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# "Dynamical Oceanography III: Stratified Fluid Dynamics"

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*Please note: These are preliminary notes intended for internal distribution only.*

# DYNAMICAL OCEANOGRAPHY III
# STRATIFIED FLUID DYNAMICS

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(*) UNDER CONSTRUCTION !

# CHAPTER 1

# A REVIEW OF BASIC EQUATIONS

## 1.1 Continuity and Momentum Equations

The continuity and momentum equations were developed in DO-I (cf. equations 2.23.$a,b$ and 3.56):

$$\frac{D\rho}{Dt}+\rho\nabla\cdot\vec{u}=0 \tag{1.1}$$

$$\frac{D\vec{u}}{Dt}+2\vec{\Omega}\times\vec{u}=\vec{g}-\frac{1}{\rho}\nabla p+\frac{1}{\rho}\nabla\cdot\mathbf{d}=\vec{g}+\frac{1}{\rho}\nabla\cdot\sigma \tag{1.2}$$

where $\rho$, $p$, and $\vec{u}$ are the density, pressure and velocity of the fluid respectively, $\vec{\Omega}$ the earths angular velocity, $\mathbf{d}$ the deviatoric stress tensor and $\vec{g}$ the gravitation vector related to the gravity $\vec{g}'$ by

$$\vec{g}=\vec{g}'-\vec{\Omega}\times\vec{\Omega}\times\vec{r} \tag{1.3}$$

including the correction resulting from the centrifugal acceleration at any $\vec{r}$ measured from the center of the earth. The stress tensor $\sigma$ is given by

$$\sigma=-p\mathbf{I}+\mathbf{d} \tag{1.4}$$

where $\mathbf{I}=\delta_{ij}$ is the identity matrix (tensor).

## 1.2 Thermodynamics

### 1.2.1 The Energy Equation

Because homogeneous fluids were studied earlier, in DO-I, the density $\rho$ was taken to be constant. In this course, we study stratified fluids with $\rho=\rho(x,y,z,t)$ in general, so that the two equations (1.1) and (1.2) are not sufficient to solve for the three unknowns $\vec{u}$, $\rho$,and $p$. Therefore, we must use additional equations, reflecting the energy balances.

The *First Law of Thermodynamics* is an energy conservation statement,

$$\frac{dE_T}{dt}=\frac{dH}{dt}+\frac{dW}{dt} \tag{1.5}$$

expressing the fact that the rate of change of total energy ($E_T$) is balanced by the rate at which heat ($H$) is supplied to the fluid and the rate of which work ($W$) is done on the fluid by extraneous sources. The total energy for a material volume $V$ bounded by surface $S$ is defined as

$$E_T=\int_V\rho\epsilon dV \tag{1.6.a}$$

where

$$\epsilon = e + \frac{1}{2}\vec{u}\cdot\vec{u} - \vec{g}\cdot\vec{x} \tag{1.6.b}$$

is the total energy per unit mass, including the internal energy (or specific enthalpy) $e$, the kinetic energy (second term) and the potential energy (third term) components. Here $\vec{x}$ is a position vector with respect to an arbitrary datum ($\vec{x}$ increasing in the opposite sense to $\vec{g}$ results in positive contribution to the potential energy).

The first term in (1.5)is be expressed follows, making use of the Leibnitz' rule, the divergence theorem and the continuity equations (1.1).

$$\begin{aligned}
\frac{dE_T}{dt} &= \frac{d}{dt}\int_V \rho\epsilon\ dV \\
&= \int_V \frac{\partial\rho\epsilon}{\partial t} + \int_S \rho\epsilon\vec{u}\cdot\hat{n}\ dS \\
&= \int_V \left(\frac{\partial\rho\epsilon}{\partial t} + \nabla\cdot\rho\epsilon\vec{u}\right) dV \\
&= \int_V \left(\rho\frac{\partial\epsilon}{\partial t} + \epsilon\frac{\partial\rho}{\partial t} + \epsilon\rho\nabla\cdot\vec{u} + \epsilon\vec{u}\cdot\nabla\rho + \rho\vec{u}\cdot\nabla\epsilon\right) dV \qquad (1.7)\\
&= \int_V \rho\frac{D\epsilon}{Dt}dv \\
+ \int_V \epsilon\left(\frac{D\rho}{Dt} + \rho\nabla\cdot\vec{u}\right) dV \\
&= \int_V \rho\frac{D\epsilon}{Dt}\ dV.
\end{aligned}$$

On the other hand, we define the total input of heat into a fluid element as

$$\begin{aligned}
\frac{dH}{dt} &= \int_V \rho Q\ dV - \int_S \vec{q}\cdot\hat{n}\ dS \\
&= \int_V (\rho Q - \nabla\cdot\vec{q})\ dV,
\end{aligned} \tag{1.8}$$

where $Q$ is the rate of internal heating (per unit mass) and (-$\vec{q}$) the heat flux (per unit area). The quantity $-\vec{q}\cdot\hat{n}$ is the flux (per unit area) entering the fluid volume $V$ through the enclosing surface $S$, opposite to the direction of the outward normal $\hat{n}$.

The rate of work done by the surrounding on the fluid is due to the surface stresses $\vec{\Sigma} = \sigma\cdot\hat{n}$ (ref. to DO-I) applied on the enclosing surface $S$ to deform it with velocity $\vec{u}$

$$\begin{aligned}
\frac{dW}{dt} &= \int_S \vec{\Sigma}\cdot\vec{u}\ dS = \int_S (\sigma\cdot\vec{n})\cdot\vec{u}\ dS \\
&= \int_S (\vec{u}\cdot\sigma)\cdot\hat{n}\ dS = \int_V \nabla\cdot(\vec{u}\cdot\sigma)\ dV.
\end{aligned} \tag{1.9}$$

By making use of the momentum equation (1.2), the integrand of (1.9) can then be written as

$$
\begin{aligned}
\nabla\cdot(\vec{u}\cdot\sigma) &= \vec{u}\cdot(\nabla\cdot\sigma)+\sigma\cdot(\nabla\cdot\vec{u})\\
&= \rho\vec{u}\cdot\left(\frac{D\vec{u}}{Dt}+2\vec{\Omega}\times\vec{u}-\vec{g}\right)+\sigma\cdot(\nabla\cdot\vec{u})\\
&= \rho\vec{u}\cdot\frac{D\vec{u}}{Dt}-\rho\vec{u}\cdot\nabla(\vec{g}\cdot\vec{x})+\sigma\cdot(\nabla\cdot\vec{u})\\
&= \rho\frac{D}{Dt}\left(\frac{1}{2}\vec{u}\cdot\vec{u}-\vec{g}\cdot\vec{x}\right)+\sigma\cdot(\nabla\cdot\vec{u})
\end{aligned}
\tag{1.10}
$$

whereas the last term of (1.10) is equivalent to (cf. 1.1 and 1.4)

$$
\begin{aligned}
\sigma\cdot\nabla\cdot\vec{u} &= -p\nabla\cdot\vec{u}+(d\cdot\nabla)\cdot\vec{u}\\
&= \frac{p}{\rho}\frac{D\rho}{Dt}+\rho\Phi
\end{aligned}
\tag{1.11}
$$

where $\Phi = (\underset{=}{d}\cdot\nabla)\cdot\vec{u}$ represent the heat generated (mechanical energy dissipated) by viscous friction.

The diffusive (conductive) heat flux is expressed by *Fourier' s Law* which relates the flux to the local temperature gradient through the linear relation (*e.g.* Batchelor, 1967)

$$\vec{q} = -K\nabla T \tag{1.12}$$

with the heat flowing from high temperature to low temperature. The constant proportionality $K$ is called the *thermal conductivity*. Substituting (1.10) and (1.11) into (1.9) and combining (1.6), (1.7) and (1.9) yields the thermodynamic energy equation

$$\frac{De}{Dt}-\frac{p}{\rho^2}\frac{D\rho}{Dt}=\frac{1}{\rho}\nabla\cdot K\nabla T+(Q+\Phi) \tag{1.13}$$

Note that the mechanical energy equation (1.10), *i.e.*

$$\frac{D}{Dt}\left(\frac{1}{2}\vec{u}\cdot\vec{u}-\vec{g}\cdot\vec{x}\right)=\vec{u}\cdot(\nabla\cdot\sigma) \tag{1.14}$$

describes the conservation of mechanical energy.

### 1.2.2 Equilibrium Thermodynamics

We will now employ concepts from *classical (equilibrium) thermodynamics* to further develop our equations. The classical theory describes the equilibrium states of uniform matter, in which all local mechanical, physical and thermal quantities are independent of position and time, *e.g.* a homogeneous fluid at rest. The non-equilibrium states are not sufficiently described by the thermodynamic theory, but observation shows that the equilibrium theory is approximately valid in practice (for example in the case of inhomogeneous fluids), since the departure from equilibrium does not greatly influence the relations between thermodynamic quantities.

The *equilibrium thermodynamic state* of a fluid with *fixed composition* (*i.e.* constituents with fixed mixing ratios) is determined by at least three *parameters of state*, which are the temperature $T$, pressure $p$ and density $\rho$. From now on, we will define the *specific volume*

$$v = \frac{1}{\rho} \tag{1.15}$$

for convenience. The state is therefore defined by the relation between any three of these parameters of state, *i.e.*

$$f(p, v, T) = 0. \tag{1.16}$$

Note that the *equation of state* defines a surface with respect to the $p$, $v$, $T$ coordinates, and can alternately be written as

$$p = f_1(v, T) \quad \text{or} \quad T = f_2(p, v) \quad \text{or} \quad v = f_3(p, T). \tag{1.17c.d}$$

Next, we consider the thermodynamic equation (1.13) and denote the right hand side, which stands for the rate of heat added to the fluid through diffusion, internal heating or frictional dissipation, as

$$\frac{1}{\rho}\nabla \cdot K\nabla T + Q + \Phi \equiv Q^* \equiv \frac{Dq}{Dt} \tag{1.18}$$

and making use of (1.15), equation (1.13) takes the following form:

$$\frac{De}{Dt} + \rho\frac{Dv}{Dt} = \frac{Dq}{Dt}. \tag{1.19}$$

Since we are considering an equilibrium state ($T$=const. and, $\vec{u}$=const.), $\nabla \cdot K\nabla T = 0$ and $\Phi = (\mathbf{d} \cdot \nabla) \cdot \vec{u} = 0$, the only contribution to (1.18) comes from the uniform internal heat rate $Q$, *i.e.*

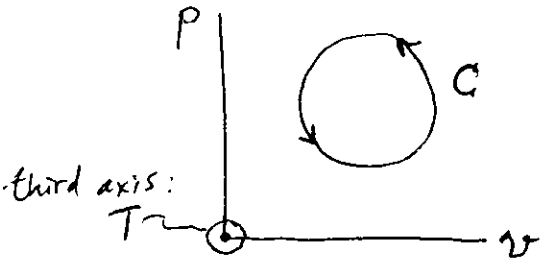
$$\frac{Dq}{Dt} = Q \tag{1.20}$$

If $Q$=0, in addition to the other terms making up (1.18), the fluid is essentially said to be *adiabatic*, *i.e.* there is no heat exchange. In any case, we can write (1.19) alternatively as

$$\delta e + p\,\delta v = \delta q \tag{1.21}$$

where $\delta e$ is the change in specific internal energy, $-p\delta v$ is the work done by compression leading to a change in volume, and $\delta q$ is the heat added.

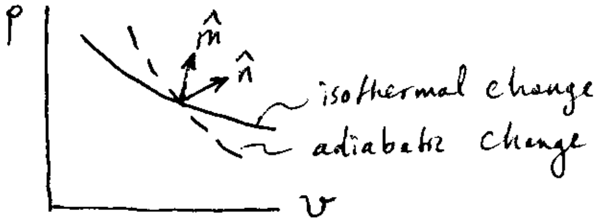
The above differentials are valid in the equilibrium case, since no temporal or spatial variations are present. The notation $\delta$ is then interpreted as the total change in the state of the fluid as a whole. Equation (1.21) describes *reversible* changes between neighoring states, *i.e.* the net change in internal evergy $\delta e$ is cancelled out when returning to the original state. However, in general, for a finite change, whether the process is *reversible* or *irreversible* is determined by the path taken, since the net change follows a contour in the $p$, $v$, $T$ space, or by showing the integral in $p-v$ plane (indicator diagram), we have:



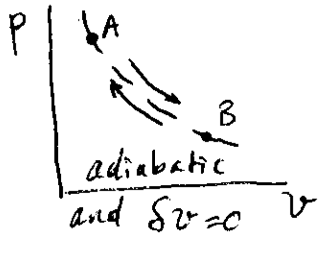
Note the state equation $f(p,v,T) = 0$ defines a surface in $p$, $v$, $T$ coordinates for fluids of fixed composition. Then,

$$\oint_c de = -\oint_c p dv + \oint_c dQ \tag{1.22}$$

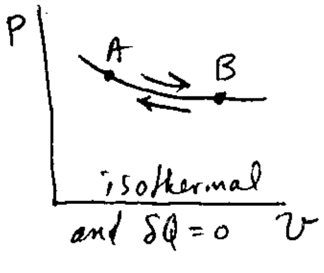
but since $p$ is not a function of $v$ alone (*i.e.* $p = p(v,T)$), the integral depends on the path. If the process is *isothermal* $p = p(v)$ only (since $T$=const.), then the first term on the r.h.s. of (1.22) vanishes, or if the process is *adiabatic* the second term on the r.h.s. vanishes, yielding *reversible* changes in each mutually exclusive case. When the path is a special combination of these, the process may still be reversible, as shown below:



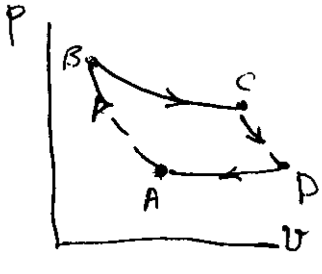
Reversible processes :



or



or



Another possibility for reversible change occurs when the first term cancels the second term on the r.h.s., *i.e.* in the case of *compressional heating*.

A practical quantity of importance is the *specific heat* $C$ of the fluid, *i.e.* the amount of heat given to unit mass of fluid per unit change in temperature during a small reversible change:

$$C = \frac{\delta q}{\delta T}. \tag{1.23}$$

The specific heat is a function of the conditions under which the reversible change takes place, *i.e.* this change can be in any direction starting from point $A$ in the above figure.

Using the chain rule to express the change in the state variable $e$ with respect to the other state veriables in (1.21), we have:

$$\delta q = \left(\frac{\partial e}{\partial p}\right)_v \delta p + \left(\frac{\partial e}{\partial v}\right)_p \delta v + p\delta v. \tag{1.24}$$

On the ofter hand, the temperature differential based on the equation of state (1.16) can be written as

$$\delta T = \left(\frac{\partial T}{\partial p}\right)_v \delta p + \left(\frac{\partial T}{\partial v}\right)_p \delta v \tag{1.25}$$

where the subscripts denote differention at constant values of the subscripts (*e.g.* $\left(\frac{\partial T}{\partial p}\right)_v$ denotes change in temperature with respect to pressure, while keeping the density $\rho$ or specific volume $v$ constant). Therefore (1.23) is expressed as

$$C = \frac{\left(\frac{\partial e}{\partial p}\right)_v \delta p + \left[\left(\frac{\partial e}{\partial v}\right)_p + p\right] \delta v}{\left(\frac{\partial T}{\partial p}\right)_v \delta p + \left(\frac{\partial T}{\partial v}\right)_p \delta v} \tag{1.26}$$

which depends on the ratio $\delta p/\delta v$ or on the choice of direction during change from point $A$. Two well-defined constants, specifying changes parallel to the axes of the $p - v$ diagram are the *principal specific heats*. The *specific heat at constant pressure* is

$$c_p = \left(\frac{\partial Q}{\partial T}\right)_{\delta p=0} = \left(\frac{\delta e}{\delta T}\right)_p + p\left(\frac{\partial v}{\partial T}\right)_p \tag{1.27.a}$$

and the specific heat at constant volume is

$$c_v = \left(\frac{\delta Q}{\delta T}\right)_{\delta v=0} = \left(\frac{\partial e}{\partial T}\right)_v \tag{1.27.b}$$

If $\hat{m}$ and $\hat{n}$ are the unit normal vectors perpendicular to the isothermal line ($\delta T = 0$) and to the adiabatic line ($\delta Q = 0$) respectively in the above figure, and if their components in $p$ and $v$ directions are defined as $\hat{m} = (m_v, m_p)$ , $\hat{n} = (n_v, n_p)$, then we can write

$$c_p = \left(\frac{\delta Q}{\delta T}\right)_{\delta p=0} = \frac{n_v(\delta Q)_{max}}{m_v(\delta T)_{max}} \tag{1.28.a}$$

$$c_v = \left(\frac{\delta Q}{\delta T}\right)_{\delta v=0} = \frac{n_p\,(\delta Q)_{max}}{m_p\,(\delta T)_{max}}. \tag{1.28.b}$$

Since $-m_v/m_p$ and $-n_v/n_p$ are the gradients of the isothermal and adiabatic lines respectively, the ratio of the principal specific heats is found to be

$$\gamma = \frac{c_p}{c_v} = \frac{n_v/n_p}{m_v/m_p} = \frac{\left(\frac{\delta p}{\delta v}\right)_{\delta Q=0}}{\left(\frac{\delta p}{\delta v}\right)_{\delta T=0}} = \frac{\left(\frac{\delta v}{\delta p}\right)_{\delta T=0}}{\left(\frac{\delta v}{\delta p}\right)_{\delta Q=0}}. \tag{1.29}$$

The specific entropy is another equilibrium property of the fluid. The *Second Law of Thermodynamics* relates the *specific entropy* $s$ to the other state variables. In a reversible transition from one equilibrium state to another, the change in entropy is proportional to the heat supplied to the fluid. The constant of proportionality, which should in principle be a function of state, depends only on temperature, the dependence chosen as $1/T$:

$$T\delta s = \delta q. \tag{1.30}$$

This relation defines the absolute scale for temperature, *i.e.* $T$ is in degrees Kelvin. An *adiabatic, reversible* change ($\delta q = 0$) is therefore also *isentropic* ($\delta s = 0$).

Note that in an *irreversible change* $q$ is defined by (1.18), so that even if the process is adiabatic ($\nabla \cdot K\nabla T + \rho Q = 0$), the heat dissipation $\Phi$ (conversion of mechanical energy to heat) is always positive, $\delta q > 0$, and since the absolute temperature $T$ is also defined to be positive ($T > 0$), we arrive at the result that *entropy must always increase* ($\delta S > 0$) in irreversible changes.

It follows from (1.21) and (1.30) that

$$\delta e = T\delta s - p\delta v. \tag{1.31}$$

This is a different way of expressing the Second Law of Thermodynamics.

Other dependent properties (state variables) can be defined, to describe equilibrium thermodynamic changes in a fluid. These are the *specific enthalpy*, $h$, the *Helmholtz free energy (Helmholtz function)*, $f$, and the *Gibbs' function* $g$, defined respectively in the following:

$$\begin{aligned} h &= e + pv \\ f &= e - Ts \\ g &= e + pv - Ts \end{aligned} \qquad (1.32.a-c)$$

The above balances of these quantities (*i.e.* energy transforms) imply "equilibrium". Differentiating the energy transforms yields the following relations:

$$\begin{aligned} \delta h &= T\delta s + v\delta p \\ \delta f &= -p\delta v - s\delta T\,. \\ \delta g &= v\delta p - s\delta T \end{aligned} \qquad (1.33a-c)$$

From (1.31) it follows that

$$\left(\frac{\delta e}{\delta v}\right)_{\delta s=0} = \left(\frac{\partial e}{\partial v}\right)_s = -p, \qquad \left(\frac{\delta e}{\delta s}\right)_{\delta v=0} = \left(\frac{\partial e}{\partial s}\right)_v = T. \qquad (1.34a,b)$$

Differentiating $(1.34a,b)$ in reverse order, we obtain two versions of the cross-derivatives of $e$:

$$\begin{aligned} \frac{\partial^2 e}{\partial v \partial s} &= \left[\frac{\partial}{\partial v}\left(\frac{\partial e}{\partial s}\right)_v\right]_s = \left(\frac{\partial T}{\partial v}\right)_s, \\ \frac{\partial^2 e}{\partial s \partial v} &= \left[\frac{\partial}{\partial s}\left(\frac{\partial e}{\partial v}\right)_s\right]_v = -\left(\frac{\partial p}{\partial s}\right)_v. \end{aligned} \qquad (1.35a,b)$$

Since $(1.35.a)$ and $(1.35.b)$ should be equal, it follows that

$$\left(\frac{\partial p}{\partial s}\right)_v = -\left(\frac{\partial T}{\partial v}\right)_s. \qquad (1.36.a)$$

This equality is the first one of the *Maxwell's thermodynamic relations.* The other three Maxwell relations can be obtained by applying reciprocal differentiation to the energy transforms $(1.33.a-c)$ to yield

$$\begin{aligned} \left(\frac{\partial v}{\partial s}\right)_p &= \left(\frac{\partial T}{\partial p}\right)_s \\ \left(\frac{\partial v}{\partial T}\right)_p &= -\left(\frac{\partial s}{\partial p}\right)_T \\ \left(\frac{\partial p}{\partial T}\right)_v &= \left(\frac{\partial s}{\partial v}\right)_T \end{aligned} \qquad (1.36.b-d)$$

Alternative expressions can be written for the specific heat, making use of (1.30):

$$C = \frac{\delta Q}{\delta T} = T\left(\frac{\delta s}{\delta T}\right), \tag{1.37}$$

so that the definition of the principal specific heats (1.27.$a,b$) are modified to become

$$c_p = T\left(\frac{\partial s}{\partial T}\right)_p, \qquad c_v = T\left(\frac{\partial s}{\partial T}\right)_v. \tag{1.38}$$

Regarding $s$ to be a function of $T$ and $v$ only, (related by the state equation (*i.e.* $f_s(s,T,v) = 0$), we can derive

$$\delta s = \left(\frac{\partial s}{\partial T}\right)_v \delta T + \left(\frac{\partial s}{\partial v}\right)_T \delta v. \tag{1.39}$$

Differentiating with respect to $T$ at constant $p$ and multiplying by $T$ yields

$$T\left(\frac{\partial s}{\partial T}\right)_p = T\left(\frac{\partial s}{\partial T}\right)_v + T\left(\frac{\partial s}{\partial v}\right)_T\left(\frac{\partial v}{\partial T}\right)_p. \tag{1.40}$$

Making repeated use of the Maxwell relations (1.36), we obtain

$$\begin{aligned} c_p - c_v &= T\left(\frac{\partial p}{\partial T}\right)_v\left(\frac{\partial v}{\partial T}\right)_p \\ &= T\left(\frac{\partial s}{\partial v}\right)_T\left(\frac{\partial v}{\partial T}\right)_p \\ &= T\left(\frac{\partial p}{\partial v}\right)_T\left(\frac{\partial s}{\partial p}\right)_T\left(\frac{\partial v}{\partial T}\right)_p \\ &= -T\left(\frac{\partial p}{\partial v}\right)_T\left(\frac{\partial v}{\partial T}\right)_p^2. \end{aligned} \tag{1.41}$$

We may now regard $s$ as a function of the other state variables $T$ and $p$, so that

$$\begin{aligned} \delta s &= \left(\frac{\partial s}{\partial T}\right)_p \delta T + \left(\frac{\partial s}{\partial p}\right)_T \delta p \\ &= \frac{c_p}{T}\,\delta T - \left(\frac{\partial v}{\partial T}\right)_p \delta p, \end{aligned} \tag{1.42}$$

and define the *coefficient of thermal expansion* of the fluid

$$\alpha_T = \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_p = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p, \tag{1.43}$$

so that (1.42) can now be written as

$$T\delta s = c_p \delta T - \alpha_T v T \delta p. \tag{1.44}$$

### 1.2.3 The Different Forms of the Thermodynamic Equation

In earlier sections, we have derived the thermodynamic equation (1.13)

$$\frac{De}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} = Q^*, \tag{1.45}$$

where the r.h.s. has been defined as $Q^* \equiv \frac{1}{\rho}\nabla \cdot K\nabla T + (Q + \Phi)$ (1.18). Using the definition $v \equiv \frac{1}{\rho}$, this equation was also written as (1.19)

$$\frac{De}{Dt} + p\frac{Dv}{Dt} = Q^*. \tag{1.46}$$

By making use of (1.31), we could also write it as

$$T\frac{Ds}{Dt} = Q^*, \tag{1.47}$$

or, using (1.44), the quation would take the following form:

$$c_p\frac{DT}{Dt} - \alpha_T T v\frac{Dp}{Dt} = Q^* \tag{1.48.a}$$

or

$$c_p\frac{DT}{Dt} - \frac{\alpha_T T}{\rho}\frac{Dp}{Dt} = Q^*, \tag{1.49.b}$$

where $Q^*$ represents the non-isentropic sources of heat due to the combined effects of diffusion, internal heating and frictional heating, with rate $Q^* = D\rho/Dt$. Note that the last equation (1.49) is more convenient because it is in terms of the measurable quantities $p$, $\rho$ and $T$. The first term is the heat storage in the fluid proportional to its heat capacity (specific heat), and the second term represents the heat storage due to compressional effects.

The relative importance of the two terms on the l.h.s. is measured by the ratio

$$-\frac{c_p \delta T}{\frac{\alpha_T T}{\rho}\delta p}$$

which can be interpreted (using equations 1.43 and 1.41) as

$$\begin{aligned} -\frac{c_p\delta T}{v\alpha_T T\delta p} &= -\frac{c_p\delta T}{\left(\frac{\partial v}{\partial T}\right)_p T\delta p} = \frac{c_p}{c_p-c_v}\left(\frac{\partial p}{\partial v}\right)_T\left(\frac{\partial v}{\partial T}\right)_p\frac{\delta T}{\delta p} \\ &= \frac{c_p}{c_p-c_v}\,\frac{\left(\frac{\partial v}{\partial T}\right)_p\delta T}{\left(\frac{\partial v}{\partial p}\right)_T\delta p}, \end{aligned} \tag{1.50}$$

measuring the relative roles of changes in volume due to temperature and pressure changes by themselves, which has then been multiplied by the factor $c_p/(c_p-c_v)$. We note that (cf. 1.43)

$$\left(\frac{\partial v}{\partial T}\right)_p = \alpha_T v \tag{1.51}$$

and define the Bulk Modulus of Elasticity $M$ of the fluid as

$$M = v\frac{\delta p}{\delta v} = -\rho\frac{\delta p}{\delta\rho} \tag{1.52.a}$$

which is dependent on the nature of the state changes. For isothermal changes

$$M_T = v\left(\frac{\partial p}{\partial v}\right)_T = -\rho\left(\frac{\delta p}{\delta\rho}\right)_T \tag{1.52.b}$$

The bulk modulus is related to the sound velocity $c_s$ through

$$M_T = \rho {c_s}^2. \tag{1.52.c}$$

Combining (1.151), (1.152.$b,c$) and noting that $\gamma = c_p/c_v$ (cf. 1.29), equation (1.50) is then written as

$$\begin{aligned} -\frac{c_p\delta T}{v\alpha_T T\delta p} &= \frac{\gamma}{\gamma-1}\alpha_T M_T\frac{\delta T}{\delta p} \\ &= \frac{\gamma}{\gamma-1}\rho\alpha_T {c_s}^2\frac{\delta T}{\delta p}. \end{aligned} \tag{1.53}$$

### 1.2.4 The Relative Roles of Compression in the Atmosphere and the Ocean

The ratio (1.53) measures the relative roles of the two terms on the l.h.s. of (1.48), *i.e.* the ratio of the heat storage due to specific heat to the heat storage due to compression.

If the atmosphere is assumed to be an ideal gas with $\gamma$=1.4, the coefficient on the r.h.s. will be $\gamma/(\gamma-1) = 3.5$. The maximum change in temperature from the surface of the earth to the top of the troposphere is $\delta T = 100^\circ K = 100^\circ C$, where the pressure change is $\delta p$=1000$mb$=1$b$=$10^5 Pa$ =$10^5 kg\ m^{-1}\ s^{-2}$. Typical values for the sound speed is $300m\ s^{-1}$, for the coefficient of thermal expansion is $\alpha_T = 3.4\times10^{-3}oK-1^{-1}$, and for the density of air is $\rho = 1.2kg\ m^{-3}$.

Putting these typical values in (1.53) yields

$$-\frac{c_p \delta T}{v \alpha_T T \delta p} = 3.5 \times 1.2 \times 3.4 \times 10^3 \times 9 \times 10^4 \times \frac{100}{10^5} \cong 1,$$

so that the two terms on the l.h.s. of (1.48) are equally important in the atmosphere.

For ocean water at typical conditions, $c_p$ and $c_v$ values are not too different, yielding a typical ratio of $\gamma$=0.997, or $\gamma/(\gamma-1)$= $3 \times 10^3$. The density value is close to $\rho = 10^3 kg/m^3$, and the coefficient of expansion for ocean water is usually close to the pure water value of $\alpha_T = 1.5x10^{-4}/{}^\circ K$. The speed of sound is typically around $c_s = 1500 m/s$, and the temperature difference from the surface to the bottom in deep basins of the ocean is on the order of $\delta T = 10^\circ C$ or more, within a pressure range of about $\delta p$=1000 $db$ = 100 $b$ = $10^7 Pa$. With these values we obtain

$$-\frac{c_p \delta T}{v \alpha_T T \delta p} = 3 \times 10^3 \times 10^3 \times 1.5x10^{-4} \times 2.25 \times 10^6 \times \frac{10}{10^7} \cong 10^3,$$

showing that the effects of compressional heating (second term of 1.48) are minimal, compared to specific-heat storage in the ocean.

Therefore we can write two versions of the thermodynamic equation. For the atmosphere

$$c_p \frac{DT}{Dt} - \frac{\alpha_T T}{\rho} \frac{Dp}{Dt} = \frac{1}{\rho} \nabla \cdot K \nabla T + (Q + \Phi) \tag{1.54}$$

and for the ocean

$$c_p \frac{DT}{Dt} = \frac{1}{\rho} \nabla \cdot K \nabla T + (Q + \Phi) \tag{1.55}$$

We can now use the equations of state for the individual fluids to complement and simplify these equations further, which is done in the following.

### 1.2.5 The Thermodynamic Equation for the Atmosphere

Let us first review the governing equations. We have the continuity and momentum equations (1.1 and 1.2) involving three unknowns $\rho$, $p$, and $\vec{u}$. To supplement these, we derived the thermodynamic equation (1.54), which introduced an additional variable $T$. To close the system, we still need to supplement another equation. This will be the equation of state

$$f(\rho, p, T) = 0. \tag{1.56}$$

In the real atmosphere, we still have another difficulty, mainly due to wet-processes, *i.e.* the influence of humidity, which introduces an additional new variable into the equations.

If we define the specific humidity

$$\sigma = \frac{\rho_v}{\rho_v + \rho_d} \tag{1.57}$$

with $\rho_v$ the density of vapor, and $\rho_d$ the density of dry air, $\sigma$ is then the concentration of vapor in the atmosphere. In this case, the ideal gas law, which constitutes the equation of state for the atmosphere becomes

$$f(\rho, p, T, \sigma) = 0 \tag{1.58.a}$$

or

$$p = \rho R(1 + 0.61\sigma)T = \rho R T_*, \tag{1.58.b}$$

where the virtual temperature $T_*$ the absorbs the effects of humidity. Since we have introduced an additional variable, $\sigma$, into the equations, we must write another equation to complete the system, expressing the conservation law for concentration $\sigma$, expressed as

$$\frac{D\sigma}{Dt} = \nabla \cdot K_\sigma \nabla\sigma + \Sigma \tag{1.59}$$

(where $K_\sigma$ is the diffusivity of moisture); typically valid for small concentrations of $\sigma \ll 1$. However, in the atmosphere this is often not true since the vapor may become saturated ($\sigma = 1$) and precipitation occurs, *i.e.*, a phase change occurs from vapor to water, releasing latent heat added as a source function in the thermodynamic equation, and acting as a sink term in the conservation of vapor, signified by $\Sigma$ in (1.59). But this becomes a highly nonlinear process, with a need for the switching on/off the source/sink terms when necessary.

The humidity effects are often ignored to avoid the above difficulties. It is then assumed that the atmosphere is an *ideal gas* with fixed composition satisfying the equation of state

$$p = \rho R T, \tag{1.60}$$

where

$$R = c_p - c_v \tag{1.61}$$

is the *universal gas constant.* The coefficient of thermal expansion becomes (cf. 1.43)

$$\begin{aligned} \alpha_T &= -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p = -\frac{1}{\rho}\frac{\partial}{\partial T}\left(\frac{p}{RT}\right)_p \\ &= +\frac{p}{\rho R}\frac{1}{T^2} = \frac{1}{T}, \end{aligned} \tag{1.62}$$

so that the thermodynamic equation (1.54) takes the following form

$$c_p \frac{DT}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt} = Q^*. \tag{1.63}$$

Furthermore, taking the logarithms of both sides in (1.60) gives

$$\ln p = \ln \rho + \ln T + \ln R \tag{1.64.a}$$

followed by differertiation yields

$$\frac{1}{p}\frac{Dp}{Dt} = \frac{1}{\rho}\frac{D\rho}{Dt} + \frac{1}{T}\frac{DT}{Dt}. \tag{1.64.b}$$

Substituting (1.65) into (1.63) gives

$$c_p \frac{DT}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} - \frac{p}{\rho T}\frac{DT}{Dt} = Q^*,$$

then, making use of (1.60) and (1.61) yields

$$c_v \frac{DT}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} = Q^*. \tag{1.65}$$

Still another form of the thermodynamic equation could be obtained by eliminating $DT/Dt$ from (1.63) and (1.65)

$$\frac{1}{\gamma p}\frac{Dp}{Dt} - \frac{1}{\rho}\frac{D\rho}{Dt} = \frac{1}{c_p T}Q^* \tag{1.66}$$

where $\gamma$ is defined by (1.29) and has the numerical value of

$$\gamma = \frac{c_p}{c_v} = 1.41 \tag{1.67}$$

for on ideal gas.

We can now define the potential temperature

$$\theta = T\left(\frac{p_*}{p}\right)^{R/c_p} = T\left(\frac{p_*}{p}\right)^{(1-\frac{1}{\gamma})} \tag{1.68.a}$$

where $p_*$ is an arbitrary (constant) reference pressure (*e.g.* using the sea-level pressure, $p_*$=1013 *mb*).

By making use of the ideal gas law (1.60), we can also write the potential temperature as

$$\theta = \left(\frac{p_*^{(1-\frac{1}{\gamma})}}{R}\right)\frac{p^{1/\gamma}}{\rho} = c_*\frac{p^{1/\gamma}}{\rho} \tag{1.68.b}$$

where $c_*$ is a constant. Logarithmic differentiation of (1.69) yields

$$\ln\theta = \ln T - \frac{R}{c_p}\ln p + \frac{R}{c_p}\ln p_*$$

and

$$\frac{1}{\theta}\frac{D\theta}{Dt} = \frac{1}{T}\frac{DT}{Dt} - \frac{R}{c_p}\frac{1}{p}\frac{Dp}{Dt}. \tag{1.69}$$

Multiplying this expression by $\rho c_p T$ results in

$$\frac{\rho c_p T}{\theta}\frac{D\theta}{Dt} = \rho c_p\frac{DT}{Dt} - \frac{Dp}{Dt},$$

and comparing with (1.63) we can write

$$\frac{D\theta}{Dt} = \frac{\theta}{c_p T}Q^*. \tag{1.70}$$

We can also make use of (1.47),

$$T\frac{Ds}{Dt} = Q^*, \tag{1.71}$$

to interpret the last equation.

An *adiabatic* atmosphere ($Q^* = 0$) is also an *isentropic* one ($\delta s = 0$) by virtue of (1.71). For the isentropic or adiabatic case (1.70)

$$\frac{D\theta}{Dt} = 0 \tag{1.72}$$

states that $\theta$ should be conserved following a fluid particle, *i.e.* $\theta$=constant, in which case (1.68) yields a simple relation between temperature and pressure, *i.e.* corresponding to compressional heating:

$$T = \theta\left(\frac{p}{p_*}\right)^{R/c_p}. \tag{1.73}$$

### 1.2.6 Thermodynamic Equation for the Ocean

We have noted that the compressibility effects are, in general, negligible in the ocean, by showing that the ratio $[\gamma/(\gamma-1)]\rho\alpha_T c_s^2\delta T/\delta p$ is on the order of $10^3$. However, in the deep ocean, the ratio can be smaller and in fact can be an $O(1)$ quantity, if we note that the temporal and spatial variations of temperature ($\delta T$ or $\frac{DT}{Dt}$) are small. Thus the two terms on the l.h.s. of (1.49) can be of comparable magnitude,

$$c_p\frac{DT}{Dt} - \alpha_T\frac{T}{\rho}\frac{Dp}{Dt} = Q^*, \tag{1.74}$$

but in the deep ocean the non-adiabatic term $Q^*$ can be neglected, $Q^* = 0$.

If we define *potential temperature* as

$$\theta = T - \int_{p_0}^{p} \frac{\alpha_T T}{\rho c_p} dp \tag{1.75}$$

with $p_0$ is a reference pressure(say pressure at the sea surface $p_0 \simeq 0$), then the material derivative of (1.74) is

$$\begin{aligned} \frac{D\theta}{Dt} &= \frac{DT}{Dt} - \frac{Dp}{Dt}\frac{D}{Dp}\int_{p_0}^{p} \frac{\alpha_T T}{\rho c_p} dp \\ &= \frac{DT}{Dt} - \frac{\alpha_T T}{\rho c_p}\frac{Dp}{Dt}. \end{aligned} \tag{1.76}$$

Comparing (1.76) with (1.74) we have

$$\frac{D\theta}{Dt} = \frac{Q^*}{c_p}, \tag{1.77}$$

and since $Q^* = 0$ in the deep ocean,

$$\frac{D\theta}{Dt} = 0, \tag{1.78}$$

or $\theta$ should be conserved (*i.e.* $\theta$=const.=$\theta_0$) which then gives a relation between temperature and pressure through (1.75). Assuming hydrostatic pressure (with $z$ pointing upwards)

$$\frac{\partial p}{\partial z} = -\rho g \tag{1.79}$$

then yields

$$\begin{aligned} T &= \theta_0 + \int_{p_0}^{p} \frac{\alpha_T T}{\rho c_p} dp \\ &= \theta_0 - \int_{z_0}^{z} \frac{g\alpha_T}{c_p} T dz \end{aligned} \tag{1.80}$$

or differentiating with respect to z

$$\frac{\partial T}{\partial z} = -\frac{g\alpha_T}{c_p} T. \tag{1.81}$$

The solution to (1.81) is

$$T = T_0 e^{-(\frac{g\alpha_T}{c_p})(z-z_0)} \tag{1.82}$$

indicating an exponential increase in temperature with depth. For typical values of $g = 10m\ s^{-2}$, $\alpha_T = 1.5 \times 10^{-4}\,{}^\circ C^{-1}$ and $c_p = 4.2 \times 10^3 Jkg\ {}^\circ C^{-1}$ , the rate of increase is characterized by

$$\frac{g\alpha_T}{c_p} = \frac{10 \times 1.5 \times 10^{-4}}{4.2 \times 10^3} = 0.3 \times 10^{-6} m^{-1}$$

*i.e.* a one percent increase in temperature takes place at a depth of about $\Delta z = 30km$ below the level $z_0$. Since the deepest part of the ocean has a depth of about 10 $km$, the increase in temperature due to compressibility of sea-water is less than 1 % everywhere.

Since we have assumed isentropic conditions, we can use the isentropic relation for the speed of sound

$$c_s{}^2 = \left(\frac{\partial p}{\partial \rho}\right)_s \tag{1.83}$$

or

$$\frac{Dp}{Dt} - c_s{}^2 \frac{D\rho}{Dt} = 0 \tag{1.84}$$

in the deep ocean. We can then define *potential density*

$$\Delta = \rho - \int_{p_0}^{p} \frac{1}{c_s{}^2} dp \tag{1.85}$$

which can be shown to satisfy (1.84)

$$\begin{aligned} \frac{D\Delta}{Dt} &= \frac{D\rho}{Dt} - \frac{Dp}{Dt}\frac{D}{Dp}\int_{p_0}^{p} \frac{1}{c_s{}^2} dp \\ &= \frac{D\rho}{Dt} - \frac{1}{c_s{}^2}\frac{Dp}{Dt} = 0 \end{aligned} \tag{1.86}$$

by virtue of (1.84); *i.e.* the potential density $\Delta$ is also conserved. Making use of the hydrostatic equation (1.79) yields

$$\rho = \Delta_0 - \int_{z_0}^{z} \frac{\rho g}{c_s{}^2} dz \tag{1.87}$$

or

$$\frac{\partial \rho}{\partial z} = -\frac{g}{c_s{}^2}\rho \tag{1.88.a}$$

or

$$\rho = \rho_0 - e^{-\int_{z_0}^{z} \frac{g}{c_s{}^2} dz} \cong \rho_0 e^{-\frac{g}{c_s{}^2}(z - z_0)} \tag{1.88.b}$$

for the deep water density variation due to compressibility. Typical values for the ocean of the rate of increase are

$$\frac{g}{{c_s}^2} = \frac{10}{(1500)^2} = 0.4 \times 10^{-5} m^{-1}$$

*i.e.* a one percent increase in in-situ density due to compressibility occurs over a depth range of 2.5 *km*.

Having shown the negligible influence of compressibility in the ocean, we return to the simplified version of the thermodynamics equation (1.55). Dividing by $c_p$, and neglecting $Q^{*\prime} + \Phi \simeq 0$, we have

$$\frac{DT}{Dt} = \nabla \cdot \kappa_T \nabla T \tag{1.89}$$

which is known as the heat diffusion equation, $\kappa_T \equiv K/\rho c_p$ defined as the *diffusivity of heat.* Again we have introduced an additional equation for a new variable $T$, in addition to the continuity and momentum equations which involve the unknowns $p$, $\rho$, $\vec{u}$. To close the system, we must invoke the equation of state. If we assume the ocean is of fixed composition, then $f(p, \rho, T)$ closes the system.

However, the ocean density is actually determined by salinity as well as temperature, since the ocean water is actually of fixed composition. The salts dissolved in water are of fixed composition but the solution is not, the total concentration of salts determining the salinity. Therefore the equation of state for sea-water is of the form

$$f(\rho, p, T, S) = 0. \tag{1.90}$$

Since the compressibility of seawater is negligible, the role of pressure in determining (1.90) is of minor importance. On the other hand, the actual empirical form of (1.90) is highly non-linear, which can be linearized around some central values $T_0$, $S_0$, $\rho_0$:

$$\rho = \rho_0(1 - \alpha_T(T - T_0) + (S - S_0)) \tag{1.91}$$

We therefore have in extra variable $S$ in our equations. To complete the system now, we need to write a conservation law for salinity (*i.e.* a diffusion equation):

$$\frac{DS}{Dt} = \nabla \cdot \kappa_S \nabla S. \tag{1.92}$$

The solutions for $\vec{u}$, $\rho$, $p$, $T$, $S$ can therefore be obtained from the complete set of equations (1.1), (1.2), (1.89), and (1.92).

The diffusivities $\kappa_T$ and $\kappa_S$ are, generally speaking, not equal in the ocean. However, if they are assumed to be equal or if the effect of salinity in the equation of state is neglected altogether, then it is possible to combine (1.89) and (1.90) in (1.91) yielding

$$\frac{D\rho}{Dt} = \nabla \cdot \kappa \nabla \rho \tag{1.93}$$

where $\kappa = \kappa_T = \kappa_s$.

Note that equation (1.93) is quite different in form and meaning form the continuity equation (1.1), although both equations involve conservation statements for $\rho$. In the continuity equation $D\rho/Dt$ stands for density changes due to either compressibility or inhomogeneity effects, which create a divergence of velocity, whereas compressibility is neglected in the thermodynamic equation, leading to equation (1.93) relating density changes to diffusions of heat and salt. Also note that by introducing (1.93), the equations (1.89) (1.91) and (1.92) become redundant. This is because equation (1.93) complements continuity and momentum equations (1.1, 1.2) to yield three equations for three unknowns $\rho$, $p$, $\vec{u}$.

As a result, we will use versions of the thermodynamic equation (1.66) or (1.70) for the atmosphere, and (1.93) for the ocean in further derivations in Chapter 2.

## 1.3 Vorticity Dynamics

We will review vorticity dynamics in rotating, stratified fluids, which was partially examined in DO-I.

The *relative vorticity* of a fluid is defined as

$$\vec{\omega} = \nabla \times \vec{u} \tag{1.94}$$

with respect to the inertial (rotating) coordinates. Since the velocity in the absolute (fixed) coordinate system is given by (DO-I)

$$\vec{u}_A = \vec{u} + \vec{\Omega} \times \vec{r}, \tag{1.95}$$

with $\vec{\Omega}$ representing earth's angular velocity and $\vec{r}$ the position vector in the rotating system, an *absolute velocity* can likewise be defined:

$$\vec{\Omega}_A = \nabla \times \vec{u} + \nabla \times \vec{\Omega} \times \vec{r} \tag{1.96}$$

by virtue of vector differentiation rules given in DO-I.

By making use of vector identities, the momentum equation (1.2) is first written as

$$\frac{\partial \vec{u}}{\partial t} + \frac{1}{2}\nabla(\vec{u}\cdot\vec{u}) - \vec{u} \times \nabla \times \vec{u} + 2\vec{\Omega} \times \vec{u} = \vec{g} - \frac{1}{\rho}\nabla p + \vec{F} \tag{1.97}$$

where $\vec{F} = \frac{1}{\rho}\nabla \cdot \mathbf{d}$ is the net frictional force. Taking the curl of the modified momentum equation (1.97), and using the definition (1.94) and vector identities of DO-I it can be shown that $\vec{\omega}$ satisfies

$$\frac{\partial \vec{\omega}}{\partial t} + \nabla \times \left[ (\vec{\omega} + 2\vec{\Omega}) \times \vec{u} \right] = -\nabla \frac{1}{\rho} \times \nabla p + \nabla \times \vec{F} \tag{1.98}$$

Noting that $\vec{\Omega}$ is constant and using equation (1.96), this can also be stated as

$$\frac{\partial \vec{\omega}_A}{\partial t} + \nabla \times (\vec{\omega}_A \times \vec{u}) = -\nabla \frac{1}{\rho} \times \nabla p + \nabla \times \vec{F} \tag{1.99}$$

Upon making use of vector identities the second term is expanded, resulting in the equation

$$\frac{D\vec{\omega}_A}{Dt} = \frac{\partial \vec{\omega}_A}{\partial t} + \vec{u} \cdot \nabla \vec{\omega}_A = \vec{\omega}_A \cdot \nabla \vec{u} - \vec{\omega}_A \nabla \cdot \vec{u} - \nabla \frac{1}{\rho} \times \nabla p + \nabla \times \vec{F} \tag{1.100}$$

Equations (1.98), (1.99) and (1.100) are different forms of the vorticity equation. In equation (1.100) the l.h.s. is the material derivative of absolute vorticity. The r.h.s. includes effects of vortex stretching(first term), vorticity changes due to compressibility(second term), *vorticity changes induced in stratified fluids* (third term) and torques generated by friction forces (fourth term). Here, the third term can be written as

$$-\nabla \frac{1}{\rho} \times \nabla p = \frac{1}{\rho^2} \nabla \rho \times \nabla p \tag{1.101}$$

and in the absence of the other effects (vortex stretching, compressibility and friction) this is the only term which induces changes in the vorticity of a stratified fluid. If either the fluid is homogeneous ($\nabla \rho = 0$), or if the fluid is stratified such that the density and pressure gradients are in the same direction every where ($p = p(\rho)$), then the above term also vanishes and the absolute vorticity $\vec{\omega}_A$ is conserved (vorticity can not be created or destroyed). If either condition is not satisfied the *overturning* tendency of a stratified fluid is represented by (1.101).

Now, consider some special quantity $\theta$ that is conserved following a fluid particle

$$\frac{D\theta}{Dt} = \frac{\partial \theta}{\partial t} + \vec{u} \cdot \nabla \theta = 0 \tag{1.102}$$

For example the quantity $\theta$ could be salinity, or temperature if diffusion, friction or internal sources, are neglected, or it could be anything else that is conserved. Through the use of vector identities involving $\nabla\theta$, $\vec{\omega}_A$, and $\vec{u}$:

$$\nabla \theta \times (\vec{\omega}_A \times \vec{u}) = (\vec{\omega}_A \cdot \nabla \theta)\vec{u} - (\vec{u} \cdot \nabla \theta)\vec{\omega}_A.$$

Substituting from (1.102) yields:

$$\nabla \theta \times (\vec{\omega}_A \times \vec{u}) = -\frac{\partial \theta}{\partial t} \vec{\omega}_A - (\vec{\omega}_A \cdot \nabla \theta)\vec{u} \tag{1.103}$$

If equation (1.99) is multiplied with $\nabla\theta$ and (1.103) is used together with vector identities, it can be written as

$$\nabla\theta \cdot \frac{\partial \vec{\omega}_A}{\partial t} + \nabla \cdot \left[ \vec{\omega}_A \frac{\partial \theta}{\partial t} + \vec{u}(\vec{\omega}_A \cdot \nabla\theta) \right] = -\nabla\theta \cdot (\nabla \frac{1}{\rho} \times \nabla p) + \nabla\theta \cdot (\nabla \times \vec{F}). \tag{1.104}$$

Further use of vector identities yields

$$\frac{D(\vec{\omega}_A \cdot \nabla\theta)}{Dt} + (\vec{\omega}_A \cdot \nabla\theta)\nabla \cdot \vec{u} = -\nabla\theta \cdot (\nabla \frac{1}{\rho} \times \nabla p) + \nabla\theta \cdot (\nabla \times \vec{F}). \tag{1.105}$$

Now, making use of the continuity equation (1.1) and dividing by $\rho$ the equation takes the form :

$$\frac{D}{Dt} \left[ \frac{\vec{\omega}_A \cdot \nabla\theta}{\rho} \right] = \frac{1}{\rho} \nabla\theta \cdot \left[ -\nabla \frac{1}{\rho} \times \nabla p + \nabla \times \vec{F} \right] \tag{1.106}$$

which is known as *Ertel's theorem.* Neglecting the frictional effects, the l.h.s. bracketed quantity must be conserved if one of the following conditions is satisfied: either $\theta$, $p$ and $\rho$ are related, or $\rho$ is constant (homogeneous fluids) or $\nabla\theta$ and $\nabla p$ are co-planar vectors. If the flow is inviscid and incompressible but stratified (satisfying $\nabla \cdot \vec{u} = 0$ or $D\rho/Dt = 0$, by virtue of 1.1), $\rho$ can replace $\theta$ in equation (1.102) and (1.106). The r.h.s. of (1.106) vanishes, yielding

$$\frac{D}{Dt} \left[ \frac{\vec{\omega}_A \cdot \nabla p}{\rho} \right] = 0. \tag{1.107}$$

Next, consider a material surface $S$ enclosed by a material curve $C$ in a fluid. The quantity

$$\Gamma(t) = \oint_C \vec{u} \cdot d\vec{r} \tag{1.108}$$

is defined as the *circulation* in the non-inertial frame, and by virtue of Stokes' theorem (DO-I)

$$\Gamma(t) = \int_S \nabla \times \vec{u} \cdot \hat{n}\ ds = \int_S \vec{\omega} \cdot \hat{n}\ ds \tag{1.109}$$

it is the vorticity flux passing through the surface S. It is also possible to define the circulation in the absolute frame of reference as

$$\Gamma_A(t) = \oint_C \vec{u}_A \cdot d\vec{r} = \Gamma(t) + \oint_C \vec{\Omega} \times \vec{r} \cdot d\vec{r} \tag{1.110}$$

by virtue of (1.95). Through the use of vector identities, and the Stokes' theorem, it can be shown that the second term equals

$$\oint_C \vec{\Omega} \times \vec{r} \cdot d\vec{r} = \int_S \nabla \times (\vec{\Omega} \times \vec{r}) \cdot \hat{n}\ ds = \int_S 2\vec{\Omega} \cdot \hat{n}\ ds, \tag{1.111}$$

yielding

$$\begin{aligned}\Gamma_A(t) &= \Gamma(t) + \int_s 2\vec{\Omega}\cdot\hat{n}\ ds \\ &= \int_S (\vec{\omega} + 2\vec{\Omega})\cdot\hat{n}\ ds = \int_S \vec{\omega}_A\cdot\hat{n}\ ds\end{aligned} \tag{1.112}$$

where $\vec{\omega}_A$ is defined in (1.96). The integrand of the second term in (1.112) is the component of $2\vec{\Omega}$ normal to the surface $S$, or alternatively

$$\int_S 2\vec{\Omega}\cdot\hat{n}\ ds = 2\Omega S_p, \tag{1.113}$$

since the $\vec{\Omega}$ vector is constant. Here, $\Omega = \mid\vec{\Omega}\mid$ and $S_p$ is the projection of the surface $S$ on the plane perpendicular to $\vec{\Omega}$. The rate of change of $\Gamma_A$ is

$$\frac{d\Gamma_A}{dt} = \frac{d}{dt}\oint_C \vec{u}_A\cdot d\vec{r} = \oint_C \frac{D\vec{u}_A}{Dt}\cdot d\vec{r} + \oint_C \vec{u}_A\cdot\frac{D}{Dt}(d\vec{r}), \tag{1.114}$$

where the second term can be shown to be

$$\oint_C \vec{u}_A\cdot d\left(\frac{D\vec{r}}{Dt}\right) = \frac{1}{2}\oint_C d(\vec{u}_A\cdot\vec{u}_A) = 0 \tag{1.115}$$

since the integral is taken for a closed curve. Utilizing (1.112), (1.114) and (1.115)

$$\frac{d\Gamma}{dt} = \oint_C \frac{D\vec{u}_A}{Dt}\cdot d\vec{r} - 2\Omega\frac{dS_p}{dt}. \tag{1.116}$$

Now, since $\vec{u}_A$ satisfies (the momentum equation for $\vec{u}_A$)

$$\frac{D\vec{u}_A}{Dt} = \vec{g} - \frac{1}{\rho}\nabla p + \vec{F}, \tag{1.117}$$

we have

$$\frac{d\Gamma}{dt} = \oint_C \vec{g}\cdot d\vec{r} - \oint_C \frac{1}{\rho}\nabla p\cdot d\vec{r} + \oint_C \vec{F}\cdot d\vec{r} - 2\Omega\frac{dS_p}{dt}. \tag{1.118}$$

Note that by expressing $\vec{g} = -\nabla\phi$ and utilizing the Stokes theorem once more,

$$\oint_C \vec{g}\cdot d\vec{r} = \oint_C \nabla\phi\cdot d\vec{r} = \int\int_S \nabla\times\nabla\phi\cdot\hat{n}\ dS = 0, \tag{1.119}$$

so that (1.118) simplifies to the *Kelvin's circulation theorem*:

$$\frac{d\Gamma}{dt} = -\int_S \nabla\frac{1}{\rho} \times \nabla p \cdot \hat{n}\; dS + \oint_C \vec{F} \cdot d\vec{r} - 2\Omega\frac{dS_p}{dt}, \tag{1.120}$$

through further use of the Stokes' theorem and vector identities. The first term on the r.h.s. is the contribution to the rate of change of circulation by the *overturning* tendency in a stratified fluid, the second term is the contribution by tangential shear stresses and the third term is the contribution by the changes in the projected area (representing conservation of angular momentum). These respective terms represent, the effects of stratification, friction and rotation on the conservation of circulation. In the absence of all three effects, Kelvin's circulation theorem states that the circulation is conserved: i.e. an *irrotational fluid* will remain to be irrotational.

Let us now consider the first term on the r.h.s. of (1.120), representing the creation or destruction of circulation (or vorticity) in stratified fluids. For this term to vanish, either of the following conditions must be met :

i) the fluid is *homogeneous* (*i.e.* $\rho=$ constant), then $\nabla\rho = 0$,

ii) the fluid is *barotropic* (*i.e.* pressure is a function of density, $p = p(\rho)$), then the corresponding term becomes

$$-\nabla\frac{1}{\rho} \times \nabla p = \frac{1}{\rho^2}\nabla\rho \times \nabla p = \frac{1}{\rho^2}\frac{\partial p}{\partial \rho}\nabla\rho \times \nabla\rho = 0.$$

If the fluid is in both *inhomogeneous* (stratified) and *baroclinic* (not barotropic), then the overturning term in (1.120) contributes to the vorticity dynamics (ref. equation 1.53).

# CHAPTER 2

# QUASIGEOSTROPHIC THEORY

## 2.1 The Existence of a "Basic State"

In developing the quasi-geostrophic theory, we will assume that the motions in the atmosphere and ocean occurs as perturbations to a basic state which is described by a static balance of the forces acting on the fluids. Therefore the basic state is one corresponding to the fluids being at rest ($\vec{u} = 0$) and to a steady-state $\left(\frac{\partial}{\partial t} = 0\right)$. As a result, $\left(\frac{D}{Dt} = 0\right)$ for all variables and presumably the basic state would only result in vertical variations in the state variables such as state p, T and $\rho$. Let us now investigate if such a state would actually exist. The equations governing the basic state are then

$$\frac{1}{\rho}\nabla p + \vec{\Omega} \times \vec{\Omega} \times \vec{r} - \vec{g} = 0 \tag{2.1.a}$$

$$\nabla^2 T = 0 \tag{2.1.b}$$

$$\rho = \rho(p, T) \tag{2.1.c}$$

where the first equation is the momentum eq" (1.2) with (1.3) substituted and the second one is the thermodynamic equation corresponding to either (1.54) or (1.55) for the atmosphere and the ocean respectively (assuming constant K): The last equation is the equation of state for either fluid when the effects of moisture in the atmosphere and salinity in the ocean respectively are neglected we should investigate the corresponding solutions on the earth in spherical coordinates. However, for the purpose of demonstration, an analogous situation on a rotating table is sufficient, and choice of coordinates is immaterial in investigating the basic state. Consider $\vec{\Omega} = \Omega\hat{k}$ and $\vec{g} = -g\hat{k}$ where $\hat{k}$ is the unit vector in z-direction (along the axis od rotation), yielding

$$\nabla p = \rho\Omega^2\vec{r} + \rho g\hat{k} \tag{2.2}$$

Taking curl of (.2) yields

$$\Omega^2\nabla \times \rho\vec{r} + g\nabla \times \rho\hat{k} = 0 \tag{2.3}$$

which can be written in cylindrical coordinates (r, $\theta$, z) as

$$g\frac{\partial\rho}{\partial r} + r\Omega^2\frac{\partial\rho}{\partial z} = 0 \tag{2.4}$$

The equation (2.1.b) satisfied by the temperature becomes

$$\frac{\partial^2 T}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial\theta^2} = 0 \tag{2.5}$$

Now, since density is a function of temperature, it would be impossible to obtain solutions (2.4) and (2.5) simultaneously. For instance, solution to (2.4) would be of the form

$$\rho = \rho\left(z - \frac{\Omega^2 r^2}{2g}\right) \tag{2.6}$$

and consequently by virtue of (2.1.c)

$$T = T\left(z - \frac{\Omega^2 r^2}{2g}\right). \tag{2.7}$$

The functional form contradicts with (2.5) to which it can not be solution. The contradiction in this simple model shows that the static equilibrium can not be achieved in a rotating-stratified fluid. Some convection (motion) is necessary in the steady state, so that $\vec{u} \neq 0$. However, if the fluid is assumed to be isentropic ($Q^* = 0$), then equations (1.54) and (1.55) become

$$c_p \frac{DT}{Dt} - \frac{\alpha_T T}{\rho}\frac{Dp}{Dt} = 0 \tag{8.a}$$

$$\frac{DT}{Dt} = 0 \tag{8.b}$$

for the atmosphere and ocean respectively and with the assumptions of $\vec{u} = 0$ (static), $\frac{\partial}{\partial t} = 0$ (steady) fluids these equations vanish completely, so that the only remaining equation is (2.4), and the solution (2.6) is then acceptable. On the other hand, since the centrifugal forces are often much smaller than gravity, the horizontal changes in density may be much smaller than that occurs in the vertical *i.e.*

$$\frac{\Omega^2 r^2}{2g} \ll z \tag{2.9}$$

So that (.6) and (.7) reduces to

$$\rho = \rho(z) \quad \text{and} \quad T = T(z). \tag{2.10.a,b}$$

For example,

$$T = T_0 + \alpha z \tag{2.11}$$

is on exact solution of (2.5).

As a result we can assume a basic - state in which the state variables are functions of $z$ only, if we neglect centrifugal acceleration as compared to gravity. Furthermore, since we will assume isentropic fluids later in the development static equilibrium can exist in the fluids, under these assumptions. We thus assume basic state variables of the form $\rho_r(z)$, $p_r(z)$, $T_r(z)$, *etc.* and as we have shown above we can take $\vec{u}_r = 0$ in this state. Since there is no motion, and only vertical variations, these variations are the result of gravitation which is the only force acting on the fluid. Therefore ( ) yields

$$\frac{1}{\rho_r}\nabla p_r = \vec{g} \tag{2.12}$$

Since we will make tangent-plane approximations we take $\vec{g} = -g\hat{k}$ where $\hat{k}$ is the unit vector in z-direction, yielding

$$\frac{dp_r}{dz} = -\rho_r g \tag{2.13}$$

which is the relation for "hydrostatic pressure" which exists in the basic state.

### 2.2 Scale Heights

Considering simple models we can estimate the vertical variation and vertical scales of change of the state variables.

For the atmosphere, we can use the ideal gas law $p_r = \rho_r R T_r$. Logarithmic differentiation yields

$$\frac{1}{p_r}\frac{dp_r}{dz} = \frac{1}{\rho_r}\frac{d\rho_r}{dz} + \frac{1}{T_r}\frac{dT_r}{dz} \tag{2.14}$$

and substituting (2.13) then gives

$$-\frac{1}{\rho_r}\frac{d\rho_r}{dz} = \frac{g}{RT} + \frac{1}{T_r}\frac{dT_r}{dz} \tag{2.15}$$

We can obtain a simple relation for an *isothermal* atmosphere ($T_r = \bar{T}$ =constant) from (.15), yielding

$$\rho_r = \rho_0 e^{-(z-z_0)/H_s} \tag{2.16}$$

where $H_s = R\bar{T}/g$ is the *density scale height.* For typical values of $R = 2.9 \times 10^6 cm^2 s^{-2 \circ} K^{-1}$, $\bar{T} = 293^\circ K, g = 981 cm s^{-2}$, $H_s = 9km$ which is almost the same order as the thickness of the troposphere.

For the ocean, we can obtained a simple relation in the isentropic case (potential density $\beta$=0). By virtue of (1.88.a)

$$\rho_r(z) = \rho_0 e^{-g\int_{z_0}^{z}\frac{1}{c_s{}^2}dz} \cong \rho_0 e^{-(z-z_0)/H_s} \tag{2.17}$$

where $H_s = \bar{c}_s^2/g$ is the density scale height, $\bar{c}_s$ is the average speed of sound ($\bar{c}_s$=constant). For typical values of $\bar{c}_s = 1500 ms^{-1}$ and $g = 9.81 ms^{-2}$, we obtain $H_s \cong 200km$. Since the deepest part of the ocean is only about 10 $km$ the basic-state density change in the vertical is in fact shown to be negligible. We can therefore use the approximation on the r.h.s. of (2.17), and equivalently, write the density scale height $H_s$ as

$$H_s = \left(-\frac{1}{\rho_r}\frac{d\rho_r}{dz}\right)^{-1} \tag{2.18}$$

where $H_s$ is typically 10 $km$ for the atmosphere and 200 $km$ for the ocean as seen from the above simplified models.

### 2.3 Perturbation Equations

Considering the motions in the geophysical fluids to be perturbations on the basic state, we can write each variable as

$$\left\{\begin{array}{c} p \\ \rho \\ T \\ \theta \\ \vec{u} \end{array}\right\} = \left\{\begin{array}{c} p_r(z) \\ \rho_r(z) \\ T_r(z) \\ \theta_r(z) \\ \\ \end{array}\right\} + \left\{\begin{array}{c} \tilde{p}(x,y,z,t) \\ \tilde{\rho}(x,y,z,t) \\ \tilde{T}(x,y,z.t) \\ \tilde{\theta}(x,y,z.t) \\ \tilde{u}(x,y,z.t) \end{array}\right\} \tag{2.19}$$

Neglecting frictional forces and writing the momentum equation (1.2) on a tangent plane results in

$$\frac{\partial \vec{u}_h}{\partial t} + \vec{u}_h \cdot \nabla_h \vec{u}_h + w\frac{\partial \vec{u}_h}{\partial z} + f\hat{k} \times \vec{u}_h = -\frac{1}{\rho}\nabla_h p, \tag{2.21.a}$$

$$\frac{\partial w}{\partial t} + \vec{u}_h \cdot \nabla_h w + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \tag{2.21.b}$$

The first equation is the horizontal component of (1.2) and the second equation represents the vertical component of the same equation. The horizontal gradient operator $\nabla_h = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ and the horizontal velocity vector $\vec{u}_h = (u, v)$ has been introduced such that

$$\vec{u} = \vec{u}_h + w\hat{k} \tag{2.22.a}$$

is the three dimensional velocity vector and

$$\nabla = \nabla_h + \hat{k}\frac{\partial}{\partial z} \tag{2.22.b}$$

is the three dimensional gradient. It has also been assumed that $\vec{g} = -g\hat{k}$, and $f = 2\Omega \sin\phi$ is the Coriolis parameter, the only contribution to (2.21.a) coming from $f\hat{k} \times \vec{u}_h$ and the vertical components of Coriolis forces being neglected since they are much smaller than $\vec{g}$ (see DOII notes).

The continuity equation becomes

$$\frac{\partial \rho}{\partial t} + \nabla_h \cdot \rho\vec{u}_h + \frac{\partial}{\partial z}\rho w = 0 \tag{2.23}$$

in the same notation.

Next we need to complement the above equations with thermodynamic counterparts. For the atmosphere we prefer to use the thermodynamics equation (1.70)

$$\frac{\partial \theta}{\partial t} + \vec{u}_h \cdot \nabla_h \theta + w\frac{\partial \theta}{\partial z} = \frac{\theta}{c_p T} Q^* \tag{2.24}$$

with an equation of state (1.68.b)

$$\theta = C_* \frac{p^{1/\gamma}}{\rho} \tag{2.25}$$

where $C_* = p_*^{(1-\frac{1}{\gamma})/R}$ is a constant

For the ocean we will prefer to use the version of thermodynamic equation written in terms of density, *i.e.* (1.93), yielding

$$\frac{\partial \rho}{\partial t} + \vec{u}_h \cdot \nabla_h \rho + w\frac{\partial \rho}{\partial z} = R^* \tag{2.26}$$

where $R^* = \nabla \cdot \kappa \nabla \rho$ is the diffusive (non-isentropic) term. This form of the equation is directly expressed in density, so that the addition of an equation of state is not needed for the ocean.

In the following developments, we will neglect the non-isentropic terms setting $Q^* = R^* = 0$ in (2.24) and (2.26).

We can now substitute (2.19) into the above equations. Equation (2.11.a) becomes, (dropping the $()_h$ notation),

$$(\rho_r + \tilde{\rho})\{\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} + w\frac{\partial \vec{u}}{\partial z} + f\hat{k} \times \vec{u}\} = -\nabla(p_r + \tilde{p}) = -\nabla \tilde{p} \tag{2.27}$$

Since $p_r = p_r(z)$ alone.

The vertical momentum equation (2.21.b) yields

$$\begin{aligned}(\rho_r + \tilde{\rho})\{\frac{\partial w}{\partial t} + \vec{u} \cdot \nabla w + w\frac{\partial w}{\partial z}\} &= -\frac{\partial(p_r + \tilde{p})}{\partial z} - (\rho_r + \tilde{\rho})g \\ &= -\frac{\partial \tilde{p}}{\partial z} - \tilde{\rho} g\end{aligned} \tag{2.28}$$

where use has been made of the hydrostatic equation (2.13).

Since $\rho_r = \rho_r(z)$ only, the continuity equation (2.23) simplifies to

$$\frac{\partial \tilde{\rho}}{\partial t} + \vec{u} \cdot \nabla \tilde{\rho} + (\rho_r + \tilde{\rho})\nabla \cdot \vec{u} + \frac{\partial}{\partial z}\rho_r w + \frac{\partial}{\partial z}\tilde{\rho} w = 0 \tag{2.29}$$

The thermodynamic equation for the atmosphere (2.24) becomes

$$\frac{\partial \tilde{\theta}}{\partial t} + \vec{u} \cdot \nabla \tilde{\theta} + w \frac{\partial \theta_r}{\partial z} + w \frac{\partial \tilde{\theta}}{\partial z} = 0 \tag{2.30}$$

and the equation of state (2.25) can be written as

$$(\theta_r + \tilde{\theta}) = C_*(p_r + p)^{1/\gamma}(\rho_r + \tilde{\rho})^{-1} \tag{2.31}$$

For the ocean (2.26) takes the form

$$\frac{\partial \tilde{\rho}}{\partial t} + \vec{u} \cdot \nabla \tilde{\rho} + w \frac{\partial \rho_r}{\partial z} + w \frac{\partial \tilde{\rho}}{\partial z} = 0 \tag{2.31}$$

## 2.4 Order of Magnitude Analysis

In order to scale our equations, we select the scales

$$\begin{array}{lll} \vec{x} \sim L, & z \sim H, & t \sim \frac{L}{U} \\ \vec{u} \sim U, & w \sim \alpha \frac{H}{L} U, & f \sim f_0 \\ p_r \sim p_0, & \rho_r \sim \rho_0, & \theta_r \sim \theta_0 \\ \tilde{p} \sim p_*, & \tilde{\rho} \sim \rho_*, & \tilde{\theta} \sim \theta_*, \end{array} \tag{2.32.a - k}$$

Here, we select the time scale as the time of travel of particle with speed $U$ over a length scale $L$. Alternatively

$$t \sim \frac{1}{f_0}\left(\frac{f_0 L}{U}\right) = \frac{1}{f_0 \delta} \tag{2.33}$$

where $f_0$ is the scale for the Coriolis parameter and

$$\delta = \frac{U}{f_0 L} \tag{2.34}$$

is the Rossby number. The meaning of (2.33) is then, that the time scale of the motion is much larger than the inertial period $2\pi/f_0$, when $\delta \ll 1$.

The inherent assumption in the scaling of the vertical is that the vertical velocity is smaller than the horizontal velocity by the ratio of the small parameter

$$\lambda = H/L \tag{2.35}$$

such that $\lambda \ll 1$ (shallow water $\lambda$ approximation). In fact, This salinity argument can directly be obtained from in the homogeneous case the continuity equation (2.23). However, in rotating fluids the vertical motion is decompled from the horizontal, and therefore the vertical cf. DOII velocities on the fluid are induced by external processes of either ekman pumping or bottom topography. Considering the bottom topography alone, the vertical velocity induced should be of the order

$$\omega = O\left(U\frac{h_b}{L}\right) = O(\alpha\lambda U) \tag{2.36}$$

where $\alpha = h_b/H$ is the small parameter ($\alpha \ll 1$) representing the ratio of the bottom topographic amplitude to the total depth. As we have done earlier in the derivation of homogeneous quasi-geostrophic theory we take $\alpha \ll 1$, although the case of $\alpha = O(1)$ can be recovered as a limiting case of the theory to be derived (as in the homogeneous case, cf DO-II). We also known that vertical motions are suppressed in rotating fluids, so that we replace the small parameter $\alpha$ with $\delta$, yielding,

$$w \sim \delta\lambda U \tag{2.37}$$

A final dimensionless parameter which appears after scaling is the (external) divergence parameter

$$\mu = \frac{f_0^2 L^2}{gH} = \left(\frac{L}{R}\right)^2 \tag{2.38}$$

where $R = \sqrt{\frac{gH}{f_0}}$ the radius of deformation.

The scales of pressure, density and temperature are detained as follows. Assuming perturbations to be small, for instance $p_* \ll p_0$, and the same follows for the other variables. Scaling (2.27) by (2.32 a-k) and (2.37) yields

$$\delta(\vec{u}_t + \vec{u}\cdot\nabla\vec{u} + \delta\ w\vec{u}_z) + f\hat{k}\times\vec{u} = -\frac{p_*}{f_0 U L(\rho_0\rho_r + \rho_*\rho)}\nabla\tilde{p} \tag{2.39}$$

so that the Coriolis term balances the pressure gradient to $O(1)$, yielding

$$p_* \sim \rho_0 f_0 U L \tag{2.40}$$

On the other hand, the hydrostatic balance of (2.13) yields

$$p_0 \sim \rho_0 g H \tag{2.41}$$

So that (2.40) and (2.41) yield

$$p_* \sim \rho_0 g H\frac{f_0 U L}{gH} = p_0\mu\delta \tag{2.42}$$

Similarly, scaling (2.28) yields

$$\lambda\delta\left(\frac{\partial w}{\partial t} + \vec{u}\cdot\nabla w + \delta w\frac{\partial w}{\partial z}\right) = \frac{1}{\left(\rho_r + \frac{\rho_*}{\rho_0}\tilde{\rho}\right)}\{-\frac{fL^2}{UH}\frac{\partial\tilde{p}}{\partial z} - \frac{\rho_* L g}{\rho_0 U^2}\} \tag{2.43}$$

Since the left hand side is small, the right hand side terms must be balanced (hydrostatic pressure in perturbation field) yielding

$$\rho_* \sim \frac{fUL}{gH}\rho_0 = \mu\delta\rho_0 \tag{2.44}$$

By similar arguments the potential temperature salinity is

$$\theta_* \sim \mu\delta\theta_0 \tag{2.45}$$

Since $\rho$ and $\theta$ are related. In fact, we can scale (2.31) with the above, yielding

$$\left(\theta_r + \delta\mu\tilde{\theta}\right) = \frac{c_*(\rho_0 gH)^{1/\gamma}}{\rho_0\theta_0}\left(p_r + \delta\mu\tilde{p}\right)^{1/\gamma}\left(\rho_r + \delta\mu\right)^{-1} \tag{2.46}$$

We note that the constant $c_* = p_*^{1-\frac{1}{\gamma}}/R$, where $p_*$ is an arbitrary reference pressure. Selecting $p_* = \rho_0 gH = p_0$, the constant in (2.46) is evaluated as

$$\begin{aligned}\frac{c_*(\rho_0 gH)^{1/\gamma}}{\rho_0\theta_0} &= \frac{p_*^{1-\frac{1}{\gamma}}}{R}\frac{(\rho_0 gH)^{1/\gamma}}{\rho_0\theta_0} = \frac{(\rho_0 gH)^{1-\frac{1}{\gamma}+\frac{1}{\gamma}}}{\rho_0 R\theta_0} \\ &= \frac{gH}{R\theta_0}\end{aligned} \tag{2.47}$$

On the other hand, we can establish a relation between the arbitrary scales $\theta_0, p_0$ and $\rho_0$, by using the relation (2.25), *i.e.*

$$\theta_0 = \frac{p_*^{1-\frac{1}{\gamma}}}{R}\frac{p_0^{1/\gamma}}{\rho_0} = \frac{p_0}{\rho_0 R} = \frac{gH}{R} \tag{2.48}$$

So that the constant in (2.46) *i.e.* the right hand side of (2.48) becomes =1, yielding (in 2.46)

$$\left(1 + \delta\mu\frac{\tilde{\theta}}{\theta_r}\right) = \left(1 + \delta\mu\frac{\tilde{p}}{p_r}\right)^{1/\gamma}\left(1 + \delta\mu\frac{\tilde{\rho}}{\rho_r}\right)^{-1} \tag{2.49}$$

Since $\delta\mu \ll 1$, we can expand the terms in Taylor Series, yielding

$$\begin{aligned}\left(1 + \delta\mu\frac{\tilde{\theta}}{\theta_r}\right) &= \left(1 + \frac{1}{\gamma}\delta\mu\frac{\tilde{p}}{p_r} + \cdots\right)\left(1 - \delta\mu\frac{\tilde{\rho}}{\rho_r} + \cdots\right) \\ &= 1 + \delta\mu\left(\frac{\tilde{p}}{\gamma p_r} - \frac{\tilde{\rho}}{\rho_r}\right) + (\delta\mu)^2\,() + \cdots\end{aligned} \tag{2.50}$$

so that to $O(\delta\mu)$ we can write

$$\frac{\tilde{\theta}}{\theta_r} = \frac{1}{\gamma}\frac{\tilde{p}}{p_r} - \frac{\tilde{\rho}}{\rho_r} \tag{2.51}$$

Similarly expanding the ideal gas law $p = \rho RT$ and scaling as $T_r \sim T_0$, $\tilde{T} \sim T_* = \delta\mu T_0$,

$$p_0\,(p_r + \delta\mu\tilde{p}) = \rho_0 RT_0\,(\rho_r + \delta\mu\tilde{\rho})\,(T_r + \delta\mu T) \tag{2.52}$$

and noting that $p_0 = \rho_0 RT_0$, to $O(\delta\mu)$ we obtain

$$\frac{\tilde{p}}{p_r} = \frac{\tilde{\rho}}{\rho_r} + \frac{\tilde{T}}{T_r} \tag{2.52}$$

These relations establish the conversion between perturbation variables $\tilde{p}$, $\tilde{\rho}$, $\tilde{\theta}$ and $\tilde{T}$.

Finally note that we scaled $f \sim f_0$, $y \sim L$ in (2.32) So that the tangent plane approximation

$$f = f_0(1 + \beta y) \tag{2.52}$$

becomes (scaling $\beta \sim \beta_0$),

$$f = 1 + (\beta_0 L)\beta y \tag{2.53}$$

in nondimensional variables. The dimensionaless parameter $\beta_0 L$ can be estimated from

$$\beta_0 L = \cot\phi_0 \left(\frac{L}{R_0}\right) \tag{2.54}$$

where $\phi_0$ is the latitude and R is the radius of the earth. In mid - latitudes $\cot\phi_0 \simeq O(1)$. Taking $R_0 = 6000km$, and $L = 100km$ (ocean) or $1000km$ (atmosphere), we find $\beta_0 L = O(10^{-2} - 10^{-1})$. We assume this small parameter to be the same order as the Rossby number $\beta L \sim \delta$, so that

$$f = 1 + \delta\beta y \tag{2.55}$$

is the appropriately scaled tangent plane function.

Making use of $(2.32.a-k)$, (2.34), (2.35), (2.37), (2.38), (2.40), (2.41), (2.42), (2.44), (2.45), (2.51) and (2.55). We can now scale the equations (2.27) - (2.31) as follows:

$$
\begin{aligned}
&(\rho_r + \delta\mu\tilde{\rho})\,\delta\left\{\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} + \delta w\frac{\partial \vec{u}}{\partial z}\right\} + (\rho_r + \delta\mu\tilde{\rho})(1+\delta\beta y)\hat{k}\times\vec{u} = -\nabla\tilde{p} \\
&(\rho_r + \delta\mu\tilde{\rho})\lambda^2\delta^2\left\{\frac{\partial w}{\partial t} + (\vec{u}\cdot\nabla)w + \delta w\frac{\partial w}{\partial z}\right\} = -\frac{\partial \tilde{p}}{\partial z} - \tilde{\rho} \\
&\delta\mu\left\{\frac{\partial \tilde{\rho}}{\partial t} + \vec{u}\cdot\nabla\tilde{\rho} + \delta\frac{\partial}{\partial z}\tilde{\rho}w\right\} + (\rho_r + \delta\mu\tilde{\rho})\,\nabla\cdot\vec{u} + \delta\frac{\partial}{\partial z}\rho w = 0 \\
&\frac{\partial \tilde{\theta}}{\partial t} + \vec{u}\cdot\nabla\tilde{\theta} + \delta w\frac{\partial \tilde{\theta}}{\partial z} + \frac{1}{\mu}w\frac{\partial \theta_r}{\partial z} = 0 \\
&\frac{\tilde{\theta}}{\theta_r} = \frac{1}{\gamma}\frac{\tilde{p}}{p_r} - \frac{\tilde{\rho}}{\rho_r} \\
&\frac{\partial \tilde{\rho}}{\partial t} + \vec{u}\cdot\nabla\tilde{\rho} + \delta w\frac{\partial \tilde{\rho}}{\partial z} + \frac{1}{\mu}w\frac{\partial \rho_r}{\partial z} = 0
\end{aligned}
\qquad (2.56a-f)
$$

The first three equations are the momentum and continuity equations which are valid for both the ocean and the atmosphere. The thermodynamic equations for the atmosphere are given by (2.56 d,e) and the corresponding equation for the ocean is (2.56 f).

### 2.4 Scale Analysis

Now, let as look at the typical values of the small parameters $\delta$, $\lambda$ and $\mu$. Taking $f_0 = 10^{-4}s^{-1}$, $g = 10ms^{-1}$ and $U = 1m/s$, $L = 100km$, $H = 10^3m$ for the ocean and $U = 10m/s$, $L = 1000km$, $H = 10^4m$ for the atmosphere, we can have typically

$$
\begin{aligned}
&\delta \sim 10^{-1} \\
&\lambda \sim 10^{-2} \\
&\mu \sim 10^{-1} - 10^{-2}
\end{aligned}
$$

for both fluids, *i.e.*

$$
\begin{aligned}
&\mu = O(\delta) - O(\delta^2) \\
&\lambda = O(\delta^2)
\end{aligned}
\qquad (2.57)
$$

so that $\delta$ is the largest of the small parameters. In fact, if we take slightly lower velocities or larger horizontal scales this parameter may become $\delta = O(1)$. Note also that the parameters appear in groups of $\delta, \mu, \delta\mu$, or $\delta\lambda$, the latter being always smaller. Therefore we decide to keep only those terms with $O(\delta)$ in the equations, yielding:

$$\begin{aligned}
&\delta\rho_r\left\{\vec{u}_t + \vec{u}\cdot\nabla\vec{u} + \beta\hat{k}\times\vec{u}\right\} + \rho_r\hat{k}\times\vec{u} = -\nabla\tilde{p} \\
&0 = -\frac{\partial\tilde{p}}{\partial z} - \tilde{\rho} \\
&\nabla\cdot\vec{u} + \frac{\delta}{\rho_r}\frac{\partial}{\partial z}(\rho_r w) = 0 \\
&\frac{\partial\tilde{\theta}}{\partial t} + \vec{u}\cdot\nabla\tilde{\theta} + \frac{1}{\mu}\frac{\partial\theta_r}{\partial z}w = 0 \\
&\frac{\tilde{\theta}}{\theta_r} = \frac{1}{\gamma}\frac{\tilde{p}}{p_r} - \frac{\tilde{\rho}}{\rho_r} \\
&\frac{\partial\tilde{\rho}}{\partial t} + \vec{u}\cdot\nabla\tilde{\rho} + \frac{1}{\mu}\frac{\partial\rho_r}{\partial z}w = 0
\end{aligned} \qquad (2.58.a-f)$$

Note that in equation (2.56.e) we have kept those terms with $O\left(\frac{1}{\mu}\right) \gg 1$, $O(1)$, and $O(\delta)$ terms. We have, purposefully divided the equation by $\mu$ beforehand, because otherwise neglecting pf the terms of $O(\mu) \simeq O(\delta^2)$ and $O(\mu\delta)$ terms would yield $w = 0$, which would be dynamically inconsistent.

The set of equations to be used for the atmosphere are $(2.58.a-e)$, while the equations for the ocean are $(2.58.a-c)$ and $(2.58.f)$. Let us now rewrite these equations, defining the new variables,

$$\begin{aligned}
\phi &= \frac{\tilde{p}}{\rho_r} \\
r &= \frac{\tilde{\rho}}{\rho_r} \\
s &= \frac{\tilde{\theta}}{\theta_r}
\end{aligned} \qquad (2.59.a-c)$$

We first divide $(2.58.b)$ by $\rho_r$ to write

$$\frac{1}{\rho_r}\frac{\partial\tilde{p}}{\partial z} + \frac{\tilde{\rho}}{\rho_r} = 0$$

and rearrange to write this as

$$\frac{\partial\left(\frac{\tilde{p}}{\rho_r}\right)}{\partial z} + \frac{1}{\rho_r}\frac{\partial\rho_r}{\partial z}\frac{\tilde{p}}{\rho_r} + \frac{\tilde{\rho}}{\rho_r} = 0 \qquad (2.60)$$

or

$$\frac{\partial\phi}{\partial z} + K(z)\phi + r = 0 \qquad (2.61)$$

where

$$K(z) = \frac{1}{\rho_r}\frac{d\rho_r}{dz}. \qquad (2.62)$$

Secondly, (2.58.$e$) is written as

$$s = L(z)\phi - r \tag{2.63}$$

where

$$L(z) = \frac{1}{\gamma}\frac{\rho_r}{p_r}. \tag{2.64}$$

Finally, noting that $\theta_r$ and $\rho_r$ are functions of $z$ only and defining

$$M(z) = \frac{1}{\theta_r}\frac{d\theta_r}{dz}, \tag{2.65}$$

equations (2.58.$d$) and (2.58.$f$) become

$$\left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right) s + \frac{1}{\mu}M(z)w = 0 \tag{2.66}$$

$$\left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right) r - \frac{1}{\mu}K(z)w = 0 \tag{2.67}$$

We can then write (2.58.$a-f$) as

$$\begin{aligned}
&\delta\left\{\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} + \beta y\hat{k}\times\vec{u}\right\} + \hat{k}\times\vec{u} = -\nabla\phi \\
&\frac{\partial \phi}{\partial z} + K(z)\phi + r = 0 \\
&\nabla\cdot\vec{u} + \delta\frac{1}{\rho_r}\frac{\partial}{\partial z}(\rho_r w) = 0 \\
&\frac{\partial s}{\partial t} + \vec{u}\cdot\nabla s + \frac{1}{\mu}M(z)w = 0 \\
&s = L(z)\phi - r \\
&\frac{\partial r}{\partial t} + \vec{u}\cdot\nabla r + \frac{1}{\mu}K(z)w = 0
\end{aligned} \tag{2.68.$a-f$}$$

## 2.5 Brunt - Vaisala (Stratification) Parameter

### 2.5.1 The Atmosphere

We will now define the dimensionless stratification parameter

$$S(z) = \frac{1}{\mu}\frac{1}{\theta_r}\frac{d\theta_r}{dz} = \frac{1}{\mu}M(z) \tag{2.69}$$

for the atmosphere. In dimensional (primed) variables,

$$\begin{aligned} S(z) &= \frac{H}{\left(\frac{f_0^2 L^2}{gH}\right)} \frac{1}{\theta_r'} \frac{d\theta_r'}{dz'} = \frac{H^2}{f_0^2 L^2} \frac{g}{\theta_r'} \frac{d\theta_r'}{dz'} \\ &= \left(\frac{H}{L}\right)^2 \left(\frac{N'(z)}{f_0}\right)^2 = \left(\frac{H}{L}\right)^2 \left(\frac{N_0}{f_0}\right)^2 N^2(z) \\ &= S_0 N^2(z) \end{aligned} \tag{2.70}$$

where the stability or Brunt Vaisala Frequency is defined as

$$N'^2(z) \equiv N_0^2 \; N^2(z) = \frac{g}{\theta_r'} \frac{d\theta_r'}{dz'}. \tag{2.71}$$

For typical values of $H = 10^4 m$, $L = 10^6 m$, $f_0 = 10^{-4} s^{-1}$ and $N_0 = 10^2 s^{-1}$ for the atmosphere,

$$S_0 = \left(\frac{H}{L}\right)^2 \left(\frac{N_0}{f_0}\right)^2 = O(1) \tag{2.72}$$

so that $\frac{1}{\mu} M(z) = S(z) = O(1)$ in equation (2.68.$d$).

We use the definition of potential temperature (2.25) for the basic state, and take logarithms of both sides,

$$\ln \theta_r' = \frac{1}{\gamma} \ln p_r' - \ln \rho_r' + \ln c_*, \tag{2.73}$$

so that (2.71) is evaluated as

$$\begin{aligned} N'^2(z) &= g \frac{d \ln \theta_r'}{dz} = \frac{g}{\gamma} \frac{1}{p_r'} \frac{dp_r'}{dz} - \frac{g}{\rho_r'} \frac{d\rho_r'}{dz} \\ &= g \left( -\frac{1}{\rho_r'} \frac{d\rho_r'}{dz} - \frac{g\rho_r'}{\gamma p_r'} \right) \end{aligned} \tag{2.74}$$

where use has been mode of the hydrostatic relation (2.13). Non-dimensionalizing with the scales $N' \sim N_0$, $\rho_r \sim \rho_0$, $p_r \sim \rho_0 g H$, $z \sim H$ yields

$$\begin{aligned} \mu S_0 N^2(z) &= -\frac{1}{\rho_r} \frac{d\rho_r}{dz} - \frac{1}{\gamma} \frac{\rho_r}{p_r} \\ &= -K(z) - L(z) \end{aligned} \tag{2.75}$$

Since $\mu \ll 1$, if follows that

$$K(z) \cong -L(z) \tag{2.76}$$

with this approximation it follows from (2.68.e) that

$$s = L(z)\phi - r \cong -K(z)\phi - r \tag{2.77}$$

and comparing these with (2.68.a-e), the governing equations for the atmosphere become

$$\begin{aligned}
&\delta\left\{\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} + \beta y\hat{k}\times\vec{u}\right\} + \hat{k}\times\vec{u} = -\nabla\phi \\
&\frac{\partial\phi}{\partial z} = s \\
&\nabla\cdot\vec{u} + \delta\frac{1}{\rho_r}\frac{\partial}{\partial z}(\rho_r w) = 0 \\
&\frac{\partial s}{\partial t} + \vec{u}\cdot\nabla s + S_0 N^2(z) w = 0
\end{aligned} \qquad (2.78.a-d)$$

### 2.5.2 The Ocean

We define the stratification parameter

$$S(z) = -\frac{1}{\mu}\frac{1}{\rho_r}\frac{\partial\rho_r}{\partial z} = -\frac{1}{\mu}K(z) \tag{2.79}$$

which is in dimensional variables

$$S(z) = \frac{H^2}{f_0 L^2}\left(-\frac{g}{\rho_r}\frac{d\rho_r}{dz}\right) = S_0 N^2(z) \tag{2.80}$$

where $N^2$ is defined as

$$N'^2(z) = N_0^2 N^2(z) = -\frac{g}{\rho_r}\frac{d\rho_r}{dz} \tag{2.81}$$

and

$$S_0 = \left(\frac{H}{L}\right)^2\left(\frac{N_0}{f_o}\right)^2 \tag{2.82}$$

Taking typical values of $H = 10^3 m$, $L = 10^5 m$, $N_0 = 10^{-2}s^{-1}$, $f_0 = 10^{-4}s^{-1}$, we obtain $S_0 = O(1)$. However by virtue of (2.79)

$$K(z) = O(\mu) \tag{2.83}$$

and we can neglect it in equation (2.68.b) yielding the following governing equations

$$\begin{aligned}
&\delta\left\{\frac{\partial\vec{u}}{\partial t}+\vec{u}\cdot\nabla\vec{u}+\beta y\hat{k}\times\vec{u}\right\}+\hat{k}\times\vec{u}=-\nabla\phi\\
&\frac{\partial\phi}{\partial z}=-r\\
&\nabla\cdot\vec{u}+\delta\frac{1}{\rho_r}\frac{\partial}{\partial z}(\rho_r w)=0\\
&\frac{\partial r}{\partial t}+\vec{u}\cdot\nabla r-S_0N^2(z)w=0
\end{aligned} \qquad (2.84.a-d)$$

## 2.7 Quasigeostrophic Theory

Note that in the above derivations of the governing equations, we have arrived at the same final set of equations for the atmosphere and the ocean. The only difference is in how we intrepret the set of variables in the equations. The set of equations are basically the same, and if we take either (2.78.a-d) or (2.84.a-d), they are both of the following form:

$$\begin{aligned}
&\delta\left\{\frac{\partial\vec{u}}{\partial t}+\vec{u}\cdot\nabla\vec{u}+\beta y\hat{k}\times\vec{u}\right\}+\hat{k}\times\vec{u}=-\nabla\phi\\
&\frac{\partial\phi}{\partial z}=s\\
&\nabla\cdot\vec{u}+\delta\frac{1}{\rho_r}\frac{\partial}{\partial z}(\rho_r w)=0\\
&\frac{\partial s}{\partial t}+\vec{u}\cdot\nabla s+S_0N^2(z)w=0
\end{aligned} \qquad (2.85a-d)$$

In fact the above set corresponds to the set of equations (2.78.a-d) developed for the atmosphere. We only need to replace the variable $s$ with $-r$ to obtain the set of equations (2.84.a-d) for the ocean. The only difference between the equations for the atmosphere and the ocean in interpreting the variable $s=\frac{\tilde{\theta}}{\theta_r}$ as the normalised perturbation potential temperature for the atmosphere, and the variable $r=\frac{\tilde{\rho}}{\rho_r}$ as the normalised perturbation density in the case of the ocean. Further, the Brunt-Vaisala stratification parameter is defined in terms of basic state potential temperature (2.71) in the case of the atmosphere, and in terms of basic state density (2.81) in the case of the ocean.

We also note that the limit of these equations for geostrophic flows (*i.e.* for Rossby number $\delta=U/f_0L\to 0$) are *degenerate*, as we have seen for the homogenous fluid case in DO-II. This is simply shown by taking *delta* $=0$ in (2.85.a), yielding $\vec{u}=\hat{k}\times\nabla\phi$, which automatically satisfies (2.85.c) *i.e.* $\nabla\cdot\vec{u}=0$. Since this means one of the above two equations is redundant, solutions to the set can not be determined uniquely. This is called *geostrophic degeneracy.*

We therefore seek solutions in power series expansions in the small parameter $\delta$, expanding the variables as follows:

$$\begin{aligned}\vec{u} &= \vec{u}_0 + \delta\vec{u}_1 + \delta^2\vec{u}_2 + \dots \\ w &= w_0 + \delta w_1 + \delta^2 w_2 + \dots \\ \phi &= \phi_0 + \delta\phi_1 + \delta^2\phi_2 + \dots \\ s &= s_0 + \delta s_1 + \delta^2 s_2 + \dots\end{aligned} \qquad (2.86a-d)$$

We then group like terms in equal powers of $\delta$. The $O(\delta^0)$ equations are

$$\begin{aligned}&\hat{k}\times\vec{u}_0 = \nabla\phi_0 \\ &\frac{\partial\phi_0}{\partial z} = s_0 \\ &\nabla\cdot\vec{u}_0 = 0 \\ &\frac{\partial s_0}{\partial z} + \vec{u}_0\cdot\nabla s_0 + S_0N^2(z)w_0 = 0.\end{aligned} \qquad (2.87a-d)$$

which we have seen to be degenerate. To the next order, the $O(\delta^1)$ equations are

$$\begin{aligned}&\frac{\partial\vec{u}_0}{\partial t} + u_0\cdot\nabla\vec{u}_0 + \beta y\hat{k}\times\vec{u}_0 + k\times\vec{u}_1 = -\nabla\phi_1 \\ &\frac{\partial\phi_1}{\partial z} = s_1 \\ &\nabla\cdot\vec{u}_1 + \frac{1}{\rho_r}\frac{\partial}{\partial z}(\rho_r w_0) = 0 \\ &\frac{\partial s_1}{\partial t} + \vec{u}_1\cdot\nabla s_0 + \vec{u}_0\cdot\nabla s_1 + S_0N^2(z)w_1 = 0\end{aligned} \qquad (2.88.a-d)$$

From the first order equation (2.87.a), we see that the first order velocity is related to a streamfunction $vecu_0 = \hat{k}\times\nabla\phi_0$, and we can then define first order vorticity as

$$\vec{\omega}_0 = \nabla\times\vec{u}_0 = \nabla\times\hat{k}\times\nabla\phi_0 \equiv \hat{k}\nabla^2\phi_0. \qquad (2.89)$$

Alternatively we can define the scalar (two-dimensional flow) vorticity $\xi_0$ such that $\vec{\omega} = \hat{k}\xi_0$, and therefore, $\xi_0 = \nabla^2\phi_0$.

Note that the result (2.89) has been obtained by using the vector identities of DO-I, *i.e.* $\nabla\times\hat{k}\times\nabla\phi_0 = (\nabla\phi_0\cdot\nabla)\hat{k} - (\hat{k}\cdot\nabla)\nabla\phi - \nabla\phi(\nabla\cdot\hat{k}) + \hat{k}(\nabla\cdot\nabla\phi_0) = \hat{k}\nabla^2\phi_0$, yielding the last term because all of the first three terms are zero.

Because the first order equations are indeterminate we must close the system by making use of the second order equations. By taking the curl of equation (2.87.a),

$$\frac{\partial\nabla\times\vec{u}_0}{\partial t} + \nabla\times(\vec{u}_0\cdot\nabla\vec{u}_0) + \nabla\times(\beta y\hat{k}\times\vec{u}_0) + \nabla\times\hat{k}\times\vec{u}_1 = -\nabla\times\nabla\phi. \qquad (2.90)$$

We then evaluate the individual terms as follows:

$$
\begin{aligned}
\vec{u}_0 \cdot \nabla \vec{u}_0 &\equiv \frac{1}{2}\nabla(\vec{u}_0 \cdot \vec{u}_0) - \vec{u}_0 \times \nabla \times \vec{u}_0 = \frac{1}{2}\nabla|\vec{u}_0|^2 + \xi_0 \hat{k} \times \vec{u}_0 \\
\nabla \times (\vec{u}_0 \cdot \nabla \vec{u}_0) &= \frac{1}{2}\nabla \times \nabla|\vec{u}_0|^2 + \nabla \times \xi_0 \hat{k} \times \vec{u}_0 = \hat{k}\nabla \cdot (\xi_0 \vec{u}_0) \\
&= \hat{k}(\vec{u}_0 \cdot \nabla \xi_0 + \xi_0 \nabla \cdot \vec{u}_0) \\
&= \hat{k}\vec{u}_0 \cdot \nabla \xi_0 \\
\nabla \times \beta y \hat{k} \times \vec{u}_0 &= \hat{k} \cdot \nabla \beta y \vec{u}_0 = \hat{k}\beta y \nabla \cdot \vec{u}_0 + \hat{k}\vec{u}_0 \cdot \nabla \beta y \\
&= \hat{k}\beta v_0 \\
\nabla \times \hat{k} \times \vec{u}_1 &= \hat{k}\nabla \cdot \vec{u}_1 .
\end{aligned}
$$

Substituting in (2.90) yields

$$
\frac{\partial \xi_0}{\partial t} + \vec{u}_0 \cdot \nabla \xi_0 + \beta v_0 + \nabla \cdot \vec{u}_1 = 0,
$$

and using (2.88.c),

$$
\left(\frac{\partial}{\partial t} + \vec{u}_0 \cdot \nabla\right) \xi_0 + \beta v_0 = \frac{1}{\rho_r}\frac{\partial}{\partial z}(\rho_r w_0). \tag{2.91}
$$

From (2.87.b) and (2.87.d), we can write

$$
\left(\frac{\partial}{\partial t} + \vec{u}_0 \cdot \nabla\right)\left(\frac{\partial \phi_0}{\partial z}\right) + S_0 N^2(z) w_0 = 0, \tag{2.92}
$$

or

$$
\rho_r w_0 = -\left(\frac{\partial}{\partial t} + \vec{u}_0 \cdot \nabla\right)\left(\frac{\rho_r}{S_0 N^2}\frac{\partial \phi_0}{\partial z}\right), \tag{2.93}
$$

and differentiating with respect to $z$ gives

$$
\frac{\partial \rho_r w_0}{\partial z} = -\left(\frac{\partial}{\partial t} + \vec{u}_0 \cdot \nabla\right)\frac{\partial}{\partial z}\left(\frac{\rho_r}{S_0 N^2}\frac{\partial \phi_0}{\partial z}\right) - \frac{\partial \vec{u}_0}{\partial z} \cdot \nabla\left(\frac{\rho_r}{S_0 N^2}\frac{\partial \phi_0}{\partial z}\right), \tag{2.94}
$$

where we note that the last term in fact vanishes because the vectors $\vec{u}_0$ and $\nabla \phi_0$ are orthogonal, $\vec{u}_0 = \hat{k} \times \nabla \phi_0$.

If we substitute (2.94) into (2.91), we have

$$
\left(\frac{\partial}{\partial t} + \vec{u}_0 \cdot \nabla\right)\left[\xi_0 + \frac{1}{\rho_r}\frac{\partial}{\partial z}\left(\frac{\rho_r}{S_0 N^2}\frac{\partial \phi_0}{\partial t}\right)\right] + \beta v_0 = 0 \tag{2.95}
$$

and substituting $\vec{u}_0 = \hat{k} \times \nabla \phi_0$, $\xi_0 = \nabla^2 \phi_0$ and $v_0 = \partial \phi_0 / \partial x$ results in

$$\left(\frac{\partial}{\partial t} + \hat{k} \times \nabla\phi_0 \cdot \nabla\right) \left[\nabla^2\phi_0 + \frac{1}{\rho_r}\frac{\partial}{\partial z}\left(\frac{\rho_r}{S_0 N^2}\frac{\partial \phi_0}{\partial z}\right)\right] + \beta\frac{\partial\phi_0}{\partial x} = 0. \qquad (2.96)$$

We have obtained a single *vorticity equation* governing $\phi_0$, which is the basis of *quasigeostrophic dynamics*.

Once this equation is solved for $\phi_0$, the solutions for $p_0$, $\vec{u}_0$, $w_0$, $\tilde{\rho}_0$, $\tilde{\theta}_0$ and $\tilde{T}_0$ can be obtained from:

$$\begin{aligned} \vec{u}_0 &= \hat{k} \times \nabla\phi_0 \\ w_0 &= -\frac{1}{S_0 N^2(z)}\left(\frac{\partial}{\partial t} + \vec{u}_0 \cdot \nabla\right)\frac{\partial\phi_0}{\partial z} \\ \frac{\tilde{p}_0}{\rho_r} &\equiv \phi_0 \\ \frac{\tilde{\theta}_0}{\theta_r} &\equiv s_0 = \frac{\partial\phi_0}{\partial z} \\ \frac{\tilde{\rho}_0}{\rho_r} &\equiv r_0 = \frac{1}{\gamma}\frac{\rho_r}{p_r}\phi_0 - s_0 \\ \frac{\tilde{T}_0}{T_s} &\equiv \tau_0 = \frac{\rho_r}{p_r}\phi_0 - r_0 \end{aligned} \qquad (2.97.e-f)$$

The dimensional equivalents of the above equations are

$$\left(\frac{\partial}{\partial t} + \frac{1}{f_0}\hat{k} \times \nabla\phi_0 \cdot \nabla\right) \left[\nabla^2\phi_0 + \frac{1}{\rho_0}\frac{\partial}{\partial z}\frac{f_0^2\rho_r}{N^2}\frac{\partial\phi_0}{\partial z}\right] + f_0\beta\frac{\partial\phi_0}{\partial x} = 0 \qquad (2.98)$$

and

$$\begin{aligned} \vec{u}_0 &= f_0\hat{k} \times \nabla\phi_0 \\ g s_0 &= \frac{\partial\phi_0}{\partial z} \\ w_0 &= -\frac{g}{N^2(z)}\left(\frac{\partial}{\partial t} + \vec{u}_0 \cdot \nabla\right) s_0 \\ r_0 &= \frac{1}{\gamma}\frac{\rho_r}{p_r}\phi_0 - s_0 \\ \tau_0 &= \frac{\rho_r}{p_r}\phi_0 - r_0. \end{aligned} \qquad (2.99a-e)$$

If we define $\psi_0 = \phi_0/f_0 = \tilde{p}_0/f_0\rho_r$, and note that

$$beta\frac{\partial\psi_0}{\partial x} = \left(\frac{\partial}{\partial t} + k \times \nabla\psi_0 \cdot \nabla\right)(\beta y)$$

then we could also write them as follows:

$$\left(\frac{\partial}{\partial t} + \hat{k}\times\nabla\psi_0\cdot\nabla\right)\left[\nabla^2\psi_0 + f_0^2\frac{1}{\rho_r}\frac{\partial}{\partial z}\frac{\rho_r}{N^2}\frac{\partial\psi_0}{\partial z}\right] + f_0\beta\frac{\partial\psi_0}{\partial x} = 0 \tag{2.100}$$

and

$$\begin{aligned}
\frac{\tilde{p}_0}{p_r} &= f_0\frac{\rho_r}{p_r}\psi_0 \\
\vec{u}_0 &= \hat{k}\times\nabla\psi_0 \\
\frac{\tilde{\theta}_0}{\theta_r} &= \frac{f_0}{g}\frac{\partial\psi_0}{\partial z} \\
w_0 &= -\frac{f_0}{N^2}\left(\frac{\partial}{\partial t} + \vec{u}_0\cdot\nabla\right)\frac{\partial\psi_0}{\partial z} \\
\frac{\tilde{\rho}_0}{\rho_r} &= \frac{1}{\gamma}\frac{\tilde{p}_0}{p_r} - \frac{\tilde{\theta}_0}{\theta_r} \\
\frac{\tilde{T}_0}{T_s} &= \frac{\tilde{p}_0}{p_r} - \frac{\tilde{\rho}_0}{\rho_r}
\end{aligned} \tag{2.101.a – f}$$

Note that (2.98) can also be written in the following form:

$$\left(\frac{\partial}{\partial t} + \hat{k}\times\nabla\psi_0\cdot\nabla\right)\left[f_0(1+\beta y) + \nabla^2\psi_0 + \frac{f_0^2}{\rho_r}\frac{\partial}{\partial z}\frac{\rho_r}{N^2}\frac{\partial\psi_0}{\partial z}\right] = 0. \tag{2.102}$$

In the above equations the stratification parameter $S(z)$ can be interpreted as follows. We note

$$S(z) = S_0N^2(z) = \left(\frac{N(z)}{f_0}\right)^2\left(\frac{H}{L}\right)^2, \tag{2.103}$$

and define an *internal radius of deformation* $L_I$

$$L_I = \frac{N(z)H}{f_0} = \sqrt{\frac{g}{\theta_r}\frac{\partial\theta_r}{\partial z}}\frac{H}{f_0} = \sqrt{\frac{1}{\theta_r}\frac{\partial\theta_r}{\partial z}}\frac{\sqrt{gH}\sqrt{H}}{f_0} = \frac{1}{f_0}\sqrt{\frac{gH}{\theta_r}\frac{\partial\theta_r}{\partial z}H} = \frac{\sqrt{g'H}}{f_0}, \tag{2.104}$$

(with a new definition of *reduced gravity* $g' \equiv \frac{gH}{\theta_r}\frac{\partial\theta_r}{\partial z}$) so that

$$S(z) = \left(\frac{L_I}{L}\right)^2, \tag{2.105}$$

*i.e.* the stratification parameter is proportional to the ratio of the internal radius of deformation to the horizontal scale.

Comparing the internal radius of deformation $L_I$ to external radius of deformation $L_E = \sqrt{gH}/f_0$,

$$\frac{L_I}{L_E} = \sqrt{\frac{g'}{g}} = \left(\frac{H}{\theta_s}\frac{\partial\theta_s}{\partial z}\right)^{1/2}. \tag{2.106}$$

Noting that

$$\begin{aligned}\frac{H}{\theta_r}\frac{\partial\theta_r}{\partial z} &= \frac{N^2H}{g} = \frac{N_0^2H}{g}N^2(z) = \frac{N_0^2H^2}{gH} \\ &= \left(\frac{N_0}{f_0}\right)^2\left(\frac{H}{L}\right)^2\left(\frac{f_0^2L^2}{gH}\right) = S_0\mu,\end{aligned} \tag{2.107}$$

with $S_0 = N_0H/f_0L = O(1)$, and $\mu = f_0^2L^2/gH \ll 1$, we thus show that

$$\frac{L_I}{L_E} = (\mu S_0)^{1/2} \ll 1. \tag{2.108}$$

Equation (2.102) is a conservation law of the form

$$\frac{Dq}{Dt} = 0, \tag{2.109.a}$$

$$q = f_0(1+\beta y) + \nabla^2\psi_0 + \frac{f_0^2}{\rho_r}\frac{\partial}{\partial z}\frac{\rho_r}{N^2}\frac{\partial\psi_0}{\partial z} \tag{2.109.b}$$

where $q$ is called the quasi-geostrophic quasi-potential vorticity. The individual terms of vorticity are

$$\begin{aligned} f = f_0(1+\beta y) &= \text{planetaryvorticity} \\ \nabla^2\psi_0 + \frac{f_0^2}{\rho_r}\frac{\partial}{\partial z}\frac{\rho_r}{N^2}\frac{\partial\psi_0}{\partial z} &= \text{fluidvorticity.}\end{aligned} \tag{2.110.a, b}$$

The second term of (2.110.b) is termed as '*thermal vorticity*' because it represents contribution to vorticity resulting from stratification.

Note that equation (2.109) can be written in either of the following ways:

$$\begin{aligned} &\frac{\partial q}{\partial t} + \vec{u}_0\cdot\nabla q = 0 \\ \text{or}\quad &\frac{\partial q}{\partial t} + \hat{k}\times\nabla\psi_0\cdot\nabla q = 0 \\ \text{or}\quad &\frac{\partial q}{\partial t} + \frac{\partial\psi_0}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial\psi_0}{\partial y}\frac{\partial q}{\partial x} = 0\end{aligned} \tag{2.111.a - c}$$

We could also write the non-linear part of the quasigeostrophic vorticity equation using the *Jacobian* of $\psi_0$ and $q$, as follows:

$$J(\psi_0, q) = \begin{vmatrix} \frac{\partial\psi_0}{\partial x} & \frac{\partial\psi_0}{\partial y} \\ \frac{\partial q}{\partial x} & \frac{\partial q}{\partial y}\end{vmatrix}. \tag{2.112}$$

## 2.8 Quasigeostrophic, Barotropic Motions

### 2.8.1 Barotropic Rossby Waves

To obtain the barotropic version of the quasigeostrophic vorticity equation, we simply neglect the vertical derivative terms, yielding

$$\left(\frac{\partial}{\partial t}+\frac{\partial\psi_0}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\psi_0}{\partial y}\frac{\partial}{\partial x}\right)\nabla^2\psi_0+f_0\beta\frac{\partial\psi_0}{\partial x}=0$$

We can assume plane-wave solutions of this equation in the form of

$$\psi_0=Re\left\{A\ e^{i(kx+ly-\omega t)}\right\}$$

Substituting this solution, and evaluating terms, it is easy to show that

$$\nabla^2\psi_0=-(k^2+l^2)\psi_0,\quad \text{then}\quad J=(\psi_0,\nabla^2\psi_0)=0,$$

leaving

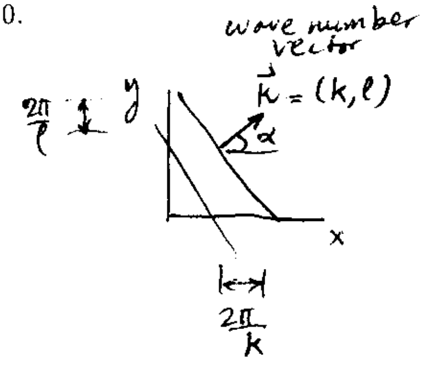
$$\frac{\partial}{\partial t}\nabla^2\psi_0+f_0\beta\frac{\partial\psi_0}{\partial x}=0$$

which then gives

$$\left\{-(k^2+l^2)(-i\omega)+f_0\beta(ik)\right\}\psi_0=0.$$

We thus have the *dispersion relation* for the waves,

$$\omega=-\frac{f_0\beta k}{k^2+l^2}$$



We define

$$f_0\beta=\beta',\quad \text{and}\quad \vec{\kappa}=k\hat{\imath}+l\hat{\jmath}$$

so that

$$|\vec{\kappa}|^2=k^2+l^2\qquad k=\vec{\kappa}\cdot\hat{\imath},\qquad l=\vec{\kappa}\cdot\hat{\jmath},$$

and then, the dispersion relation takes the form of

$$\omega=-\frac{\beta'}{|\vec{\kappa}|}\cos\alpha=\beta'\frac{\vec{\kappa}\cdot\hat{\imath}}{|\vec{\kappa}|}.$$

Note that if $\beta=0$, we have $\omega\to 0$, and no waves can exist. It is therefore clear that the waves are supported by the presence of the *beta-effect*. Such waves are called *Rossby waves*.

We now define a phase velocity vector

$$\vec{C} = \left(\frac{\omega}{k}, \frac{\omega}{l}\right) = (C_x, C_y)$$

and we have

$$C_x = \frac{\omega}{k} = -\left(\frac{\beta'}{k^2 + l^2}\right) < 0$$

$$C_y = \left(\frac{k}{l}\right) C_x$$

This shows one of the basic characteristics of Rossby waves. The zonal phase speed $C_x$ is always negative, so that the phase always propagates to the west (*i.e.* $\omega > 0$, $k < 0$*orvicaversa*). The azimuthal phase speed $C_y$ can be either positive or negative.

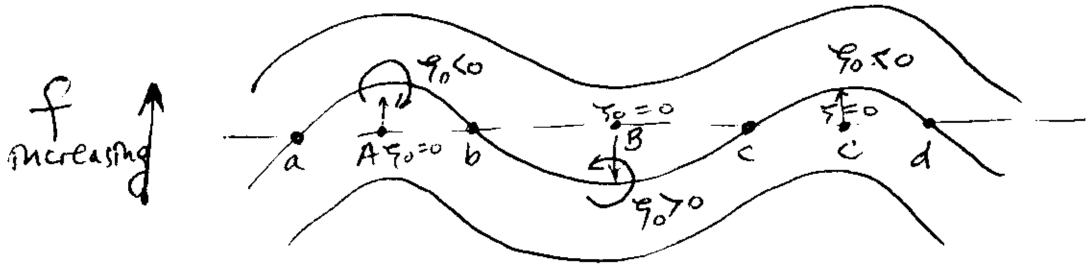
Conservation of potential vorticity requires that

$$\left(\frac{\partial}{\partial t} + \frac{\partial \psi_0}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \psi_0}{\partial y}\frac{\partial}{\partial x}\right)(f_0 + \beta' y + \nabla^2 \psi_0) = 0$$

or

$$\left(\frac{\partial}{\partial t} + \vec{u} \cdot \nabla\right)(f + \xi_0) = \frac{D}{Dt}(f + \xi_0) = 0$$

This requires that $f + \xi_0$=const. following the motion of material (fluid) elements. Consider zonally oriented chains of fluid. If we then deform these into a sinusoidal form,



The velocity field at $A'$ will cause $a$ to go up, $b$ to go down, and the velocity field at $B'$ will cause $b$ to go down, $c$ to go up, so that the entire wave will go towards the left (west). We can also think of the flow pattern as a vortex street which self-advects to the west.

The group velocity $\vec{C}_g$ is defined as

$$\vec{C}_g = \left(\frac{\partial \omega}{\partial k}\,,\ \frac{\partial \omega}{\partial l}\right) = \hat{i}\frac{\partial \omega}{\partial k} + \hat{j}\frac{\partial \omega}{\partial l}$$

and its components for Rossby waves can be calculated from the dispersion relation as follows:

$$\begin{aligned} C_{g_x} &= +\beta\frac{k^2+l^2}{(k^2-l^2)} \qquad \begin{cases} >0 \text{ if } k > l \\ <0 \text{ if } k < l \end{cases} \\ C_{g_y} &= 2\beta\frac{kl}{(k^2+l^2)^2} \end{aligned}$$

### 2.8.2 Barotropic Rossby Waves in Basic Zonal Flows

Let us now consider the case when the Rossby waves are superposed on a basic state with a constant zonal flow. Again, we start with the vorticity equation

$$\frac{\partial}{\partial t}(\nabla^2\psi_0) + J(\psi_0, \nabla^2\psi_0) + \beta'\frac{\partial \psi_0}{\partial x} = 0,$$

but, now, we separate the flow into a superposition of basic and perturbed states

$$\psi_0 = \Psi + \epsilon\psi$$

such that $\epsilon \ll 1$, and $\Psi$ is a function of $y$ only

$$\Psi = \Psi(y), \quad U(y) = -\frac{\partial \Psi}{\partial y}.$$

Substituting in the equation gives

$$\frac{\partial}{\partial t}(\nabla^2\Psi) + \epsilon\frac{\partial}{\partial t}(\nabla^2\psi) + J\left(\Psi + \epsilon\psi, \nabla^2(\Psi + \epsilon\psi)\right) + +\beta'\frac{\partial \Psi}{\partial x} + \epsilon\beta'\frac{\partial \psi}{\partial x} = 0.$$

In the above, the term $\beta'\partial\Psi/\partial x$ term drops out because $\Psi = \Psi(y)$ only. For example, expanding the Jacobian,

$$\begin{aligned} J\left(\Psi + \epsilon\psi, \nabla^2(\Psi + \epsilon\psi)\right) &= \frac{\partial(\Psi + \epsilon\psi)}{\partial x}\frac{\partial}{\partial y}\nabla^2(\Psi + \epsilon\psi) - \frac{\partial(\Psi + \epsilon\psi)}{\partial y}\frac{\partial}{\partial x}\nabla^2(\Psi + \epsilon\psi) \\ &= \frac{\partial \Psi}{\partial x}\frac{\partial \nabla^2\psi}{\partial y} - \frac{\partial \nabla^2\Psi}{\partial x}\frac{\partial \Psi}{\partial y} \\ &+ \epsilon\left(\frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\nabla^2\Psi - \frac{\partial}{\partial x}\nabla^2\Psi\frac{\partial \psi}{\partial y}\right) + \epsilon\left(\frac{\partial \Psi}{\partial x}\frac{\partial}{\partial y}\nabla^2\Psi - \frac{\partial}{\partial x}\nabla^2\psi\frac{\partial \Psi}{\partial y}\right) \\ &+ \epsilon^2(\ldots) \end{aligned}$$

and dropping $x$ derivatives applying to the $\Psi$ terms,

$$J = \epsilon \left( \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} \nabla^2 \Psi - \frac{\partial}{\partial x} \nabla^2 \psi \frac{\partial \Psi}{\partial y} \right) + \epsilon^2 (\ldots).$$

Substituting these into the vorticity equation and collecting equal powers of $\epsilon$ gives

$$\frac{\partial}{\partial t}(\nabla^2 \Psi) + \epsilon \left\{ \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} \nabla^2 \Psi - \frac{\partial}{\partial x} \nabla^2 \psi \frac{\partial \Psi}{\partial y} + \frac{\partial}{\partial t}(\nabla^2 \Psi) + \beta' \frac{\partial \psi}{\partial x} \right\} + \epsilon^2 (\ldots) + \ldots = 0.$$

To first order ($O(\epsilon^0)$), we have

$$\frac{\partial}{\partial t} \nabla^2 \psi = 0,$$

and since $\Psi = \Psi(y)$ is not a function of time, therefore the above is identically satisfied.

We note,

$$\begin{aligned} \frac{\partial \psi}{\partial y} &= -U(y) \\ \nabla^2 \psi &= -\frac{\partial U}{\partial y} \\ \frac{\partial}{\partial y} \nabla^2 \psi &= -\frac{\partial^2 U}{\partial y^2} \end{aligned}$$

and therefore, to the next order ($O(\epsilon^1)$), we obtain

$$\frac{\partial}{\partial t}(\nabla^2 \psi) + U \frac{\partial}{\partial x}(\nabla^2 \psi) + \left( \beta' - \frac{\partial^2 U}{\partial y^2} \right) \frac{\partial \psi}{\partial x} = 0$$

This equation will serve as the vorticity equation for the perturbation streamfunction $\psi$ on a basic flow $U$. We can simplify the notation by writing

$$\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \nabla^2 \psi + (\beta' - U'') \frac{\partial \psi}{\partial x} = 0$$

where

$$U'' \equiv \partial^2 U / \partial y^2$$

.

To obtain a plane wave solution for the above equation, let $\psi = Re\left( A e^{i(kx + ly - \omega t)} \right)$, and substitution yields

$$(-i\omega + ikU)(-k^2 - l^2) + (\beta' - U'' ik) = 0$$

or

$$\omega = -\frac{(\beta' - U'')k}{(k^2 + l^2)} + kU$$

as the dispersion relation. To compare with the earlier case without the basic flow, we calculate the zonal phase speed as follows:

$$C_x = \frac{\omega}{k} = -\frac{(\beta' - U'')}{k^2 + l^2} + U$$

The above is valid for any zonal basic flow $U(y)$. For simplicity, suppose a uniform basic current $U = U_0$=constant, $U'' = 0$. Then we see that the phase speed $C_x = 0$ when $U_0 = \beta'/|\vec{\kappa}|^2$. With $|\vec{\kappa}| = \kappa = 2\pi/\lambda_x$, where $\lambda_x$ is the zonal wavelength, we find that this occurs when

$$\lambda_x = \lambda_* \equiv (2\pi)\sqrt{U_0/\beta'}$$

.

We will call $\lambda_*$ wavelength of the *standing Rossby wave*, because when $\lambda_x = \lambda_*$ there is no propagation.

For typical values of $U_0 \simeq 25\ m\ s^{-1}$, $\beta' = 1.6 \times 10^{-13}\ cm^{-1}\ s^{-1}$, we find $\lambda_* = 7000\ km$. The Rossby waves propagate from west to east for $\lambda_x < \lambda_*$, and from east to west for $\lambda_x > \lambda_*$.

In mid-latitudes, (*e.g.* the European area) the observed wavelengths of storm systems are usually less than 7000 $km$, so the atmospheric waves often travel from west to east.

Note that for $U =< 0$ (easterly jets: *e.g.* the TEJ - Tropical Easterly Jet) putting $U = -U_0$, we see that

$$C_x = -\frac{\beta'}{k^2 + l^2} - U_0 < 0$$

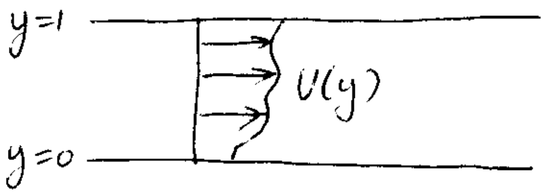
so that the propagation is always to the west. In the tropics such waves are often seen superposed on the 'easterlies'.

### 2.8.3 Barotropic Instability

To study the stability of basic zonal flows we consider the non-dimensional version of the vorticity equation developed in the last section:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2\psi + (\beta - U'')\frac{\partial\psi}{\partial x} = 0$$

where $U = U(y)$ is the zonal velocity profile.



For simplicity, we will consider the idealized case of flows confined between two rigid vertical walls at $y = 0$ and $y = 1$ (channel flow), where we apply the boundary conditions

$$v_0 = \frac{\partial \psi}{\partial x} = 0 \quad \text{on} \quad y = 0, 1$$

We try the solution $\psi(x, y, t) = P(y)e^{ik(x-ct)}$, where $c = C_x$ is the zonal phase speed. Substituting into the equation,

$$(-ikc + Uik)\left(\frac{d^2 P}{\partial y^2} - k^2 P\right) + (\beta - U'')ikP = 0,$$

or

$$\left\{\frac{d^2}{dy^2} - k^2 + \frac{\beta - U''}{U - c}\right\} P(y) = 0.$$

The boundary conditions at the walls are

$$\begin{cases} P(0) = 0, \\ P(1) = 0. \end{cases}$$

We now redefine the phase speed as $c = c_r + ic_i$. Allowing a complex phase speed will give us the possibility of having a time dependent wave amplitude. In particular, a positive imaginary part will give a growing wave:

$$\psi = P(y)e^{+kc_i t}e^{ik(x-c_r t)}.$$

Since $k < 0$ for Rossby waves, $e^{+kc_i t}$ corresponds to a growing wave. We can thus investigate stability by substituting this into the equation:

$$\left\{\frac{d^2}{dy^2} - k^2 + \frac{\beta - U''}{(U - c_r) - ic_i}\right\} P(y) = 0$$

or

$$\left\{\frac{d^2}{dy^2} - k^2 + \frac{(\beta - U'')\left[(U - c_r) + ic_i\right]}{(U - c_r)^2 + c_i^2}\right\} P = 0.$$

Now, defining $\lambda \equiv (\beta - U'')(U - c_r)$ and $\mu \equiv (\beta - U'')c_i$,

$$\left[\frac{d^2}{dy^2} - k^2 + \frac{\lambda + iu}{|U - c|^2}\right] P = 0,$$

and letting $P = P_r + P_i$ alllows splitting the equation into real and imaginary parts:

$$\left[\frac{d^2}{dy^2} - k^2 + \frac{\lambda}{|U-c|^2}\right] P_r - \frac{\mu}{|U-c|^2} P_i = 0$$
$$\left[\frac{d^2}{dy^2} - k^2 + \frac{\lambda}{|U-c|^2}\right] P_i + \frac{\mu}{|U-c|} P_r = 0$$

which is then combined as

$$P_i \frac{d^2 P_r}{dy^2} - P_r \frac{d^2 P_i}{dy^2} - \frac{\mu}{|U-c|^2}\left(P_i^2 + P_i^2\right) = 0.$$

Integrating across the channel,

$$\int_0^1 P_i \frac{d^2 P_r}{dy^2} dy - \int_0^1 P_r \frac{d^2 P_i}{dy^2} dy - \int_0^1 \frac{\mu}{|U-c|^2}\left(P_r^2 + P_i^2\right) dy = 0,$$

and carrying out the integrations by parts

$$\left(P_i \frac{dP_r}{dy}\Big|_0^1 - \int_0^1 \frac{dP_r}{dy}\frac{dP_i}{dy} dy\right) - \left(P_r \frac{dP_i}{dy}\Big|_0^1 - \int_0^1 \frac{dP_i}{dy}\frac{dP_r}{dy} dy\right) - \int_0^1 \frac{\mu}{|U-c|^2}(P_r^2 + P_i^2) dy = 0.$$

With boundary conditions $P_r(0) = P_i(0) = 0$, $P_r(1) = P_i(1) = 0$, the first terms in the brackets vanish, and the second terms in the brackets cancel each other, so that we are left with:

$$\int_0^1 \frac{\mu}{|U-c|^2}(P_r^2 + P_i^2) dy = 0$$

or

$$c_i \int_0^1 (\beta - U'')\left\{\frac{P_r^2 + P_i^2}{|U_c|^2}\right\} dy = 0$$

where $\mu = c_i(\beta - U'')$ has been substituted. Note that the term in the braces is always positive, and since $c_i \neq 0$ (for unstable solutions), we observe that the only way an instability can occur is *if* $(\beta - U'')$ *changes sign at least once in the domain* $0 < y < 1$. Note that this is a *necessary* (but not *sufficient*) condition for *barotropic instability.* If the velocity profile $U(y)$ is such that the above integral does not integrate to zero, we can still have stable solutions with $c_i = 0$. If there was no $\beta$-effect present, then, for an unstable solution, the necessary condition for instability is to have an inflection point in $U(y)$.

## 2.9 Quasi-geostrophic, Baroclinic Motions

### 2.9.1 Rossby Waves in Quasigeostrophic Stratified Fluids

To study Rossby waves in a quasigeostrophic stratified fluid, we first simply the equations a little, for a simpler case where the $z$-dependent coefficients in the original vorticity equations are made constant.

For this we consider the case with

$$S = S_0 N^2 = \text{constant} = \hat{S},$$

and therefore the scale height defined in (2.18) is

$$H_s^{-1} \equiv -\frac{1}{\rho_r}\frac{\partial l_s}{\partial z} \equiv gN^2 = \text{constant}.$$

Taking $H_s^{-1}$=constant, or $S = \hat{S}$=constant, we can write

$$\begin{aligned}
\frac{1}{\rho_r}\frac{\partial}{\partial z}\frac{\rho_r}{\hat{S}}\frac{\partial \phi_0}{\partial z} &= \frac{1}{\hat{S}}\left\{\frac{1}{\rho_r}\frac{\partial \rho_r}{\partial z}\frac{\partial \phi_0}{\partial z} + \frac{\partial^2 \phi_0}{\partial z^2}\right\} \\
&= \frac{1}{\hat{S}}\frac{\partial^2 \phi_0}{\partial z^2} - \frac{1}{\hat{S}H_s}\frac{\partial \phi_0}{\partial z,}
\end{aligned}$$

and then, the quasigeostrophic vorticity equation reads

$$\left\{\frac{\partial}{\partial t} + \frac{\partial \phi_0}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \phi_0}{\partial y}\frac{\partial}{\partial x}\right\}\left\{\frac{1}{\hat{S}}\frac{\partial^2 \phi_0}{\partial z^2} - \frac{1}{\hat{S}H_s}\frac{\partial \phi_0}{\partial z} + \frac{\partial^2 \phi_0}{\partial x^2} + \frac{\partial^2 \phi_0}{\partial y^2}\right\} + \beta\frac{\partial \phi_0}{\partial x} = 0.$$

The above equation is now a simplified form of the more general equation with constant coefficients. Assume a plane-wave solution of the form:

$$\begin{aligned}
\phi_0 &= Ae^{\frac{z}{2H_s}}\cos(kx + ly + mz - \omega t) \\
&= Re\left\{Ae^{\frac{z}{2H_s}}e^{i(kx+ly+mz-\omega t)}\right\},
\end{aligned}$$

then, we calculate fluid vorticity

$$\begin{aligned}
Q &\equiv \frac{1}{S}\frac{\partial^2 \phi_0}{\partial z^2} - \frac{1}{\hat{S}H_s}\frac{\partial \phi_0}{\partial z} + \frac{\partial^2 \phi_0}{\partial x^2} + \frac{\partial^2 \phi_0}{\partial y^2} \\
&= \left\{\frac{1}{\hat{S}}\left(\frac{1}{2H_s} + im\right)^2 - \frac{1}{\hat{S}H_s}\left(\frac{1}{2H_s} + im\right) - k^2 - l^2\right\}Ae^{\frac{z}{2H_s}}e^{i(kx+ly+mz-\omega t)} \\
&= \left\{\frac{1}{4H_s^2\hat{S}} + \frac{2im}{2H_s\hat{S}} - \frac{m^2}{\hat{S}} - \frac{1}{2H_s^2\hat{S}} - \frac{im}{H_s\hat{S}} - k^2 - l^2\right\}\phi_0 \\
&= -\left(\frac{1}{4H_sS} + \frac{m^2}{\hat{S}} + k^2 + l^2\right)\phi_0.
\end{aligned}$$

Therefore, the Jacobian of $\phi_0$ and $Q$ vanishes:

$$\frac{\partial \phi_0}{\partial x}\frac{\partial Q}{\partial y} - \frac{\partial \phi_0}{\partial y}\frac{\partial Q}{\partial x} = -\left(\frac{1}{4H_s^2\hat{S}} + \frac{m^2}{S} + k^2 + l^2\right)\left\{\frac{\partial \phi_0}{\partial x}\frac{\partial \phi_0}{\partial y} - \frac{\partial \phi_0}{\partial y}\frac{\partial \phi_0}{\partial x}\right\} = 0.$$

With this simplification, we arrive at a linear equation

$$\frac{\partial Q}{\partial t} + \beta\frac{\partial \phi_0}{\partial x} = 0$$

which has to be satisfied by the proposed plane wave solution if

$$-\left(\frac{1}{4H_s^2\hat{S}} + \frac{m^2}{\hat{S}} + k^2 + l^2\right)(-i\omega)\phi_0 + \beta(ik)\phi_0 = 0,$$

or

$$\omega = -\frac{\beta k}{k^2+l^2+\frac{m^2}{\hat{S}}+\frac{1}{4H_s^2\hat{S}}}.$$

The above is the dispersion relation for the Rossby waves. Note that it is very similar to the barotropic Rossby wave dispersion relation, except for an addition of vertical wave number term scaled by $\hat{S}^{-1}$, and a constant term.

### `2.9.2 The Boussinesq Approximations`

Remember that before making the perturbation analysis leading to the quasigeostrophic approximation, we initially had the equations

$$\begin{aligned}
&\left\{\frac{\partial \vec{u}}{\partial z} + \vec{u}\cdot\nabla\vec{u} + \beta y\hat{k}\times\vec{u}\right\} + \hat{k}\times\vec{u} = \nabla\phi \\
&\frac{\partial \phi}{\partial z} = S \\
&\nabla\cdot\vec{u} + \delta\frac{1}{\rho_r(z)}\frac{\partial}{\partial z}\rho_r(z)w = 0 \\
&\frac{\partial s}{\partial t} + \vec{u}\cdot\nabla s + S_0N^2(z)w = 0
\end{aligned}
\quad ,$$

with $\delta = \frac{U}{f_0L}$ =*Rossby number*, and

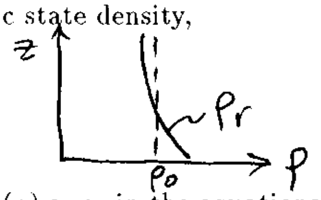
$$\phi = \frac{\tilde{p}}{\rho_r}, \quad s = \frac{\tilde{\theta}}{\theta_r}.$$

We also had the simplified equation of state

$$\frac{\tilde{\theta}}{\theta_r} = \frac{1}{\gamma}\frac{\tilde{p}}{p_r} - \frac{\tilde{\rho}}{\rho_r}, \quad \frac{\tilde{p}}{p_r} = \frac{\tilde{\rho}}{\rho_r} + \frac{\tilde{T}}{T_r}$$

Further approximations could in fact be made with the following shortcut assumptions:

First assume that the pertirbation density is much smaller than the basic state density,



$$\rho_r(z) = \rho_0 + (\Delta\rho)\rho''(z), \quad \left(\frac{\Delta\rho}{\rho_0}\right) \ll 1$$

or more specifically, $O(\delta^2) < \frac{\Delta\rho}{\rho_0} < O(\delta)$, and therefore we can take $rho_r(z) \simeq \rho_0$ in the equations. Secondly we assume the effect of compression is small on the perturbations, *i.e.*, and neglect $\frac{\tilde{p}}{p_r}$ in the state equations, so that they can become

$$\frac{\tilde{\theta}}{\theta_0} = \frac{\tilde{T}}{T_0} = -\frac{\tilde{\rho}}{\rho_0}$$

replacing $\theta_r(z)$, $T_r(z)$, $\rho_r(z)$ by constants $\theta_0$, $T_0$, $\rho_0$ respectively, using the fact that deviations are small.

We thus ignore the effects of basic state stratification and compression in the continuity, momentum, and state equations, but keep the effects in the thermodynamic equation, where we lump them in $N^2(z)$ ! The above are called *Boussinesq approximations.* The heart of the Boussinesq approximation is stating that stratification constitutes the restoring force in the dynamics, rather than compression.

The Boussinesq version of the governing equations are thus

$$\begin{aligned}
&\partial\vec{u}\partial t + \vec{u}\cdot\nabla\vec{u} + f\hat{k}\times\vec{u} = \frac{1}{\rho_0}\nabla\tilde{p} \\
&\frac{\partial\tilde{p}}{\partial z} = -g\tilde{\rho} \\
&\nabla\cdot\vec{u} + \frac{\partial w}{\partial z} = 0 \\
&\frac{\partial\tilde{\rho}}{\partial t} + \vec{u}\cdot\nabla\tilde{\rho} - \frac{\rho_0}{g}N^2 w = 0
\end{aligned}$$

The corresponding Boussinesq version of the vorticity equation would be

$$\left[\frac{\partial}{\partial t} + \left(\hat{k}\times\nabla\psi\right)\cdot\nabla\right]\left[\nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial x}\right)\right] + \beta'\frac{\partial\psi}{\partial x} = 0,$$

where

$$\tilde{\rho} = -\left(\frac{\rho_0}{T_0}\right)\tilde{T}, \quad \text{and} \quad \psi = \frac{\tilde{p}}{\rho_0 f_0}$$

have been used. An alternative form of the Boussinesq vorticity equation is

$$\left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right)\left(\nabla^2\psi + f_0^2\frac{\partial}{\partial z}\frac{1}{N^2}\frac{\partial\psi}{\partial z}\right) + \beta v = 0.$$

Remember that in obtaining the vorticity equation, we made use of the following equations:

$$\frac{1}{\rho_r}\frac{\partial}{\partial z}(\rho_r w_0) = -\left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right)\frac{\partial}{\partial z}\left(\frac{\rho_r}{S_0 N^2}\frac{\partial \phi_0}{\partial z}\right)$$
$$\left(\frac{\partial}{\partial t} + \vec{u}_0\cdot\nabla\right)\xi_0 + \beta v_0 = \frac{1}{\rho_r}\frac{\partial}{\partial z}(\rho_r w_0),$$

where $\xi = \nabla^2\psi$.

In dimensional form, the second equation reads (dropping subscripts):

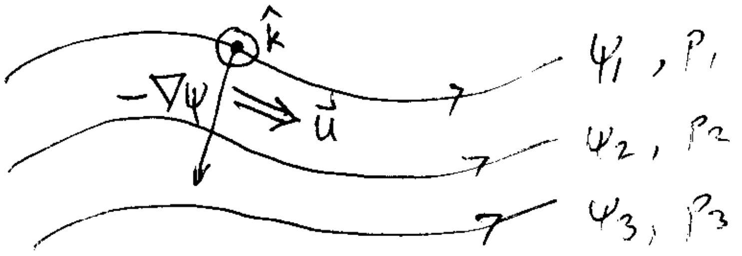
$$\left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right)\xi + \beta v = \frac{f_0}{\rho_r}\frac{\partial}{\partial z}(\rho_r w),$$

and the Boussinesq version reads:

$$\left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right)\xi + \beta v = f_0\frac{\partial w}{\partial z}.$$

### 2.9.3 Physics Represented by Quasigeostrophic Equations

To leading order, the horizontal velocity is related to a *streamfunction* $\psi = \tilde{p}/\rho_0 f_0$, such that $\vec{u} = -\hat{k}\times\nabla\psi$. Thsi means that at any vertical level of the atmosphere or the ocean ($z$ =constant) pressure patterns (isobars) serve as the streamlines.



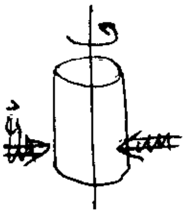
Note that near ground in the atmosphere, and the surface or the bottom of the ocean, the approximation is not strictly valid, because friction affects the motion in the Ekman boundary layers (thickness $\simeq 1km$ in the atmosphere and $\simeq 50m$ in the ocean).

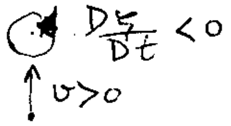
The vorticity conservation

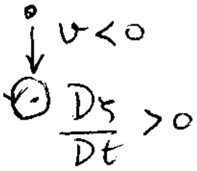
$$\frac{D\zeta}{Dt} \equiv \left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right)\xi = -\beta v + f_0\frac{\partial w}{\partial z}$$
$$= -\beta v - f_0\nabla\cdot\vec{u}$$

states that the vorticity $\xi = \nabla^2\psi = \hat{k}\cdot(\nabla\times\vec{u})$ is changed either by ($i$) the *beta-effect* (first term) or ($ii$) the *divergence effect* (second term)

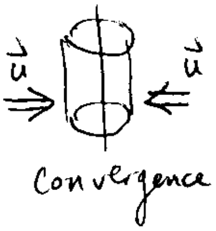
($i$) The *beta-effect* contributes to vorticity when fluid elements move to a position with different planetary vorticity from their earlier ambient value. For example, for $v > 0$ or northerly transport (fluid moving to a region with higher planetary vorticity), we find that the fluid vorticity decreases ($D\xi/Dt < 0$). Similarly, for $v < 0$ or southerly transport (fluid moving to a region with lower planetary vorticity), we find that the fluid vorticity inreases ($D\xi/Dt > 0$).

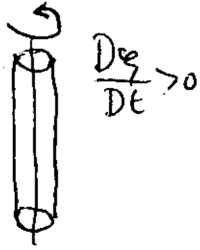


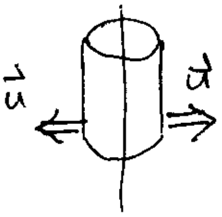


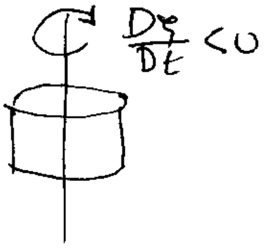


(*ii*) The *divergence effect* changes the vorticity or equivalently the angular momentum of a finite fluid volume. In the case of a fluid cylinder with fixed volume, we can see the following consequences: When there is convergence (*i.e.*, $\partial w/\partial z > 0$, or $\nabla \cdot \vec{u} < 0$), the fluid column gets thinner and rotates faster, *i.e.* $D\xi/Dt > 0$. Similarly, when there is divergence (*i.e.*, $\partial w/\partial z < 0$, or $\nabla \cdot \vec{u} > 0$), the fluid column gets fatter and rotates slower, *i.e.* $D\xi/Dt < 0$.



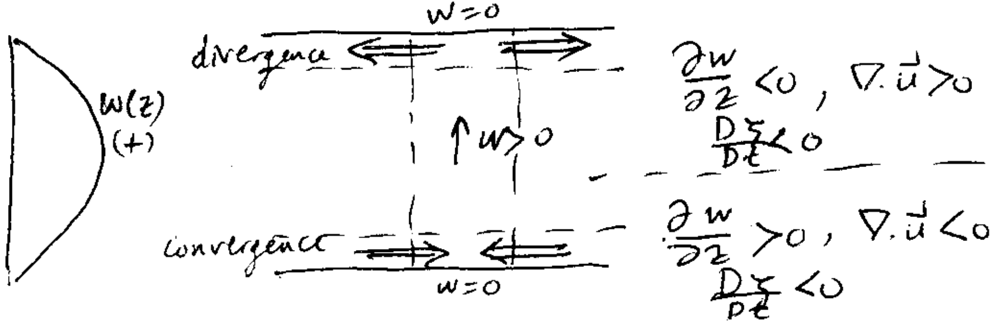




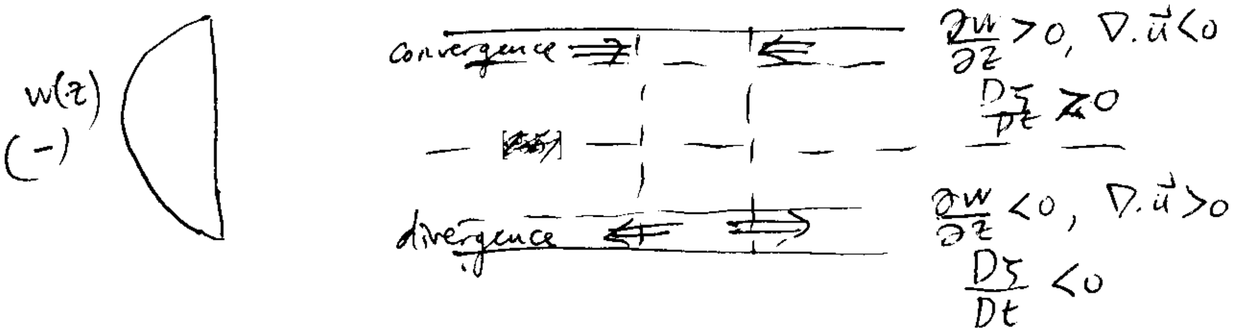


This is also called the *ballerina effect*, after a ballerina controlling its speed of rotation by extending or contracting her arms.

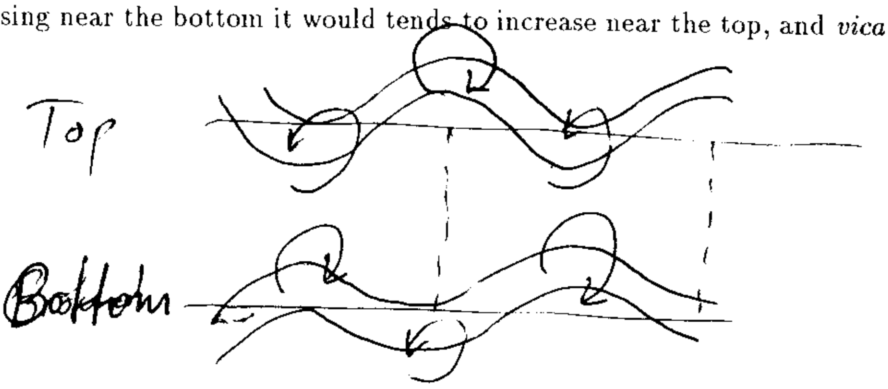
We can visualise the effect of top and bottom boundaries as follows. For simplicity, let us consider flat boundaries, where vertical velocity has to vanish. In the case of ascending motion we must have $\frac{\partial w}{\partial z} < 0$ or $\nabla \cdot \vec{u} > 0$ in the upper part, and the opposite in the lower part. As a result, vorticity decreases $\frac{D\xi}{Dt} < 0$ in the upper part, and increases in the lower part:



In the case of descending motion we must have the opposite case for upper lower parts, so that vorticity increases in the upper part, and decreases in the lower part:



Therefore, if the fluid is bounded between two horizontal surfaces, we expect *phase shifts* between motions near top and bottom parts of the fluid motion. This *slanting* of motion results from different rates of the change of vorticity at different vertical levels. While the vorticity of a wave motion may be decreasing near the bottom it would tends to increase near the top, and *vica versa.*



Finally, let us study the changes that occur in density during the motion. The thermodynamic equation states that

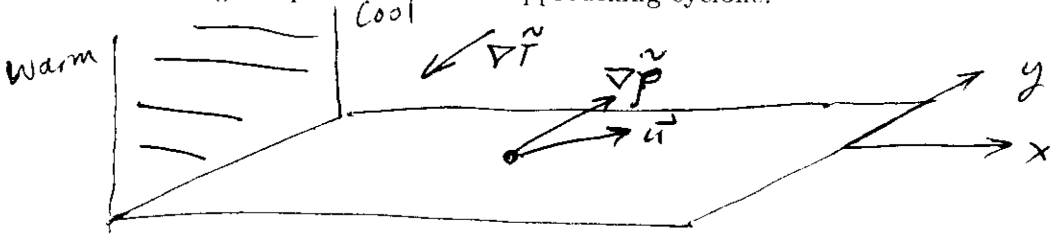
$$\frac{D\tilde{\rho}}{Dt} = \left(\rho_0 \frac{N^2}{g}\right) w$$

This equation shows that the perturbation density following a fluid particle changes only by vertical motions and in relation to the basic state density stratification profile. In particular, for a fluid parcel moving downward ($w < 0$), the density decreases $\frac{D\tilde{\rho}}{Dt} < 0$ (*i.e.* temperature increases). Similarly, for a fluid parcel moving upward ($w > 0$), the density increases $\frac{D\tilde{\rho}}{Dt} < 0$ (*i.e.* temperature decreases). (Note that this result runs contrary to the popular belief that a warming parcel would ascend, and a cooling parcel would descend !)

What happens near the top and bottom boundaries ? For eaxample, let us consider what happens in the atmosphere near the ground. We first note that near the boundary $w \simeq 0$, so that the above thermodynamic equation yields

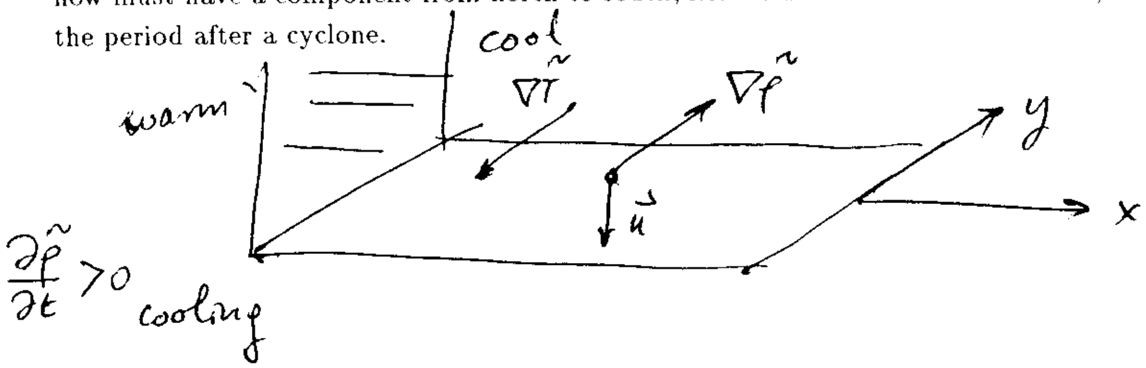
$$\frac{\partial\tilde{\rho}}{\partial t} \simeq -\vec{u}\cdot\nabla\tilde{\rho}.$$

Therefore, if there is a warming ($\partial\tilde{T}/\partial t > 0$ or $\frac{\partial\tilde{\rho}/}{\partial t} < 0$) situation at a fixed position, we must have $\vec{u}\cdot\nabla\tilde{\rho} > 0$, i.e., the vectors $\vec{u}$ and $\nabla\tilde{\rho}$ must have components pointing in the same direction. But since $\nabla\tilde{\rho}$ must have a component towards north (it is usually warmer in the south than north), this means that the flow also must have a component from south to north, i.e. we must have *warm advection*, such as during the period before an approaching cyclone.



if $\frac{\partial\tilde{\rho}}{\partial t} < 0$
warming period

On the other hand if there is a cooling situation ($\partial\tilde{\rho}/\partial t > 0$), we must have $\vec{u}\cdot\nabla\tilde{\rho} < 0$, i.e., the vectors $\vec{u}$ and $\nabla\tilde{\rho}$ must have components pointing in the opposite direction. This means that the flow must have a component from north to south, i.e. we must have *cold advection*, such as during the period after a cyclone.



One of the most basic relationships is that of *thermal wind*, simply combining the geostrophic and hydrostatic relations:

$$\vec{u} = \hat{k}\times\nabla\psi = \hat{k}\times\nabla\left(\frac{\tilde{p}}{f_0\rho_0}\right), \quad \frac{\partial\tilde{p}}{\partial z} = -\tilde{\rho}g$$

yielding

$$\frac{\partial\vec{u}}{\partial z} = \frac{1}{f_0\rho_0}\hat{k}\times\nabla\left(\partial\tilde{p}over\partial z\right) = -\frac{g}{\rho_0 f_0}\hat{k}\times\nabla\tilde{\rho} = \frac{g}{\rho_0 f}\hat{k}\times\nabla\tilde{T}.$$

Integrating in the vertical gives

$$\vec{u} = \vec{u}_0 - \frac{g}{\rho_0 f_0}\int_0^z \hat{k}\times\nabla\tilde{T}\,dz$$

where $\vec{u}_0$ is the velocity at $z = 0$.