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I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

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"Course on Shallow Water and Shelf Sea Dynamics"
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"Shallow Water and Shelf Sea Dynamics"

S. BRENNER
Israel Oceangraphic and Limnological Research, LtD
Haifa
Israel

Please note: These are preliminary notes intended for internal distribution only.

Shallow water & shelf sea dynamics

I. Introduction (pre-dynamics)

Shallow and shelf seas cover a relatively small area of the ~~world~~ world ocean (only a few percent) yet in recent years they have become the focus of intense study. Why?

- Interaction with land surface
- Importance to man - economic exploitation (fishing, shipping, Tourism, recreation); military
- Part of the ocean most immediately effected by anthropogenic activities - eutrophication; pollution

Definition of shallow and shelf seas

No hardfast, clear cut definition. Depends on particular situation. Scientific definition vs. legal definition.

The "mythical" continental shelf - slope of 1:500 (0.11° or 6.9'), width of 65 km, extends from shore to ~~contin~~ shelf break at 130 m where slope becomes ~ 1:20 (continental slope). In reality shelf width can be from several hundred meters to several hundred km.

Driving forces

— Atmospheric forcing

Wind forcing – direct forcing of currents; imparts turbulent KE leading to mixing and entrainment (forced convection).

Usually consider Wind stress $\underset{\sim}{\tau} = \rho_a C_D |U| \underset{\sim}{U}$

ρ_a: air density; C_D = drag coefficient; U = wind speed

$\underset{\sim}{(\,)}$ = vector quantity. Units of N m^{-2}

Heat flux – four components

Q_s = solar (shortwave) heating; penetrative radiation

Q_r = longwave (IR) cooling $\sim \varepsilon \sigma T_s^4$

Q_e = latent heat flux (evaporation) $= \rho_a C_e |U| (q_a - q_s)$

Q_h = sensible heat flux $= \rho_a C_h |U| (T_a - T_s)$

all in units of W m^{-2} $(\,)_a \rightarrow$ air; $(\,)_s$ - ocean sfc

P: precipitation

Net heating → stratification and stability

Net cooling → instability, free convection; deepening of the mixed layer

Can also be expressed in terms of buoyancy ($-\rho g$) flux

— Tidal forcing – tidal circulation vs. residual circulation

Very often can be described by relatively few harmonics

— Runoff from land – river discharge ("large" areas)

vs man-made discharges (outfalls) - local effects

— Bottom friction – bottom or benthic boundary layer

— Interaction with deep ocean circulation. shoaling of

Our main goal is to understand the steady and transient response of the system, mainly to "external" forcing.
This course will focus primarily on the physical and dynamical aspects of the sea, but it is clear that we are ultimately ~~[illegible]~~ interested in understanding the physical-chemical-biological system.

Approach must be two pronged - observations combined with theoretical/mathematical models

Observations - modern approach combines in-situ and remotely sensed measurements. Ideally we would like a ~~[illegible]~~ synoptic description of the three dimensional structure of the sea including the mass (i.e., density) and motion fields. In practice we are far short of this due to expense ~~[illegible]~~ and/or other limitations of the observing system and instruments.

Examples of in-situ measurements (physical)

CTD/STD - temperature and salinity profiles

XBT, AXBT - temperature profiles

current meters (mechanical, electromagnetic, acoustic doppler)

subsurface drifters

Examples of remotely sensed measurements (mainly satellites)
IR - sea surface temperature
Microwave - (altimeter) - sea surface height; sfc winds
Visible - pigments (chlorophyll); suspended matter.

Theoretical/mathematical models
analytic or quasi-analytic - unique solutions that isolate and describe a particular process or phenomenon.

~~nume~~ numerical - can range in complexity from fairly simple to full 3-D

data assimilation - use model as a dynamic interpolator to fill in gaps in data while data constrains model solution

II. Basic thermodynamics and hydrostatics (fluid at rest)

State of a fluid described by state or thermodynamic variables, most often p, T, composition (salinity)

Two samples have same state if when in contact there is no transfer of properties.

If p not equal, one does work on other
If T not equal, transfer of heat
If composition not equal, transfer of concentration.

Equation of state – equation that gives other state variables in terms of basic variables (p, T, S).
Mainly interested in equation for density
$\rho = \rho(p, T, S)$.

For seawater it is given by an empirically ~~derived~~ derived polynomial (UNESCO eq of state)

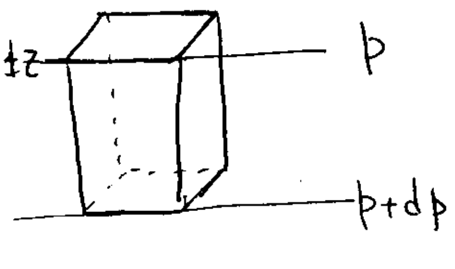
In many cases, can be closely approximated by

$$\rho = \rho_0 \left[1 - \alpha (T - T_0) + \beta (S - S_0) \right]$$

where subscript 0 is a reference value and
$\alpha = -\frac{1}{\rho} \left(\frac{\partial \rho}{\partial T} \right)_{p,S}$, $\beta = \frac{1}{\rho} \left(\frac{\partial \rho}{\partial S} \right)_{p,T}$ are the thermal and salinity expansion coefficients, respectively

Hydrostatic equation

Here will derive it from consideration of forces on fluid at rest. Later will derive it from scaling of the equations of motion.



For column shown at rest, there is a net pressure gradient force $-\frac{dp}{dz}$ which in equilibrium must be balanced by the buoyancy or weight of the fluid ρg. This immediately gives the hydrostatic equation

$$\boxed{\frac{dp}{dz} = -\rho g}$$

In order to find the equilibrium p distribution, we must know $\rho(z)$. For the oceans, ρ varies by $\lesssim \pm 2\%$ around constant value of $\rho_0 = 1035\ \mathrm{kg\,m^{-3}}$ in which case we can solve the equation $\rightarrow$ $\boxed{p(z) = p_A - \rho_0 g z}$

At what depth will $p(z) = 2p_A$, i.e. how thick a layer of sea water will have same weight as entire atmospheric column?

Using $p_A = 1013.25\ \mathrm{hPa} = 1.01325 \times 10^5\ \mathrm{N\,m^{-2}}$, $g = 9.8\ \mathrm{m\,s^{-2}}$

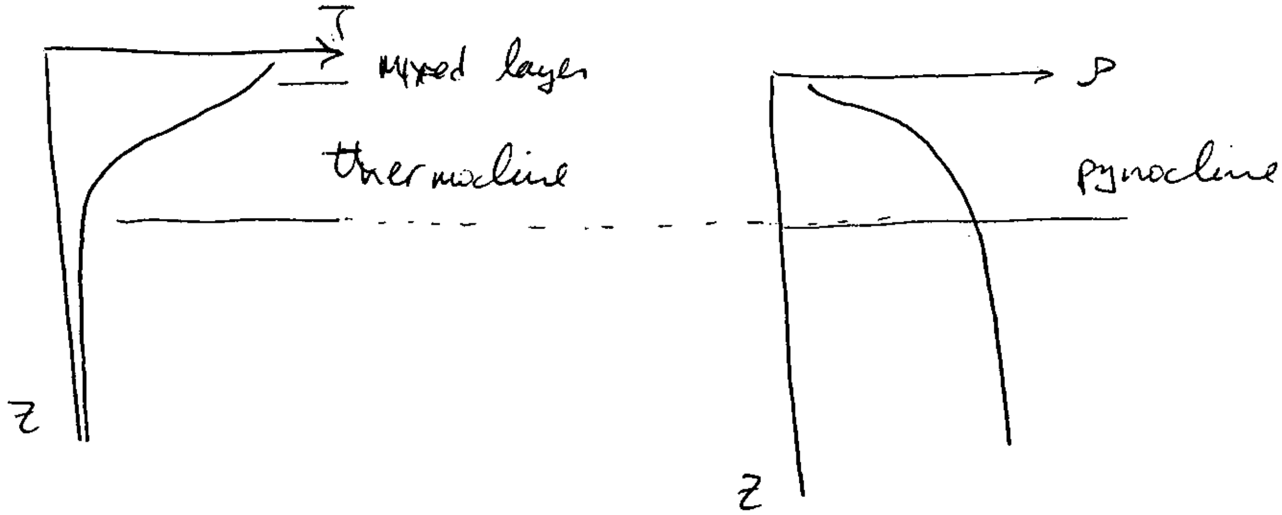
we find $\boxed{z = 9.9896\ \mathrm{m}}$

For this reason, depth and pressure are often used interchangably where 1 dbar: $\frac{10^5 N m^{-2}}{\rho \cdot g} \approx \frac{10^5}{1035 \times 9.8} = 9.86 m$

Note that dbar is not an SI unit

Stratification and static stability

For oceans, typical stratification looks like



Gravitational stability - situation in which the lighter (i.e. less dense, if g = const) fluid is above.

Thermodynamic equation gives us

$$T\,de = c_p\,dT - T\left(\frac{\partial v_s}{\partial T}\right)_{p,s} dp$$

where e: specific entropy, c_p: specific heat capacity and v_s: ρ^{-1} is the specific volume

Now for adiabatic or isentropic processes

$$dT = \frac{T}{c_p}\left(\frac{\partial v_s}{\partial T}\right)_{p,s} dp = -\frac{T}{\rho^2 c_p}\left(\frac{\partial \rho}{\partial T}\right)_{p,s} = \frac{\alpha T}{\rho c_p}\, dp$$

where $\alpha = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_{p,s}$

Now using hydrostatic equation we find that

$$\frac{dT}{dz} = -\frac{\alpha T g}{c_p} = \Gamma \quad \text{adiabatic lapse rate}$$

Can show that for parcel displaced isentropically, density change will be

$$d\rho = \rho\left[-g\left(\frac{\partial \rho}{\partial p}\right)_{T,s} + \alpha \Gamma\right) dz$$

which for stability must exceed change in the environment leading to

$$\boxed{\alpha\left(\frac{dT}{dz} + \Gamma\right) - \beta\frac{dS}{dz} > 0}$$

or

$$\boxed{\alpha\frac{dT}{dz} + \frac{g\alpha^2}{c_p}T - \beta\frac{dS}{dz} > 0}$$

Alternatively we could directly derive this from the vertical eq. of motion of a displaced parcel

$$\frac{dw}{dt} = -g - \frac{1}{\rho}\frac{dp}{dz}$$

which if expanded and solved gives

$$\frac{d^2 z}{dt^2} + N^2 z = 0 .$$

where $N^2 = g\alpha\left(\frac{dT}{dz} + \Gamma\right) - g\beta\,\frac{dS}{dz}$ Brunt-Vaisala frequency

(buoyancy freq.)

Examination of this equation shows that

$N^2 > 0$ solution is a sinusoidal oscillation (stable) with frequency N

$N^2 < 0$ exponential growth (unstable).

Typical values of N in the upper ocean $\simeq 0.01\ s^{-1}$

(period $\frac{2\pi}{N} \simeq 10$ min)

while in deep ocean $\lesssim 0.001\ s^{-1}$

Two additional variables related to stability are

potential temperature Θ = temp parcel would have if moved adiabatically to some reference pressure, p_r, usually 1 Pa ($10^5\ N\,m^{-2}$)

and potential density ρ_θ which is density parcel would have if moved adiabatically to p_r

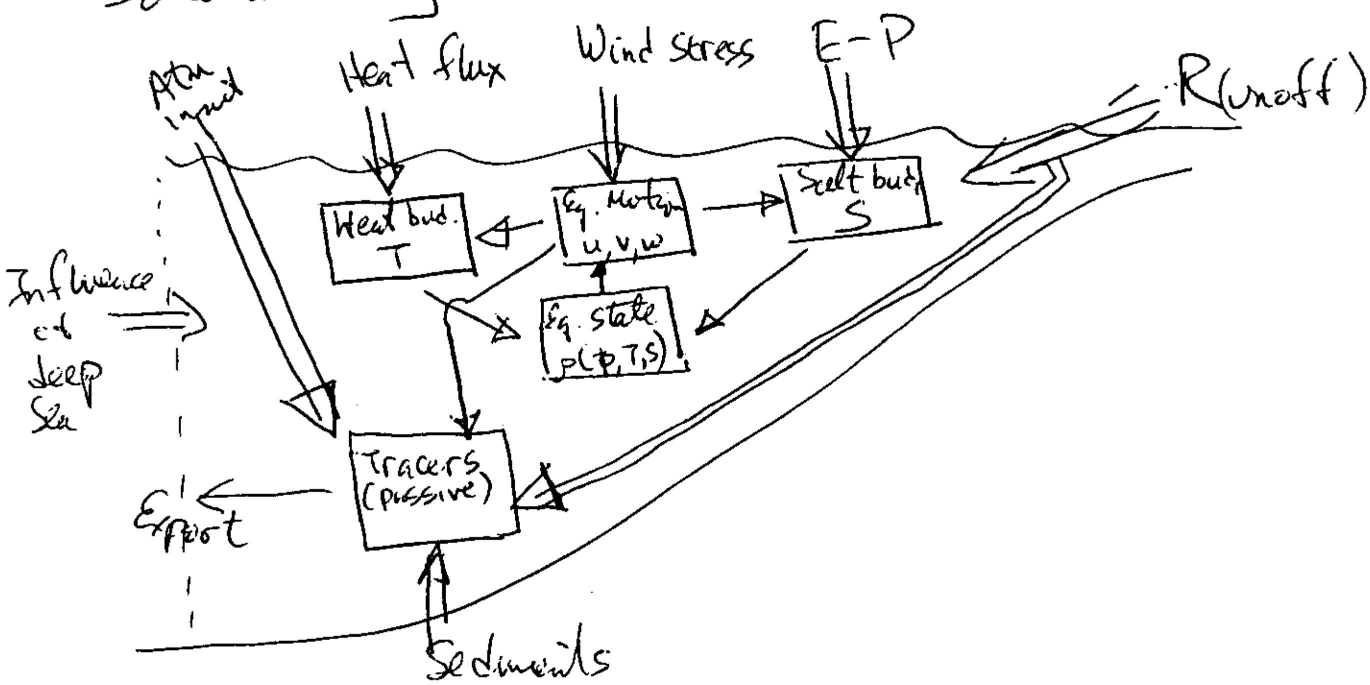
i.e. $\rho_\theta = \rho(p_r, S, \theta)$

Equations of Motion

Objective - to derive a "generic" set of equations that describe dynamics of shelf and shallow seas.

"Generic" since it is starting point. Equations can then be tailored (simplified, approximated) to specific situations

Schematically we have:

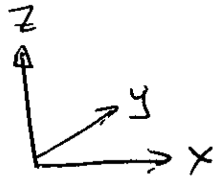


Tom Ukstrang

In deriving the equations must consider fluid in motion with respect to frame of reference. Often will need to consider relationship between a scalar field following the fluid motion (Lagrangian description) as compared to motion relative to a fixed point (Eulerian description).

Former expressed as individual or total derivative
latter " " partial or local derivative

For scalar ψ varying in space (x, y, z)



and in time $\psi = \psi(x, y, z, t)$

change following motion is

$$d\psi = \frac{\partial \psi}{\partial t} dt + \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial z} dz$$

or

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial x}\frac{dx}{dt} + \frac{\partial \psi}{\partial y}\frac{dy}{dt} + \frac{\partial \psi}{\partial z}\frac{dz}{dt}$$

and if we define; u, v, w as velocity components in x, y, z directions, respectively

$u = \frac{dx}{dt}$, $v = \frac{dy}{dt}$, $w = \frac{dz}{dt}$

we get

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + u\frac{\partial \psi}{\partial x} + v\frac{\partial \psi}{\partial y} + w\frac{\partial \psi}{\partial z}$$

or in vector form with

$$\underset{\sim}{V} = (u, v, w) = \underset{\sim}{i}\,u + \underset{\sim}{j}\,v + \underset{\sim}{k}\,w$$

we have

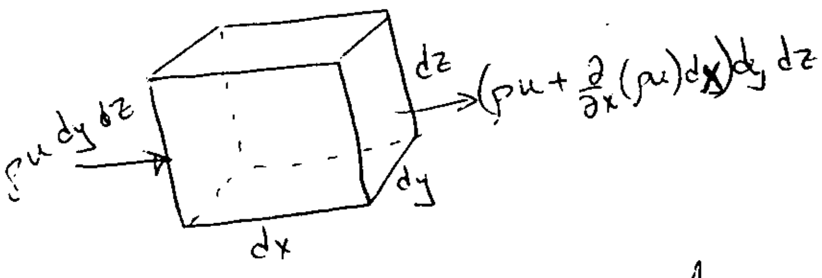
$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \underset{\sim}{V} \cdot \underset{\sim}{\nabla} \psi$$ where

gradient operator $\underset{\sim}{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

① Continuity equation (mass conservation)

This one is easy and straightforward

Consider a small volume fixed in space



net mass flux into volume must equal local rate of change of ρ

Net flux $= \left(\rho u - \left[\rho u + \frac{\partial}{\partial x}(\rho u)\, dx\right]\right) dy\, dz = -\frac{\partial(\rho u)}{\partial x}\, dx\, dy\, dz$

or per unit volume: $-\frac{\partial(\rho u)}{\partial x}$

Adding up contributions in all ~~other~~ 3 directions and equating the local rate of change of $\frac{\text{mass}}{\text{vol}} = \rho$

gives $\frac{\partial \rho}{\partial t} = -\frac{\partial \rho u}{\partial x} - \frac{\partial \rho v}{\partial y} - \frac{\partial \rho w}{\partial z}$

or in vector form $\boxed{\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{V}) = 0}$

or can expand to give

$$\frac{\partial \rho}{\partial t} + \rho \underset{\sim}{\nabla} \cdot \underset{\sim}{V} + \underset{\sim}{V} \cdot \underset{\sim}{\nabla} \rho = 0 \quad \Rightarrow \quad \frac{d\rho}{dt} = -\rho \underset{\sim}{\nabla} \cdot \underset{\sim}{V}$$

For an incompressible fluid (ρ is independent of p) or Boussinesq approximation ($\rho = \rho_0 + \rho'(x,y,z,t)$; $\rho' << \rho_0$) Continuity equation reduces to simple statement regarding 3 dimensional divergence, i.e.

$$\boxed{\underset{\sim}{\nabla} \cdot \underset{\sim}{V} = 0}$$

(2) Conservation equation for any general scalar, ψ

Following same description of a ~~fluid~~ volume fixed in space used for continuity equation, we can show that for any conservative scalar $\psi(x,y,z,t)$, the time rate of change per unit volume must be balance by the net flux into the volume,

$$\frac{\partial \rho \psi}{\partial t} + \underset{\sim}{\nabla} \cdot (\rho \psi \underset{\sim}{V}) = 0 \qquad [\text{Flux form}]$$

If $\psi = 1$ (special case) we retrieve the continuity equation. ψ can be S (salinity), nutrients, etc.

If we expand flux form of the conservation equation we get

$$\rho\left(\frac{\partial \psi}{\partial t} + \underset{\sim}{V}\cdot\nabla\psi\right) + \psi\underbrace{\left(\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla}\cdot\rho\underset{\sim}{V}\right)}_{0} = 0$$

since this is just continuity eq.

which gives us the

<u>advection form</u> $\frac{\partial \psi}{\partial t} + \underset{\sim}{V}\cdot\nabla\psi = 0$

For the <u>nonconservative case</u> there may some other transport mechanism besides advection (eg. diffusion) or internal sources or sinks of ψ.

Nevertheless we can still write a balance equation with the latter written on R.H.S. of the equation.

For case of diffusive flux (i.e. down gradient transport) we get

$$\frac{\partial \rho\psi}{\partial t} + \underset{\sim}{\nabla}\cdot(\rho\underset{\sim}{V}\psi) = \nabla\cdot(\rho K \underset{\sim}{\nabla}\psi)$$

where K is a diffusion coefficient (molecular, eddy ..)

and for $\rho K = \text{constant}$ we get the familiar

$$\nabla\cdot(\rho K\nabla\psi) = \rho K\nabla^2\psi$$

(3) First law of thermodynamics

change of internal energy = internal heating + work due to compression

$$\rho \frac{dE_I}{dt} = -\rho p \frac{dV_s}{dt} + \rho Q$$

or alternatively (depending on which state variables used)

$$\rho C_p \frac{dT}{dt} - \alpha T \frac{dp}{dt} = \rho Q$$

Now ρQ may be divided into diffusive flux + other properties (e.g. radiative heating, Q_R)

which leads to

$$\rho C_p \frac{dT}{dt} - \alpha T \frac{dp}{dt} = \nabla \cdot (\rho k \nabla T) + \rho Q_R$$

and for incompressible fluid we can neglect second term on LHS

$$\frac{dT}{dt} = K \nabla^2 T + \frac{Q_R}{C_p}$$ (also assume k = const and $K = \frac{k}{C_p}$)

K is thermal diffusivity

For Q_R a radiative flux, generally write

$$I(z) = I_o \exp(-kz)$$

and $Q_R = \frac{\partial I}{\partial z}$

In case of cooling water outfall, Q_R is an internal heat source and other heat fluxes enter as upper boundary conditions

IV. Equations II (Coriolis force)

④ Equation of motion

Newton's second law → rate of change of momentum (acceleration) ~~[illegible]~~ equals net force acting ~~[illegible]~~

For an inertial (non rotating) frame of ref

$$\rho \frac{d\underset{\sim}{V}}{dt} = -\nabla p + \rho \nabla \phi + \rho \underset{\sim}{F}$$

→ other nonconservative forces (only consider friction)

$-\nabla p$: pressure gradient

$\rho\nabla\phi$: body force (represented by potential ϕ) for us this is g

or

$$\frac{d\underset{\sim}{V}}{dt} = -\frac{1}{\rho}\nabla p + \nabla\phi + \underset{\sim}{F}$$

Now in reality we live on the rotating earth so it is desirable for ~~[illegible]~~ us to use a coordinate system fixed to the earth (i.e., noninertial or rotating frame of reference)

Heuristically, we want to know in what cases is rotation important. We will call these situations "large scale" circulation.

Assume length scale L, velocity scale U characteristic of the motion

Derived time scale is $\tau = \frac{L}{U}$

Also consider rotation rate $\Omega \Rightarrow$ time scale Ω^{-1}

We expect rotation to be important (fluid feels rotation rate) if $\tau \gtrsim \frac{\Omega^{-1}}{2} \Rightarrow \frac{L}{U} \gtrsim \frac{1}{2\Omega}$

or $R_o = \frac{U}{2\Omega L} \lesssim 1$ (Rossby number)

Note that R_o depends on ratio $\frac{U}{L}$ (not absolute L)

If L gets smaller, flow still qualifies as "large scale" if velocity scale is smaller.

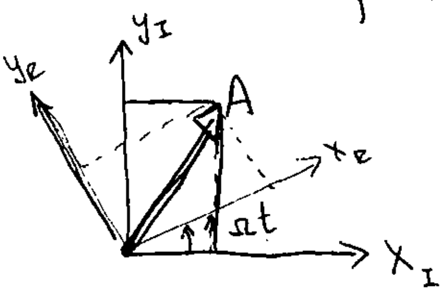
e.g. for Atm we use $U \sim 10$ m/s $L \sim 1000$ km (synoptic scales)

$\Omega = 7.29 \times 10^{-5} \Rightarrow R_o = 0.1$

While for strong ocean currents (Gulf Stream) upper limit of $U \sim 1$ m/s and if we want $R_o = 0.1$

$\Rightarrow L \simeq 100$ km

Let's consider an arbitrary vector $\underset{\sim}{A}$ rotating at $\underset{\sim}{\Omega}$ and how to describe it in ~~an~~ inertial frame I and in rotating frame R (i.e. how will observers see it?)



We can show that

$$\left(\frac{d\underset{\sim}{A}}{dt}\right)_I = \left(\frac{d\underset{\sim}{A}}{dt}\right)_R + \underset{\sim}{\Omega} \times \underset{\sim}{A}$$

$\Rightarrow$ I observer sees change in both $\underset{\sim}{A}$ and unit vectors in R

If we take $\underset{\sim}{A}$ as the position vector ~~in rotating frame~~

$$\underset{\sim}{A} = \underset{\sim}{r} = (x, y, z)$$

then $$\underset{\sim}{V}_I = \left(\frac{d\underset{\sim}{r}}{dt}\right)_I = \left(\frac{d\underset{\sim}{r}}{dt}\right)_R + \underset{\sim}{\Omega} \times \underset{\sim}{r} = \underset{\sim}{V}_R + \underset{\sim}{\Omega} \times \underset{\sim}{r}$$

$\Rightarrow$ velocity seen by inertial observer is velocity seen by rotating observer plus imparted by solid body rotation $\underset{\sim}{\Omega} \times \underset{\sim}{r}$

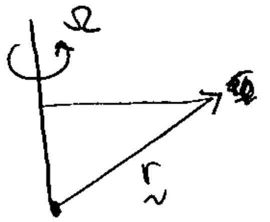
Applying our operator again (to get acceleration)

$$\left(\frac{d\underset{\sim}{V}_I}{dt}\right)_I = \left(\frac{d\underset{\sim}{V}_I}{dt}\right)_R + \underset{\sim}{\Omega} \times \underset{\sim}{V}_I = \frac{d}{dt}\left(\underset{\sim}{V}_R + \underset{\sim}{\Omega} \times \underset{\sim}{r}\right) + \underset{\sim}{\Omega} \times \left(\underset{\sim}{V}_R + \underset{\sim}{\Omega} \times \underset{\sim}{r}\right)$$

$$\Rightarrow \left(\frac{d\underset{\sim}{V}_I}{dt}\right)_I = \left(\frac{d\underset{\sim}{V}_R}{dt}\right)_R + \underbrace{2\underset{\sim}{\Omega} \times \underset{\sim}{V}_R}_{\text{Coriolis acceleration}} + \underbrace{\underset{\sim}{\Omega} \times (\underset{\sim}{\Omega} \times \underset{\sim}{r})}_{\text{centripetal acceleration}}$$

We can rewrite centripetal ~~the~~ acceleration in terms of the radius vector of the latitude circle

say $\underset{\sim}{r}' \Rightarrow |\underset{\sim}{r}'| = a\cos\phi$



$$\underset{\sim}{\Omega} \times \underset{\sim}{r} = \underset{\sim}{\Omega} \times \underset{\sim}{r}' \Rightarrow$$

$$\underset{\sim}{\Omega} \times (\underset{\sim}{\Omega} \times \underset{\sim}{r}) = -\Omega^2 \underset{\sim}{r}'$$

magnitude not vector

This in turn can be written ~~but~~ in terms of a potential function ϕ_c

$$\Omega^2 \underset{\sim}{r} = \nabla \phi_c \qquad \text{where } \phi_c = \frac{1}{2}|\underset{\sim}{\Omega} \times \underset{\sim}{r}|^2$$

So going back to the equation of motion we have

$$\rho\left[\frac{d\underset{\sim}{V}}{dt} + 2\underset{\sim}{\Omega} \times \underset{\sim}{V}\right] = -\nabla p + \rho\nabla\Phi + \underset{\sim}{F}$$

where $\Phi = \phi + \phi_c$ and we have dropped R

Finally we introduce apparent gravity $\underset{\sim}{g} = \underset{\sim}{g}' + \Omega^2 \underset{\sim}{r}$

and we have the familiar

$$\boxed{\frac{d\underset{\sim}{V}}{dt} = -2\underset{\sim}{\Omega} \times \underset{\sim}{V} - \frac{1}{\rho}\nabla p + \underset{\sim}{g} + \underset{\sim}{F}}$$

We have ~~ove~~ moved Coriolis accel to rhs so appears to observer in R as an additional force. It is always $\perp$ to velocity and does no work and parallel to plane of equator. Causes deflection to right in N.H. and to left in S.H.

For scalars the ~~separation~~ total derivative $\frac{d\psi}{dt}$ is unaffected by rotating frame.

Expanding vector form of ~~of~~ momentum equation in spherical coordinates gives

$$\frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} = fv - f'w - \frac{1}{\rho}\frac{\partial p}{\partial x} + A_H\frac{\partial^2 u}{\partial x^2}$$

$$\frac{dv}{dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} = -fu - \frac{1}{\rho}\frac{\partial p}{\partial y} + A_H\frac{\partial^2 v}{\partial y^2}$$

$$\frac{dw}{dt} \underbrace{- \frac{u^2}{a} - \frac{v^2}{a}}_{} = +f'u - \frac{1}{\rho}\frac{\partial p}{\partial z} - g + A_v\frac{\partial^2 w}{\partial z^2}$$

curvature terms due to change in direction of unit vectors; they do no work and do not generate kinetic energy

also, $f = 2\Omega\sin\phi$; $f' = 2\Omega\cos\phi$

Now for scale analysis

$L \sim 10^5$ m (100 km)

$H \sim 10^3$ m (1 km)

$U \sim 1$ m s^{-1}

$W \sim \frac{H}{L}U = 10^{-2}$ m s^{-1}

$\Delta p_H \sim 10^4$ Pa (1 m sfc height)

$\tau \sim \frac{L}{U} = 10^6$ s

$2\Omega \sim 10^{-4}$ s^{-1}

$\rho \sim 10^3$ kg m^{-3}

$g \sim 10$ m s^{-1}

$a \sim 10^7$ m

$A_H \sim 10^2$ m^2 s^{-1}

$A_V \sim 10^{-4}$ m^2 s^{-1}

$\Delta p_V \sim 10^7$ Pa

Apply this to u equation

$$\frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} = fv - f'w - \frac{1}{\rho}\frac{\partial p}{\partial x} + A_H\frac{\partial^2 u}{\partial x^2}$$

$\frac{U^2}{L}$	$\frac{U^2}{a}$	$\frac{UW}{a}$	$2\Omega U$	$2\Omega W$	$\frac{\Delta p_H}{\rho L}$	$\frac{A_H U}{L^2}$
10^{-5}	10^{-7}	10^{-9}	10^{-4}	10^{-6}	10^{-4}	10^{-8}

Applying 10^{-4} filter gives steady state equation

$$\boxed{fv = \frac{1}{\rho}\frac{\partial p}{\partial x} \quad ; \quad -fu = \frac{1}{\rho}\frac{\partial p}{\partial y}}$$ Geostrophic approximation

Applying 10^{-5} filter adds time derivative (local + advection)

$$\frac{du}{dt} = fv - \frac{1}{\rho}\frac{\partial p}{\partial x} \quad ; \quad \frac{dv}{dt} = -fu - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

or

$$\boxed{\frac{d\underset{\sim}{v}}{dt} = -f\underset{\sim}{k}\times\underset{\sim}{v} - \frac{1}{\rho}\nabla p}$$ primitive equations

Doing same for vertical equation

$$\frac{dw}{dt} - \frac{u^2}{a} - \frac{v^2}{a} = f'u - \frac{1}{\rho}\frac{\partial p}{\partial z} - g + A_V\frac{\partial^2 w}{\partial z^2}$$

$\frac{WU}{L}$	$\frac{U^2}{a}$		$2\Omega U$	$\frac{\Delta p_V}{\rho H}$	g	
10^{-7}	10^{-7}		10^{-4}	10	10	10^{-9}

which immediately gives hydrostatic equation $\boxed{\frac{1}{\rho}\frac{dp}{dz} = -g}$

So to summarize what we have so far

① x-momentum $\quad \frac{\partial u}{\partial t} + \underset{\sim}{V} \cdot \nabla u = fv - \frac{1}{\rho}\frac{\partial p}{\partial x}$

② y-momentum $\quad \frac{\partial v}{\partial t} + \underset{\sim}{V} \cdot \nabla v = -fu - \frac{1}{\rho}\frac{\partial p}{\partial y}$

③ hydrostatic $\quad \frac{\partial p}{\partial t} = -\rho g$

④ continuity $\quad \frac{\partial \rho}{\partial t} + \underset{\sim}{V} \cdot \nabla \rho + \rho \nabla \cdot \underset{\sim}{V} = 0$

⑤ thermodynamic $\quad \frac{\partial T}{\partial t} + \underset{\sim}{V} \cdot \nabla T = \kappa \nabla^2 T + \frac{Q_R}{C_p}$

⑥ eq of state $\quad \rho = \rho(p, T, S)$

⑦ tracers $\quad \frac{\partial(\rho\psi)}{\partial t} + \nabla \cdot (\rho \psi \underset{\sim}{V}) = \rho \kappa \nabla^2 \psi$

And if we now apply Boussinesq approximation which says $\rho = \rho_0$ except in ~~buoy~~ buoyancy force

① + ② remain same except replace ρ by ρ_0

③ remains same

④ $\nabla \cdot \underset{\sim}{V} = 0$

⑤ remains same (already assumed incompressible)

⑥ remains same

⑦ $\frac{\partial \psi}{\partial t} + \underset{\sim}{V} \cdot \nabla \psi = \kappa \nabla^2 \psi$