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"Course on Shallow Water and Shelf Sea Dynamics"
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"Barotropic Flow over Topography"

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Please note: These are preliminary notes intended for internal distribution only.

BAROTROPIC FLOW OVER TOPOGRAPHY

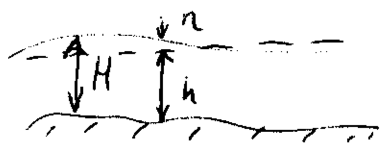
— NONDIVERGENT —
— FRICTIONAL —

SHALLOW WATER EQUATIONS :

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} + f\hat{k}\times\vec{u} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho H}\left(\vec{\tau}^{(s)} - \vec{\tau}^{(b)}\right) + \frac{1}{H}\nabla.H\underline{F}$$

$$\frac{\partial \eta}{\partial t} + \nabla.H\vec{u} = 0$$

eqns (2.89)-(2.90) DOSI



Simplifications :

Rigid lid : $\frac{\partial \eta}{\partial t} = 0$

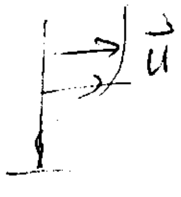
Lateral friction isotropic, simple $\frac{1}{H}\nabla.H\underline{F} \simeq \nu\nabla^2\vec{u}$

Bottom friction (Linearised)

Turbulent (well-stirred) $\frac{\vec{\tau}^{(b)}}{\rho H} \sim \frac{\rho\, \overbrace{C_D|\vec{u}|}^{r}\, \vec{u}}{\rho H} \sim \underset{\sim}{r}\frac{\vec{u}}{H}$

dimensional constant $\sim\left[\frac{L}{T}\right]$

Turbulent/laminar (bottom boundary layer)



$\frac{\vec{\tau}^{(b)}}{\rho H} \sim \underset{\sim}{k}\vec{u}$

dimensional constant $\sim\left[\frac{1}{T}\right]$

Use the turbulent well-stirred case here.

$$\boxed{\frac{D\vec{u}}{Dt} + f\hat{k}\times\vec{u} = -\frac{1}{\rho}\nabla p + \frac{\vec{\tau}}{\rho H} - \frac{r}{H}\vec{u} + \nu\nabla^2\vec{u} \quad (1) \\ \nabla . H\vec{u} = 0 \quad (2)}$$

Define a <u>transport stream function</u> ψ

$$\boxed{H\vec{u} = \hat{k}\times\nabla\psi} \qquad (3)$$

which satisfies (2).

i.e. $\nabla . H\vec{u} = \nabla . \hat{k}\times\nabla\psi$

$$= \nabla\psi . \underbrace{\nabla\times\hat{k}}_{0} - k\,\underbrace{\nabla\times\nabla\psi}_{\equiv 0} \equiv 0$$

[using $\nabla . \vec{a}\times\vec{b} = \vec{b} . \nabla\times\vec{a} - \vec{a} . \nabla\times\vec{b}$]

Define <u>vorticity</u>

$$\zeta = \hat{k} . \nabla\times\vec{u} = \hat{k} . \nabla\times\frac{1}{H}\hat{k}\times\nabla\psi$$

$$= \hat{k} . \hat{k}\ \nabla . \frac{1}{H}\nabla\psi$$

[using $\nabla\times\vec{a}\times\vec{b} = (\vec{b} . \nabla)\vec{a} - (\vec{a} . \nabla)\vec{b} + \vec{a}(\nabla . \vec{b}) - \vec{b}(\nabla . \vec{a})$] $\boxed{\zeta = \nabla . \frac{1}{H}\nabla\psi}$

Take curl of (1) $[\nabla \times (1):]$

$$\frac{\partial}{\partial t}\underbrace{\nabla\times\vec{u}}_{\zeta\hat{k}} + \nabla\times(\vec{u}.\nabla\vec{u}) + f\,\nabla\times\hat{k}\times\vec{u}$$

$$= -\frac{1}{\rho}\cancel{\nabla\times\nabla p}^{\equiv 0} + \frac{1}{\rho}\nabla\times\left(\frac{\vec{\tau}}{H}\right) - r\,\nabla\times\left(\frac{\vec{u}}{H}\right) + \nu\nabla^2\underbrace{\nabla\times\vec{u}}_{\zeta\hat{k}}$$

Note: $\vec{u}.\nabla\vec{u} = \nabla\left(\frac{1}{2}\vec{u}.\vec{u}\right) - \vec{u}\times\nabla\times\vec{u}$

so that $\nabla\times(\vec{u}.\nabla\vec{u}) = \underbrace{\cancel{\nabla\times\nabla}\left(\frac{1}{2}\vec{u}.\vec{u}\right)}_{\equiv 0} - \nabla\times[\vec{u}\times\overbrace{\nabla\times\vec{u}}^{\zeta}]$

$$= \nabla\times\hat{k}\zeta\times\vec{u}$$

Then:

$$\frac{\partial}{\partial t}\hat{k}\zeta + \nabla\times[\hat{k}(\zeta+f)\times\vec{u}]$$

$$= \frac{1}{\rho}\nabla\times\left(\frac{\vec{\tau}}{H}\right) - r\,\nabla\times\left(\frac{\vec{u}}{H}\right) + \nu\hat{k}\nabla^2\zeta$$

next note that $\nabla\times[(\zeta+f)\hat{k}\times\vec{u}] = \vec{u}.\nabla[(\zeta+f)\hat{k}] + (\zeta+f)\hat{k}\,\nabla.\vec{u}$

[using $\nabla\times\vec{a}\times\vec{b} = (\vec{b}.\nabla)\vec{a} - (\vec{a}.\nabla)\vec{b} + \vec{a}(\nabla.\vec{b}) - \vec{b}(\nabla.\vec{a})$]

[using $\nabla.H\vec{u} = H\nabla.\vec{u} + \vec{u}.\nabla H = 0$] $\left(-\frac{1}{H}\frac{DH}{Dt}\right.$

$\frac{\partial H}{\quad}$ $\rightarrow \nabla.\vec{u} = \frac{1}{H}\vec{u}.\nabla H = -\frac{1}{H}\frac{DH}{}$

or

$$\frac{\partial (\zeta+f)\hat{k}}{\partial t} + \vec{u}\cdot\nabla\left[(\zeta+f)\hat{k}\right] - \frac{(\zeta+f)\hat{k}}{H}\frac{DH}{Dt}$$
$$= \frac{1}{\rho}\nabla\times\frac{\vec{\tau}}{H} - r\,\nabla\times\frac{\vec{u}}{H} + \hat{k}\,\nu\nabla^2\zeta$$

note $\quad \nabla\times\frac{\vec{u}}{H} = \nabla\times\frac{1}{H}\hat{k}\times\frac{1}{H}\nabla\psi = \hat{k}\,\nabla\cdot\frac{1}{H^2}\nabla\psi$

Multiply by $\hat{k}$ and divide by H:

$$\frac{1}{H}\Big(\overbrace{\frac{\partial(\zeta+f)}{\partial t} + \vec{u}\cdot\nabla(\zeta+f)}^{\frac{D(\zeta+f)}{Dt}}\Big) - \frac{\zeta+f}{H^2}\frac{DH}{Dt}$$
$$= \frac{1}{\rho}\hat{k}\cdot\frac{1}{H}\nabla\times\frac{\vec{\tau}}{H} - \frac{r}{H}\nabla\cdot\frac{1}{H^2}\nabla\psi + \nu\frac{\nabla^2\zeta}{H}$$

i.e.

$$\boxed{\begin{aligned} &\frac{D}{Dt}\left(\frac{\zeta+f}{H}\right) = \frac{1}{\rho}\hat{k}\cdot\frac{1}{H}\nabla\times\frac{\vec{\tau}}{H} - r\frac{1}{H}\nabla\cdot\frac{1}{H^2}\nabla\psi + \frac{\nu}{H}\nabla^2\zeta \\ &\nabla\cdot\frac{1}{H}\nabla\psi = \zeta \end{aligned}}$$

note $\quad \nabla\cdot\frac{1}{H^2}\nabla\psi = \frac{1}{H^2}\nabla^2\psi - \frac{2}{H^3}\nabla H\cdot\nabla\psi$

Alternative form:

$$\underbrace{\frac{\partial}{\partial t}\left(\frac{\xi+f}{H}\right)}_{\frac{1}{H}\frac{\partial \xi}{\partial t}} + \underbrace{\vec{u}\cdot\nabla\left(\frac{\xi+f}{H}\right)}_{\underbrace{\vec{u}\cdot\nabla\left(\frac{\xi}{H}\right)}_{\text{nonlinear term}} + \underbrace{\vec{u}\cdot\nabla\left(\frac{f}{H}\right)}_{\text{linear term.}}} = \dots$$

~~Mult~~ Mult by H:

$$\frac{\partial \xi}{\partial t} + H\vec{u}\cdot\nabla\frac{\xi}{H} + H\vec{u}\cdot\nabla\frac{f}{H} = \frac{1}{\rho}\hat{k}\cdot\nabla\times\frac{\vec{\tau}}{H} - r\nabla\cdot\frac{1}{H^2}\nabla\psi + \frac{\nu}{\cancel{H}}\nabla^2\xi$$

$$\nabla\cdot\frac{1}{H}\nabla\psi = \xi$$

PDE2D: coupled eqns

$$\begin{cases} C_i\frac{\partial u_i}{\partial t} = \frac{\partial A_i}{\partial x} + \frac{\partial B_i}{\partial y} - F_i \\ + \text{natural B.C.} \end{cases}$$

Can write:

$$\underbrace{\frac{1}{r}}_{C_1}\frac{\partial \xi}{\partial t} = \frac{\partial}{\partial x}\underbrace{\left(-\frac{\psi_x}{H^2}\right)}_{A_1} + \frac{\partial}{\partial y}\underbrace{\left(-\frac{\psi_y}{H^2}\right)}_{B_1} \overbrace{+\frac{1}{r\rho}\hat{k}\cdot\nabla\times\frac{\vec{\tau}}{H} - \frac{1}{r}\vec{u}\cdot\nabla\left(\frac{\xi+f}{H}\right)}^{F_1} + \frac{\nu}{r\cancel{H}}\nabla^2\xi$$

$$\underbrace{0}_{C_2} = \frac{\partial}{\partial x}\underbrace{\left(\frac{\psi_x}{H}\right)}_{A_2} + \frac{\partial}{\partial y}\underbrace{\left(\frac{\psi_y}{H}\right)}_{B_2} - \underbrace{\xi}_{F_2}$$

Non-dimensional form $\hat{x} \sim L_0$, $t \sim f_0^{-1}$, $H \sim H_0$

$u \sim U_0$, $\psi \sim U_0 H_0 L_0$, $\zeta \sim \frac{U_0}{L_0}$

$\vec{\tau} \sim \Pi_0$

$$\frac{\partial \zeta}{\partial t} + \hat{k} \times \nabla\psi \cdot \nabla\left(\frac{R_o \zeta + 1}{H}\right) = S_0 \hat{k} \cdot \nabla \times \frac{\vec{\tau}}{H} - \mu \nabla \cdot \frac{1}{H^2} \nabla\psi + \mathbb{E} \nabla^2 \zeta$$

$$\nabla \cdot \frac{1}{H} \nabla\psi = \zeta$$

where $R_o = \dfrac{U_0}{f_0 L_0}$, $\mathbb{E} = \dfrac{\nu}{f_0 L^2}$

Rossby # Ekman #

$\mu = \dfrac{r}{f_0 H_0}$, $S_0 = \dfrac{\Pi_0}{\rho_0 f_0 U_0 H_0}$

typical values for shelf:

$U_0 = 1\,m/s$, $f_0 = 10^{-4}$ $L_0 = 100\,km = 10^5\,m$

$r = 10^{-3}\,m/s$ $\nu = 100\,m^2/s$ $H_0 = 100\,m$

$$R_o = \frac{1}{10^4\, 10^5} \sim 0.1 \qquad E = \frac{10^2}{10^{-4}\, 10^{10}} \sim 10^{-4}$$

$$\mu = \frac{10^{-3}}{10^{-4}\, 10^2} \sim 0.1$$

$$S_0 = \frac{\Pi_0}{\rho_0 f_0 U_0 H_0} = \frac{\rho_a C_D U_a^2}{\rho_0 f_0 U_0 H_0}$$

$$= \left(\frac{\rho_a}{\rho_0}\right)\left(\frac{U_a}{U_0}\right)^2 R_o \frac{1}{(H_0/L_0)}$$

$$= 10^{-3} (10)^2\, 0.1\, \frac{1}{10^{-3}} = 10$$

$\sim O(1)$

STEADY FLOW – unforced $\vec{\tau} = 0$

$\nu = 0$

$Ro \to 0$

<u>balance</u> :

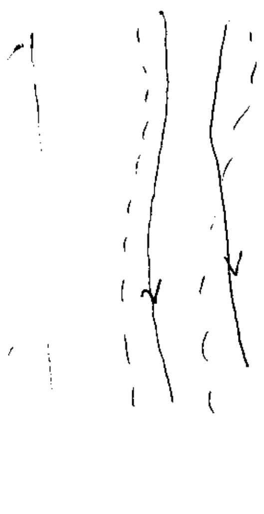
$$\overbrace{H\vec{u}}^{\hat{k}\times\nabla\psi} \cdot \nabla \frac{f_0}{H} = -r \, \nabla \cdot \frac{1}{H^2} \nabla \psi$$

$$-\frac{f_0}{H^2}\, \hat{k} \times \nabla\psi \cdot \nabla H = -r \left\{ \frac{1}{H^2} \nabla^2 \psi - \frac{2}{H^3} \nabla H \cdot \nabla \psi \right\}$$

$$f_0\, \hat{k} \cdot \nabla\psi \times \nabla H = r \left\{ \nabla^2 \psi - \frac{2}{H} \nabla H \cdot \nabla \psi \right\}$$

Analogy with heat diffusion

$$\underbrace{\nabla^2 \psi}_{\text{diffusion}} = \underbrace{\frac{2}{rH} \nabla H \cdot \nabla \psi + \frac{f_0}{r}\, \hat{k} \cdot \nabla \psi \times \nabla H}_{\text{Sources}}$$



contributed by depth variations aligned with flow

Contributed by depth variations crossing the flow

If flow (streamlines) does not cross depth contours, then

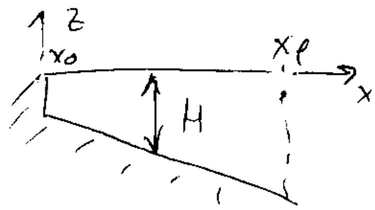
$$\nabla^2 \psi - \frac{2}{H} \nabla H \cdot \nabla \psi = 0$$

or $$\nabla \cdot \frac{1}{H^2} \nabla \psi = 0$$

$$\frac{1}{H^2} \nabla \psi = \text{const.} \quad \Rightarrow \quad \underline{\nabla \psi = \text{const.}\, H^2}$$

$$\left\{ \begin{array}{l} \text{if } H = H(x) \\ \psi = C \int_0^x H^2 \, dx \end{array} \right.$$

Example: $H = h_0 + a x$



$$\psi = C \int_{\cancel{x}x_0}^{x} (h_0 + a x)^2 \, dx$$

$$= \frac{C}{3a} \left[(h_0 + a x)^3 - (h_0 + a x_0)^3 \right]$$

$$= \frac{C}{3a} \left[(\cancel{h_0^3} + 3 h_0^2 a x + 3 h_0 a^2 x^2 + a^3 x^3) - (\cancel{h_0^3} + \dots) \right]$$

$$= C \left[h_0^2 (x - x_0) + h_0 a (x^2 - x_0^2) + \frac{a^2}{3} (x^3 - x_0^3) \right]$$

Note $$Q = \int_{x_0}^{x_\ell} H v \, dx = \psi_\ell - \psi_0$$

$\rightarrow$ determines C!