eta_b(t)$

$$[\eta_\alpha(t), \eta^+_\beta(t')] = \langle \eta_\alpha(t)\eta^+_\beta(t')\rangle = \delta_{\alpha\beta}\,\delta(t-t') ,$$

$$\langle \eta_\alpha(t)\rangle = \langle \eta^+_\alpha(t)\eta_\beta(t')\rangle = \langle \eta_\alpha(t)\eta_\beta(t')\rangle = 0 .$$

These forces guarantee that

$$[a(t), a^+(t)] = [b(t), b^+(t)] = 1 .$$

Adiabatic elimination of the passive mode

$$\dot{b} = (i/\hbar)[H, b] + (\partial b/\partial t)_{irr} =$$

$$= -g_{01} S_{01} - \varkappa_b b + \sqrt{2\varkappa_b}\,\eta_b(t)$$

We assume that $\varkappa_b \gg \varkappa_a$, put $\dot{b} = 0$ and obtain

$$b(t) = -\frac{g_{01}}{\varkappa_b} S_{01}(t) + \sqrt{\frac{2}{\varkappa_b}}\, \eta_b(t) ,$$

$b(t)$ adiabatically follows the collective atomic polarization on $1 \leftrightarrow 0$ transition

Now we substitute $b(t)$ into the Heisenberg equations for S_{ij}

For example: $\dot{S}_{02} = \frac{i}{\hbar}[H, S_{02}] =$

$$= -\Omega(S_{22} - S_{00}) + g_{12}\, a\, S_{01} - g_{01} S_{12}\, b =$$

$$= -\Omega(S_{22} - S_{00}) + g_{12}\, a\, S_{01} - S_{12}\left[-\frac{g_{01}^2}{\varkappa_b} S_{01} + \sqrt{\frac{2 g_{01}^2}{\varkappa_b}}\, \eta_b\right]$$

$g_{01}^2/\varkappa_b = \gamma$ — collective atomic relaxation rate

$$\dot{S}_{02} = -\Omega(S_{22} - S_{00}) + g_{12}\, a\, S_{01} +$$

$$+\gamma S_{12} S_{01} - \sqrt{2\gamma}\, S_{12}\, \eta_b$$

$\gamma S_{12} S_{01}$ → nonlinear term

$\sqrt{2\gamma}\, S_{12}\, \eta_b$ → multiplicative noise

Ordering of operators is important!

I can write $b S_{12}$ instead of $S_{12} b$ because b and S_{12} commute.

Then I have to substitute b together with the Langevin force η_b:

$$-g_{01} b S_{12} = -\left[-\frac{g_{01}^2}{\varkappa_b} S_{21} + \sqrt{\frac{2 g_{01}^2}{\varkappa_b}}\, \eta_b\right] S_{12} =$$

$$= \gamma S_{21} S_{12} + \sqrt{2\gamma}\, \eta_b S_{12}$$

These two forms are equivalent.

Total set of Heisenberg-Langevin equations for the superradiant laser

$$\dot S_{02} = g_{12}\, a S_{01} - \Omega\left(S_{22} - S_{00}\right) + \gamma S_{12} S_{01} - \sqrt{2\gamma}\, S_{12}\, \eta_b ,$$

$$\dot S_{12} = -g_{12}\, a\left(S_{22} - S_{11}\right) + \Omega S_{10} - \gamma S_{10} S_{02} - \sqrt{2\gamma}\, \eta_b^{+} S_{02} ,$$

$$\dot S_{01} = -g_{12}\, a^{+} S_{02} - \Omega S_{21} + \gamma\left(S_{11} - S_{00}\right) S_{01} - \sqrt{2\gamma}\left(S_{11} - S_{00}\right)\eta_b ,$$

$$\dot S_{00} = -\Omega\left(S_{02} + S_{20}\right) + 2\gamma S_{10} S_{01} - \sqrt{2\gamma}\left(S_{10}\eta_b + \eta_b^{+} S_{01}\right) ,$$

$$\dot S_{11} = -g_{12}\left(a S_{21} + a^{+} S_{12}\right) - 2\gamma S_{10} S_{01} + \sqrt{2\gamma}\left(S_{10}\eta_b + \eta_b^{+} S_{01}\right) ,$$

$$\dot S_{22} = \Omega\left(S_{02} + S_{20}\right) + g_{12}\left(a S_{21} + a^{+} S_{12}\right) ,$$

$$\dot a = -g_{12} S_{12} - \kappa a + \sqrt{2\kappa}\, \eta_a .$$

Constants of the motion

a) invariance under a phase shift ϕ of the form

$$a \to e^{i\phi} a \,, \quad S_{12} \to e^{i\phi} S_{12} \,, \quad S_{10} \to e^{i\phi} S_{10} \,,$$

$$\eta_a \to e^{i\phi} \eta_a \,, \quad \eta_b \to e^{-i\phi} \eta_b \,;$$

b) three operators

$$C_1 = \sum_i S_{ii} \,,$$

$$C_2 = \sum_{i,j} S_{ij} S_{ji} \,,$$

$$C_3 = \sum_{i,j,k} S_{ij} S_{jk} S_{ki} \,,$$

commute with the Hamiltonian and, therefore, are constants of the motion; conservation of C_1 = conservation of the number of atoms;

C_2 and C_3 are the Casimir operators of the group $U(3)$,

$$[C_k, S_{ij}] = 0 \,.$$

We shall confine ourselves to the semiclassical limit, $N \gg 1$

$$S_{ij},\ a, a^{+} \rightarrow X = \bar{X} + \delta X ,$$

$\bar{X} \sim N$ - classical term

δX - operator-valued "small" fluctuation

To within corrections of relative order $1/N$

$$\bar{C}_1 = \sum_i \bar{S}_{ii} = N , \quad \bar{C}_2 = \sum \bar{S}_{ij} \bar{S}_{ji} = c_2 N^2 ,$$

$$\bar{C}_3 = \sum_{ijk} \bar{S}_{ij} \bar{S}_{jk} \bar{S}_{ki} = c_3 N^3 ,$$

c_2, c_3 - cooperativity parameters.

Admissible cooperativity parameters

To be physically acceptable the classical solutions $\bar{S}_{ij}$ must satisfy two requirements of quantum-mechanical origin

1) the matrix $\bar{S}$ formed from $\bar{S}_{ij}/N$ must be nonnegative Hermitian

2) $\sum_i \bar{S}_{ii} = N \leftrightarrow tr\,\bar{S} = 1$

These requirements imply three Schwartz' inequalities,

$$\bar{S}_{ii}\bar{S}_{jj} - \bar{S}_{ij}\bar{S}_{ji} \geq 0 \qquad \text{and}$$

$$0 \leq \bar{S}_{ii}/N \leq 1$$

Moreover, they put some restrictions on the cooperativity parameters c_2, c_3.

First, we can write c_2, c_3 as

$$c_2 = tr\,\bar{S}^2, \quad c_3 = tr\,\bar{S}^3.$$

The eigenvalues l_i, $i=1,2,3$ of $\bar{S}$ must be real and nonnegative and satisfy $l_1+l_2+l_3=1$.

Then

$$c_2 = l_1^2+l_2^2+l_3^2, \quad c_3 = l_1^3+l_2^3+l_3^3.$$

From here we immediately obtain that

$$\frac{1}{3} \leq c_2 \leq 1, \quad \frac{1}{9} \leq c_3 \leq 1$$

More than that! For given c_2 and N c_3 is allowed to range within an admissible region. To find this region we write the characteristic polynomial of $\bar{s}$,

$$F(\ell) = (\ell-\ell_1)(\ell-\ell_2)(\ell-\ell_3) ,$$

and use the Newton formulas [Mostowski, Stark, 1964] to express the coefficients of this polynomial via c_2 and c_3,

$$F(\ell) = \ell^3 - \ell_2 + \frac{1}{3}(1-r^2)\ell - \frac{1}{3}(c_3 - r^2)$$

where $r = \sqrt{\frac{3c_2 - 1}{2}}$, $0 \leq r \leq 1$

We shall call r the participation parameter.

The three roots of $F(\ell)$ are real only if its discriminant

$$D = -\frac{1}{9} r^6 + 9\left(\frac{c_3}{2} - \frac{r^2}{3} - \frac{1}{18}\right)^2 \leq 0 .$$

This gives

$$\frac{1+6r^2-2r^3}{9} \le c_3 \le \frac{1+6r^2+2r^3}{9}$$

In addition, applying the Hurwitz criterion [Marden, 1966] to the polynomial $F(\ell)$, we find that all roots ℓ_i are nonnegative only if

$$r^2 \le c_3$$

Hence the admissible region of the cooperativity parameter c_3 is

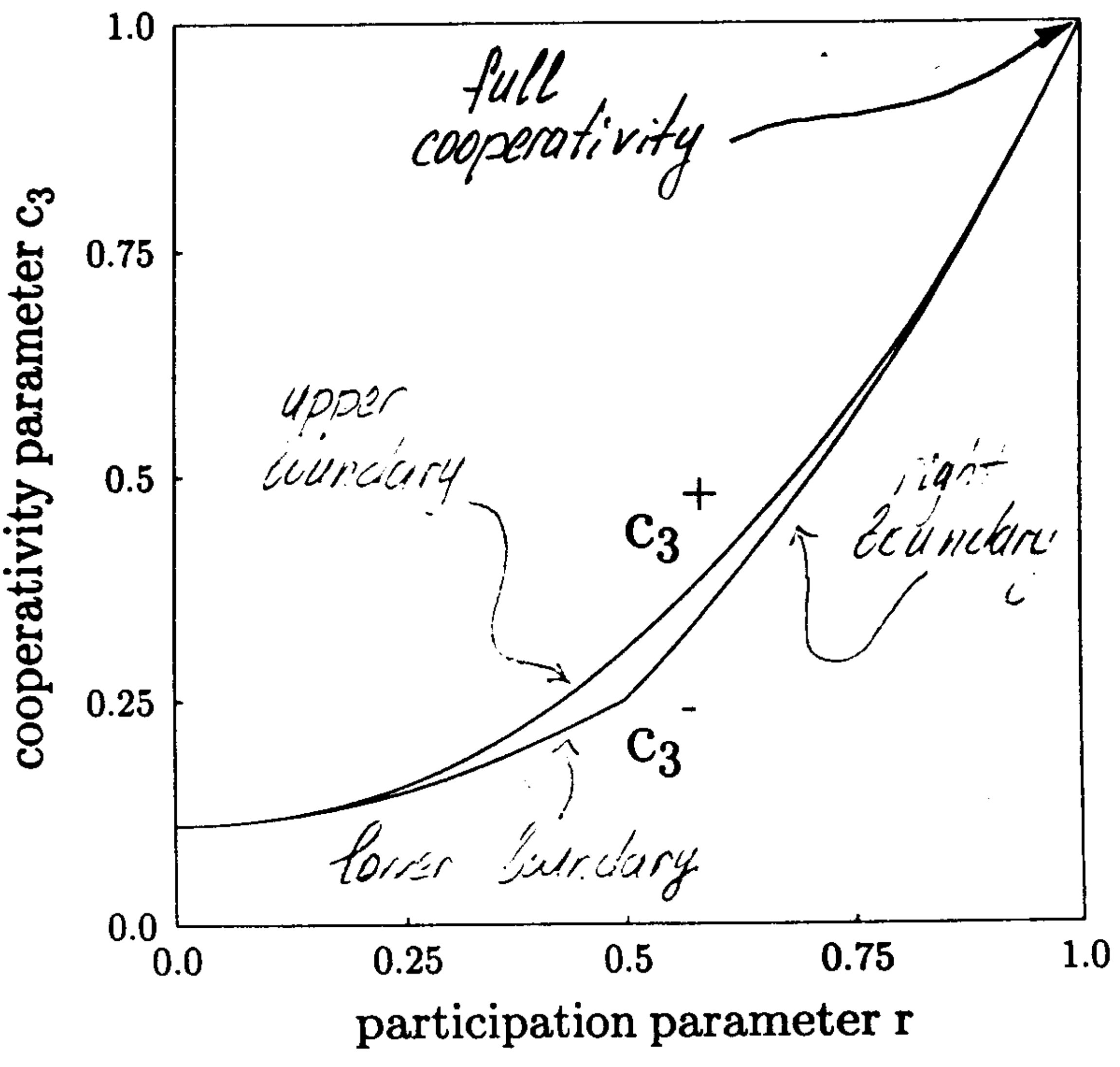
$$c_3^-(r) \le c_3 \le c_3^+(r) \quad ,$$

$$c_3^-(r) = \begin{cases} (1+6r^2-2r^3)/9 \quad , & 0 \le r \le 1/2 \\ r^2 \quad , & 1/2 \le r \le 1 \end{cases}$$

$$c_3^+(r) = (1+6r^2+2r^3)/9 \quad , \quad 0 \le r \le 1$$

$$c_3^-(r) = \begin{cases} (1 + 6r^2 - 2r^3)/9, & 0 \le r \le 1/2, \\ r^2, & 1/2 \le r \le 1, \end{cases}$$

$$c_3^+(r) = (1 + 6r^2 + 2r^3)/9, \quad 0 \le r \le 1.$$



Full cooperativity: quantum fluctuations

Introduce three dimensionless parameters

$c = \frac{g_{12}^2}{\gamma_a \gamma}$ – dimensionless coupling strength,

$p = \frac{\Omega}{N\gamma\sqrt{c}}$ – effective pump strength,

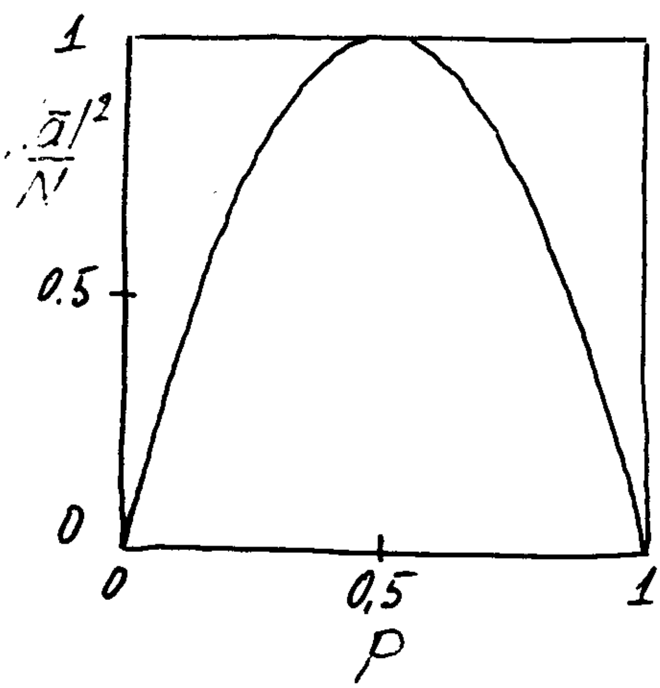
$\zeta = \frac{N\gamma}{\gamma_a}$ – time scale ratio.

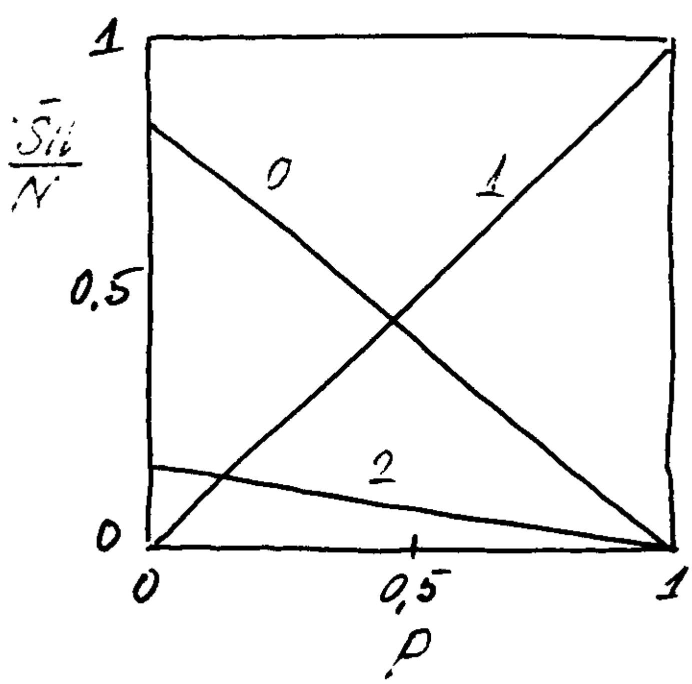
Stationary classical solutions $\bar{S}_{ij}, \bar{a}$:

$$\bar{S}_{01}/N \equiv x = \sqrt{\frac{cp(1-p)}{1+c}}, \quad \bar{S}_{02}/N = \frac{x^2}{p\sqrt{c}}, \quad \bar{S}_{12}/N = \frac{x}{\sqrt{c}},$$

$$\bar{S}_{00}/N = \frac{c(1-p)}{1+c}, \quad \bar{S}_{11}/N = p, \quad \bar{S}_{22}/N = \frac{1-p}{1+c},$$

$$\bar{a} = -N\sqrt{\frac{p(1-p)}{1+c}}$$





Linearization of fluctuations

$\delta a(t) = \delta u(t) + i\delta v(t)$,

$\delta S_{ij}(t) = \delta u_{ij}(t) + i\delta v_{ij}(t)$,

the real and imaginary parts of fluctuations are decoupled

$S_{ij}(t)\,\eta_b(t) \rightarrow \bar{S}_{ij}\,\eta_b(t)$ inhomogeneous term

We perform the Fourier transform

$$\delta X(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, \delta X(t) \quad ,$$

and obtain the linear algebraic equations for $\delta u(\omega)$ and $\delta v(\omega)$

Finally, input-output transform

$$a_{out}(t) = \sqrt{2\kappa_a}\, a(t) - \eta_a(t)$$

I consider here the adiabatic limit,

$$\xi \gg 1 \quad \text{or} \quad N\gamma \gg \kappa_a$$

We calculate two correlation functions

$$\langle \delta v_{out}(\omega)\delta v_{out}(\omega')\rangle \to \delta(\omega+\omega')\,|\bar{a}|^2\,\Delta\nu/\omega^2$$

for $\omega \to 0$

$\Delta\nu$ – the linewidth of the laser

$\sim 1/\omega^2$ – phase diffusion

$$\Delta\nu = \frac{\varkappa_a}{|\bar{a}|^2}\,\frac{p^2(1+c)^2+(1-p)^2(1-c)^2}{c-1+2p},$$

$$\Delta\nu \sim 1/N^2 \quad !!$$

The amplitude fluctuation spectrum

$$\langle \delta u_{out}(\omega)\delta u_{out}(\omega')\rangle = (1/4)\,\delta(\omega+\omega')\left\{1-\frac{S(c,p)}{1+\omega^2\tau_a^2}\right\}$$

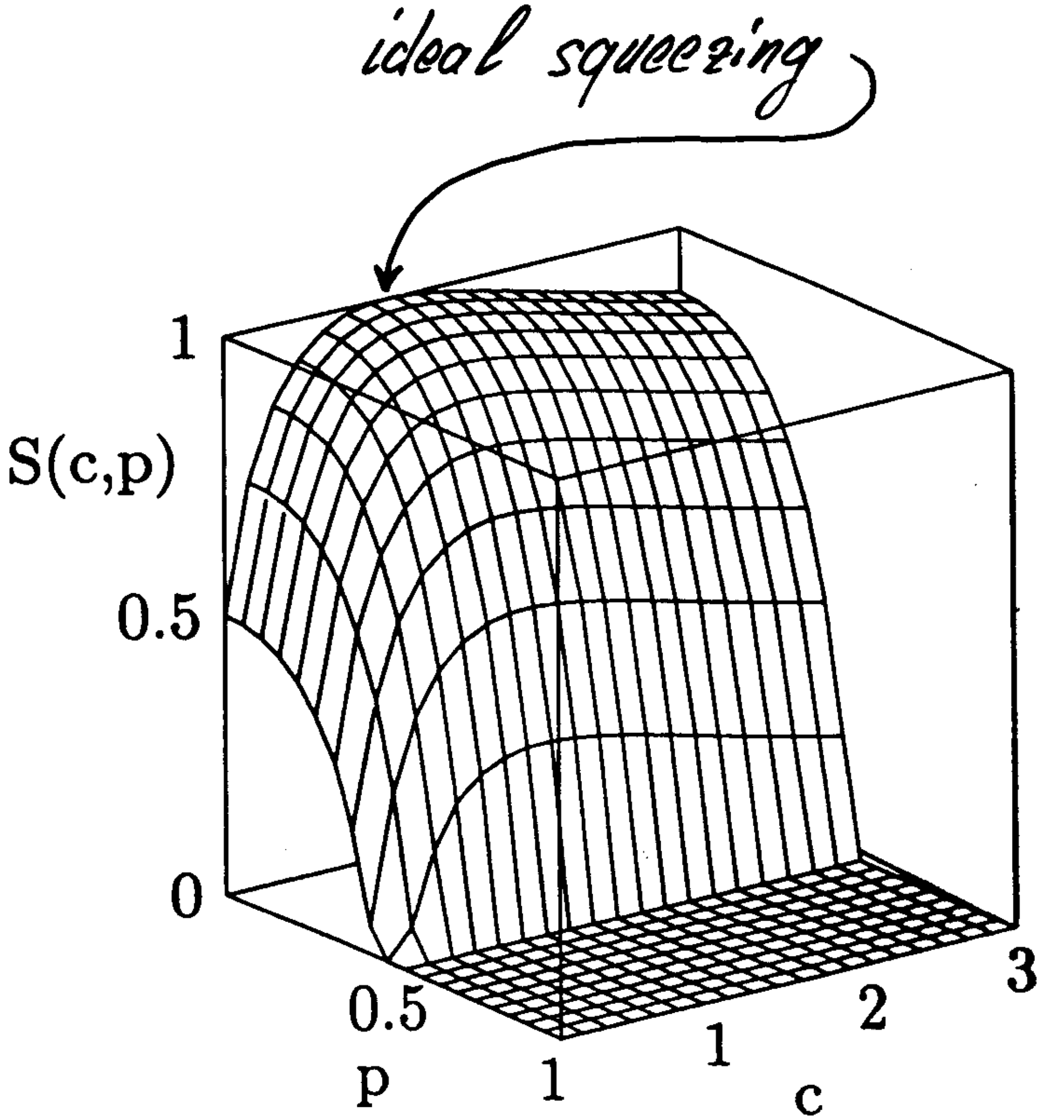
$$\frac{1}{\tau_a} = 4\varkappa_a\,\frac{(1-p)(1+c)}{3+c-2p} \quad \text{the width},$$

$$S(c,p) = \frac{1}{2}+\frac{2c}{(1+c)^2}-\frac{p^2}{2(1-p)} \quad -$$

the squeezing function;

$p \to 0$, $c=1$, ideal squeezing !!

The squeezing function



S = 0 coherent state

S > 0 squeezing

S = 1 ideal squeezing

Partial cooperativity

The system of equations of motion for SRL has a particular symmetry. From a solution $S_{ij}(t), a(t)$ for $\tilde{N}$ atoms in full cooperativity, i.e. $\tilde{C}_1 = \tilde{N}$, $\tilde{C}_2 = \tilde{N}^2$, $\tilde{C}_3 = \tilde{N}^3$, we can construct two other solutions as follows:

$$S_{ii}^e(t) = \frac{1-er}{3r}\tilde{C}_1 + e\,S_{ii}(t), \quad a^e(t) = e\,a(t), \qquad e = \pm 1,$$

$$S_{02}^e(t) = S_{02}(t), \quad S_{12}^e(t) = S_{12}(t), \quad S_{01}^e(t) = S_{01}(t), \qquad e = +1,$$

$$S_{02}^e(t) = S_{02}^+(t), \quad S_{12}^e(t) = -S_{12}^+(t), \quad S_{01}^e(t) = S_{01}^+(t), \qquad e = -1.$$

We require $\sum_i S_{ii}^e = N$ and obtain $\tilde{N} = rN$, i.e. new solutions are constructed from solutions for smaller number of atoms in full cooperativity.

For $e=+1$ we obtain

a) $\bar{S}_{ii}^{e} \geq (1-r)N/3$,

b) $c_2=(1+r^2)/3$, $c_3=c_3^{+}(r)$, i.e.

the upper boundary ;

For $e=-1$,

a) $(1-2r)N/3 \leq \bar{S}_{ii}^{e} \leq (1+r)N/3$,

$0 \leq r \leq 1/2$,

b) $c_2=(1+r^2)/3$, $c_3=c_3^{-}(r)$ –

the lower boundary ;

Stationary solutions

Rescaled parameters $\tilde{N}=rN$, $\tilde{p}=p/r$,

$$\bar{S}_{01}^{e}/\tilde{N} \equiv \tilde{x} = \sqrt{\frac{c\tilde{p}(1-\tilde{p})}{1+c}}, \quad \bar{S}_{02}^{e}/\tilde{N} = \frac{\tilde{x}^2}{\tilde{p}/c}, \quad \bar{S}_{12}^{e}/\tilde{N} = \frac{\tilde{x}}{\sqrt{c}},$$

$$\bar{S}_{00}^{e}/\tilde{N} = \left[\frac{1-er}{3r} + e\,\frac{c(1-\tilde{p})}{1+c}\right], \quad \bar{S}_{11}^{e}/\tilde{N} = \left[\frac{1-er}{3r} + e\tilde{p}\right],$$

$$\bar{S}_{22}^{e}/\tilde{N} = \left[\frac{1-er}{3r} + e\,\frac{1-\tilde{p}}{1+c}\right],$$

$$\bar{a}^{e} = -e\tilde{N}\sqrt{\frac{\tilde{p}(1-\tilde{p})}{1+c}} .$$

the linewidth and the squeezing spectrum for partial cooperativity are related to those for full cooperativity by simple rescaling

$$\Delta\nu_e = \frac{\kappa_a}{|\bar{a}^e|^2}\,\frac{\tilde{p}^2(1+c)^2+(1-\tilde{p})^2(1-c)^2}{c-1+2\tilde{p}},$$

$$\Delta\nu_e \sim 1/\tilde{N}^2 \quad , \quad \tilde{N} = rN$$

squeezing

$$\langle \delta u_{out}(\omega)\,\delta u_{out}(\omega')\rangle = (1/4)\,\delta(\omega+\omega')\left\{1-\frac{S(\tilde{p},c)}{1+\omega^2\tilde{\tau}_a'^2}\right\},$$

$$\frac{1}{\tilde{\tau}_a'} = 4\kappa_a\,\frac{(1-\tilde{p})(1+c)}{3+c-2\tilde{p}},$$

$$S(\tilde{p},c) = \frac{1}{2}+\frac{2c}{(1+c)^2}-\frac{\tilde{p}^2}{(1-\tilde{p})^2},$$

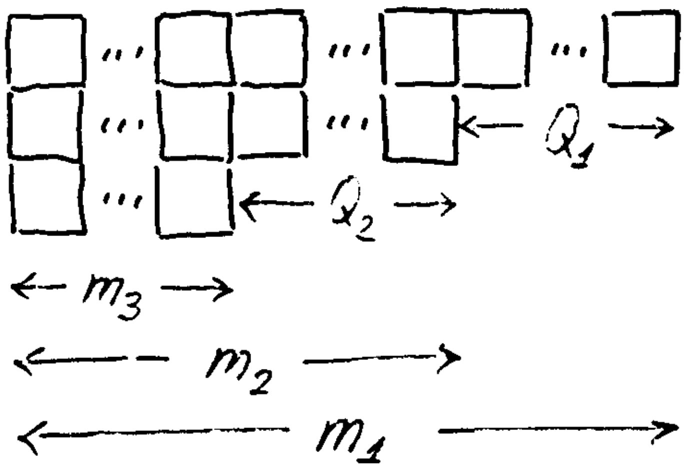
still ideal squeezing for $\tilde{p}\to 0,\ c=1$!

$i(S_{ij}+S_{ji})$ and $S_{ij}-S_{ji}$ are nine anti-Hermitian generators of the group $U(3)$; C_2 and C_3 are the Casimir operators of $U(3)$, commute with all generators, $[C_K, S_{ij}] = 0$.

The atomic Hilbert space:

3^N states $|K_1\rangle_1 |K_2\rangle_2 \dots |K_N\rangle_N$, $K_i = 0,-1,1$,

may be split into orthogonal eigenspaces of the Casimir operators (invariant subspaces). Each invariant subspace is uniquely specified by a Young frame:



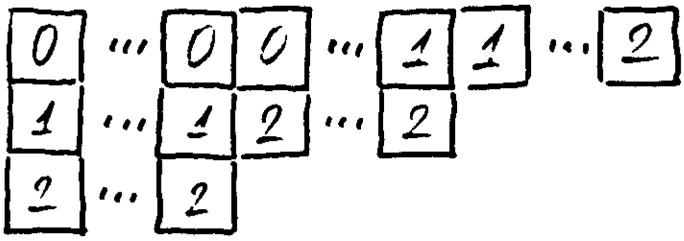
$Q_1 = m_1 - m_2$

$Q_2 = m_2 - m_3$

$m_1 + m_2 + m_3 = N$

$m_1 \geq m_2 \geq m_3$ - highest weights.

The basis states in a given invariant subspace are represented by Young tableaux:



The dimension d of a given inv. subspace is equal to [Barut, Rączka, 1986],

$$d = \frac{1}{2}(Q_1+1)(Q_2+1)(Q_1+Q_2+2)$$. i.e.

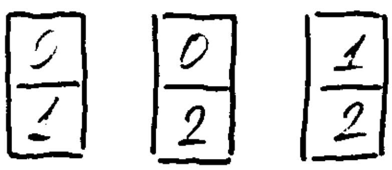
function only of Q_1 and Q_2.

Three-box columns physically irrelevant.

Subspaces with Q_1, Q_2 and $\tilde{Q}_1 = Q_2$, $\tilde{Q}_2 = Q_1$ have the same dimensions. However, physically different.

Example:

$Q_1 = 1, \; Q_2 = 0, \quad d = 3$

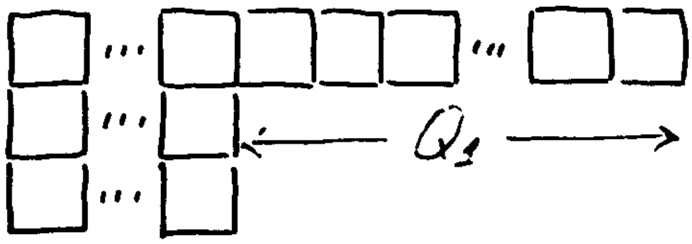


$Q_1 = 0, \; Q_2 = 1, \quad d = 3.$

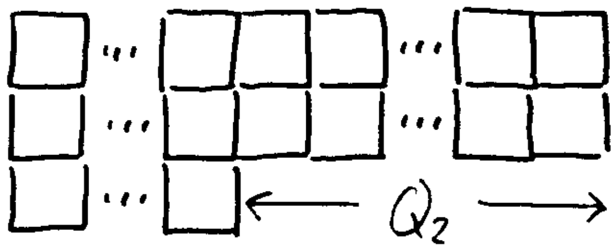
fundamental subspaces

Semiclassical limit, $N \gg 1$

The Young frames corresponding to the upper, lower, and right boundaries of C_3:

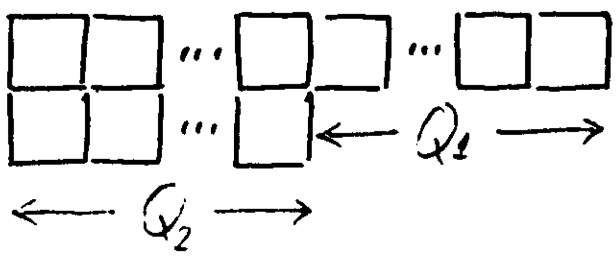


$Q_1 = rN$,
$Q_2 = 0$,



$Q_1 = 0$
$Q_2 = rN$, $0 \leq r \leq \frac{1}{2}$

because need $2rN$ atoms



$Q_1 + 2Q_2 = N$

need both fundam. representations

Three-box columns are physically irrelevant. The evolution takes place outside these blocs. Only $N - 3m_3 = Q_1 + 2Q_2$ atoms are responsible for the evolution.

Spontaneous emission

From the upper level 2 to level 1 with the rate γ_s, $\gamma_s \ll N\gamma, \mathcal{L}_a$.

$$\left(\partial S_{02}/\partial t\right)_{sp} = -\gamma_s S_{02} + \sqrt{2\gamma_s}\,\bar{\sum_\mu} S_{01}^{\mu}\eta_\mu ,$$

$$\left(\partial S_{12}/\partial t\right)_{sp} = -\gamma_s S_{12} - \sqrt{2\gamma_s}\,\bar{\sum_\mu}\left(S_{22}^{\mu} - S_{11}^{\mu}\right)\eta_\mu ,$$

$$\left(\partial S_{01}/\partial t\right)_{sp} = -\sqrt{2\gamma_s}\,\bar{\sum_\mu}\eta_\mu^{+} S_{02}^{\mu} ,$$

$$\left(\partial S_{00}/\partial t\right)_{sp} = 0 ,$$

$$\left(\partial S_{11}/\partial t\right)_{sp} = -\left(\partial S_{22}/\partial t\right)_{sp} = 2\gamma_s S_{22} - \sqrt{2\gamma_s}\,\bar{\sum_\mu}\left(\eta_\mu S_{21}^{\mu} + \eta_\mu^{+} S_{12}^{\mu}\right).$$

coll. relaxation $\sim N^2$, spont. em. $\sim N$,
for $N \gg 1$ we may confine ourselves to $\gamma_s \to 0$.
However not taking this limit blindly by $\gamma_s = 0$ from the onset.
Spont emission breaks $U(3)$ symmetry.

$$\frac{d}{dt}\bar{C}_2 = 4\gamma_s\left[\bar{S}_{02}\bar{S}_{20} + \bar{S}_{21}\bar{S}_{12} + \bar{S}_{22}\left(\bar{S}_{11} - \bar{S}_{22}\right)\right] ,$$

$$\frac{d}{dt}\bar{C}_3 = 6\gamma_s\Big[\left(\bar{S}_{22} + \bar{S}_{11}\right)\bar{S}_{21}\bar{S}_{12} + \bar{S}_{01}\bar{S}_{12}\bar{S}_{02} + \bar{S}_{10}\bar{S}_{02}\bar{S}_{21} + \bar{S}_{20}\bar{S}_{02}\bar{S}_{00} + \bar{S}_{22}\left(\bar{S}_{11}^{\,2} - \bar{S}_{22}^{\,2} + \bar{S}_{01}\bar{S}_{10}\right)\Big].$$

stationary solution, $d\bar{C}_2/dt = 0$, $d\bar{C}_3/dt = 0$; for $\gamma_s = 0$ trivially fulfilled; for $\gamma_s \neq 0$ obtain two conditions previously played by conserv. of C_2, C_3; then let $\gamma_s \to 0$.

$\bar{C}_{01}^{e}/N = x$ et cetera, obtain for x (from $d\bar{C}_2/dt = 0$)

$$2(1+c)\,x^4 + cp(e+cp)\,x^2 + p^3c^2(p-e) = 0 \,,$$

$$\left(x_{1,2}^{e}\right)^2 = \frac{cp(1+cpe)}{4(1+c)}\left\{-e \pm \sqrt{1 - \frac{8e(1+c)p(1-pe)}{(1+cpe)^2}}\right\}$$

x_1^e "+" , x_2^e "−" ;

$(x_2^+)^2$ – negative, nonphysical,

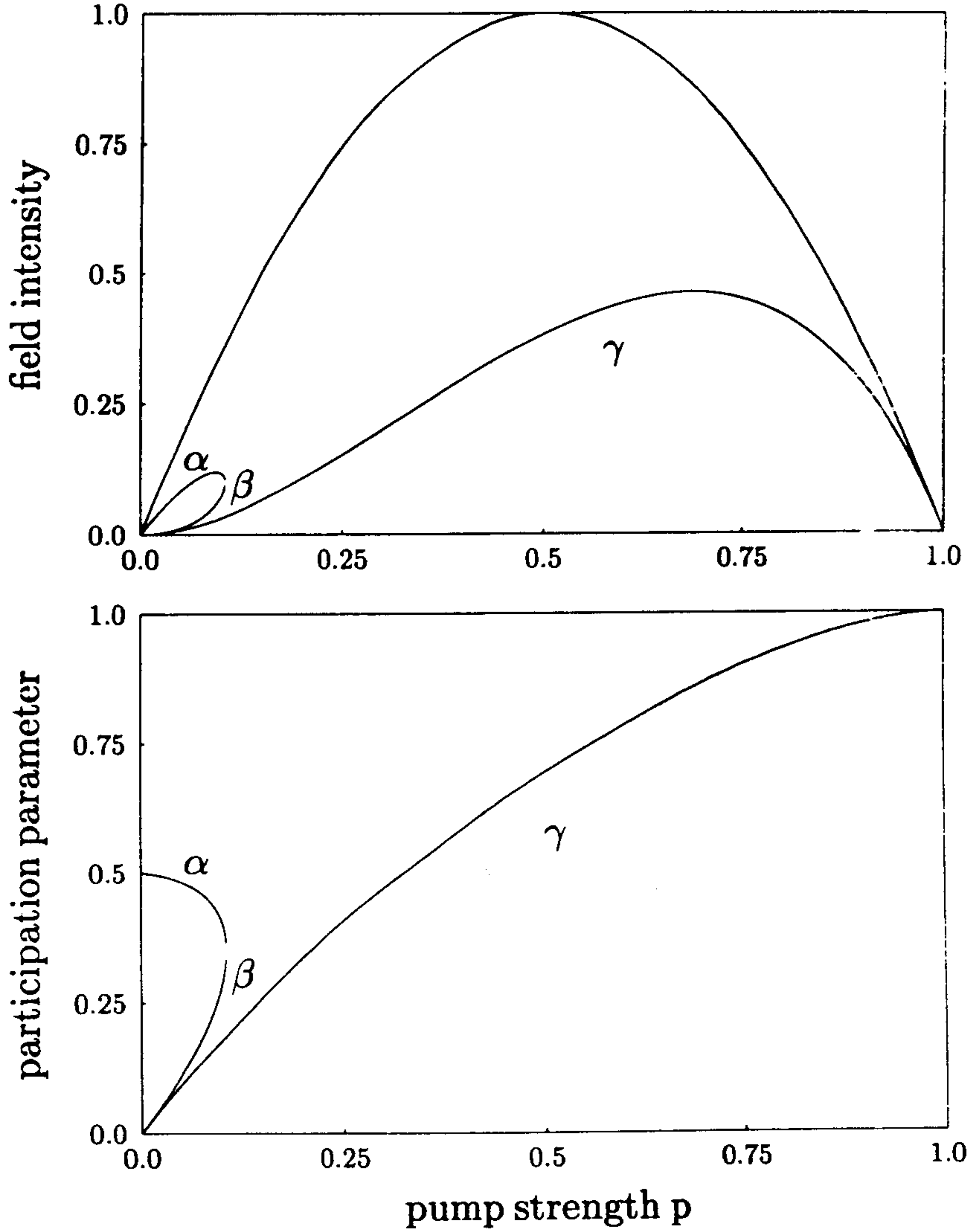
three solutions: $x_1^- \equiv x_\alpha$, $x_2^- \equiv x_\beta$, $x_1^+ \equiv x_\gamma$

these can be viewed as stationary solutions with partial cooperativity with some particular value of r

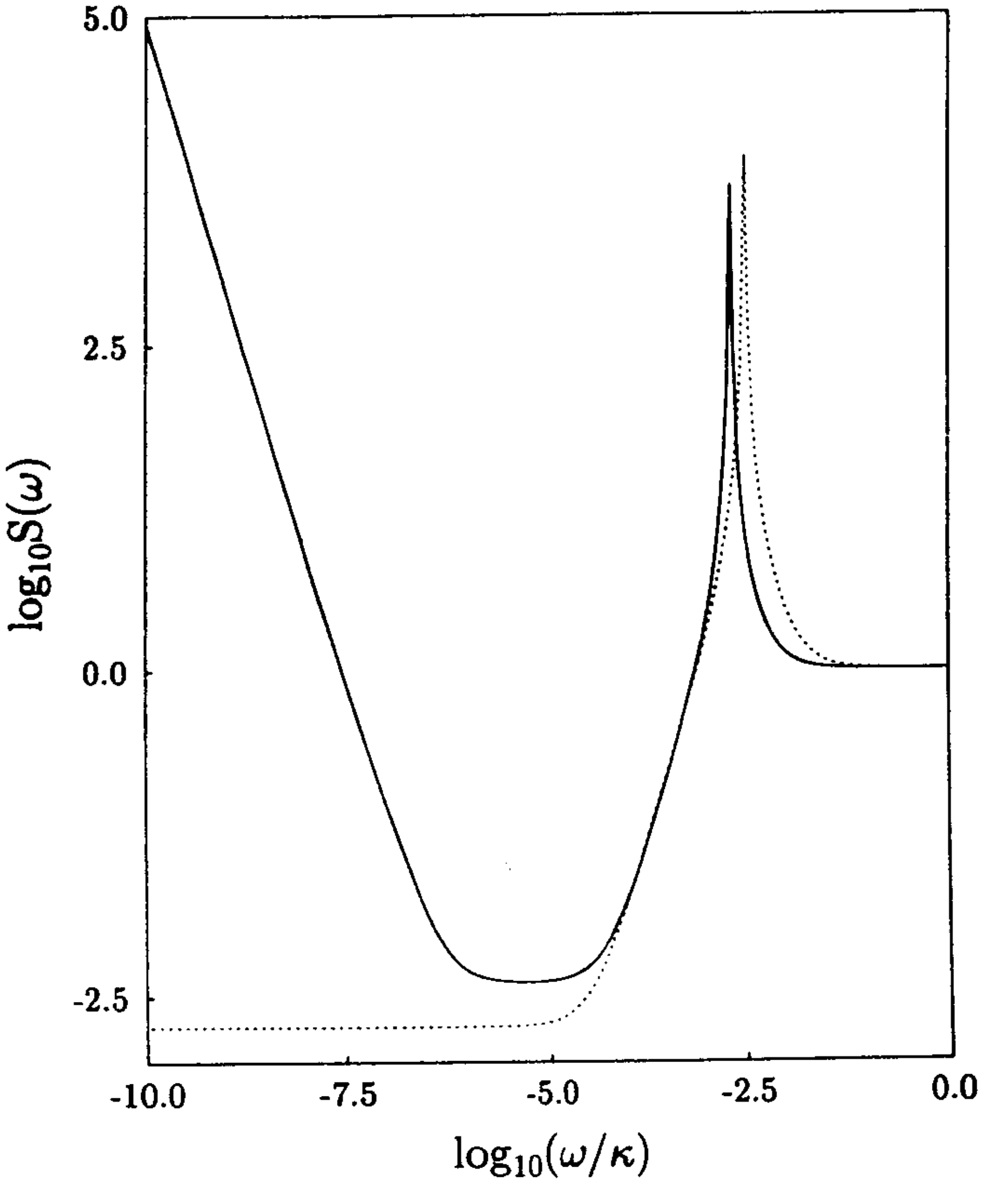
$$x = \sqrt{\frac{pc(r-p)}{1+c}} \;\to\; r_a = p + \frac{1+c}{pc}\,x_a^2 \,,$$

$$a = \alpha, \beta, \gamma \,.$$

Stationary intracavity field intensity and participation parameter vs pump strength p without and with spontaneous emission



Squeezing spectrum for the amplitude quadrature component without (dots) and with (solid) spontaneous emission for the "alpha" solution: $\gamma_s/N\gamma = 10^{-10}$, $N\gamma/\kappa_a = 0.01$, $c = 0.9$, $p = 0.03$



Summary

superradiant laser : $I \sim N^2$,

$\Delta\nu \sim 1/N^2$ (Schawlow-Townes),

100% squeezing at low frequencies ;

partial cooperativity :

region of admissible cooperativity parameters ; two stationary solutions (upper and lower boundaries) ; two fundamental representations of $U(3)$; rescaling

$$N \to rN \; ; \; p \to p/r \; ;$$

spont emission :

symmetry breaking of $U(3)$; three solutions $x_\alpha, x_\beta, x_\gamma$; high narrow peak in the noise spectrum around $\omega = 0$; squeezing still possible for $\omega \neq 0$.

References

F. Haake, M. Kolobov, C. Fabre, E. Giacobino, and S. Reynaud, PRL 71, 995 (1993);

F. Haake, M. Kolobov, C. Seeger, C. Fabre, E. Giacobino, and S. Reynaud, PRA 54, 1625 (1996);

C. Seeger, M. Kolobov, M. Kuś, and F. Haake, PRA 54, 4440 (1996);

R. Dicke, PR 93, 493 (1954);

M. Gross and S. Haroche, Phys. Rep. 93, 302 (1982);

H. Ritsch, M. Marte, and P. Zoller, Europhys. Lett. 19, 7 (1992);

A. Mostowski and M. Stark, Introduction to Higher Algebra (Pergamon Press, Oxford, 1964);

M. Marden, Geometry of Polynomials (Amer. Mathematical Society, Providence, 1966);

A. O. Barut and R. Rączka, Theory of Group Representation and Applications (World Scientific, Singapore, 1986).