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I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

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Free Electron Lasers

C. Pellegrini

Physics Department, University of California at Los Angeles, U.S.A.

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Claudio Pellegrini
Physics Department, University of California at Los Angeles
Los Angeles, California, 90095

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Abstract

We review the physics of Free Electron Lasers, and the collective FEL instability that characterizes this system. We show that the instability can be described with a single parameter, the FEL parameter, that determines the instability growth rate (or FEL gain), and the saturation length and power. We also discuss the physics and technology issues relevant for the operation of an FEL at a wavelength of about 0.1nm, and the expected peak power, average power, and brightness of such a source. We show that based on our present understanding of FEL physics, and using existing technology, we can build a 0.1nm FEL with peak power of about 10 GWatt, pulse duration of about 0.1 ps, and bandwidth of about 0.1%.

1 INTRODUCTION

Free Electron Lasers (FELs) are powerful sources of coherent electromagnetic radiation, operating in the microwave to UV region of the spectrum. Recent progress in the FEL physics and technology makes now possible to extend their wavelength into the Soft X-ray and X-ray region, with peak power and brightness many orders of magnitude larger than that obtainable from other sources.

In this paper we review the basic properties of FELs, starting from the FEL collective instability [1], and relate the FEL growth rate to the electron beam phase-space density. We then discuss the development of FELs as a high power, coherent source of Soft X-ray and X-ray radiation. This development is made possible by the recent progress in the production of high brightness electron beams. and by the use of the Self Amplified Spontaneous Emission (SASE) mode of operation of the FEL [1], [2], [3].

In the SASE mode lasing is produced in a single pass of a high phase-space density electron beam through a long undulator, eliminating the need for optical cavities, difficult to build in the Soft X-ray or X-ray spectral region. However in the SASE approach the requirements on the electron beam peak current, emittance, and energy spread are very stringent [5], [4] and until recently difficult to satisfy. This situation has been changed by the recent development of high-brightness radio frequency photocathode electron guns [6], and the progress in accelerating and compressing these beams without spoiling their brightness resulting from the work on linear colliders [7],[8], [9],[10], [11]. As a result there is now the possibility to make a major extension of FEL operation, from the shortest wavelength yet achieved -240 nm- to 0.1 nm [12], [13], [14], [15], [16], [17].

In this paper we will first review the spontaneous emission of radiation from an electron crossing an undulator magnet, the basic element of FEL physics. We will then discuss the production of coherent undulator radiation, and its connection to FELs. The next section will be a review of the 1-dimensional model of the FEL, with a discussion of the FEL collective instability, the derivation of the exponential gain, and of the effect of the beam energy spread on the gain.

2 Principle of operation

2.1 Undulator radiation from one electron

The FEL is based on the emission of radiation from relativistic electrons moving in an undulator magnetic field (undulator radiation). We will first review the basic characteristics of the undulator radiation from a single electron; we will then discuss how the undulator radiation intensity is increased in an FEL, using high brightness electron beams. We assume for simplicity the undulator to have a helical magnetic field, of amplitude B_u, and period λ_u. A more detailed discussion, including the case of planar undulators, can be found in reference [18], to which we refer the reader. If z is the undulator axis, and x, y, the two perpendicular directions, the undulator field, near to the axis, is approximately

$$B_x = B_u \cos(2\pi \frac{z}{\lambda_u}),\ B_y = B_u \sin(2\pi \frac{z}{\lambda_u}). \tag{1}$$

One relativistic electron of energy $E = mc^2\gamma$, and momentum $p_z >> p_x, p_y$, traversing the undulator, executes a helical trajectory, with constant velocity V_z, and transverse velocity

$$\frac{V_x}{c} = \frac{a_u}{\gamma} \sin(2\pi \frac{z}{\lambda_u}), \frac{V_y}{c} = \frac{a_u}{\gamma} \cos(2\pi \frac{z}{\lambda_u}), \tag{2}$$

where

$$a_u = \frac{eB_u\lambda_u}{2\pi mc^2}, \tag{3}$$

is called the undulator parameter, and is the undulator vector potential normalized to the electron rest energy. The helix radius is

$$R = 2\pi \frac{a_u\lambda_u}{\gamma}. \tag{4}$$

For a relativistic beam the periodic magnetic field of the undulator appears approximately, using the Weiszacker-Williams approximation, as a circularly polarized plane wave, of wavelength $\lambda = \lambda_u/\gamma_z$, where $\gamma_z = \gamma/\sqrt{1+a_u^2}$ is the relativistic factor for the Lorentz transformation to the frame where the longitudinal momentum of the electron is zero. Some of the photons in this plane wave can be backscattered by the electrons, and in the laboratory frame they will appear as spontaneous radiation, emitted in a narrow line centered at the wavelength

$$\lambda = \frac{\lambda_u}{2\gamma^2}(1+a_u^2). \tag{5}$$

The additional factor γ_z in (5) appears when transforming again to the Laboratory system

The spontaneous radiation is emitted in a narrow cone of aperture $1/\gamma$ around the axis. The width of the radiation line (bandwidth) is related to the number of undulator periods N_u [18] by

$$\frac{\Delta\omega}{\omega} = \frac{1}{2N_u}. \tag{6}$$

The undulator is an extended linear source, but it can be approximately described as an equivalent source at the undulator center, with angular aperture

$$\theta == \sqrt{\frac{\lambda}{\lambda_u N_u}}, \tag{7}$$

and an effective source radius (diffraction limited)

$$a = \frac{1}{4\pi}\sqrt{\lambda\lambda_u N_u}. \tag{8}$$

Notice that the product

$$a\theta = \frac{\lambda}{4\pi}, \tag{9}$$

gives the minimum phase space for a diffraction limited photon beam.

The intensity of the radiation emitted on axis, and at the wavelength (5) is [18]

$$\frac{d^2I}{d\omega d\Omega} = 2N_u^2\frac{e^2}{c}\gamma^2\frac{a_u^2}{(1+a_u^2)^2}. \tag{10}$$

The coherent intensity is obtained by multiplying (10) by the solid angle corresponding to (7) and the bandwidth (6). Dividing this intensity by the energy of a photon with the wavelength (5) we can also rewrite the coherent intensity as the number of photons per electron within the solid angle corresponding to (7) and bandwidth (6) as

$$N_{ph} = \pi\alpha\frac{a_u^2}{1+a_u^2}, \tag{11}$$

where α is the fine structure constant. For a typical value $a_u \approx 1$ one obtains $N_{ph} \approx 10^{-2}$, showing that the undulator radiation process is rather inefficient.

The number of photons per electron is the basic number determining the brightness of an undulator source. In fact assuming that the electron beam transverse radius and angular divergence are smaller than that of the effective radiation source defined previously, the brightness is given by

$$B_\lambda = 4\pi^2 \frac{\dot{N}_e N_{ph}}{\lambda^2}, \tag{12}$$

where $\dot{N}_e$ is the number of electrons per second, proportional to the average electron current. In the opposite case, which can be characterized by the condition that the beam emittance ϵ, the product of the electron beam transverse radius and angular divergence, is larger than the corresponding quantity for the radiation beam, $\lambda/4\pi$, the brightness becomes

$$B_\lambda = \frac{\dot{N}_e N_{ph}}{\epsilon^2}. \tag{13}$$

The conventional definition of brightness considers also the frequency spread of the radiation, assuming a 0.1% spread as the reference. The previous definition then becomes

$$B_\lambda = \frac{\dot{N}_e N_{ph}}{\epsilon^2} \frac{10^{-3}}{\Delta\omega/\omega}. \tag{14}$$

The transverse electron beam brightness is defined in a similar way as

$$B_{e,T} = e\frac{\dot{N}_e}{\epsilon^2}. \tag{15}$$

In defining the brightness it is important to distinguish between the peak brightness and the average brightness, depending on wether we use the peak or average value of $\dot{N}_e$. For instance the peak electron beam current is what determines the FEL gain and performance. Similarly the peak photon brightness is important for X-ray imaging, while the average brightness might be important for other applications.

2.2 FEL radiation

To increase the peak brightness for a given wavelength we can either increase the electron current or increase the number of photons produced per electrons. An FEL achieves the second goal, by increasing the number of photons per electron by about seven orders of magnitude.

How do we increase the number of photons emitted per electron? If we have many electrons, say N_e , and they are all grouped within a small fraction of a wavelength, the total intensity would be the single particle intensity times N_e, and the number of photons per electron would be increased by a factor N_e. In practical cases the electrons are in a bunch much longer than the radiation wavelength, and their position distribution on a scale of λ is completely random. As a result the radiation fields emitted by different electrons have a random relative phase, the total intensity is proportional to N_e , and the number of photons per electrons is still given by (12).

However also in this case we can increase the number of photons emitted per electron if we take advantage of a collective instability of the electron beam-EM radiation field-undulator system [1]. This instability works as follows:

1. the electron beam interacts with the electric field of the radiation; the electric field is perpendicular to the direction of propagation of the beam (the undulator axis), and is parallel to the wiggling (transverse) velocity (2) of the electrons produced by the undulator magnet, of amplitude a_u/γ; the interaction produces an electron energy modulation, on the scale λ;

2.the electron energy modulation modifies the electron trajectory in the undulator, in a such a way to produce bunching of the electrons at the scale λ;

3. electrons bunched within a wavelength emit radiation in phase, thus producing a larger intensity; the larger intensity leads to more energy modulation and more bunching, leading to exponential growth of the radiation; the intensity can reach the limit $I \sim N_e^2$ for the case of extreme bunching (superradiance) [19]

The FEL instability will be discussed in detail in the following section, in the simple case of a one dimensional theory. Here we will summarize some of the most important results for the FEL physics. For the collective instability to occur there are some conditions that must be satisfied . These conditions depend on the FEL parameter [1],

$$\rho = (\frac{a_u}{4\gamma}\frac{\Omega_p}{\omega_u})^{2/3}, \tag{16}$$

and on the beam emittance, ϵ [4], [5]. The quantities in (16) are: $\omega_u = 2\pi c/\lambda_u$ is the frequency associated to the undulator periodicity,

$$\Omega_p = (\frac{4\pi r_e c^2 n_e}{\gamma})^{1/2}, \tag{17}$$

is the beam plasma frequency, n_e is the electron density, and r_e is the classical electron radius. The FEL parameter characterizes the instability, giving the instability growth rate, or gain length,

$$L_G \approx \frac{\lambda_u}{2\sqrt{3}\pi\rho} \tag{18}$$

The amplitude of the radiation field grows exponentially along the undulator axis z, as $A \approx A_0 \exp(z/L_G)$. The conditions are:

a. beam emittance smaller than the wavelength:

$$\epsilon < \frac{\lambda}{4\pi} \tag{19}$$

b. beam energy spread smaller than the FEL parameter:

$$\sigma_E < \rho \tag{20}$$

c. undulator length larger than the gain length:

$$N_u \lambda_u >> L_G \tag{21}$$

d. the gain length must be shorter than the radiation Raleigh range:

$$L_G < L_R, \tag{22}$$

where the Raleigh range is defined in terms of the radiation beam radius, w_0, and the wavelength by $\pi w_0^2 = \lambda L_R$.

Condition a says that for the instability to occur the electron beam must match the angular and transverse space characteristics of the radiation emitted by one electron in traversing the undulator, equations (7, 8, 9). This is also the condition to obtain from the electron beam diffraction limited spontaneous radiation. Notice that for nanometer wavelength this condition cannot be met at present by storage ring based synchrotron radiation sources, but it can be satisfied by electron beams produced by a Radio Frequency laser driven electron guns (photoinjectors), as we will discuss in the following sections.

Condition b limits the beam energy spread to a value such that the width of the spontaneous radiation line is not increased. Conditions c introduces a requirement on the minimum undulator length for this process to become significant. Condition d requires that more radiation is produced by the beam than what is lost through diffraction out of the finite radius beam. Conditions a and d both depend on the beam radius and the radiation wavelength , and are not independent. If they are satisfied diffraction and 3-dimensional effects are not important, and we can use with good approximation the 1-dimensional model.

If these conditions are satisfied the radiation field emitted by the beam will grow exponentially along the undulator length, with a growth rate given by (18). The field will saturate after an undulator length (saturation length,) of the order of ten gain length. At saturation the radiation power is given by [1]

$$P_{sat} = \rho I_{beam} E_{beam}, \tag{23}$$

where I_{beam} is the beam current, and E_{beam} the beam energy, and the number of photons per electron is

$$N_{sat} = \rho \frac{E}{E_{ph}} \tag{24}$$

If we consider a case of interest to us for a Soft X-ray FEL, with $E_{ph} \approx 250eV$, $E \approx 3GeV$, $\rho \approx 10^{-3}$, we obtain $N_{ph} \approx 10^4$, i.e. an increase of almost 6 orders of magnitude in the number of photons produced per electron. This increase is reflected in a much larger brightness.

3 The 1-dimensional FEL theory

AS we discussed in the previous section the system consisting of an electron beam, an electromagnetic field and an undulator, is an unstable system if the beam has a longitudinally uniform distribution. The system will evolve toward a state in which the electrons are bunched at the radiation wavelength, i.e. they are equally spaced in the longitudinal direction and separated by a distance equal to λ. To see how this transition from the initial to the final state can take place we have to study the Maxwell equations for the electromagnetic field, in combination with the equation of motion for the electrons in the combined field of the undulator magnet and of the radiation field. we will follow closely the work of references [1], [18]. Let us look at the equation describing the electron energy change in an E-M field of amplitude A and phase Ψ,

$$mc^2\frac{d\gamma}{dt} = eA\frac{a_u}{\gamma}\sin(\Phi + \Psi) \tag{25}$$

where the phase is

$$\Phi = 2\pi(\frac{1}{\lambda} + \frac{1}{\lambda_u})\beta_z - \omega. \tag{26}$$

If $z = \beta_z ct + z_0$, and the distribution of z_0 is uniform, as is usually the case for an electron beam produced in an accelerator, then the phase covers all values between 0 and 2π, and some particles will gain energy, while some will loose energy. The change in energy will result in a change of velocity, which will produce a change in phase. Hence the system will evolve.

An equilibrium state for the system exists, and corresponds to electrons being at the phase $n\pi$, which gives zero energy change, and also which corresponds to all electrons having the resonant energy. Hence if we start from the initial state with a uniform distribution in phase and in longitudinal electron distribution, the system will evolve toward the "bunched" equilibrium state. The speed at which the system will evolve will define the gain length.

To describe this situation and this evolution we need to introduce the dynamical variables describing the electron-electromagnetic field system. For simplicity we consider only the case of a helical undulator.

We also make the following approximations:

1. we assume that the beam transverse size is much larger than the radiation wavelength, and use a 1-dimensional picture, neglecting diffraction; the E-M field is described as a plane wave and the beam transverse density distribution is assumed constant;

2. We assume that the field is propagating in vacuum;

3. we assume that the E-M field, oscillating at the frequency $\omega = 2\pi c/\lambda$, has an amplitude and a phase that change slowly, and simplify Maxwell equations neglecting their second derivatives;

4. we neglect the slippage, $S = N_u\lambda$, between electrons and the E-M field due to the difference in velocity, assuming that it is smaller than the bunch length, L_e, or $S/L_e << 1$;

5. We neglect the effects due to the beam emittance.

The E-M Field is described with the vector potential of the undulator and of the radiation field,

$$\vec{A}=\vec{A}_u + \vec{A}_R \, . \tag{27}$$

Using circularly polarized waves and a helical undulator we have

$$\vec{A}_u= -\frac{B_u}{k_u}(\cos(k_u z)\ \vec{x} + \sin(k_u z)\ \vec{y}), \tag{28}$$

where $k_u = 2\pi/\lambda$, and

$$\vec{A}_R= -\frac{A(z,t)}{k_R}(-\cos(\phi_R)\ \vec{x} + \sin(\phi_R)\ \vec{y}), \tag{29}$$

where $k_R = 2\pi/\lambda$,

$$\phi_R = k_R(z - ct) + \psi(z,t), \tag{30}$$

and $A(z,t)$, and $\psi(z,t)$ are the slowly varying amplitude and phase of the wave.

The system Hamiltonian is

$$H = c[(\vec{P} - \frac{e}{c}\ \vec{A})^2 + m^2c^2]^{1/2}, \tag{31}$$

and does not depend on x and y. Hence the corresponding canonical momenta are constants of the motion. From these constants we can obtain the electron transverse velocity

$$\beta_x = \frac{e}{mc^2\gamma}(A_{u,x} + A_{R,x}) + \frac{P_{x,0}}{mc\gamma}, \tag{32}$$

and

$$\beta_y = \frac{e}{mc^2\gamma}(A_{u,y} + A_{R,y}) + \frac{P_{y,0}}{mc\gamma}. \tag{33}$$

Introducing the quantities

$$a_u = \frac{eB_u}{mc^2 k_u}, \tag{34}$$

$$a_R = \frac{eA(z,t)}{mc^2 k_R}, \tag{35}$$

we also have

$$\beta_x = \frac{1}{\gamma}[a_u \cos(k_u z) - a_R \cos(\phi_R)] + \frac{P_{x,0}}{mc\gamma}, \tag{36}$$

$$\beta_x = \frac{1}{\gamma}[a_u \sin(k_u z) + a_R \sin(\phi_R)] + \frac{P_{y,0}}{mc\gamma} \tag{37}$$

Notice that for typical cases $a_u \approx 1, a_R << 1$.

In the approximation of neglecting emittance effects, we will assume for the time being that the two initial values of the canonical momenta, $P_{x,0}, P_{y,0}$ can be neglected.

We could now use the Hamiltonian to obtain the equations of motion for z, and P_z , but it is simpler to use instead the equivalent equations for the energy change, and for the phase change

$$mc^2 \frac{d\gamma}{dt} = \frac{eA(z,t)a_u}{\gamma} \sin(\Phi + \psi(z,t)), \tag{38}$$

$$\frac{d\Phi}{dt} = (k_R + k_u)z - \omega, \tag{39}$$

and obtain the longitudinal velocity from the energy, using the relationship

$$\beta_z = (1 - \frac{1}{\gamma^2} - \beta_x^2 - \beta_y^2)^{1/2}. \tag{40}$$

For relativistic velocities this can be approximated as

$$\beta_z \approx 1 - \frac{1}{2\gamma^2} - \frac{\beta_x^2 + \beta_y^2}{2} = 1 - \frac{1}{2\gamma^2}(a_u^2 + a_R^2 - 2a_u a_R \cos(\Phi + \psi). \tag{41}$$

Next we need the equations for the field. We write Maxwell equations in the form

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}) \vec{A} = -\frac{4\pi}{c} \vec{J}, \tag{42}$$

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})V = -4\pi\rho. \tag{43}$$

The current and charge density are

$$\rho = e\sum \delta(\vec{r} - \vec{r_i(t)}), \tag{44}$$

$$\vec{J} = ec\sum \vec{\beta_i}\, \delta(\vec{r} - \vec{r_i(t)}). \tag{45}$$

We consider first the effects of the transverse fields, and introduce the quantities

$$\hat{J} = J_x - iJ_y = ec\sum \delta(\vec{r} - r_i(t))\frac{1}{\gamma_i}[a_u e^{-ik_u z} - a_R e^{i\phi_R}], \tag{46}$$

$$\hat{A} = A_x - iA_y = i\frac{\alpha}{k_R}e^{ik_R(z-ct)}, \text{ with } \alpha(z,t) = -iA(z,t)e^{i\psi}. \tag{47}$$

In our approximation of a plane wave the sum over particles can be rewritten as

$$\sum \delta(\vec{r} - r_i(t)) \approx \sigma_0 e \vec{\sum} \delta(z - z_i(t)), \tag{48}$$

where σ_0 is the transverse beam density.

With these definitions, and using the slowly varying amplitude and phase approximation, the Maxwell equations for the complex amplitude α become

$$(\frac{\partial}{\partial z} - \frac{1}{c}\frac{\partial}{\partial t})\alpha = 2\pi\sigma_0 \sum_{l=1}^{N} \frac{1}{\gamma_l} e \sum \delta(z - z_l(t))(a_u e^{-i\Phi_l} - \frac{ie\alpha}{mc^2 k_R}). \tag{49}$$

We write now the complete set of 1-D equations, using complex notations, and introducing the "resonant longitudinal velocity" β_{zR} and the resonant energy γ_R such that

$$\lambda = \lambda_u \frac{1 - \beta_{zR}}{\beta_{zR}} = \lambda_u \frac{1 + a_u^2}{2\gamma_R^2}. \tag{50}$$

.Notice that to establish the relationship between the resonant longitudinal velocity and the resonant energy we assume that the electrons are ultrarelativistic, and that the transverse velocity is determined only by the undulator magnetic field and not by the radiation field. The set of equations is

$$\frac{d}{dt}\Phi_l = k_u(1 - \frac{\gamma_R^2}{\gamma^2}) + \frac{k_R}{2\gamma^2}\{a_u(i\alpha e^{i\Phi_l} - cc) - |\alpha|^2\}, \tag{51}$$

$$\frac{d}{dt}\gamma_l = \frac{a_u k_R}{2\gamma_l}(\alpha e^{i\Phi_l} + cc), \tag{52}$$

$$(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t})\alpha = 2\pi\sigma_0 \sum_{l=1}^{N} \frac{1}{\gamma_l}\delta(z - z_l)(a_u e^{-i\Phi_l} - \frac{ie\alpha}{mc^2 k_R}). \tag{53}$$

3.1 The small slippage regime

The field equations can be simplified in the small slippage regime. Due to the difference in velocity between the electrons and the radiation field, the field will advance in front of the electrons by one wavelength per undulator period. We call slippage the quantity

$$S = N_u \lambda, \tag{54}$$

N_u being the number of periods in the undulator. To characterize different FEL regimes we use the ratio of the slippage to the electron bunch length L_e.

The FEL regimes can be characterized according to wether this ratio is larger or smaller than one. In this paper we will consider only the case in which the ratio is much less than one. For a discussion of the general case the reader is referred to reference [19].

To consider the effect of the slippage we write the electromagnetic field amplitude as a function of t and $z - V_g t$, where V_g is the group velocity

$$\alpha = \alpha(t, \frac{z - V_G t}{L}). \tag{55}$$

In what follows we will assume to simplify our discussion that $V_g = c$. The dependence on $z - ct$ is used to describe the shape of the wave packet, which we assume to have a length L, on the order of the electron bunch length, L_e , and the field amplitude can be written as

$$\alpha(t, z - ct) = \sum \alpha_n(t) e^{i2\pi n(z-ct)/L}. \tag{56}$$

If we evaluate the argument $z - ct$, at the particle position $z = z_l(t) = Vt + z_{l0}$ we have, using (5,54),

$$\frac{z_l - ct}{L} = -\frac{ct(1-\beta_z)}{L} + \frac{z_{l0}}{L} < \frac{N\lambda_u}{2L\gamma^2} + \frac{z_{l0}}{L} \approx \frac{S}{L} + \text{ constant.} \tag{57}$$

If we are in the small slippage case, $S/L << 1$, the argument $z - ct$ in (55) is a constant, and we can approximate $\alpha(t, z - ct) \approx \alpha(t, \text{constant})$ and the only important component of the expansion in (56) is $\alpha_0(t)$.

The field equation now becomes

$$\frac{\partial}{\partial t}\alpha_0(t) = 2\pi \frac{ec\sigma_0}{L}\{a_u N_e B(t) - \frac{ie\alpha_0}{mc^2 k_R}\sum_{l=1}^{N_e}\frac{1}{\gamma_l}\}, \tag{58}$$

where

$$B(t) = \frac{1}{N}\sum_{l=1}^{N_e} e^{-i\Phi_l} \tag{59}$$

is the bunching factor, which plays a key role in the FEL physics.

3.2 The normalized FEL equation

We now put the FEL equations in a form that facilitate their analysis, using the notations introduced in ([1]).

We introduce the beam plasma frequency

$$\Omega_p = (\frac{4\pi r_e c^2 n_e}{\gamma})^{1/2}, \tag{60}$$

the FEL parameter

$$\rho = (\frac{a_u}{4\gamma}\frac{\Omega_p}{\omega_u})^{2/3}, \tag{61}$$

and the detuning parameter

$$\delta = \frac{\gamma^2 - \gamma_R^2}{2\rho\gamma_R^2}, \tag{62}$$

where γ, γ_R are the beam energy and the resonant energy. Notice that using the relationship between energy and wavelength of the emitted radiation, equations (5, 52), the quantity δ can also be written as

$$\delta = \frac{\lambda - \lambda_R}{4\lambda_R\rho}, \tag{63}$$

and represents the detuning of the FEL measured in units of the FEL parameter ρ.

We then introduce the scaled variables

$$\tau = 4\pi\rho\frac{\gamma_R^2}{\gamma^2}\frac{ct}{\lambda_u}, \tag{64}$$

$$\Gamma_l = \frac{\gamma_l}{\rho\gamma}, \tag{65}$$

$$\Psi_l = \Phi_l - \omega_u(1 - \frac{\gamma_R^2}{\gamma^2})t, \tag{66}$$

$$\hat{A} = \frac{ea_u}{4mc^2\gamma_R^2 k_u\rho^2}\alpha_0 e^{i\omega_u(1-\frac{\gamma_R^2}{\gamma^2})t}. \tag{67}$$

With these notations the FEL equations are:
the phase equation

$$\frac{d\Psi_l}{d\tau} = \frac{1}{2\rho}(1 - \frac{1}{\rho^2\Gamma_l^2}) + \frac{i}{\rho\Gamma_l^2}(\hat{A}\, e^{i\Psi_l} - cc) - 2\rho\frac{(1+a_u^2)}{a_u^2\Gamma_l^2}|\hat{A}|^2, \tag{68}$$

the energy exchange equation

$$\frac{d\Gamma_l}{d\tau} = -\frac{1}{\rho\Gamma_l}\{\hat{A}\, e^{i\Psi_l} + cc\}, \tag{69}$$

the field equation

$$\frac{d\hat{A}}{d\tau} = < \frac{e^{-i\Psi}}{\Gamma} > + i\delta\,\hat{A} - 2i\,\hat{A}\,\rho\frac{1+a_u^2}{a_u^2} < \frac{1}{\Gamma} >, \tag{70}$$

where $<> = (1/N_e)\sum_{l=1}^{N_e}$.

In most cases the FEL parameter is small and we can simplify the equations as

$$\frac{d\Psi_l}{d\tau} = \frac{1}{2\rho}(1 - \frac{1}{\rho^2\Gamma_l^2}), \tag{71}$$

$$\frac{d\Gamma_l}{d\tau} = -\frac{1}{\rho\Gamma_l}(\hat{A}\, e^{i\Psi_l} + cc), \tag{72}$$

$$\frac{dA}{d\tau} = < \frac{e^{-i\Psi}}{\Gamma} > + i\, \hat{A}\, \delta, \tag{73}$$

which we will use as the basic FEL equations.

3.3 Hamiltonian and conservation laws.

The FEL equations have two conserved quantities:

$$E = \mid \hat{A} \mid^2 + < \Gamma >, \tag{74}$$

and

$$H = \sum_l \{ \frac{1}{2\rho^2}(\rho\Gamma_l + \frac{1}{\rho\Gamma_l}) - \frac{i}{\rho^2\Gamma_l}(\hat{A}\, e^{i\Psi_l} + cc) - \mid \hat{A} \mid^2 (\delta - 2\rho\frac{1 + a_u^2}{a_u^2}\frac{1}{\Gamma_l})\}. \tag{75}$$

The first quantity is the total energy, which can also be written as

$$E = mc^2 n_e \gamma + \frac{1}{4\pi}|A|^2 = constant, \tag{76}$$

telling us that the beam kinetic energy density plus the field energy density is a constant;

The FEL equations of motion can be obtained from the Hamiltonian in the standard form, considering $\Gamma, \Psi, \hat{A}, \hat{A}^*$ as conjugate variables.

3.4 The FEL collective instability

To determine the behavior of the FEL when the initial state is an electron beam with a uniform longitudinal phase distribution, a given energy spread, and zero initial field amplitude, we linearize the FEL equations, assuming

$$\Gamma = 1 + \eta,\ \eta << 1, \tag{77}$$

and neglecting terms in the square of the field, to obtain

$$\frac{d\Psi_l}{d\tau} = \frac{\eta_l}{\rho} + i\rho(\hat{A}\, e^{i\Psi_l} - cc), \tag{78}$$

$$\frac{d\eta_l}{d\tau} = -\rho(1 - \eta_l)(\hat{A}\, e^{i\Psi_l} + cc), \tag{79}$$

$$\frac{d\hat{A}}{d\tau} = i\delta\, \hat{A} + < (1 - \eta_l)e^{i\Psi_l} > . \tag{80}$$

This equation can be obtained from the Hamiltonian

$$H = \sum_l \{ \frac{\eta_l^2}{2\rho} - i\rho(1-\eta_l)(\hat{A}\, e^{i\Psi_l} - cc)\} - \delta\, \hat{A}\hat{A}^* \tag{81}$$

We characterize the beam by a distribution function $f = f(\Psi, \eta, \tau)$, satisfying Vlasov equation

$$(\frac{\partial}{\partial\tau} + \frac{d\Psi_l}{d\tau}\frac{\partial}{\partial\Psi} + \frac{d\eta}{d\tau}\frac{\partial}{\partial\eta})f = 0. \tag{82}$$

We assume the distribution function to be

$$f = \frac{1}{2\pi} f_0(\eta) + f_1(\eta) e^{i\Psi + i\mu\tau}, \tag{83}$$

and use it to evaluate the current term in the field equation. We also assume the field to be proportional to $e^{i\mu\tau}$. We obtain

$$\hat{A} = -\frac{2\pi i e}{\mu - \delta} e^{i\mu\tau} \int f_1(\eta)(1-\eta) d\eta. \tag{84}$$

Notice that $\hat{A} = 0, f = f_0$ is a solution of the Vlasov equation. Substituting the expression for the field in the Vlasov equation we obtain the dispersion relation

$$\mu - \delta + \rho \int d\eta \frac{(1-\eta)^2}{\mu + (\eta/\rho)} \frac{\partial f_0}{\partial\eta} = 0. \tag{85}$$

For a monochromatic beam,the dispersion relation becomes

$$\mu^3 - \delta\mu^2 + 2\rho\mu + 1 = 0. \tag{86}$$

The system is unstable when the dispersion relation has solutions with $\text{Im}\,\mu < 0$. For a monochromatic beam such solutions exist if $\delta < 1.93$. The $\text{Im}\,\mu$ has a maximum when $\delta = 0$. If ρ is small this maximum is given by the root of $\mu^3 + 1 = 0$, and is $\sqrt{3}/2$.

To see the effect of a spread in energy of the electron beam we have evaluated the dispersion relation for the case of a Lorentzian distribution

$$f_0(\eta) = \frac{1}{\pi} \frac{\Delta}{\eta^2 + \Delta^2} \tag{87}$$

The results are shown in figure 1, where we show the imaginary part of the eigenvalue, the growth rate, for three different values of the ratio of the energy spread, Δ, to ρ.

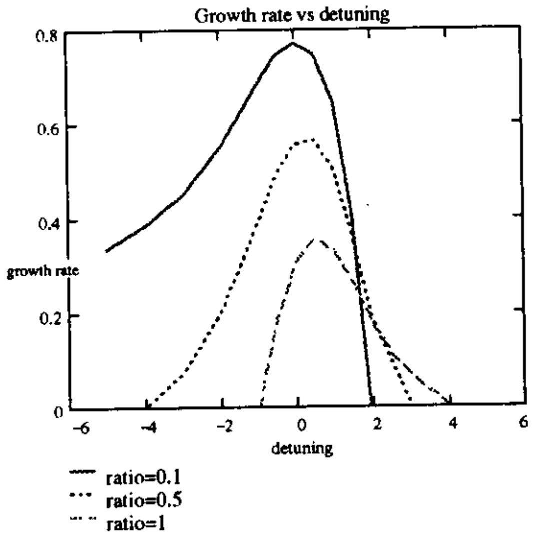

Figure 1.

The figure shows that the growth rate decreases when the energy spread becomes of the order of the FEL parameter ρ. One can also notice that for a small energy spread there is threshold for the instability at $\delta \approx 2$. This threshold disappears for larger values of the energy spread. When the energy spread becomes larger than ρ, the growth rate becomes proportional to the derivative of the unperturbed distribution function, $\mathrm{Im}\,\mu \approx \partial f_0/\partial\eta$ [18]. The dependence of the growth rate on the slope of the distribution function can be considered analogous to the population inversion in quantum lasers.

One can see from this picture that for an energy spread small compared to ρ, the growth rate, in terms of the normalized time τ, is of the order of one. Going back to the laboratory time, and using (64) with the usual case $\gamma \approx \gamma_R$,we have an exponential growth rate that for ρ small, and $\delta = 0$ is

$$L_G = (ct)_G = \frac{\lambda_u}{2\sqrt{3}\pi\rho} \tag{88}$$

in agreement with (18).

The exponential growth described in the linear approximation, must saturate at some field level. For small values of the FEL parameter ρ, and assuming the field to be nearly constant, as is the case at saturation, the equations (81,82), describing the electron motion in the energy-phase plane, are like a pendulum equation. The FEL saturates when the growth rate becomes of the order of the period of rotation of the electrons in the energy-phase plane. Evaluating this condition we obtain that saturation occurs after about ten gain lengths.

At saturation the normalized field amplitude $\hat{A}$ is also of the order of one, and the beam energy spread is of the order of ρ [1]. From the result $\hat{A} \approx 1$, and using (74) we also obtain eq. (23) for the saturation power.

3.5 Small signal gain

While for long undulators, $N_u \lambda_u > L_G$, only the eigenvalue of the dispersion relation with negative imaginary part is important, for short undulators, $N_u \lambda_u < L_G$, all three eigenvalues of the dispersion relation (87) are important in determining the field amplitude, and in fact the interference between these three waves determines the change in the field amplitude. In this case, which is called the small signal regime case, the gain, defined as the change in field intensity, I, over the intensity for one undulator crossing,

$$G_{ssg} = \frac{\Delta I}{I}, \tag{89}$$

is given by ([20])

$$G_{ssg} = \frac{4}{\delta^3}(1 - \cos(\delta\tau) - \frac{\delta\tau}{2}\sin(\delta\tau)) \tag{90}$$

where, using (65, 66)

$$\delta\tau = 4\pi N_u \frac{\Delta\omega}{\omega} \tag{91}$$

The small signal gain vs δ is shown in figure 2 for the case $\tau = 1$.

Notice that this curve is antisymmetric respect to δ, while the similar curve for the gain length in the high gain regime is not.

4 X-ray FELs

At the present time, the most intense sources of X-rays are undulators based on synchrotron radiation sources. Let us take as an example a storage ring like the APS at Argonne, with a beam energy of 8 GeV, average beam current of about 100 mA, and peak current of about 500A, with a pulse length of 20 ps, and a beam emittance of $2 \times 10^{-9} mrad$. Assume also that we use an undulator with $a_u = 1, \lambda_u = 1.6cm$ and $N_u = 100$ to produce radiation at $\lambda = 0.1nm$. Notice that since the emittance is larger than the wavelength the radiation production is diffraction limited. Using (14) we obtain an average and a peak brightness $B_{\lambda,ave} \approx 10^{20} ph/mm^2 mrad^2 0.1\%$, $B_{\lambda,p} \approx 10^{23} ph/mm^2 mrad^2 0.1\%$.

An X-ray FEL can provide many order of magnitudes larger peak brightness, because of the larger peak current and smaller emittance that we can obtain today using a new generation of electron sources, and the techniques for emittance

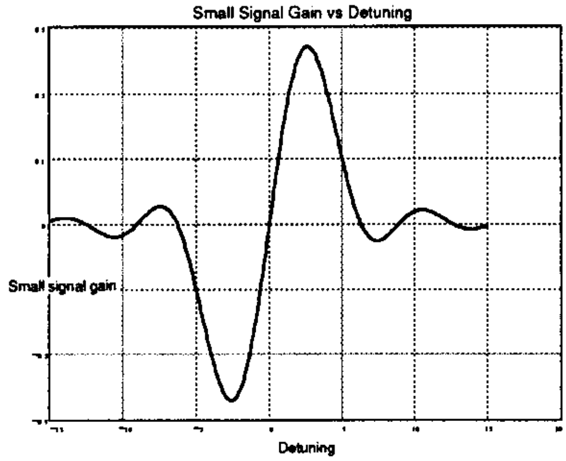

Figure 1:

preservation during acceleration developed for electron-positron linear colliders. As an example of an X-ray FEL we consider a possible system based on the SLAC linac; a list of beam and undulator parameters is given in Table 1.

Energy, and relative energy spread	15 GeV,0.0002
Emittance, rms at 15 GeV	3×10^{-11}mrad
Peak current, and bunch length	5kA, 24μm(80fs)
Undulator period, and undulator parameter	2.67cm, 2.91
Focusing wavelength in undulator	38m
FEL parameter	10^{-3}
Gain length, Saturation length	4m, 42m
Peak brightness and Average brightness	10^{32}, and 10^{23}ph/mm^{2}mrad^2/0.1%
Peak power	> 10GW

Table 1: X-ray FEL list of parameters.

The possibility of operating an FEL at such short wavelength follows from the favourable scaling laws for this system. To obtain and discuss the FEL scaling laws let us write the FEL parameter using two quantities, the emittance, ϵ, and the longitudinal brightness, B_L, to characterize the electron beam. The longitudinal brightness is defined as (??)

$$B_L = \frac{eN_e}{2\pi\gamma\sigma_L\sigma_\gamma}, \tag{92}$$

where σ_γ is the rms relative energy spread and σ_L is the rms bunch length. The quantity in the denominator of (92) is the beam longitudinal phase-space area. Using the emittance and the longitudinal brightness the FEL parameter can be written as

$$\rho = (\frac{\sqrt{2\pi}}{8\pi^2}\frac{a_u^2}{1+a_u^2}\lambda^2\gamma^2\sigma_\gamma\frac{B_L/I_A}{\varepsilon\beta_u})^{1/2}, \tag{93}$$

where I_A is the Alfven current ($\approx 17kA$).

We also require that we satisfy the conditions for the validity of the 1-D model, equations (17, 18, 20), and assume $\sigma_\gamma = \rho/k_1, \varepsilon = k_2\lambda/4\pi$. The condition, (20), on the optical focusing can be shown to follow from the other two. We also use additional focusing to produce through the undulator a betatron oscillation wavelength of the order of the gain length, $\lambda_\beta \approx 2\pi L_G$. Using these conditions and rewriting the FEL parameter in terms of emittance and longitudinal brightness, we obtain (??)

$$\rho = \frac{1}{k_1 k_2}\frac{a_u^2}{1+a_u^2}\frac{B_L}{I_A}. \tag{94}$$

Using typical values, $k_1 \approx 4$, and $k_2 \approx 6$, we can see that to obtain a value of ρ of the order of 10^{-3}, the minimum compatible with a practical undulator, we need a longitudinal brightness of the order of 500A, a value which has been exceeded in photinjector electron sources. The scaling law (94) does not depend directly on the radiation wavelength, but only indirectltly through the requirements (17). This weak dependence of the FEL scaling law on the radiation wavelength is an important property, which can be used to develop an X-ray FEL.

5 Conclusions

We have shown that the main properties of an FEL are determined by the collective instability, and can be characterized by the FEL parameter ρ. The properties of the FEL, combined with recent progress in the production of high brightness electron beams, are such that this system promises to become a very powerful source of coherent X-rays.

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