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"Course on Shallow Water and Shelf Sea Dynamics"
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"Coastal Dynamics: Introduction"

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Please note: These are preliminary notes intended for internal distribution only.

Various approximations and simplifications of equations

Let's recall complete Boussinesq set of equation previously derived. We will focus on the momentum and continuity equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

$$0 = -\frac{\partial p}{\partial z} - \rho g$$

x, y, z momentum

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

continuity

Geostrophic approximation

Also recall that the lowest order approximation to the x & y momentum equations ~~was~~ is the steady state geostrophic balance

$$-fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \quad ; \quad fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

This means that the geostrophic velocity vector is perpendicular to the pressure gradient, i.e. geostrophic flow is parallel to the isobars.

Now let us assume f = constant (f-plane approx) and compute the divergence of the geostrophic vel

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{1}{\rho_0 f}\frac{\partial^2 p}{\partial x \partial y} - \frac{1}{\rho_0 f}\frac{\partial^2 p}{\partial x \partial y} = 0$$

which by continuity implies $\frac{\partial w}{\partial z} = 0$ so w is independent of z and if w vanishes at some level (e.g. at a rigid bottom or surface) the $w \equiv 0$.

meaning that flow is two dimensional. Also means that the steady state equations are degenerate i.e. they are not independent and therefore do not supply enough information to allow us to find a unique solution, i.e. any pressure ~~[illegible]~~ distribution will give a stream function that ~~[illegible]~~ satisfies all of the steady-state equations.

Nevertheless, as a diagnostic tool the geostrophic approximation is an ~~[illegible]~~ extremely valuable tool for diagnosing the motion field from the mass field (i.e. computation of dynamic height field and relative geostrophic flow).

Now back to geostrophic flow: let's also assume the fluid is fully incompressible (not Boussinesq): in which case p ~~[illegible]~~ can be written as reference ~~[illegible]~~ profile (function of z only) plus a fluctuation due to the motion

$$p = p_0(z) + p'(x,y,z,t)\,;$$ ~~[illegible]~~

and the hydrostatic equation becomes

$$\underbrace{\frac{\partial p_0}{\partial z} + \frac{\partial p'}{\partial z} = -\rho_0 g}_{\text{hydrostatic balance}} \implies \frac{\partial p'}{\partial z} = 0$$

This means that $\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$ which means that fluid will move as "rigid" vertical columns. (Taylor-Proudman theorem).

This has profound implications for flow over ~~[illegible]~~ bottom topography.

① For a flat bottom, geostrophic flow can have an arbitrary pattern dependendent upon initial conditions

② For an isolated bump, flow must be around the bump. Fluid cannot rise over the bump and fluid over the bump is trapped (Taylor column)

③ For sloping bottom flow must be along the depth contours

Note that above is true for rigid lid. For free surface we find that restriction is that fluid depth must remain ~~[illegible]~~ constant, i.e. fluid can move up and down bathymetry as long as it is not squeezed or stretched vertically.

Shallow water approximation

Let's relax the ~~[illegible]~~ geostrophic constraint and go to the next order of approximation (i.e. include advection) but still for fully incompressible flow &

$$\frac{\partial u}{\partial t} + \underset{\sim}{V} \cdot \nabla u - fv = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}$$

$$\frac{\partial v}{\partial t} + \underset{\sim}{V} \cdot \nabla v + fu = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}$$

(note here that $\underset{\sim}{V} \cdot \nabla$ is 3-dim)

$$\frac{\partial p'}{\partial z} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

If the initial horizontal flow is independent of z then vertical advection will be zero, and incompressibility implies p independent of z so the flow will remain independent of z, i.e. flow will be barotropic and equations reduce to

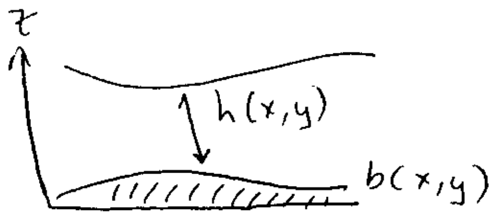
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

Although the flow ~~has~~ has no vertical structure, unlike geostrophic flow, it can now ~~be horizontally~~ ~~be~~ be horizontally non divergent, i.e. $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \neq 0$

which means there can now be a vertical velocity and flow can cross isobars and/or bathymetry.

Vertical velocity can be computed by vertically integrating the continuity equation:



$b(x,y)$ - bathymetry

$h(x,y)$ - height of the sfc above b

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\int_b^{b+h} dz + w\Big|_b^{b+h} = 0$$

Boundary conditions are that fluid particles on bottom stay on bottom, and those on sfc stay on sfc

$$w(b) = u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y} \quad ; \quad w(b+h) = \frac{\partial}{\partial t}(b+h) + u\frac{\partial}{\partial x}(b+h) + v\frac{\partial}{\partial y}(b+h)$$

Which upon substitution into the vertically integrated continuity equation gives

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0$$

(Some authors use $H+\eta$ where H is mean depth and η is free surface displacement)

The dynamic pressure field is given by $p = \rho_0 g(h+b)$ (independent of depth), i.e. we can replace p by $h+b$, and if we are considering a flat bottom then the shallow water equations become:

$$\boxed{\begin{aligned} &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial h}{\partial x} \\ &\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial h}{\partial x} \\ &\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0 \end{aligned}}$$

which is a set of 3 equations for the three unknowns u, v, h

For the linearized equations, we can combine them into a single 2nd order wave equation for h with phase speed $c = \sqrt{gH}$ (barotropic or external gravity wave speed)

Vorticity of shallow water flow

By taking $\frac{\partial}{\partial x}$(y-momentum eq) $-$ $\frac{\partial}{\partial y}$(x-momentum eq)
we get

$$\frac{d}{dt}(f+\zeta) + (f+\zeta)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) = 0$$

where $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ is the relative vorticity, f: planetary vorticity

and $\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$ $\qquad f+\zeta$: absolute vorticity

Expanding the continuity equation gives

$$\frac{dh}{dt} + h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) = 0$$

and combining these two $\quad$ (remember $\frac{df}{dt} = \overset{0}{\cancel{\frac{\partial f}{\partial t}}} + \overset{0}{\cancel{u\frac{\partial f}{\partial x}}} + v\frac{\partial f}{\partial y}$)
gives

$$h\frac{d(f+\zeta)}{dt} - (f+\zeta)\frac{dh}{dt} = 0$$

or

$$\boxed{\frac{d}{dt}\left(\frac{f+\zeta}{h}\right) = 0}$$

Conservation of potential vorticity

For shallow water flow this is ratio of ~~circulation~~ ~~depth~~
$\frac{\text{circulation}}{\text{volume}}$ $\Rightarrow$ if fluid is squeezed or stretched
it's relative vorticity must change so as to
conserve PV.

In the geostrophic limit, $\zeta \ll f \Rightarrow PV = \frac{f}{h}$

which for f-plane implies that each column must preserve its height (same conclusion from geostrophic discussion above). But now the conservation of PV provides the information needed to determine a unique steady state solution

In the Rossby adjustment problem, the steady state solution contains a length scale $\frac{c}{|f|} = \frac{\sqrt{gH}}{|f|}$ which is the (external) Rossby radius of deformation, L_R.

For $f \to 0$, $L_R \to \infty$ implying $L < L_R$ rotation effect are small
$L \gtrsim L_R$ rotation effects important

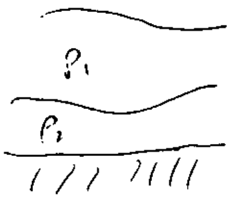
For the ~~new~~ stratified case ~~we can treat the whole layer as a stack of layers each having uniform density the same~~ the potential vorticity is $PV = (f+\zeta)\frac{\partial \rho}{\partial z} + \left(\frac{\partial u}{\partial z}\frac{\partial \rho}{\partial y} - \frac{\partial v}{\partial z}\frac{\partial \rho}{\partial x}\right)$

and quite often can ~~scale~~ neglect second ( ) on RHS giving $PV \simeq (f+\zeta)\frac{\partial \rho}{\partial z}$

Also stratified case can be approximated by stack of layers each with its own uniform ρ.

To represent baroclinic effects, the simplest system is a two layer system where we can write the ~~....~~ analogous

shallow water equation for each layer separately.
The equations for the upper layer look like SW eq.
while for lower layer, the pgf ~~weakness~~ depends on



reduced gravity $g' = \left(\frac{\rho_2 - \rho_1}{\rho_2}\right) g$

Alternatively can use equivalent depth H_e

Now the B.V. freq becomes $N^2 = -\frac{g}{\rho}\frac{d\rho}{dz} \simeq \frac{g'}{H}$

(where H is a typical thickness, usually of ~~upper~~ upper layer)

and the corresponding radius of deformation is

$$L_R = \frac{\sqrt{g'H}}{f}$$

typically $g' \simeq 0.03$

$\Rightarrow c_1 = \sqrt{g'H} \sim 2-3 \ \mathrm{m\,s^{-1}}$

and L_R internal $\lesssim \frac{L_R \text{ external}}{10}$

The shallow water equations, ~~so~~ or reduced gravity equations, ~~will~~ especially in linearized form (i.e. neglect advection or at most include uniform basic current) ~~will~~ appear throughout the literature as the basic mathematical tools for studying the oceans in general and shelf motions (barotropic and baroclinic) in particular.

Numerical models

In modern oceanography, the term "model" generally refers to a dynamic model in which we attempt to define the behavior of the system based on known physical principals, and to ultimately develop a predictive capability. While models can range in their complexity, they all share a common characteristic of simplifying or approximating reality. In any event, this leads to a set of pde's that are solved numerically. Furthermore, these models are deterministic.

In designing and building a model, one must go through a sequence of steps beginning with a clear definition of the problem we wish to solve. Subsequent steps are

- identify problem
- building a model or adapting an existing model
- simple or gross validation
- sensitivity analysis
- calibration (tuning)
- verification of predictive capability

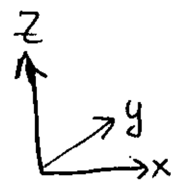
I cannot overemphasize the importance of data or observations for model development. This is specially crucial in operational models.

More often than not, today tendency is to adapt an existing model. In this case we run the danger of treating the model as a black box which can be dangerous. It is important to understand what is in the model, what it is doing, what its ~~capa~~ capabilities are, and what its limitations are. Otherwise one cannot rationally interpret and assess the results. This is very important if the model is to be used as an operational or predictive tool within the context of some type of management system.

As a starting point lets look at the 3 dimensional primitive equations derived in previous lectures (Boussinesq) (eg. as used in POM)

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(uu) + \frac{\partial}{\partial y}(vu) + \frac{\partial}{\partial z}(wu) - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + F_u$$

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(uv) + \frac{\partial}{\partial y}(vv) + \frac{\partial}{\partial z}(wv) + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + F_v$$

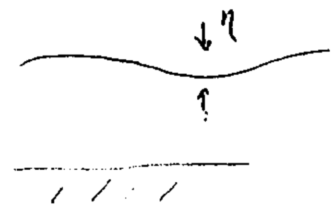


$$\frac{\partial T}{\partial t} + \frac{\partial}{\partial x}(uT) + \frac{\partial}{\partial y}(vT) + \frac{\partial}{\partial z}(wT) = F_T + \frac{Q}{\rho C_p}$$

$$\frac{\partial S}{\partial t} + \frac{\partial}{\partial x}(uS) + \frac{\partial}{\partial y}(vS) + \frac{\partial}{\partial z}(wS) = F_S$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{p}{\rho_0} = g(\eta - z) + \int_z^{\eta}\left(\frac{\rho - \rho_0}{\rho_0}\right) g\,dz$$



and if including other tracers

$$\frac{\partial \psi}{\partial t} + \frac{\partial}{\partial x}(u\psi) + \frac{\partial}{\partial y}(v\psi) + \frac{\partial}{\partial z}(w\psi) = F_\psi$$

~~[illegible]~~

Where $F_u, F_v, F_T, F_S, F_\psi$ ~~[illegible]~~ include all of the processes not explicitly represented by the mathematical formalism of the model (eg. mixing, sources, sinks, etc...). Often referred to as "subgrid scale" or "parameterized" ~~[illegible]~~ processes

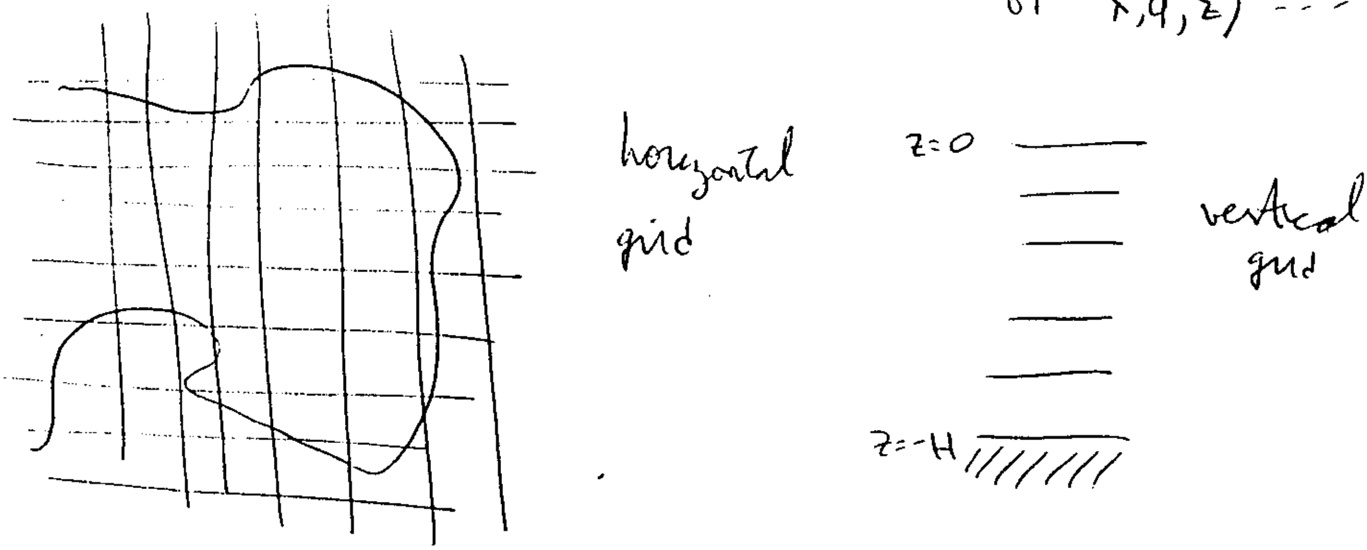
The overall objective is to design a numerical method of solution in which the continuous pde's are approximated by a set of algebraic equations which will give solutions at discrete "grid points"

Three most methods are

1. finite differencing
2. spectral (or semi-spectral)
3. finite element

where by far finite differencing is the most widely used.

First step is to "replace" the continuous fluid with a discrete grid system (could be x, y, z or λ, ϕ, z) ---



Finite differencing

For illustrative purposes consider the time and space derivatives

$$\frac{\partial F}{\partial t} \qquad \begin{matrix} n+1 \\ n \\ n-1 \end{matrix} \longrightarrow \frac{\partial F}{\partial t} \approx \begin{cases} \dfrac{F^{n+1} - F^{n}}{\Delta t} + \varepsilon_f & \text{forward} \\ \dfrac{F^{n+1} - F^{n-1}}{2\Delta t} + \varepsilon_c & \text{leap-frog (centered)} \end{cases}$$

$$\frac{\partial F}{\partial x} \qquad i-1 \quad i \quad i+1 \qquad \frac{\partial F}{\partial x} \begin{cases} \dfrac{F_{i+1} - F_{i}}{\Delta x} + \varepsilon_f & \text{forward (upwind)} \\ \dfrac{F_{i+1} - F_{i-1}}{2\Delta x} + \varepsilon_f & \text{centered} \end{cases}$$

Can show that for one-side (forward or backward) differencing, error $\varepsilon_f \sim O(\Delta t, \Delta x)$ while for centered differencing $\varepsilon \sim (\Delta t^2, \Delta x^2)$, i.e. centered differencing is more accurate.

There several factors that one must bear in mind when designing the finite diffence scheme — linear (computational) instability, nonlinear instability, and preservation of integral constraints.

<u>Linear (computational) instability</u> — refers to the amplification or damping properties of the solution to the finite difference equations. In an unstable scheme, the magnitude of the solution increases without bound with time, i.e.

the magnitude is amplified at each successive time step. This can generally be controlled by maintain the proper relationship between the grid spacing and time step. In particular, need to ensure something like $C \frac{\Delta t}{\Delta x} \leq 1$ where C is the phase speed of the fastest moving wave in the solution to the equations, i.e. the relationship between Δt and Δx must be able to resolve C. Note however, this does not necessarily mean the finite difference equations give the correct solution. This is only a guideline.

Let's consider the stability of various finite difference approximations to the simple wave equation

$$\frac{\partial F}{\partial t} + c\frac{\partial F}{\partial x} = 0 \quad \text{where } C = \text{constant}$$

centered in space - centered in time

$$F_i^{n+1} = F_i^{n-1} - r\left(F_{i+1}^n - F_{i-1}^n\right)$$

$$r = c\frac{\Delta t}{\Delta x}$$

stable for $r \leq 1$

centered in space forward in time

$$F_i^{n+1} = F_i^n - \frac{r}{2}\left(F_{i+1}^n - F_{i-1}^n\right)$$

unstable for all r

forward in space
forward in time
$$F_i^{n+1} = F_i^n - r\left(F_{i+1}^n - F_i^n\right)$$

depends on sign (direction) of c

$$c \begin{cases} >0 & \text{unstable (amplify)} \quad \text{downstream} \\ <0 & \text{stable (damp)} \quad \text{upstream} \end{cases}$$

All schemes considered until now were explicit, i.e. can march ahead in time based only on information from present and/or earlier time steps. $F^{n+1} = F^{n-1} + L(F^n)$

Implicit schemes depend also on future time step $F^{n+1} = F^{n-1} + L(F^{n+1}$

Consider

$$\frac{F_i^{n+1} - F_i^n}{\Delta t} = -c\left[\alpha\left(\frac{F_{i+1}^{n+1} - F_{i-1}^{n+1}}{2\Delta x}\right) + (1-\alpha)\left(\frac{F_{i+1}^{n} - F_{i-1}^{n}}{2\Delta x}\right)\right]$$

where $0 \leq \alpha \leq 1$

$\alpha = 1$ fully implicit - stable for all r

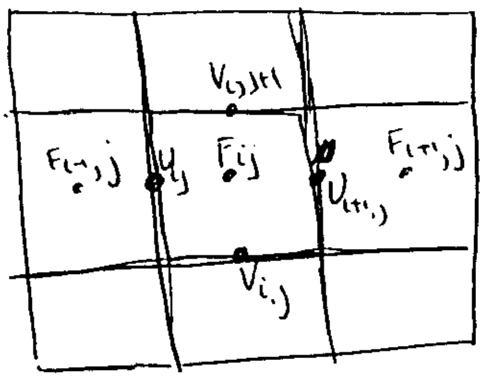
$\alpha = \frac{1}{2}$ Crank-Nicholson (semi-implicit) stable for all r

Nonlinear instability - cascading of energy to smallest resolved scales (related to aliasing). N. Phillips late 1950's. showed that simply reducing Δt and Δx (i.e. computational instability) was not enough.

Can be controlled by (1) including a "diffusion term" $A\nabla^2\psi$, (2) using a damping time scheme, or (3) occasional smoothing.

Preserving integral constraints - It is desirable to design finite difference schemes that preserve certain integral properties of the continuous equations (e.g. conservation of energy). This is desirable although not mandatory for obtaining numerical solutions whereas controlling linear and nonlinear instability is a necessary condition. Often accomplished by "proper" arrangement or ~~[illegible]~~ staggering of the dependent variables on the grid, specifically the way you write the the advection (Jacobian) operator. These can also control the problem of nonlinear instability.

For example, POM is based on the Arakawa-C scheme is as follows:



$F = T, S, \eta, \psi$

The velocity components are shifted $\frac{1}{2}$ grid space relative to the mass or scalar fields.

Scheme is quadratic conserving and has added benefit that ~~when~~ advection is second order accurate.

Vertical coordinate or "How do we represent topography?"

Consider the generalized vertical coordinate

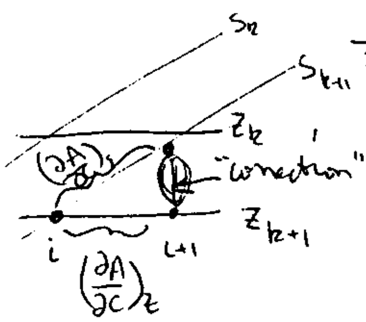
$$S = s(x, y, z, t)$$

where s is single valued and ~~[illegible]~~ monotonic.

We can then invert to get $z = z(x, y, s, t)$ and for any scalar A ~~[illegible]~~ we have

$$\left(\frac{\partial A}{\partial C}\right)_s = \left(\frac{\partial A}{\partial C}\right)_z + \frac{\partial A}{\partial z}\left(\frac{\partial z}{\partial C}\right)_s \qquad C = x, y, t$$

$$\equiv \left(\frac{\partial A}{\partial C}\right)_z + \frac{\partial s}{\partial z}\left(\frac{\partial z}{\partial C}\right)_s \frac{\partial A}{\partial s} \qquad \left(\text{since } \frac{\partial A}{\partial z} = \frac{\partial s}{\partial z}\frac{\partial A}{\partial s}\right)$$



If make all necessary substitutions we get a momentum equation that says

$$\frac{d\underset{\sim}{V}}{dt} + f\underset{\sim}{k} \times \underset{\sim}{V} = -\frac{1}{\rho_0}\nabla_s p - g\nabla_s z + F$$

where $\underset{\sim}{V} = (u, v)$ and $\frac{d}{dt} = \frac{\partial}{\partial t} + \underset{\sim}{V}\cdot\nabla_s + \dot{s}\frac{\partial}{\partial s}$

~~[illegible]~~ and $\dot{s}$ is the vertical velocity in the S - system.

Notice that main difference compared to Z-system is in the p.g.f.

~~Ultforma~~
We can now define our S-system so that the lower boundary is a coordinate surface in which case, the bc is no mass flux through the surface, i.e. $\dot{S}=0$. ~~Thus~~

If I use a terrain following coordinate (Phillips 1957)

$\sigma = \frac{z-\eta}{H+\eta}$ Then I can easily include topography.

Main drawback is the "σ-problem" over steep topography. i.e. pgf is now small difference between two large terms.

Sub grid scale physics - F_u, F_v, ...

Focus on the vertical mixing term

$$\frac{\partial}{\partial z}\left(K_m \frac{\partial u}{\partial z}\right), \quad \frac{\partial}{\partial z}\left(K_m \frac{\partial v}{\partial z}\right), \quad \frac{\partial}{\partial z}\left(K_T \frac{\partial T}{\partial z}\right), \quad \frac{\partial}{\partial z}\left(K_S \frac{\partial S}{\partial z}\right)$$

and the main question is how to represent the "eddy" diffusion coefficients. (i.e. closure scheme).

Simplest is to use K_m = constant
Higher order schemes determine K as a function of turbulence (e.g. Ri dependent, Mellor-Yamada)

Boundary Conditions

At vertical boundaries, minimally need w or $\dot{\sigma}$ but if including $\frac{\partial}{\partial z}(\kappa \frac{\partial}{\partial z})$ terms then need additional boundary conditions.

Top: external forcing - wind stress and heat flux

Issues to be considered - type, source, and quality of data (spatial & temporal resolution)

relaxation, flux, or "interactive" flux (feedback)

atmospheric model vs. reanalysis

, time mean (climatology) or synoptic variability

Bottom: bottom drag or friction.

Lateral boundaries - interaction with or influence of the "external" ocean on the open boundary (i.e. this is a mathematical boundary not a physical one).

Specify external information (T, S profiles, η)

Sommerfeld radiation condition

Nesting and/or two way interaction

Data assimilation - the realm of model $\Leftrightarrow$ data feedback

Two goals of data assimilation
1. To provide a "diagnostic" time series
2. To provide initial conditions for prediction / simulation

Two main components are the model and the objective analysis system. Purpose of the OA scheme is to take randomly distributed data and to interpolate onto a fixed grid

In general analysis is computed as a first guess field plus the weighted sum of data and / or model results. Computing the weights can be as simple as distance weighting or can be based on statistical properties of the data