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"Dynamical Oceanography II: Rotating Fluid Dynamics"

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DYNAMICAL OCEANOGRAPHY II
ROTATING FLUID DYNAMICS

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CHAPTER 1

ROTATING, HOMOGENEOUS, INCOMPRESSIBLE FLUIDS

1.1. The Equations Governing the Motion of a Fluid

In the first course of this series (Dynamical Oceanography I - Basic Fluid Dynamics, hereafter referenced as DOI), the governing equations which can be simplified for a homogeneous ($\rho = constant$), incompressible ($D\rho/Dt = 0$) fluid in a rotating, non-inertial frame of reference were derived. These equations are

$$\nabla \cdot \vec{u} = 0 \qquad (1.1)[DOI - 3.1.b]$$

and

$$\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{u} \qquad (1.2)[DOI - 3.56]$$

As a side product of these governing equations, an equation for the vorticity balance was also derived. Simplifying again for homogeneous, incompressible fluids this equation reads:

$$\frac{D\vec{\omega}_A}{Dt} = \vec{\omega}_A \cdot \nabla \vec{u} + \nu \nabla^2 \vec{\omega}_A \qquad (1.3)[DOI - 3.68]$$

where $\vec{\omega}_A = \vec{\omega} + 2\vec{\Omega}$ is the absolute vorticity. In relation to vorticity balance, an equation governing the circulation around a closed curve was also derived. For a homogeneous fluid this equation is

$$\frac{d\Gamma}{dt} = \nu \int_S \nabla^2 \vec{\omega} \cdot \hat{n}\, dS - 2\Omega \frac{dS_p}{dt} \qquad (1.4)[DOI - 3.80]$$

where Γ is the circulation around the closed curve C enclosing the surface S, and S_p is the projection of the surface S on the plane perpendicular to the angular velocity vector $\vec{\Omega}$. The above result stated the Kelvin's circulation theorem.

Finally, Bernoulli's theorem was stated for a *steady flow* of a homogeneous incompressible fluid, based on

$$\nabla H = \vec{u} \times \vec{\omega}_A + \nu \nabla^2 \vec{u} \qquad (1.5.a)[DOI - 3.83.a]$$

where

$$H = \frac{1}{2}\vec{u} \cdot \vec{u} + \frac{p'}{\rho} - \vec{g} \cdot \vec{x} - \frac{1}{2}(\vec{\Omega} \times \vec{x}) \cdot (\vec{\Omega} \times \vec{x}). \qquad (1.5.b)[DOI - 3.83.b]$$

Note that p' represents the *fluid pressure* in (1.5.b), while the notation p in equation(1.2) represents the *modified pressure*, replacing the last three terms of (1.5.b).

The above will be considered as the basic equations in this course, since our subject will only cover the motion of homogeneous, incompressible fluids with respect to a rotating frame of reference. We will elaborate the effects of rotation, leading to important new beahaviour in *geophysical fluids* (*i.e.* the fluid systems on a rotating earth). The effects due to density inhomogeneity of the fluid (stratification) are deliberately omitted in this course, and will be studied later in the third course in these series (Stratified Fluids).

1.2. The Restoring Effect of Coriolis Forces

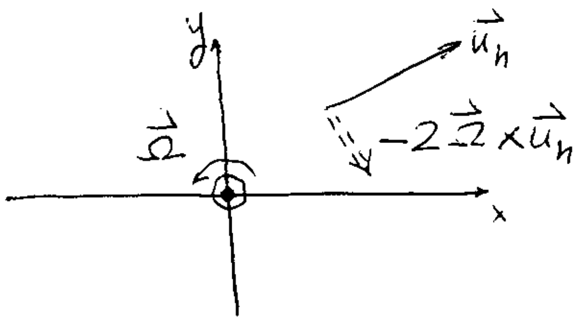
The effects of rotation are expressed by the Coriolis term (second term in equation 1.2), since the centrifugal force has already been included in the modified pressure p of equation (1.2). We will now demonstrate novel effects in rotating fluid motion arising due to this apparently minor modification of the governing equations. One of the most important of these effects is the elasticity created in rotating fluid motion. This effect is important, because the presence of a restoring mechanism allows particular types of wave motions to be supported.

1.2.1. Inertial Motion

To see the restoring mechanism of the Coriolis term in more detail, consider an inviscid fluid ($\nu = 0$) with vanishing pressure gradients ($\nabla p = 0$). Equation (1.2) becomes

$$\frac{D\vec{u}}{Dt} = -2\vec{\Omega} \times \vec{u}, \tag{1.6}$$

i.e. the fluid acceleration is balanced only by the restoring Coriolis force (per unit mass) $-2\vec{\Omega} \times \vec{u}$. Without loss of generality, we assume that the angular velocity vector is aligned with the z-axis in a Cartesian coordinate system (x, y, z) with unit vectors $(\hat{\imath}, \hat{\jmath}, \hat{k})$, *i.e.*, $\vec{\Omega} = \Omega\hat{k}$. Then, the only component of velocity $\vec{u} = (u, v, w)$ contributing to the right hand side of (1.6) is $\vec{u}_h = (u, v, 0)$ in the plane perpendicular to $\vec{\Omega} = \Omega\hat{k}$), so that $\vec{u}$ can be replaced by $\vec{u}_h$. The direction of the restoring force is at right angles to the lateral component of fluid velocity $\vec{u}_h$, and its sense is to the right of this vector:



For a uniform flow in infinite domain, the nonlinear advection terms can be neglected, so that (1.6) becomes

$$\frac{\partial \vec{u}_h}{\partial t} + 2\Omega \hat{k} \times \vec{u}_h = 0 \tag{1.7}$$

Cross multiplying with $-2\Omega\hat{k}$ and adding with the time derivative of the above equation yields

$$\frac{\partial^2 \vec{u}_h}{\partial t^2} + (2\Omega)^2 \vec{u}_h = 0 \tag{1.8}$$

This equation has sinusoidal solutions (harmonic motion), analogous to a spring-mass system. For example, we can use the initial conditions:

$$\vec{u}_h(0) = \vec{U}_0 \tag{1.9.a}$$

$$\frac{\partial \vec{u}_h}{\partial t}(0) = -2\Omega\hat{k} \times \vec{U}_0 \tag{1.9.b}$$

The solution follows as

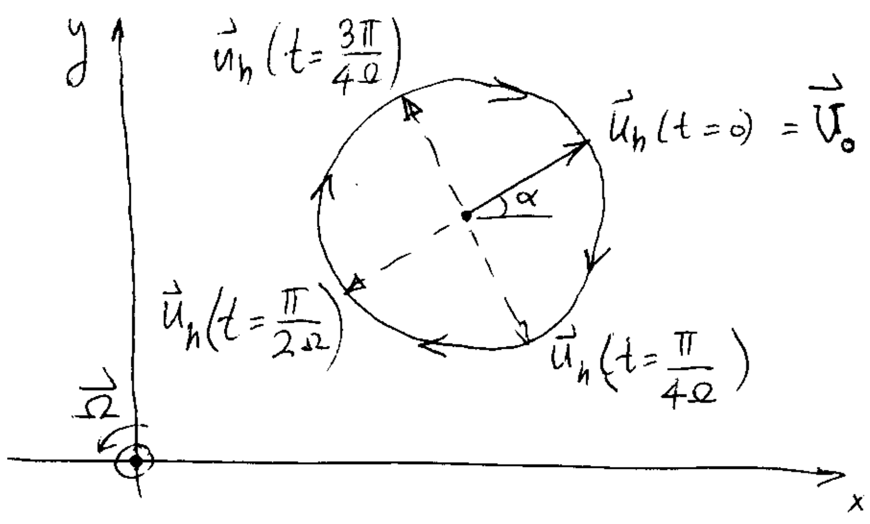
$$\vec{u}_h = \vec{U}_0 \cos 2\Omega t - \hat{k} \times \vec{U}_0 \sin 2\Omega t. \tag{1.10}$$

The components (u, v), of the velocity $\vec{u}_h = (u, v, 0)$ are

$$u = U_0 \cos(2\Omega t - \alpha) \tag{1.11.a}$$

$$v = -U_0 \sin(2\Omega t - \alpha) \tag{1.11.b}$$

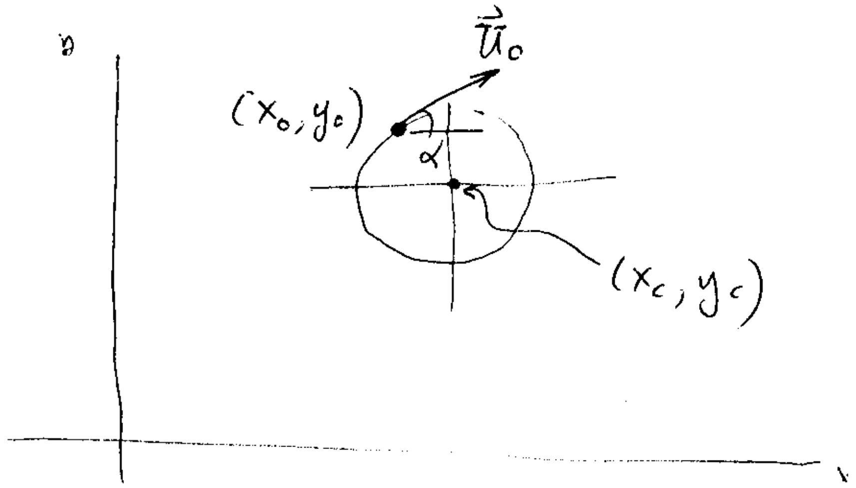
where α is the angle that the initial velocity vector $\vec{U}_0$ makes with the x-axis, and $U_0 = ||\vec{U}_0||$.



For small amplitude motions, the displacements (x, y) of a material point (fluid particle) with respect to its initial position (x_0, y_0) can be obtained by integrating (1.11.a,b) with respect to time:

$$(x - x_0) = \frac{U_0}{2\Omega}[\sin(2\Omega t - \alpha) + \sin\alpha] \tag{1.12.a}$$

$$(y - y_0) = \frac{U_0}{2\Omega}[\cos(2\Omega t - \alpha) - \cos\alpha] \tag{1.12.b}$$



These can be combined by eliminating t between (1.12.a,b), to yield

$$(x - x_c)^2 + (y - y_c)^2 = \left(\frac{U_0}{2\Omega}\right)^2 \tag{1.13}$$

where x_c, y_c are appropriate constants determined from (1.12).

In the above solutions, both the sense of rotation of the velocity vector $\vec{u}_h$ and the trajectory $(x(t), (y(t))$ in (1.12) are in the *clockwise* direction. Each particle rotates clockwise, and comes to its initial position after one *inertial period* $T_I = \frac{2\pi}{2\Omega} = \frac{\pi}{\Omega}$. This *inertial motion* demonstrates the restoring effects of the Coriolis force. Because it arises due to the inertia, without any external (surface or body) forces, this motion is considered to be a free oscillation in a rotating fluid, corresponding to a natural frequency of 2Ω.

1.2.2. "Elasticity" in Rotating Fluids

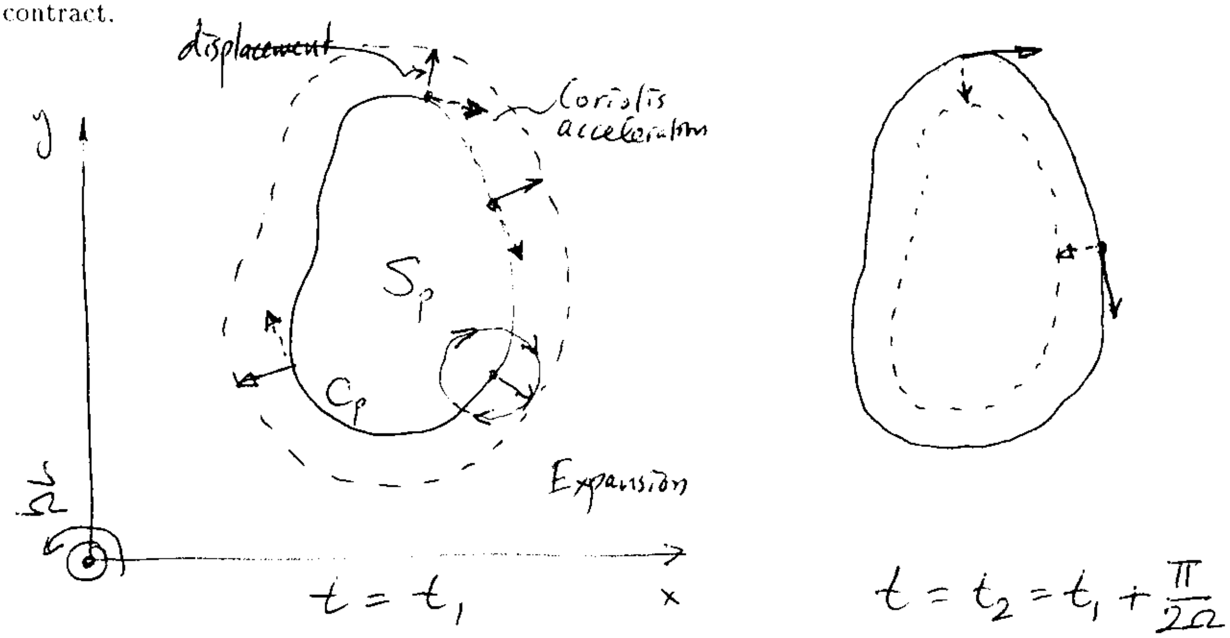
The previous example on inertial motion illustrates the restoring mechanism rotating fluids. Since particles displayed return to their initial positions after one characteristic (inertial) period, the fluid acts as if it has some special form of elasticity, whereby particles are forced into closed circular trajectories.

To further demonstrate the elastic behavior, consider a closed material curve C whose projection in the lateral plane ($\perp$ *to* $\vec{\Omega}$) is C_p:

Suppose that a motion is generated in the fluid such that it will cause a positive rate of expansion in the lateral plane, *i.e.* with

$$\nabla_h \cdot \vec{u}_h = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} > 0.$$

This outward motion along the material curve C is going to give rise to Coriolis forces in the clockwise direction along the curve since the induced force is to the right hand side of the motion (in the northern hemisphere). This is also seen exactly by equation (1.4) (Kelvin's theorem) since an increase in the projected area S_p enclosed by curve C leads to a negative contribution to the circulation. On the other hand, clockwise motion along the material curve will give rise to Coriolis forces in the inward direction (*i.e.* with $\nabla_h \cdot \vec{u}_h < 0$) then the material line C will then tend to contract.



Thus the fluid is seen to resist elastically to any motion that would cause displacement of fluid elements leading to a change in the projection of an are enclosed by a curve of such elements.

The relative importance of Coriolis effects is determined by the inverse of the Rossby number $R_0 = U_0/L_0\Omega_0$ measuring the ratio of Coriolis terms to other inertial terms (cf. equation [DOI - 4.4]). When $Ro << 1$, the elasticity effect of rotation is expected to be dominant.

1.2.2. Steady Flow at Small Rossby Number (Geostrophic Flow)

When the flow is *steady* ($\frac{\partial \vec{u}}{\partial t} = 0$), *inviscid* ($\nu = 0$), and if the Rossby number $Ro << 1$, then the nonlinear term $\vec{u} \cdot \nabla \vec{u}$ is negligibly small compared to the Coriolis term $2\vec{\Omega} \times \vec{u}$). In this limit the momentum equation (1.2) becomes

$$2\vec{\Omega} \times \vec{u} = -\frac{1}{\rho}\nabla p \tag{1.14}$$

Without loss of generality we can let $\vec{\Omega} = \Omega\hat{k}$ be aligned with the z-axis of a Cartesian coordinate system (x, y, z) with unit vectors $(\hat{\imath}, \hat{\jmath}, \hat{k})$, such that

$$2\Omega\hat{k} \times \vec{u} = -\frac{1}{\rho}\nabla p. \tag{1.15}$$

The continuity equation

$$\nabla \cdot \vec{u} = 0 \tag{1.16}$$

complements the momentum equation. In principle (1.15) and (1.16) should be sufficient to solve for the unknowns $\vec{u}$ and p. However, it turns out that the so-called *geostrophic motion* by these equations have some very special characteristics.

First, by taking the curl of (1.15) we can show that

$$\nabla \times \hat{k} \times \vec{u} = -\frac{1}{2\Omega\rho}\nabla \times \nabla p \equiv 0 \tag{1.17}$$

by virtue of (DOI-1.27.i). Then, by making use of (DOI-1.27.d) the *l.h.s.* is

$$\nabla \times \hat{k} \times \vec{u} \equiv \hat{k}\nabla \cdot \vec{u} - \hat{k} \cdot \nabla\vec{u} = 0, \tag{1.18}$$

of which, the first term on the *r.h.s.* vanishes by (1.16). Then (1.18) states that

$$\hat{k} \cdot \nabla\vec{u} = \frac{\partial \vec{u}}{\partial z} = 0 \tag{1.19}$$

expressing the fact that the velocity field has to be two-dimensional; $\vec{u} = \vec{u}(x, y)$ only. On the other hand, (1.15) dictates that

$$2\Omega\hat{k} \cdot (\hat{k} \times \vec{u}) + \frac{1}{\rho}\hat{k} \cdot \nabla p = 0 \tag{1.20}$$

so that $p = p(x, y)$ only.

The above results, namely that none of the flow variables depend on the vertical coordinate z alone, indicates that the flow is essentially two-dimensional and occurs in the (x, y) plane.

By virtue of the above results, (1.15) and (1.16) can equivalently be written as

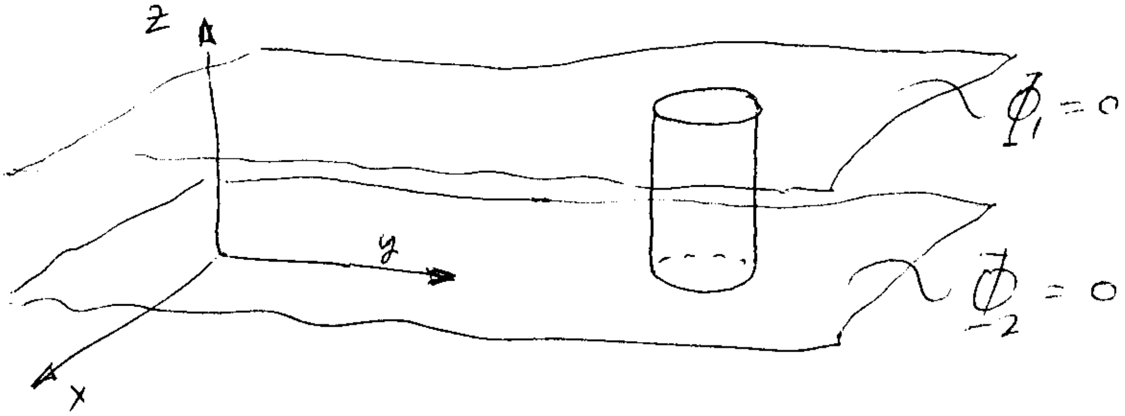
$$2\Omega\hat{k} \times \vec{u}_h = -\frac{1}{\rho}\nabla_h p \tag{1.21}$$

$$\nabla_h \cdot \vec{u}_h = 0, \tag{1.22}$$

where $\vec{u}_h = (u, v, 0)$ is the velocity vector in the lateral plane and $\nabla_h = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0\right)$ is the gradient operator in the same plane.

The result, that the rotating flow at the limit $Ro \to 1$ must be two dimensional, is known as the *Taylor-Proudman theorem.* Consider the flow bounded by two rigid surfaces

$$\Phi_1 = z - f_1(x,y) = 0, \Phi_2 = z - f_2(x,y) = 0. \qquad (1.23.a, b)$$



Since the flow is two-dimensional, any fluid column that is initially vertical will remain vertical. However, while moving, the net height of the column h would have to adjust itself to the distance of separation between the two surfaces, requiring that

$$\frac{D\Phi_1}{Dt} = 0, \frac{D\Phi_2}{Dt} = 0 \qquad (1.24)$$

Since Φ_1, Φ_2 are material surfaces according the (DOI-1.41). Substituting (1.23.a,b):

$$\frac{D\Phi_1}{Dt} = \vec{u} \cdot \nabla\Phi_1 = \vec{u}_h \cdot \nabla_h f_1 - w = 0, \text{ on } z = f_1 \qquad (1.25.a)$$

$$\frac{D\Phi_2}{Dt} = \vec{u} \cdot \nabla\Phi_2 = \vec{u}_h \cdot \nabla_h f_2 - w = 0, \text{ on } z = f_2 \qquad (1.25.b)$$

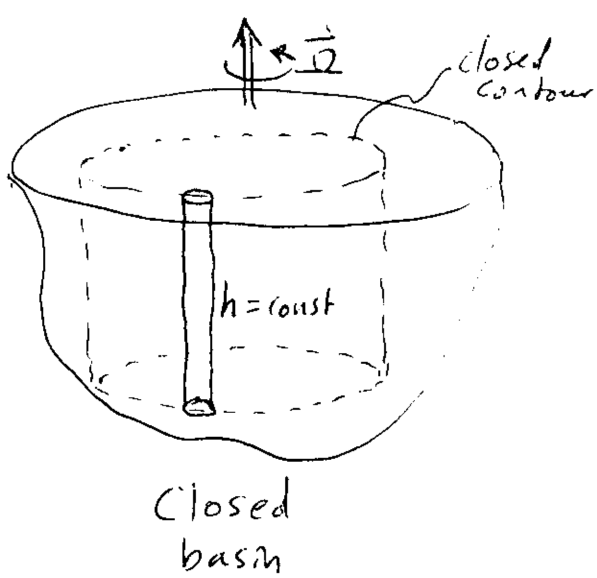
Subtracting(1.25.b) from (1.25.a) and since $\vec{u}_h = \vec{u}_h(x, y)$ only, we have

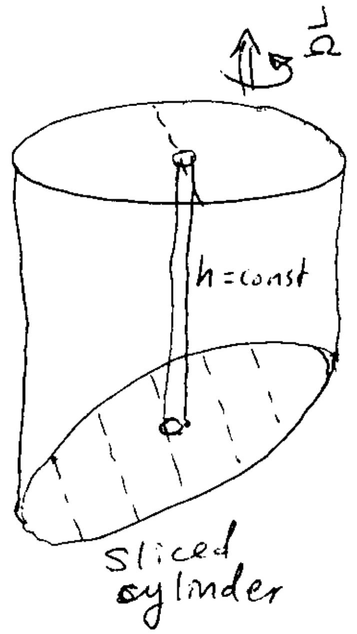
$$\begin{aligned} w\,|_{z=f_1} - w\,|_{z=f_2} &= \vec{u}_h \cdot \nabla(f_1 - f_2) \\ &= \vec{u}_h \cdot \nabla h \\ &= \frac{Dh}{Dt} \end{aligned} \qquad (1.26)$$

On the other hand, since $w = w(x, y)$ only ($\frac{\partial w}{\partial z} = 0$, the vertical velocity w at the upper surface can not be different from that at the lower surface, *i.e.* the *l.h.s.* of (1.26) must vanish, so that

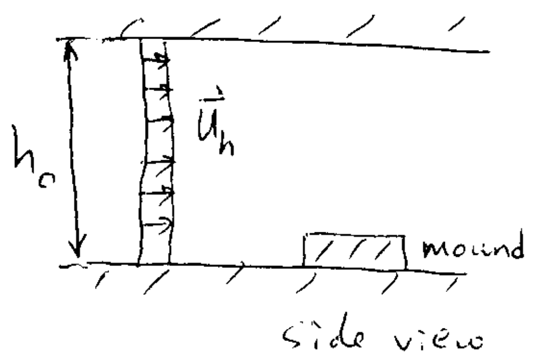
$$\frac{Dh}{Dt} = 0 \qquad (1.27)$$

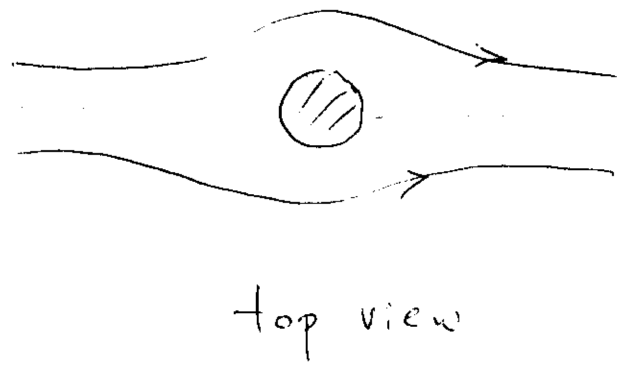
This result indicates that any moving fluid column must preserve its height in geostrophic motion, *i.e.* the fluid column moves along a very special trajectory that would make h=constant. In a closed container, this would mean that fluid columns could only move along closed contours having h=const, if such closed contours exist. If there are no such closed contours, geostrophic motion would not be possible





If a mound was placed in an otherwise constant depth motion, would bypass the mound.

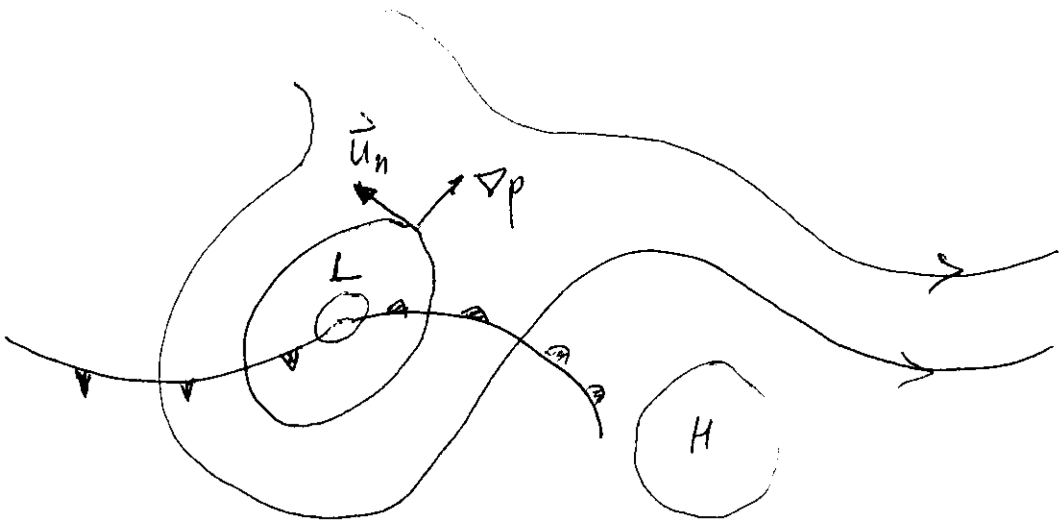




Such columns which are identified with their constant thicknesses in geostrophic motion are called Taylor columns, since *Taylor* was the first to discover them. The flow modelled by equations (1.15) and (1.16) [equivalently (1.21) and (1.22)] is called *geostrophic flow*. It is a steady approximation to the governing equations for inviscid, homogeneous, incompressible rotating fluids in the limit $Ro \to 0$.

Since we have shown that the statements in equivalent to the statement in (1.22), one of the two equations is *redundant*, *i.e.* it is dependent on the other equation and does not supply additional information. Therefore, it is impossible to solve the two equations simultaneously, illustrating the fact that geostrophic flow is indeterminate or degenerate.

Since there are two unknowns $\vec{u}_h$ and p in (1.21) and (1.22) it is only possible to infer one of these fields from given values of the other field. For example if pressure is given we can infer the velocity distribution, or vice versa. This is what is usually done in interpreting weather charts or interpreting hydrographic fields in oceanohraphy.



On the other hand, it can immediately be seen that *geostrophic flow* is in fact degenerate or indeterminate, *i.e.* while such a flow would exist, it is impossible to obtain a "solution" to equations. This is seen if pressure is eliminated from (1.21), by first rearranging such that

$$2\Omega \vec{u}_h = \frac{1}{\rho}\hat{k} \times \nabla p$$

then taking divergence of both sides

$$\nabla \cdot \vec{u}_h = \frac{1}{2\Omega\rho}\nabla \cdot (\hat{k} \times \nabla P).$$

Utilizing [DOI-1.27.c and 1.27.i], the above equation is equivalent to

$$\nabla \cdot \vec{u}_h = -\frac{1}{2\Omega\rho}\hat{k} \cdot (\nabla_h \times \nabla_h p) \equiv 0, \tag{1.27}$$

i.e. the same thing as equation (1.22).

Equation (1.21) can be put into the form

$$\begin{aligned} \vec{u}_h &= \frac{1}{2\Omega\rho}\hat{k} \times \nabla_h p \\ &= \hat{k} \times \nabla_h \left(\frac{p}{2\Omega\rho}\right), \end{aligned} \tag{1.28}$$

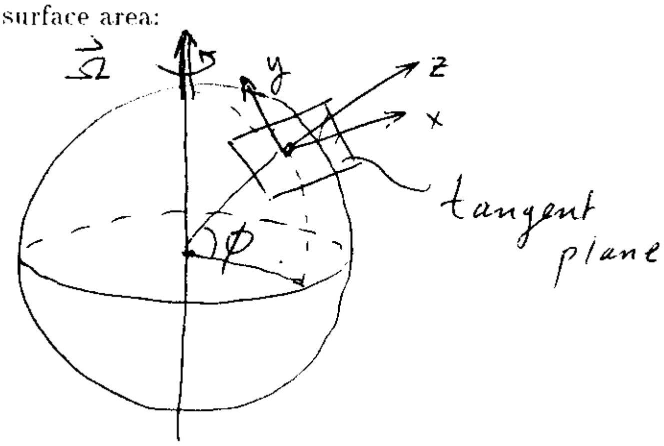
so that the velocity vector is perpendicular to the pressure gradient, and its sense is such that it takes high pressure to its right hand side. In fact, $\Psi = (\frac{p}{2\Omega\rho})$ acts as the stream function for the two dimensional flow; comparing (1.28) with [DOI-2.34]. Around low-pressure centers L, the flow is *cyclonic*, *i.e.* it rotates in the *anti-clockwise* sense; and around high pressure centers H it is *anti-cyclonic* (*i.e.*, rotation in clockwise sense). We must finally note that, to remove the geostrophic indeterminacy, we must include other effects in the dynamics, such as friction, unsteady variations, etc. The inclusion of these effects can be in the form of small corrections if $Ro << 1$, but nevertheless they would render the equations determinate.

CHAPTER 2

SHALLOW WATER THEORY

2.1. The Tangent Plane Approximation

Since we are dealing with motions on a spherical earth we need to use equations (1.1) and (1.2) written in spherical coordinates.However, these are often complicated, and therefore their solutions would be difficult. In order to overcome this difficulty we often employ the "tangent plane " approximation. We envision the motions to take place on a plane that is tangent to the earth near the region of interest. This is a feasible approximation if the horizontal domain of interest is only a small portion of the earth's surface area:



The selected plane is tangent to the earth at points O, which has a latitude angle of ϕ.

The Cartesian coordinates on this tangent plane are chosen such that x, y axes point in the east and north directions,and the z-axis points in the vertical direction (perpendicular to the tangent plane). We can conveniently decompose the velocity vector $\vec{u} = (u, v, w)$ and the angular velocity vector $\vec{\Omega} = (\Omega_x, \Omega_y, \Omega_z)$ as follows:

$$\vec{u} = \vec{u}_h + w\hat{k} \qquad (2.1.a)$$

$$\vec{\Omega} = \vec{\Omega}_h + \Omega_z\hat{k} \qquad (2.1.b)$$

where

$$\vec{u}_h = (u, v) \qquad (2.1.c)$$

$$\vec{\Omega}_h = (\Omega_x, \Omega_y) \qquad (2.1.d)$$

Now the Coriolis terms in equation(1.2) can be written as

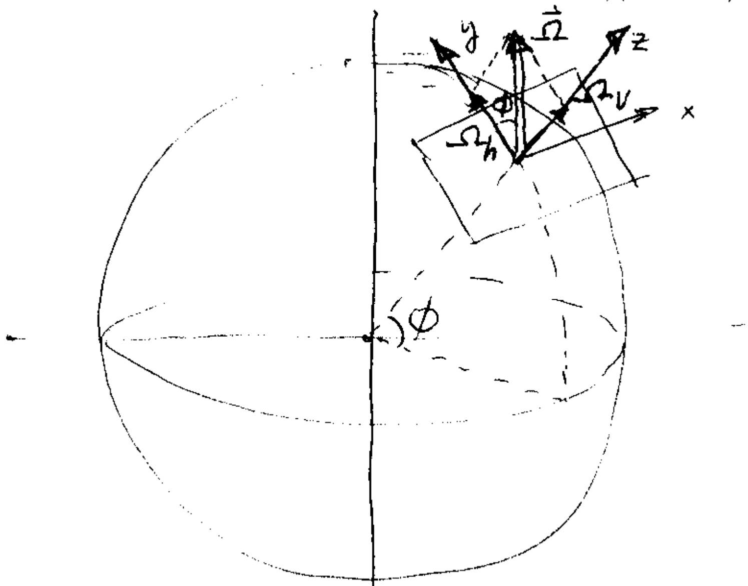
$$\begin{aligned} 2\vec{\Omega} \times \vec{u} &= 2(\vec{\Omega}_h + \Omega_z\hat{k}) \times (\vec{u}_h + w\hat{k}) \\ &= 2\Omega_z\hat{k} \times \vec{u}_h + 2\vec{\Omega}_h \times \vec{u}_h + 2\vec{\Omega}_h \times \hat{k}w \end{aligned} \qquad (2.2)$$

Next, we note that because of the selected orientation of axes,

$$\vec{\Omega}_h = \Omega_y \hat{\jmath} = \Omega(\cos\phi)\hat{\jmath}\Omega_z = \Omega\sin\phi$$

where $\Omega = \mid \vec{\Omega} \mid$, so that (2.2) becomes

$$\begin{aligned} 2\vec{\Omega}\times\vec{u} &= 2\Omega_z\hat{k}\times\vec{u}_h + 2\Omega_y\hat{\jmath}\times\vec{u}_h + 2\Omega_y\hat{\jmath}\times\hat{k}w \\ &= (2\Omega\sin\phi)\hat{k}\times\vec{u}_h + (2\Omega\cos\phi)(w\hat{\imath} - u\hat{k}) \end{aligned} \tag{2.4}$$



We can now substitute this form into equations (1.2). First we utilize the definition of the "del" operator decomposed into horizontal and vertical components:

$$\nabla = \nabla_h + \hat{k}\frac{\partial}{\partial z} \tag{2.5}$$

where

$$\nabla_h = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y},$$

to write the horizontal and vertical components of (1.2):

$$\frac{\partial\vec{u}_h}{\partial t} + \vec{u}_h\cdot\nabla_h\vec{u}_h + w\frac{\partial\vec{u}_h}{\partial z} + (2\omega\sin\phi)\hat{k}\times\vec{u}_h + (2\omega\cos\phi)w\hat{\imath} = -\frac{1}{\rho}\nabla_h p + \nu\left(\nabla_h^2\vec{u}_h + \frac{\partial^2\vec{u}_h}{\partial z^2}\right) \tag{2.6.a}$$

and

$$\frac{\partial w}{\partial t} + \vec{u}_h \cdot \nabla_h w + w\frac{\partial w}{\partial z} - (2\omega \cos\phi)u = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\nabla_h^2 w + \frac{\partial^2 w}{\partial z^2}\right) \tag{2.6.b}$$

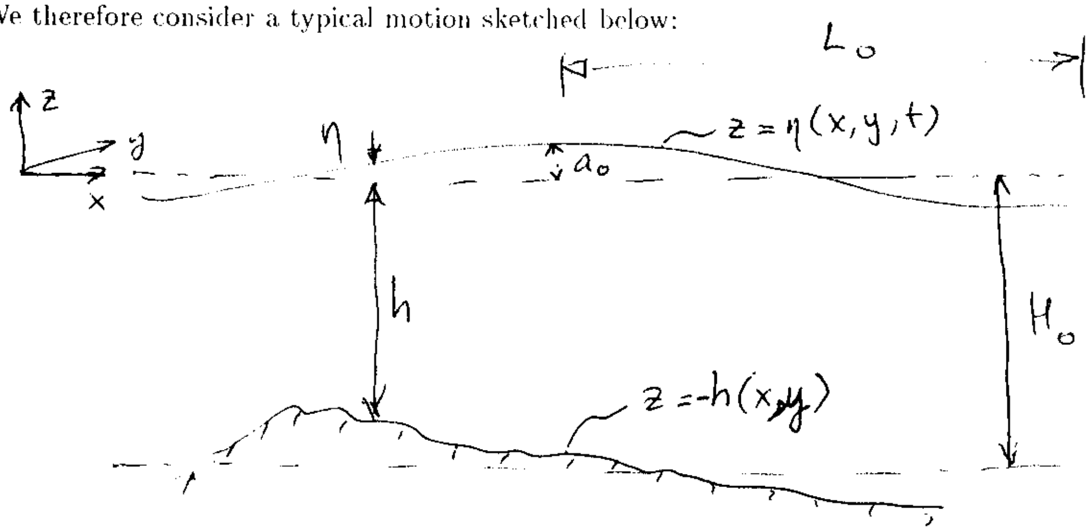
The continuity equation (1.1) can also be written as

$$\nabla_h \cdot \vec{u}_h + \frac{\partial w}{\partial z} = 0 \tag{2.7}$$

The equations (3.6.a,b) and (3.7) constitute the governing equations written in tangent plane coordinates.

We can therefore make use of this fact in simplifying our equations. The so-called shallow-water approximation $(\lambda \ll 1)$ arises from the fact that the oceans and the atmosphere have essentially a small thickness as compared to the earth's radius.

We therefore consider a typical motion sketched below:



The lower surface $z = -h(x,y)$ describes the bottom topography in the ocean or (*i.e.*, the sea-bottom topography in the ocean or the earth's surface topography in the case of the atmosphere). The upper surface $z = \eta(x,y,t)$ describes the displacement of the sea-surface (or an equivalent,imaginary "tropospheric upper surface" in the atmosphere) as a result of the motion. The magnitude of this displacement is characterized by the scale a_0, which is typically small as compared to the total depth H_0:

2.2. Shallow Water Approximations

2.2.1. Scaling of the equations (A cursory Examination)

A common feature of geophysical motions (*i.e.*, the oceanic and atmospheric flows) is that they are too often characterized by horizontal length scales that are much larger than the vertical scales. For example,the largest depth to be found in the world's oceans is about 10 km. Similarly the thickness of the troposphere (the lowest atmospheric layer in which most of the weather processes take place) is also of the order of 10 km. The horizontal scale of a typical domain of study extends from 100 km (meso-scale) to 1000 km (synoptic scale) or more. Therefore if L_0 represents the horizontal length scale and H_0 represents the vertical length scale, the dimensionless ratio λ (aspect ratio) is typically:

$$\lambda = \frac{H_0}{L_0} = O(10^{-1}) - O(10^{-2}) \ll 1, \tag{2.8}$$

$$\mu = \frac{a_0}{H_0} \ll 1. \tag{2.9}$$

In order to non-dimensionalize equations (2.6.a,b) and (2.7) for a cursory examination, we choose the following scales :

$$\begin{aligned} (x, y) &\sim L_0 \\ z &\sim H_0 \\ t &\sim T_0 \\ \vec{u}_h = (u, v) &\sim U_0 \\ w &\sim \left(\frac{a_0}{L_0}\right) U_0 \\ p &\sim \rho g H_0 \end{aligned} \tag{2.10.a - f}$$

The vertical velocity scale is selected as $(a_0/L_0)\,U_0$ since the vertical motion should be proportional to the displacement of the upper surface $z = \eta(x, y, t)$, whose scale is a_0 To give it correct dimensions we divide by the time scale L_0/U_0. The pressure scale is selected as $\rho g H_0$, since the symbol p represents the modified pressure.

$$p = p' - \rho \vec{g} \cdot \vec{x} - \frac{1}{2}\rho \mid \vec{\Omega} \times \vec{x} \mid^2 \tag{2.11}$$

(ref. equation [DOI-3.55]), where p' stands for fluid pressure. Since $\vec{g} = -g\hat{k}$, $p = O(\rho g z)$.

If we use the scales (2.10.a.f) in equations (2.6.a,b) and (2.7), the non-dimensional equations can be written as

$$Ro\left(\epsilon_T \frac{\partial \vec{u}_h}{\partial t} + \vec{u}_h \cdot \nabla \vec{u}_h + \mu w \frac{\partial \vec{u}_h}{\partial z}\right) + (2\sin\phi)\hat{k} \times \vec{u}_h + \mu\lambda(2\cos\phi)w$$

$$= -s\lambda \nabla_h p + E^2 \left(\lambda^2 \nabla_h^2 \vec{u}_h + \frac{\partial^2 \vec{u}_h}{\partial z^2}\right) \tag{2.12a}$$

$$\mu\lambda Ro\left(\epsilon_T \frac{\partial w}{\partial t} + \vec{u}_h \cdot \nabla_h w + \mu w \frac{\partial w}{\partial z}\right) - (2\cos\phi)u = -\frac{s\partial P}{\partial z} + \mu\lambda E^2 \left(\lambda^2 \nabla_h^2 w + \frac{\partial^2 w}{\partial z^2}\right) \tag{2.12.b}$$

$$\nabla_h \cdot \vec{u}_h + \mu \frac{\partial w}{\partial z} = 0 \tag{2.13}$$

where the variables are all non-dimensional and the following non-dimensional parameters are defined:

$$\begin{aligned} \mu = \frac{a_0}{H_0}, \quad \lambda = \frac{H_0}{L_0}, \quad \epsilon_T = \frac{L_0}{U_0 T_0} \\ Ro = \frac{U_0}{\Omega L_0}, \quad E^2 = \frac{\nu}{\Omega H_0^2}, \quad s = \frac{g}{U_0 \Omega}. \end{aligned} \tag{2.14.a – f}$$

Now we can use these non-dimensional parameters to estimate the relative orders of magnitude of the various terms. We know that $\mu \ll 1$ and $\lambda \ll 1$. The parameter ϵ_T is the ratio of the length scale L_0 to particle excurtion length $U_0 T_0$ and is usually $O(1)$. We can also assume that the Rossby number $Ro = O(1)$ or smaller. The Ekman number E can also be estimated as $E = O(1)$ or smaller, by using typical values. On the other hand, the parameters can be estimated as follows:

$$\begin{aligned} S = \frac{g}{U_0 \Omega} &= \frac{g H_0}{U_0^2} \cdot \frac{U_0}{\Omega L_0} \cdot \frac{L_0}{H_0} \\ &= \frac{1}{F^2} \cdot Ro \cdot \frac{1}{\lambda} \end{aligned} \tag{2.15}$$

Where $F = U_0/\sqrt{g H_0}$ is the Froude number. The Froude number is typically $O(1)$ or smaller. We therefore find that since $\lambda \ll 1$.

$$S = \frac{Ro}{F^2} \frac{1}{\lambda} \gg 1. \tag{2.16}$$

With these typical estimates of the parameters, it can be seen that in equation (2.12.b) the dominating term is the vertical gradient of pressure. Neglecting all other terms, we therefore have

$$\frac{\partial p}{\partial z} = 0, \tag{2.17}$$

i.e. the modified pressure p is independent of the vertical coordinate, $p = p(x, y, t)$ only.

If the same scales are used for a cursory examination of equation (2.12.a), it is first seen that the coefficient of the pressure gradient term

$$S\lambda = O(1) \tag{2.18}$$

by virtue of (2.16). All the other terms are also of $O(1)$, except the term arising due to the horizontal component of earth's angular speed which is multiplied by $\mu\lambda \ll 1$. Therefore this term is neglected. If we also neglect the $w\frac{\partial \vec{u}_h}{\partial z}$ term ($\mu \ll 1$) and the $\nabla_h^2 \vec{u}_h$ term ($\lambda^2 \ll 1$), the equation becomes:

$$Ro\left(\epsilon_T\frac{\partial \vec{u}_h}{\partial t}+\vec{u}_h\cdot\nabla_h\vec{u}_h\right)+(2\sin\phi)\hat{k}\times\vec{u}_h=-S\lambda\nabla_h p+E^2\frac{\partial^2\vec{u}_h}{\partial z^2} \tag{2.19}$$

the dimensional equivalent of which is

$$\frac{\partial \vec{u}_h}{\partial t}+\vec{u}_h\cdot\nabla_h\vec{u}_h+(2\Omega\sin\phi)\hat{k}\times\vec{u}_h=-\frac{1}{\rho}\nabla_h p+\nu\frac{\partial^2\vec{u}_h}{\partial z^2}. \tag{2.20}$$

The obvious result of equation (2.19) is that since $p = p(x, y, t)$ only (cf. equation 2.17) then $\vec{u}_h = \vec{u}_h(x, y, t)$ only (if frictional terms are neglected). The situation similar to that found in geostrophic flow, and the *flow is essentially two-dimensional.*

Finally we can observe that, to the same approximation (2.13) becomes

$$\nabla_h\cdot\vec{u}_h=0 \tag{2.21}$$

which implies that

$$\frac{\partial w}{\partial z}=0 \tag{2.22}$$

and therefore w is also independent of depth $w = w(x, y, t)$.

We have shown that the *modified* pressure is independent of z (cf. equation 2.17). The actual fluid pressure can be found from (2.11). First noting that we can write *modified gravity* or *gravitation* (cf. [DOI-3.52]) as

$$\vec{g}'=\vec{g}-\vec{\Omega}\times\Omega\times\vec{x} \tag{2.23}$$

we can write (2.11) as

$$\begin{aligned} p&=p'+\rho\left(\vec{g}\cdot\vec{x}+\frac{1}{2}(\vec{\Omega}\times\vec{x})\cdot(\vec{\Omega}\times\vec{x})\right)\\ &=p'+\rho\left(\vec{g}\cdot\vec{x}-\frac{1}{2}\Omega\times(\Omega\times\vec{x})\cdot\vec{x}\right)\\ &=p'+\rho\vec{g}'\cdot\vec{x}\end{aligned} \tag{2.24}$$

Now the tangent plane is actually perpendicular to the gravitation vector $\vec{g}'$. In fact the difference between the $\vec{g}$ and $\vec{g}'$ vectors is only minor, arising due to centrifugal forces. If the earth had uniform density, it would take the form of an ellipsoid where the tangent plane would be exactly perpendicular to the $\vec{g}'$ vector. In this case, substituting

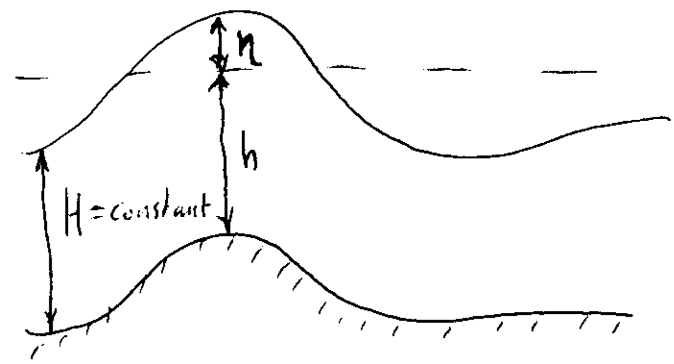
$$\begin{aligned}\vec{g}'&=-g'\hat{k}\\ \vec{x}&=x\hat{\imath}+y\hat{\jmath}+z\hat{k}\end{aligned} \tag{2.25}$$

in (2.24) gives

$$p = p' - \rho g' z. \tag{2.26}$$

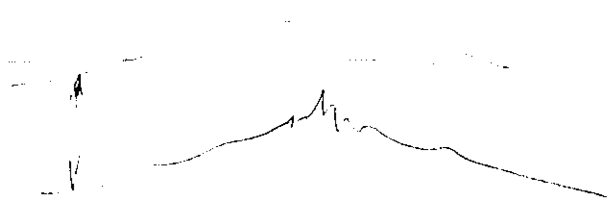
By virtue of (2.17) the modified pressure $p = p(x, y, t)$, and therefore p is constant with respect to z. Equation (2.26) is said to express *hydrostatic pressure*, since the pressure *at any horizontal position* (x, y) and *at any instant* t depends on z as if it was for a static fluid. Note that in this approximation, $\vec{u}_h = \vec{u}_h(x, y, t)$, $p = p(x, y, t)$, $w = w(x, y, t)$, the fluid motion is essentially two-dimensional and bears much similarity to the characteristics of geostrophic motion. Fluid columns which are initially vertical remain vertical more like Taylor columns. Since $\frac{\partial w}{\partial z} = 0$ the total height of these columns would not change. A fluid column moving over topography (if it does) would therefore adjust its surface elevation such that

$$H \equiv h + \eta = \text{constant}$$



Since the upper surface is adjusted accordingly it is not necessarily implied that the fluid should follow constant bottom depth (h) contours as it would in geostrophic flow with a rigid upper surface.

It seems that the above approximations are in fact too rigid and will be somewhat relaxed in later sections. However, in spite of the excessive rigidity of the present approximation, some of its features have been observed in the ocean. In recent years satellite altimetry methods have allowed the measurement of the ocean surface elevation from space. It has been often found that the surface of the ocean takes almost the same shape as the underlying topography especially in the mid-ocean regions. It is not necessarily true that $H = h + \eta = constant$, but nevertheless the topography is often "impressed" on the sea surface.



2.2.2. Continuity of Surface Forces (Dynamic Boundary Conditions)

We will consider an element of the upper surface at $z = \eta(x, y, t)$ and investigate the continuity of surface forces across this surface. In short, we should insist that the surface stress $\vec{\Sigma}$ (force per unit area of the upper surface) be continuous across the surface

$$\vec{\Sigma}\mid_{z=\eta^-} = \vec{\Sigma}\mid_{z=\eta^+} \qquad (2.27)$$

where $\eta^- = \eta - \delta$, $\eta^+ = \eta + \delta$, such that $\delta \to 0$; *i.e.* the surface stress just above the surface should be balanced by that just below. We had seen in DOI Section 2.4 that the body forces could be neglected if the fluid volume considered as infinite small in size. Another way to write (2.27) is to state it as the jump condition

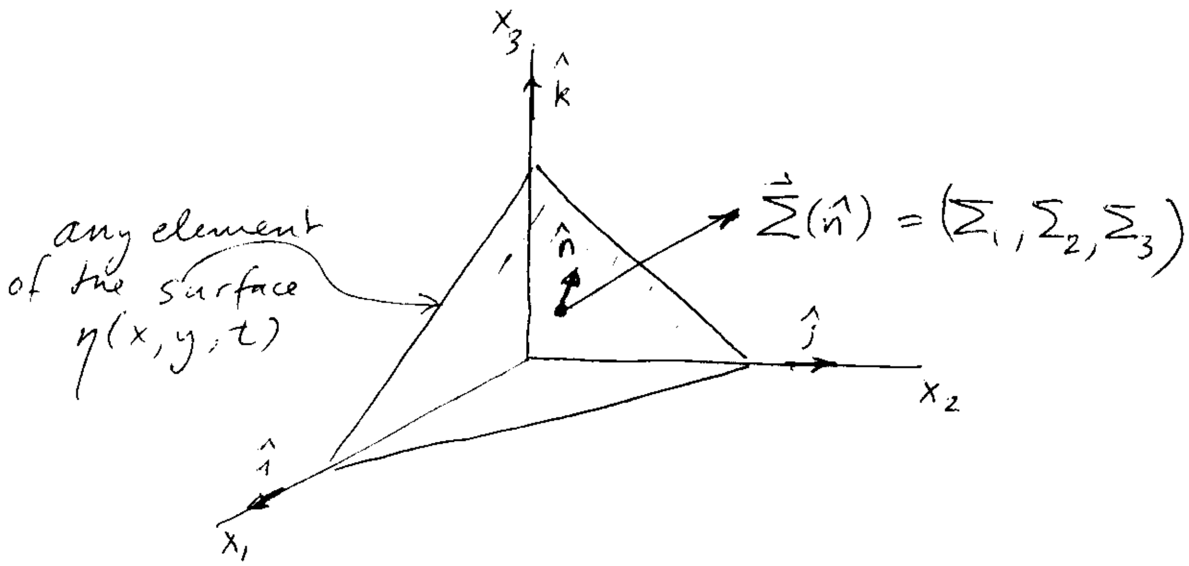
$$\left[\vec{\Sigma}\right]_{z=\eta^-}^{z=\eta^+} = 0 \qquad (2.28)$$

i.e., there will be no jump in the value of $\vec{\Sigma}$ across the surface.

In DOI Section 2.4 the surface stress vector $\vec{\Sigma}$ was expressed as the dot product of the stress tensor with the normal vector

$$\Sigma_i(\hat{n}) = \sigma_{ij} n_j = \sigma \cdot \hat{n} \qquad (2.29)[DOI - 2.4]$$

(cf. equation [DOI-1.15]). Here we take any arbitrary element of the surface oriented perpendicular to the normal vector $\hat{n}$,and $\Sigma_i(\hat{n})$ represents the i-th component of the stress on this surface element



In a moving fluid the stress tensor σ was then expressed as (DOI-Section 2.4.2)

$$\sigma_{ij} = -p\delta_{ij} + d_{ij} \qquad (2.30)[DOI - 2.17]$$

where $\mathbf{d}=d_{ij}$ was the *deviatoric stress tensor* expressed later in DOI Section 3.3 as

$$\mathbf{d} = d_{ij} = \begin{bmatrix} \sigma'_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma'_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma'_{33} \end{bmatrix} \qquad (2.31)[DOI-3.22]$$

whose off-diagonal elements are shear stresses arising only due to the motion of the fluid. Now using (2.29) and (2.30) the components of the stress on the upper surface are then

$$\begin{aligned} \Sigma_i &= (-p\delta_{ij} + d_{ij})n_j \\ &= -pn_j + d_{ij}n_j \end{aligned} \qquad (2.32)$$

or in vector form

$$\vec{\Sigma} = -p\hat{n} + \mathbf{d} \cdot \hat{n} \qquad (2.33)$$

This vector has components in the x, y, z directions referred to $1, 2, 3$ in index notation. The x and y-components of (2.33) are

$$\Sigma_x = -p\hat{n} \cdot \hat{\imath} + (\mathbf{d} \cdot \hat{n}) \cdot \hat{\imath} = -pn_1 + (d_{1j}n_j)e_1 \qquad (2.34.a)$$

$$\Sigma_y = -p\hat{n} \cdot \hat{\jmath} + (\mathbf{d} \cdot \hat{n}) \cdot \hat{\jmath} = -pn_2 + (d_{2j}n_j)e_2 \qquad (2.34.b)$$

Instead of writing the z-component of the vector $\vec{\Sigma}$, we choose to write its component perpendicular to the surface in direction n, since the vector $\hat{n}$ is a linear combination of the $(\hat{\imath}, \hat{\jmath}, \hat{k})$ vectors. The n-component is

$$\Sigma_n = -p\hat{n} \cdot \hat{n} + (\mathbf{d} \cdot \hat{n}) \cdot \hat{n} = -p + d_{ij}n_jn_i \qquad (2.34.c)$$

In these equations the components of the normal vector is

$$\begin{aligned} n_1 &= (\hat{n} \cdot \hat{\imath}) \\ n_2 &= (\hat{n} \cdot \hat{\jmath}) \\ n_3 &= (\hat{n} \cdot \hat{k}) \end{aligned} \qquad (2.35a-c)$$

Using (2.35a-c) and (2.31), (2.34a-c) can be written as

$$\Sigma_x = -p(\hat{n} \cdot \hat{\imath}) + \sigma'_{xx}(\hat{n} \cdot \hat{\imath}) + \tau_{xy}(\hat{n} \cdot \hat{\jmath}) + \tau_{xz}(\hat{n} \cdot \hat{k})$$

$$\Sigma_y = -p(\hat{n} \cdot \hat{\jmath}) + \tau_{y}x(\hat{n} \cdot \hat{\imath}) + \sigma_{yy}(\hat{n} \cdot \hat{\jmath}) + \tau_{yz}(\hat{n} \cdot \hat{k})$$

$$\Sigma_z = -p + (\hat{n}\cdot\hat{\imath})^2\sigma'_{xx} + (\hat{n}\cdot\hat{\jmath})^2\sigma'_{yy} + (\hat{n}\cdot\hat{k})^2\sigma'_{zz} + 2(\hat{n}\cdot\hat{\imath})(\hat{n}\cdot\hat{\jmath})\tau_{xy} + 2(\hat{n}\cdot\hat{\imath})(\hat{n}\cdot\hat{k})\tau_{xz} + 2(\hat{n}\cdot\hat{\jmath})(\hat{n}\cdot\hat{k})\tau_{yz} \qquad (2.36a-c)$$

where use has been made of the symmetry property of the deviatoric stress tensor, $d_{ij} = d_{ji}$. Defining the surface $z = \eta(x, y, t)$ by the equation

$$\phi = z - \eta(x, y, t) = 0 \qquad (2.37)$$

and its gradient by

$$\nabla\phi = -\frac{\partial\eta}{\partial x}\hat{\imath} - \frac{\partial\eta}{\partial y}\hat{\jmath} + \hat{k}, \qquad (2.38.a)$$

the normal vector $\hat{n}$ is found to be (cf. [DOI-1.39])

$$\hat{n} \equiv \frac{\nabla\phi}{|\nabla\phi|} = \frac{-\frac{\partial\eta}{\partial x}\hat{\imath} - \frac{\partial\eta}{\partial y}\hat{\jmath} + \hat{k}}{\sqrt{\left(\frac{\partial\eta}{\partial x}\right)^2 + \left(\frac{\partial\eta}{\partial y}\right)^2 + 1}}. \qquad (2.39)$$

In order to shorten the expressions, let

$$S = \sqrt{1 + \left(\frac{\partial\eta}{\partial x}\right)^2 + \left(\frac{\partial\eta}{\partial y}\right)^2}. \qquad (2.40)$$

Now, the components of the unit normal vector $\hat{n}$ are

$$(\hat{n}\cdot\hat{\imath}) = -\frac{1}{S}\frac{\partial\eta}{\partial x}, \quad (\hat{n}\cdot\hat{\jmath}) = -\frac{1}{S}\frac{\partial\eta}{\partial y}, \quad (\hat{n}\cdot\hat{k}) = \frac{1}{S} \qquad (2.41a-c)$$

Which, upon substituting into (2.36a-c) yield

$$\Sigma_x = \left(\frac{1}{S}\frac{\partial\eta}{\partial x}\right)p - \left(\frac{1}{S}\frac{\partial\eta}{\partial x}\right)\sigma'_{xx} - \left(\frac{1}{S}\frac{\partial\eta}{\partial y}\right)\tau_{xy} + \left(\frac{1}{S}\right)\tau_{xz} \qquad (2.41a)$$

$$\Sigma_y = \left(\frac{1}{S}\frac{\partial\eta}{\partial y}\right)p - \left(\frac{1}{S}\frac{\partial\eta}{\partial x}\right)\tau_{xy} - \left(\frac{1}{S}\frac{\partial\eta}{\partial y}\right)\sigma'_{yy} + \left(\frac{1}{S}\right)\tau_{yz} \qquad (2.41b)$$

$$\begin{aligned}\Sigma_z = -p &+ \left(\frac{1}{S}\frac{\partial\eta}{\partial x}\right)^2\sigma'_{xx} + \left(\frac{1}{S}\frac{\partial\eta}{\partial y}\right)^2\sigma'_{yy} + \left(\frac{1}{S}\right)^2\sigma'_{zz} \\ &+ \frac{2}{S^2}\left(\frac{\partial\eta}{\partial x}\right)\left(\frac{\partial\eta}{\partial y}\right)\tau_{xy} - \frac{2}{S^2}\left(\frac{\partial\eta}{\partial x}\right)\tau_{xz} - \frac{2}{S^2}\left(\frac{\partial\eta}{\partial y}\right)\tau_{yz}\end{aligned} \qquad (2.41c)$$

We can now use the scales introduced in Section (2.2.1), namely that $(x,y) \sim L_0$ and $\eta \sim a_0$. Then, we find that

$$\begin{aligned} \frac{\partial \eta}{\partial x} &= O\left(\frac{a_0}{L_0}\right) = O\left(\frac{a_0}{H_0} \cdot \frac{H_0}{L_0}\right) = O(\mu\lambda) \ll 1 \\ \frac{\partial \eta}{\partial y} &= O(\mu\lambda) \ll 1 \\ S &= \sqrt{1 + |\nabla_h \eta|^2} = \sqrt{1 + 2O(\mu^2\lambda^2)} \simeq 1 = O(1) \end{aligned} \qquad (2.42a-c)$$

Since the $O(\mu\lambda)$ terms are very small, neglecting these and setting S=1 in equations (2.41a-c) to the same order used in shallow water approximations yields:

$$\begin{aligned} \Sigma_x &\simeq \tau_{xz} \\ \Sigma_y &\simeq \tau_{yz} \\ \Sigma_n &\simeq -p + \sigma'_{zz}. \end{aligned} \qquad (2.43a-c)$$

In the last equation (2.43.c) σ'_{zz} stands for the vertical normal stress component (*i.e.*, stress pointing in direction z on the (x,y) plane) arising only due to the motion of the fluid. In fact

$$-p + \sigma'_{zz} = \sigma_{zz} \qquad (2.44)$$

is the total normal stress. This term is usually small as compared to the fluid pressure ($\sigma'_{zz} \ll p$) at the surface and can be neglected altogether:

$$\Sigma_n \approx -p \qquad (2.45)$$

After these simplifications, the continuity of surface forces across the surface in (2.28) requires

$$\begin{aligned} [\tau_{xz}]_{z=\eta^-}^{z=\eta^+} &= 0 \\ [\tau_{yz}]_{z=\eta^-}^{z=\eta^+} &= 0 \\ [p]_{z=\eta^-}^{z=\eta^+} &= 0 \end{aligned} \qquad (2.46.a-c)$$

i.e. the horizontal components of the vertical shear and the fluid pressure must be continuous across the surface. At the surface of the ocean the shear stress occur due to the stresses applied by wind. Assuming that the wind only applies horizontal forces (a horizontal vector $\vec{\tau}^{(s)}$) defined as

$$\vec{\tau}^{(s)} = \left(\tau_{xz}\mid_{z=\eta^+}\right)\hat{\imath} + \left(\tau_{yz}\mid_{z=\eta^+}\right)\hat{\jmath} \qquad (2.47)$$

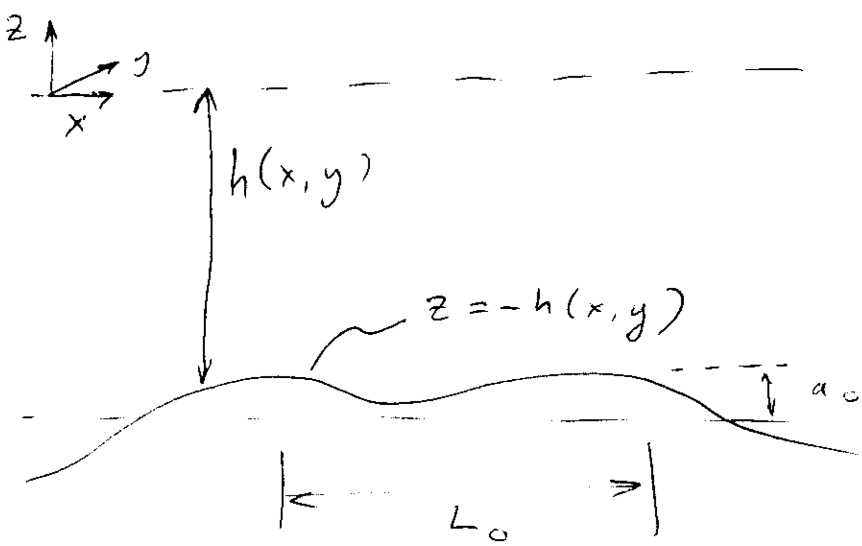
and writing the fluid shear in terms of the velocity gradient a boundary condition is found:

$$\mu \frac{\partial \vec{u}_h}{\partial z} \mid_{z=\eta^-} = \vec{\tau}^{(s)} \tag{2.48}$$

The pressure at the sea surface is the atmospheric pressure $p^{(s)} = p \mid_{z=\eta^+}$ so that (2.46.c) implies another surface boundary condition:

$$p \mid_{z=\eta^-} = p^{(s)} \tag{2.49}$$

At the sea bottom similar conditions may be applied, the mean bottom surface is assumed to be horizontal (constant depth) with slowly varying small undulations superimposed:



i.e., again assuming $a_0/L_0 = \mu\lambda \ll 1$, so that the shear stresses must be continuous. Defining similarly a bottom stress vector:

$$\vec{\tau}^{(b)} = (\tau_{xz} \mid_{z=-h^-})\,\hat{i} + (\tau_{yz} \mid_{z=-h^-})\,\hat{j} \tag{2.50}$$

we can write

$$\mu \frac{\partial \vec{u}_h}{\partial z} \mid_{z=-h^+} = \vec{\tau}^{(b)} \tag{2.51}$$

It is shown in the above derivations that horizontal components of vertical shear and the pressure is transmitted into the fluid purely when the shallow water approximation $\mu\lambda \ll 1$ holds, *i.e.* when the surface across which these forces are transmitted is almost flat.

2.2.3. Hydrostatic Pressure

We can now combine (2.26) and (2.49) since P is constant by virtue of (2.17). At the surface

$$p \mid_{z=\eta^-} = p' - \rho g' \eta^- = p^{(s)} \tag{2.52}$$

so that

$$p'(x,y,t) = p^{(s)}_{(x,y,t)} + \rho g' \eta^-(x,y,t) \tag{2.53}$$

Substituting into (2.26) yields

$$p = p^{(s)}(x,y,t) + \rho g[\eta(x,y,t) - z] \tag{2.54}$$

where prime and minus signs have been dropped and will not be used hereafter.

The above expression (2.54) simply states *hydrostatic pressure*, *i.e.* that the pressure at any depth $-z$ is the weight per unit area of the overlying fluid plus the atmospheric pressure. This is true for any horizontal position (x,y) and instant t.

Furthermore, by using (2.53), the horizontal pressure gradients appearing in equation (2.6.a) appear as

$$-\frac{1}{\rho}\nabla_h p = -\frac{1}{\rho}\nabla_h p^{(s)} + g\nabla_h \eta \tag{2.55}$$

i.e. horizontal pressure gradients are partly caused by gradients in the atmospheric pressure or may be manifested as the gradients of the undulations in the upper surface. The latter of these, *i.e.* the surface elevation gradients impose a horizontal pressure gradient in the fluid through the action of *gravity*.

2.2.4. Kinematic Boundary Conditions

In addition to the dynamic boundary conditions reviewed in Section 2.2.2, we can derive kinematic boundary conditions utilizing the fact that the upper and lower surfaces of the fluid are material surfaces, *i.e.*

$$\frac{D\phi_1}{Dt} = 0, \quad \frac{D\phi_2}{Dt} = 0 \tag{2.56a, b}$$

where

$$\phi_1 = z - \eta(x,y,t) = 0$$
$$\phi_2 = z + h(x,y) = 0 \tag{2.57a, b}$$

describe these surfaces. Substituting (2.57) into (2.26) gives

$$-\frac{\partial \eta}{\partial t} - \vec{u}_h \cdot \nabla_h \eta + w = 0 \quad \text{on} \quad z = \eta(x,y,t) \tag{2.58.a}$$

$$\vec{u}_h \cdot \nabla_h h + w = 0 \quad \text{on} \quad z = -h(x,y) \tag{2.58.b}$$

or equivalently

$$w\mid_{z=\eta}=\frac{\partial\eta}{\partial t}+\vec{u}_h\mid_{z=\eta}\cdot\nabla_h\eta \tag{2.59.a}$$

$$w\mid_{z=-h}=-\vec{u}_h\mid_{z=-h}\cdot\nabla_h h \tag{2.59.b}$$

2.2.5. Shallow Water Equations

In Section 2.2.1 we used estimated scales for a cursory examination of the governing equations. Then, making the approximations $\mu\to 0$, $\lambda\to 0$ and $S=Ro/(F^2\lambda)\to\infty$, $S\lambda\to O(1)$ resulted in purely two-dimensional equations, with all flow variables becoming independent of the vertical coordinate. However we saw that this was not very realistic, and it was concluded that the assumptions used were rather restrictive. We now relax these assumptions a little, especially with regard to vertical velocity. The vertical velocity scale chosen was

$$w\sim\frac{a_0}{L_0}U_0=\frac{a_0}{H_0}\frac{H_0}{L_0}U_0=\mu\lambda U_0$$

as compared to the horizontal velocity scale of

$$\vec{u}_h\sim U_0$$

so that

$$\frac{w}{\mid\vec{u}_h\mid}=O(\mu\lambda) \tag{2.60}$$

where it was assumed that $\mu\ll 1$, $\lambda\ll 1$.

Since we did not have any prior knowledge of the vertical velocity scale we choose the above scales arbitrarily. However if we re-consider equation(2.13)

$$\nabla_h\cdot\vec{u}_h+\mu\frac{\partial w}{\partial z}=0$$

we see that if the horizontal divergence is to be balanced by the vertical gradient term, we should have $\mu=O(1)$ or that $w\sim(\lambda U_0)=O(\frac{H_0}{L_0}U_0)$, *i.e.* vertical velocity should be smaller than the horizontal velocity by only the ratio H_0/L_0 of depth to horizontal scale of motion. This essentially means that perhaps we should have scaled vertical velocity as $w\sim(H_0/L_0)U_0$ at the beginning. In actuality, vertical velocity is at most balanced by the horizontal divergence is to be balanced by the horizontal divergence as indicated above, or somewhat smaller, so that

$$\mu\leq O(1). \tag{2.61}$$

in other words, in the previous scaling we assumed both of the two small parameters $\mu,\ \lambda \to 0$ without stating which one of these two parameters is actually smaller. Here we assume $\lambda \ll \mu$ by virtue of (2.61). With approximation (2.61) and the other previous approximations in Section 2.2.1, the non dimensional equations (2.12 a,b) and (2.13) can be simplified accurate to $O(\lambda)$ as follows:

$$Ro\left(\epsilon_T \frac{\partial \vec{u}_h}{\partial t} + \vec{u}_h \cdot \nabla_h \vec{u}_h + \mu w \frac{\partial \vec{u}_h}{\partial z}\right) + (2\sin\phi)\hat{k} \times \vec{u}_h = -S\lambda \nabla_h p + E^2 \frac{\partial^2 \vec{u}_h}{\partial z^2} \tag{2.62.a}$$

$$0 = -s\frac{\partial p}{\partial z} \tag{2.62.b}$$

$$\nabla_h \cdot \vec{u}_h + \mu \frac{\partial w}{\partial z} = 0 \tag{2.63}$$

The dimensionless equivalents of the above are :

$$\frac{\partial \vec{u}_h}{\partial t} + \vec{u}_h \cdot \nabla_h \vec{u}_h + w\frac{\partial \vec{u}_h}{\partial z} + f\hat{k} \times \vec{u}_h = -\frac{1}{\rho}\nabla_h p + \nu \frac{\partial^2 \vec{u}_h}{\partial z^2} \tag{2.63.a}$$

$$\frac{\partial p}{\partial z} = 0 \tag{2.63.b}$$

$$\nabla_h \cdot \vec{u}_h + \frac{\partial w}{\partial z} = 0 \tag{2.64}$$

where the *Coriolis parameter* f has been defined as

$$f = 2\Omega \sin\phi. \tag{2.65}$$

Note that equations (2.63a,b)and (2.64) can alternatively be written as

$$\frac{D\vec{u}_h}{Dt} + f\hat{k} \times \vec{u}_h \equiv \frac{\partial \vec{u}_h}{\partial t} + \vec{u} \cdot \nabla \vec{u}_h + f\hat{k} \times \vec{u}_h = -\frac{1}{\rho}\nabla_h p + \nu \frac{\partial^2 \vec{u}_h}{\partial z^2} \tag{2.66.a}$$

$$\frac{\partial p}{\partial z} = 0 \tag{2.66.b}$$

$$\nabla \cdot \vec{u} = 0 \tag{2.67}$$

Equation (2.66.b) indicates that the former approximation (that modified pressure is independent of z) in Section 2.2.1 and 2.2.3 are still valid. That is the fluid pressure is *hydrostatic*, even with the new form of approximations. Earlier arguments have shown that the horizontal velocity $\vec{u}_h$ is also expected to be *approximately uniform* in the vertical:

$$\vec{u}_h \simeq \vec{u}_h(x, y, t) \tag{2.68}$$

More exactly, we can decompose horizontal velocity into two components: one having no dependence on z, and the other with a dependence on z:

$$\vec{u}_h(x, y, z, t) = \bar{\vec{u}}_h(x, y, t) + \vec{u}_h'(x, y, z, t) \tag{2.69}$$

where $\bar{\vec{u}}_h$ is the vertically averaged horizontal velocity

$$\bar{\vec{u}}_h \equiv \frac{1}{h+\eta} \int_{z=-h}^{\eta} \vec{u}_h(x, y, z, t) dz, \tag{2.70}$$

and $\vec{u}_h'$ is the deviation from this average velocity.

Using these approximations we can now integrate the governing equations (2.66) and (2.67) vertically to derive equations for the vertically averaged component $\bar{\vec{u}}_h$. First, we integrate the continuity equation (2.67) or (2.64), to yield

$$\int_{-h(x,y)}^{\eta(x,y,t)} \nabla_h \cdot \vec{u}_h dz + w \mid_{z=\eta} - w \mid_{z=-h} = 0. \tag{2.71}$$

Since the limits of integration are functions of (x, y), the Leibnitz' rule [DOI-1.44.a] is used to write

$$\nabla_h \cdot \int_{-h}^{\eta} \vec{u}_h dz - \vec{u}_h \mid_{z=\eta} \cdot \nabla_h \eta - \vec{u}_h \mid_{z=-h} \cdot \nabla_h + w \mid_{z=\eta} - w \mid_{z=-h} = 0.$$

We can now insert the kinematic boundary conditions (2.59 a,b) and the definition (2.70) in the above, to yield

$$\nabla_h \cdot (h+\eta) \bar{\vec{u}}_h + \frac{\partial \eta}{\partial t} = 0.$$

Defining the total depth

$$H = h + \eta \tag{2.72}$$

this result becomes

$$\frac{\partial \eta}{\partial t} + \nabla_h \cdot (H \bar{\vec{u}}_h) = 0. \tag{2.73}$$

Similarly, the momentum equation is integrated. However, first we make some manipulations in equation (2.66.a).

The identity [DOI-1.28] will be of some use:

$$\begin{aligned} \nabla \cdot (\vec{a} \circ \vec{b}) &= \vec{a}(\nabla \cdot \vec{b}) + \vec{b} \cdot (\nabla \circ \vec{a}) \\ &= \vec{a}(\nabla \cdot \vec{b}) + (\vec{b} \cdot \nabla) \vec{a} \end{aligned} \qquad [DOI - 1.28.a]$$

[Proof:

$$\begin{aligned}
\nabla \cdot (\vec{a} \circ \vec{b}) &= \hat{e}_i \frac{\partial}{\partial x_j} a_i b_j \\
&= \hat{e}_i a_i \frac{\partial b_j}{\partial x_j} + \hat{e}_i b_j \frac{\partial a_i}{\partial x_j} \\
&= \vec{a}(\nabla \cdot \vec{b}) + \vec{b} \cdot (\nabla \circ \vec{a}) \\
&= \hat{e}_i a_i \frac{\partial b_j}{\partial x_j} + b_j \frac{\partial a_i \hat{e}_i}{\partial x_j} \\
&= \vec{a}(\nabla \cdot \vec{b}) + (\vec{b} \cdot \nabla)\vec{a}.
\end{aligned} \tag{]}$$

Applying this identity to the second term on the left hand side of (2.66.a) gives

$$\begin{aligned}
(\vec{u} \cdot \nabla)\vec{u}_h &= \nabla \cdot (\vec{u}_h \circ \vec{u}) - \vec{u}_h(\nabla \cdot \vec{u}) \\
&= \nabla \cdot (\vec{u}_h \circ \vec{u})
\end{aligned} \tag{2.74}$$

where the second term on the *r.h.s.* vanishes by virtue of (2.67). The divergence of the dyadic product $\vec{u}_h \circ \vec{u}$ in equation (2.74) is simply

$$\begin{aligned}
\nabla \cdot (\vec{u}_h \circ \vec{u}) &= \hat{e}_i \frac{\partial}{\partial x_j} u_{hi} u_j \\
&= \hat{i} \left(\frac{\partial}{\partial x} u^2 + \frac{\partial}{\partial y} uv + \frac{\partial}{\partial z} uw \right) \\
&+ \hat{j} \left(\frac{\partial}{\partial x} vu + \frac{\partial}{\partial y} v^2 + \frac{\partial}{\partial z} vw \right) + \hat{k}\ (0) \\
&= \nabla_h \cdot (\vec{u}_h \circ \vec{u}_h) + \frac{\partial}{\partial z}(\vec{u}_h w).
\end{aligned} \tag{2.75}$$

Substituting (2.74) and (2.75) into equation (2.66.a) the momentum equation becomes

$$\frac{\partial \vec{u}_h}{\partial t} + \nabla \cdot (\vec{u}_h \circ \vec{u}) + f\hat{k} \times \vec{u}_h = -\frac{1}{\rho}\nabla_h p^{(s)} - g\nabla_h \eta + \nu \frac{\partial^2 \vec{u}_h}{\partial z^2}, \tag{2.76}$$

which will next be integrated vertically. Note that the variables $p^{(s)} = p^{(s)}(x, y, t)$, $\eta = \eta(x, y, t)$ are constants with respect to vertical integration. Integrating from $z = -h$ to $z = \eta$, we have

$$\int_{-h}^{\eta} \frac{\partial \vec{u}_h}{\partial t} dz + \int_{-h}^{\eta} \nabla \cdot (\vec{u}_h \circ \vec{u}) dz + f\hat{k} \times \int_{-h}^{\eta} \vec{u}_h dz = -\frac{H}{\rho}\nabla_h p^{(s)} - gH\nabla\eta + \nu \left[\frac{\partial \vec{u}_h}{\partial z} \right]_{z=-h}^{z=\eta}$$

Using the definition (2.70) and substituting dynamic boundary conditions (2.48) and (2.51) derived in Section (2.2.2) we have

$$\int_{-h}^{\eta} \frac{\partial \vec{u}_h}{\partial t} dz + \int_{-h}^{\eta} \nabla.(\vec{u}_h \circ \vec{u}) dz + fH\hat{k} \times \vec{u}_h = -\frac{H}{\rho}\nabla_h p^{(s)} - gH\nabla\eta + \frac{1}{\rho}\left(\vec{\tau}^{(s)} - \vec{\tau}^{(b)}\right), \tag{2.77}$$

where, the definition $\nu = \mu/\rho$ has also been used. The integration of the first term on the left hand side is carried out, using the Leibnitz' rule [DOI-1.44.a]:

$$\begin{aligned}\int_{-h(x,y)}^{\eta(x,y,t)} \frac{\partial \vec{u}_h}{\partial t} &= \frac{\partial}{\partial t}\int_{-h}^{\eta} \vec{u}_h dz - u_h \mid_{z=\eta} \frac{\partial \eta}{\partial t} \\ &= \frac{\partial}{\partial t} H \bar{\vec{u}}_h - u_h \mid_{z=\eta} \frac{\partial \eta}{\partial t}.\end{aligned} \tag{2.78}$$

The second term on the left hand side of (2.77) can similarly be integrated, this time making use of (2.75):

$$\begin{aligned}&\int_{-h}^{\eta} \nabla \cdot (\vec{u}_h \circ \vec{u}_h) dz = \int_{-h}^{\eta} \nabla_h \cdot (\vec{u}_h \circ \vec{u}_h) dz + \int_{-h}^{\eta} \frac{\partial}{\partial z}(\vec{u}_h w) dz \\ &= \nabla_h \cdot \int_{-h}^{\eta} (\vec{u}_h \circ \vec{u}_h) dz - (\vec{u}_h \circ \vec{u}_h) \mid_{z=\eta} \cdot \nabla_h \eta \\ &- (\vec{u}_h \circ \vec{u}_h) \mid_{z=-h} \cdot \nabla_h h + \vec{u}_h w \mid_{z=\eta} - \vec{u}_h w \mid_{z=-h} .\end{aligned} \tag{2.79}$$

Here, we can make use of the identity

$$(\vec{a} \circ \vec{a}) \cdot \vec{b} = (a_i a_j) b_j = a_i (a_j b_j) = \vec{a}(\vec{a} \cdot \vec{b}) \tag{2.80}$$

to write (2.79) as

$$\begin{aligned}&\int_{-h}^{\eta} \nabla \cdot (\vec{u}_h \circ \vec{u}) dz = \nabla_h \cdot \int_{-h}^{\eta} (\vec{u}_h \circ \vec{u}_h) dz \\ &- (\vec{u}_h \mid_{z=\eta})(\vec{u}_h \mid_{z=\eta} \cdot \nabla_h \eta) - (\vec{u}_h \mid_{z=-h})(\vec{u}_h \mid_{z=-h} \cdot \nabla_h h) \\ &+ \vec{u}_h w \mid_{z=\eta} - \vec{u}_h w \mid_{z=-h} .\end{aligned} \tag{2.81}$$

Substituting (2.78) and (2.81), equation (2.77) takes the following form:

$$\begin{aligned}&\frac{\partial H \bar{\vec{u}}_h}{\partial t} + \nabla_h \cdot \int_{-h}^{\eta} (\vec{u}_h \circ \vec{u}_h) dz + f H \hat{k} \times \bar{\vec{u}}_h \\ &- \left[\vec{u}_h \frac{\partial \eta}{\partial t}\right]_{z=\eta} - [\vec{u}_h(\vec{u}_h \cdot \nabla_h \eta)]_{z=\eta} - [\vec{u}_h(\vec{u}_h \cdot \nabla_h h)]_{z=-h} + [\vec{u}_h w]_{z=\eta} - [\vec{u}_h w]_{z=-h} \\ &= -\frac{H}{\rho} \nabla_h p^{(s)} - g H \nabla \eta + \frac{1}{\rho}(\vec{\tau}^{(s)} - \vec{\tau}^{(b)})\end{aligned} \tag{2.82}$$

Note that a number of cancellations occur because of the kinematic boundary conditions (2.59.a) and (2.59.b). The second term left hand side can be further simplified by utilizing (2.69) to write

$$
\begin{aligned}
\int_{-h}^{\eta} (\vec{u}_h \circ \vec{u}_h) dz &= \int_{-h}^{\eta} (\bar{\vec{u}}_h + \vec{u}'_h) \circ (\bar{\vec{u}}_h + \vec{u}'_h) dz \\
&= \int_{-h}^{\eta} (\bar{\vec{u}}_h \circ \bar{\vec{u}}_h) dz + \int_{-h}^{\eta} (\bar{\vec{u}}_h \circ \vec{u}'_h) dz \\
&+ \int_{h}^{\eta} (\vec{u}'_h \circ \bar{\vec{u}}_h) dz + \int_{-h}^{\eta} (\vec{u}'_h \circ \vec{u}'_h) dz.
\end{aligned}
\tag{2.83}
$$

On the other hand, by virtue of (2.69) and (2.70)

$$
\begin{aligned}
\int_{-h}^{\eta} \vec{u}'_h dz &= \int_{-h}^{\eta} \vec{u}_h dz - \bar{\vec{u}}_h \int_{-h}^{\eta} dz \\
&= H \bar{\vec{u}}_h - H \bar{\vec{u}}_h = 0,
\end{aligned}
\tag{2.84}
$$

so that the second and third terms of (2.83) vanish, leaving

$$
\int_{-h}^{\eta} (\vec{u}_h \circ \vec{u}_h) dz = H \bar{\vec{u}}_h \circ \bar{\vec{u}}_h + \int_{-h}^{\eta} (\vec{u}'_h \circ \vec{u}'_h) dz. \tag{2.85}
$$

Now, substituting (2.85) into (2.82) and defining a second order tensor

$$
\mathbf{F} = -\frac{1}{H} \int_{-h}^{\eta} (\vec{u}'_h \circ \vec{u}'_h) dz, \tag{2.86}
$$

the momentum equations become:

$$
\begin{aligned}
&\frac{\partial H \bar{\vec{u}}_h}{\partial t} + \nabla_h \cdot (H \bar{\vec{u}}_h \circ \bar{\vec{u}}_h) + f H \hat{k} \times \bar{\vec{u}}_h \\
&= -g H \nabla \eta - \frac{H}{\rho} \nabla_h p^{(s)} + \frac{1}{\rho} \left(\vec{\tau}^{(s)} - \vec{\tau}^{(b)} \right) + \nabla_h \cdot H \mathbf{F}.
\end{aligned}
\tag{2.87}
$$

The first two terms can be expanded by making use of [DOI-1.28a] as

$$
\begin{aligned}
&\frac{\partial H \bar{\vec{u}}_h}{\partial t} + \nabla_h \cdot (\bar{\vec{u}}_h \circ H \bar{\vec{u}}_h) \\
&= H \frac{\partial \bar{\vec{u}}_h}{\partial t} + \bar{\vec{u}}_h \frac{\partial H}{\partial t} + \bar{\vec{u}}_h \left(\nabla_h \cdot H \bar{\vec{u}}_h \right) + \left(H \bar{\vec{u}}_h \cdot \nabla_h \right) \bar{\vec{u}}_h, \\
&= H \left\{ \frac{\partial \bar{\vec{u}}_h}{\partial t} + \bar{\vec{u}}_h \cdot \nabla_h \bar{\vec{u}}_h \right\} + \bar{\vec{u}}_h \left\{ \frac{\partial \eta}{\partial t} + \nabla_h \cdot \bar{\vec{u}}_h \right\}
\end{aligned}
\tag{2.88}
$$

the second term of which vanishes by virtue of (2.73). Therefore, using (2.88) in (2.87), the momentum equation becomes (dropping overbars and the subscript h hereafter):

$$
\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} + f \hat{k} \times \vec{u} = -g \nabla \eta - \frac{1}{\rho} \nabla p^{(s)} + \frac{1}{\rho H} \left(\vec{\tau}^{(s)} - \vec{\tau}^{(b)} \right) + \frac{1}{H} \nabla \cdot H \mathbf{F}, \tag{2.89}
$$

supplemented by (2.73), written in the same manner as

$$\frac{\partial \eta}{\partial t} + \nabla \cdot H\vec{u} = 0 \tag{2.90}$$

Equations (2.89) and (2.90) are called the *shallow water equations*.

In the momentum equation, the left hand side and the first term on the right hand side are familiar terms. The second term (left hand side) represents the pressure gradient arising due to variations of atmospheric pressure, and vanishes if it is uniformly distributed. The third term is the difference in surface and bottom shear stresses distributed per unit mass of the fluid column (divided by ρH). The bottom stresses can often be neglected and the remaining term represents forcing by wind stress on the sea surface. The last term is a weighed divergence of a tensor **F** with a form similar to that appearing in [DOI-3.8.a]. Therefore the tensor **F** defined in (2.86) is called ***Reynold's stress tensor***. Since this term arises because of the vertical averaging of the $\vec{u}'_h \circ \vec{u}'_h$ tensor, and since it is expected that $|\vec{u}'_h| \ll |\vec{u}_h|$, often it can be neglected.

2.2.6. Conservation Properties

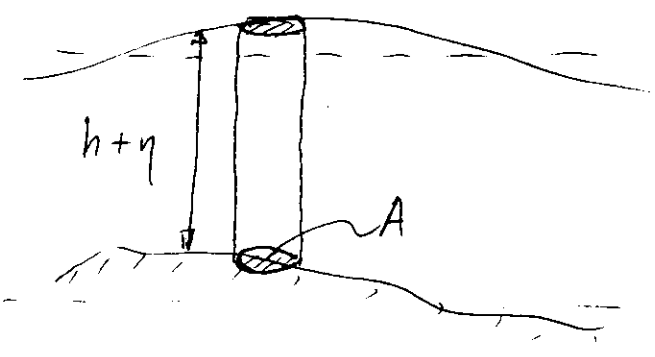
2.2.6.1. Mass Conservation

Since $H(x,y,t) = h(x,y) + \eta(x,y,t)$, we can write continuity equation (2.90) as

$$\frac{\partial H}{\partial t} + \nabla \cdot H\vec{u} = 0 \tag{2.91}$$

or by regrouping the terms as

$$\frac{DH}{Dt} = \frac{\partial H}{\partial t} + \vec{u} \cdot \nabla H = -H \nabla \cdot \vec{u}. \tag{2.92}$$



Here, the horizontal divergence of the horizontal velocity represents the relative rate of change of the horizontal cross-sectional area of a material element :

$$\nabla \cdot \vec{u} = \frac{1}{A}\frac{dA}{dt} \tag{2.93}$$

so that (2.92) becomes

$$\frac{1}{H}\frac{DH}{Dt} + \frac{1}{A}\frac{dA}{dt} = \frac{D}{Dt}ln H + \frac{d}{dt}ln A = \frac{d}{dt}ln(HA) = 0 \tag{2.94}$$

Equation (2.94) expresses the fact that the total volume HA of any fluid column is conserved. For an increase to occur in total depth, there must be a corresponding reduction in cross-sectional area and vice versa.

We can also integrate the continuity equation (2.67) vertically to obtain vertical velocity,assuming that the horizontal velocity $\vec{u}$ is approximately independent of z:

$$w(x,y,z,t) = \int \frac{\partial w}{\partial z} dz \simeq -z(\nabla \cdot \vec{u}) + c(x,y,t) \tag{2.95}$$

where c is a constant of integration with respect to z. In the above equation $\nabla \cdot \vec{u}$ stands for $\nabla_h \cdot \vec{u}_h \simeq \nabla_h \cdot \bar{\vec{u}}_h$, but since subscripts have been dropped earlier, we choose this notation.

Equation (2.95) indicates that vertical velocity is approximately a linear function of z. In fact this is the only possibility to be able to satisfy kinematic boundary conditions (2.59 a,b). Making use of (2.59 b) the integration constant c is evaluated and

$$w = -(z+h)\nabla \cdot \vec{u} - \vec{u} \cdot \nabla h. \tag{2.96}$$

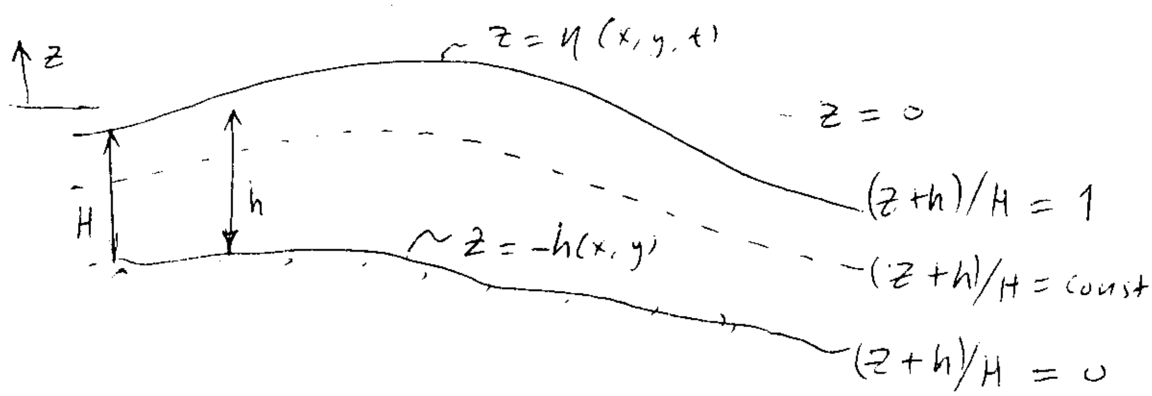
Substituting (2.92) and nothing that $h = h(x, y)$, we can write

$$w \equiv \frac{Dz}{Dt} = -\frac{z+h}{H}\frac{DH}{Dt} - \frac{Dh}{Dt} \tag{2.97}$$

or

$$\frac{D}{Dt}\left(\frac{z+h}{H}\right) = 0. \tag{2.98}$$

This result (equation 2.98) shows that the vertical position of any material point measured relative to $z = -h$ and normalized by the total depth, *i.e.* $(z+h)/H$ is conserved following the motion.



Consider the flow contained by vertical fixed side walls on a boundary C enclosing a region A.

At the side-walls there will be no normal fluxes,

$$\vec{u} \cdot \hat{n} = 0. \tag{2.99}$$

Integration of (2.90) across the area A yields

$$\int_A \frac{\partial \eta}{\partial t} dA = -\int_A \nabla \cdot H\vec{u} dA = -\oint_C H\vec{u} \cdot \hat{n} dl \tag{2.100}$$

through the use of the divergence theorem [DOI-1.29]. For an open ocean, or a semi-enclosed basin, this yields

$$\frac{\partial}{\partial t} \int \eta dA = -\oint_C H\vec{u} \cdot \hat{n} dl, \tag{2.101.a}$$

relating the mean sea-level to the volume flux through the open boundaries (*i.e.* a statement of mass conservation).

On the other hand, for a totally enclosed basin, (2.99) requires the right hand side of (2.100) to vanish, so that

$$\frac{\partial}{\partial t} \int \eta dA = 0. \tag{2.101.b}$$

This equation states that the surface displacements integrated over the enclosed area should be constant at all times.

2.2.6.2. Vorticity Conservation

Neglecting the last three terms of (2.89) and making use of the vector identity [DO I-1.27.e], namely

$$\vec{u} \cdot \nabla \vec{u} = \frac{1}{2} \nabla \vec{u} \cdot \vec{u} - \vec{u} \times \nabla \times \vec{u} \tag{2.102}$$

the simplified shallow water momentum equation can be re-written as

$$\frac{\partial \vec{u}}{\partial t} = -\nabla(g\eta + \frac{1}{2}\vec{u} \cdot \vec{u}) + \vec{u} \times (\nabla \times \vec{u} + f\hat{k}) \tag{2.103}$$

Here the "del" operator is one that is horizontal $\nabla = \nabla_h$, because the fields are only two-dimensional; and $\vec{u} = \vec{u}_h$ is the horizontal velocity, with subscripts dropped earlier. The term $\nabla \times \vec{u} = \nabla_h \times \vec{u}_h$ is the vertical component of the vorticity vector defined by

$$\hat{k} \cdot \vec{\omega} = \hat{k} \cdot (\nabla_h \times \vec{u}_h) = \zeta \tag{2.104}$$

so that substituting (2.104) into (2.103) and taking curl yields

$$\frac{\partial \zeta \hat{k}}{\partial t} = \nabla \times (\vec{u} \times \hat{k}(\zeta + f)) \tag{2.105}$$

The second term of (2.103) vanishes upon taking the curl. Expanding the right hand side of (2.105) yields

$$\hat{k}\frac{\partial \zeta}{\partial t} = -\vec{u}\cdot\nabla\hat{k}(\zeta+f) - \hat{k}(\zeta+f)\nabla\cdot\vec{u} \tag{2.106}$$

or

$$\frac{\partial(\zeta+f)}{\partial t} + \vec{u}\cdot\nabla(\zeta+f) + (\zeta+f)\nabla\cdot\vec{u} \tag{2.107}$$

Substituting from (2.92) then gives

$$\frac{D(\zeta+f)}{Dt} - \frac{(\zeta+f)}{H}\frac{DH}{Dt} = 0 \tag{2.108}$$

or

$$\frac{D}{Dt}\left(\frac{\zeta+f}{H}\right) = 0 \tag{2.109}$$

Equation (2.109) states that the *potential vorticity* defined as $\frac{\zeta+f}{H}$ is conserved following any fluid column. ζ is often referred to as *relative vorticity* , and f as *planetary vorticity* which together make up the *absolute vorticity* $\zeta+f$. A fluid column will preserve its initial relative vorticity only if f and H are constant.

Integration of (2.103) along the boundary gives

$$\begin{aligned}\frac{\partial}{\partial t}\oint_C \vec{u}\cdot d\vec{r} &= -\oint_C \nabla(g\eta + \frac{1}{2}\vec{u}\cdot\vec{u})\cdot d\vec{r} + \oint_C [\vec{u}\times(\zeta+f)\hat{k}]\cdot d\vec{r}\\ &= -\oint_C (\zeta+f)\hat{k}\cdot\vec{u}\times d\vec{r},\\ &= -\oint_C (\zeta+f)\vec{u}\cdot\hat{n}dl\end{aligned} \tag{2.110}$$

yielding a relation between the circulation $\Gamma = \oint_C \vec{u}\cdot d\vec{r}$ for a closed path C, and the flux of vorticity through the boundaries.

Similarly, if equation (2.106) is integrated over the area A enclosed by the closed curve C, and if the divergence theorem [DOI-1.29] is used, we obtain

$$\frac{\partial}{\partial t}\int_A \zeta dA = -\int_A \nabla\cdot(\zeta+f)\vec{u}dA = -\oint_C (\zeta+f)\vec{u}\cdot\hat{n}dA. \tag{2.111}$$

Note that the *l.h.s.* of either equation (2.110) or (2.111) are equal by *Stokes' theorem* [DOI-3.71 and 3.72], written in the present context. The average vorticity in A is equal to the circulation of C, and changes only by fluxes of vorticity across the boundary C.

Consider now, a region enclosed by side-walls on boundary C, where $\vec{u}\cdot\hat{n} = 0$ (or $\vec{u}$ is parallel to $d\vec{r}$); in this case, the *r.h.s.* of (2.110) or (2.111) vanish:

$$\frac{\partial}{\partial t}\oint_C \vec{u}\cdot d\vec{r} = \frac{\partial}{\partial t}\int_A \zeta dA = 0. \tag{2.112}$$

Since there is no normal velocity across the solid boundary C the circulation around any rigid boundary, or equivalently, the area-averaged relative vorticity in an enclosed region A is conserved.

2.2.6.3. Energy conservation

To derive the energy conservation equation, again we consider a region A enclosed by a boundary C. We first multiply (2.103) by $H\vec{u}$ to obtain

$$H\vec{u}\cdot\frac{\partial\vec{u}}{\partial t} + H\vec{u}\cdot\nabla\frac{1}{2}\vec{u}\cdot\vec{u} + H\vec{u}\cdot(\zeta+f)\hat{k}\times\vec{u} = -gH\vec{u}\cdot\nabla\eta \tag{2.113}$$

where, the last term on the *l.h.s* vanishes because $u\cdot\hat{k}\times\vec{u}\equiv 0$. We note, by making use of (2.92), that

$$H\vec{u}\cdot\frac{\partial\vec{u}}{\partial t} = \frac{\partial}{\partial t}\frac{1}{2}H\vec{u}\cdot\vec{u} - \frac{1}{2}\vec{u}\cdot\vec{u}\frac{\partial H}{\partial t} = \frac{\partial}{\partial t}\frac{1}{2}H\vec{u}\cdot\vec{u} + \frac{1}{2}\vec{u}\cdot\vec{u}\nabla\cdot H\vec{u} \tag{2.114}$$

so that (2.113) becomes (using 2.92 once again)

$$\begin{aligned}\frac{\partial}{\partial t}(\frac{1}{2}H\vec{u}\cdot\vec{u}) + \nabla\cdot[(\frac{1}{2}\vec{u}\cdot\vec{u})H\vec{u}] &= -gH\vec{u}\cdot\nabla\eta \\ &= -g\nabla\cdot\eta H\vec{u} + g\eta\nabla\cdot H\vec{u} \\ &= -g\nabla\cdot\eta H\vec{u} - g\eta\frac{\partial H}{\partial t}\end{aligned} \tag{2.115}$$

Reorganizing (2.115),

$$\frac{\partial}{\partial t}(\frac{1}{2}H\vec{u}\cdot\vec{u} + \frac{1}{2}g\eta^2) = -\nabla\cdot[(\frac{1}{2}\vec{u}\cdot\vec{u})H\vec{u} + g\eta H\vec{u}] \tag{2.116}$$

and, integrating over the domain A, using the divergence theorem [DOI-1.29] yields

$$\frac{\partial}{\partial t}\int_A(\frac{1}{2}H\vec{u}\cdot\vec{u} + \frac{1}{2}g\eta^2)dA = \oint_C(\frac{1}{2}\vec{u}\cdot\vec{u} + g\eta)H\vec{u}\cdot\hat{n}dl. \tag{2.117}$$

The individual terms on the *l.h.s* are defined as

$$\begin{aligned}KE &= \frac{1}{2}H\vec{u}\cdot\vec{u}, \\ PE &= \int_0^\eta gzdz = \frac{1}{2}g\eta^2,\end{aligned} \tag{2.118.a, b}$$

representing the kinetic and potential energy per unit mass of the fluid column, so that (2.117) states the conservation of total mechanic energy. For open or semi-enclosed domains, the total energy in the region A changes by fluxes of kinetic and potential energy (the first and second terms on the *r.h.s* respectively), across the boundary C.

For an enclosed domain, with $\vec{u}\cdot\hat{n}=0$ on the solid boundary C, we have

$$\frac{\partial}{\partial t}\int_A (KE+PE)dA = 0. \qquad (2.119)$$

2.3. The f-plane and the β-plane Approximations

We already have assumed in Section 2.1 that the spherical geometry of earth can be reasonably approximated by a *tangent-plane* fitted to the region of interest. It should be noted however that the coriolis parameter

$$f = f(\phi) = 2\Omega\sin\phi$$

is a function of the latitude angle ϕ. With respect to a fixed point on the earth (at latitude angle ϕ_0, where the tangent plane contacts the earth) the Coriolis parameter is expressed as

$$f = 2\Omega\sin(\phi_0+\Delta\phi) = 2\Omega(\sin\phi_0\cos\Delta\phi + \cos\phi_0\sin\Delta\phi) \qquad (2.120)$$

If the angle $\Delta\phi$ (which measures deviations from ϕ_0) is small, we can approximate

$$\cos\Delta\phi = 1 - \frac{(\Delta\phi)^2}{2!} + \frac{(\Delta\phi)^4}{4!} - \cdots \qquad (2.121.a)$$

$$\sin\Delta\phi = \Delta\phi - \frac{(\Delta\phi)^3}{3!} + \frac{(\Delta\phi)^5}{5} - \cdots \qquad (2.121.b)$$

and, neglecting terms of $O(\Delta\phi)$ and smaller, (2.120) approximates to

$$f = f_0 \equiv 2\Omega\sin\phi_0 = \text{constant}. \qquad (2.122)$$

This approximation for f is referred to as the *f-plane approximation.* Since $f=f_0$ is taken as constant, the effects of the latitudinal change in the coriolis parameter are not incorporated in the dynamics.

As the next level of approximation we can neglect terms of $O(\Delta\phi^2)$ and smaller,which yields

$$f = 2\Omega(\sin\phi_0 + \Delta\phi\cos\phi_0) \qquad (2.123)$$

and since angle $\Delta\phi$ is small, it can be interchanged with

$$\Delta\phi = \frac{y}{r_0} \tag{2.124}$$

where y is the horizontal coordinate on the tangent plane pointing towards the north and r_0 is the earth's radius. Then equation (2.123) becomes

$$f = f_0 + \beta_0 y \tag{2.125}$$

where

$$\beta_0 = \frac{2\Omega\cos\phi_0}{r_0} = \frac{f_0\cot\phi}{r_0} \tag{2.126}$$

and f_0 is given by (2.122). The variation of the Coriolis parameter f with latitude has been approximated by a linear function in (2.126), in order to incorporate this variation in the equations. This is called the β-plane approximation.

2.4. Simple Applications of the Potential Vorticity Conservation

Among the conservation laws derived in Section 2.2, the conservation of potential vorticity (equation 2.109) is one of the most important and useful results in understanding the fundamental behavior of geophysical flows. We will consider several simple applications to emphasize the use of potential vorticity conservation. First we write (2.109) as

$$\frac{D}{Dt}\left(\frac{\zeta + f_0 + \beta y}{H}\right) = 0 \tag{2.127}$$

by making use of (2.125), *i.e.* including variations of the Coriolis parameter at the β-plane approximation level.

2.4.1. Geostrophic Flow

The geostrophic approximation (cf. Section 1.2.3) for shallow water equations (2.89) and (2.90) excluding the forcing terms of the momentum equation and unsteady terms of the continuity equation are:

$$f\hat{k} \times \vec{u} = -g\nabla\eta \tag{2.128.a}$$

$$\nabla \cdot H\vec{u} = 0 \tag{2.128.b}$$

The conservation of vorticity can be re-derived for these equations, first by taking care of (2.128.a)

$$\nabla \times f\hat{k} \times \vec{u} = -g\nabla \times \nabla\eta \tag{2.129}$$

then expanding the left hand side as

$$\vec{u} \cdot \nabla\hat{k}f + \hat{k}f\nabla \cdot \vec{u} = 0 \tag{2.130}$$

then utilizing (2.128.b)

$$\nabla \cdot H\vec{u} = H\nabla \cdot \vec{u} + \vec{u} \cdot \nabla H = 0 \tag{2.131}$$

(2.130) can be written as

$$\vec{u} \cdot \nabla f - \frac{f}{H}\vec{u} \cdot \nabla H = 0 \tag{2.132}$$

or first dividing by H and collecting terms,

$$\vec{u} \cdot \nabla \left(\frac{f}{H}\right) = 0 \tag{2.133}$$

Now, since the motion is steady.This is equivalent to writing

$$\frac{D}{Dt}\left(\frac{f}{H}\right) = \frac{D}{Dt}\left(\frac{f_0 + \beta y}{H}\right) = 0 \tag{2.134}$$

Note that the relative vorticity ζ does not enter the conservation law (2.134). Since the inertial terms have been neglected in (2.128.a). This does not actually imply that the vorticity is zero, since through (2.128.a)

$$\zeta = \hat{k} \cdot \nabla \times \vec{u} = \hat{k} \cdot \nabla \times (\frac{g}{f}\hat{k} \times \nabla\eta) = \nabla\eta \cdot \nabla\frac{g}{f} + \frac{g}{f}\nabla \cdot \nabla\eta - \frac{g}{f^2}\nabla\eta \cdot \nabla f + \frac{g}{f}\nabla^2\eta \tag{2.135}$$

and in the case that $f = f_0$ =constant, this reduces to

$$\zeta = \frac{g}{f_0}\nabla^2\eta \tag{2.136}$$

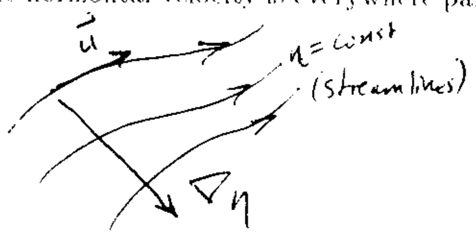
which is consistent with the definition of pressure $(g\eta/f_0)$ as stream function in geostrophic flow (as shown in Section 1.2.3). In fact (2.128.a) or

$$\vec{u} = \frac{g}{f}\hat{k} \times \nabla\eta \tag{2.137}$$

implies that

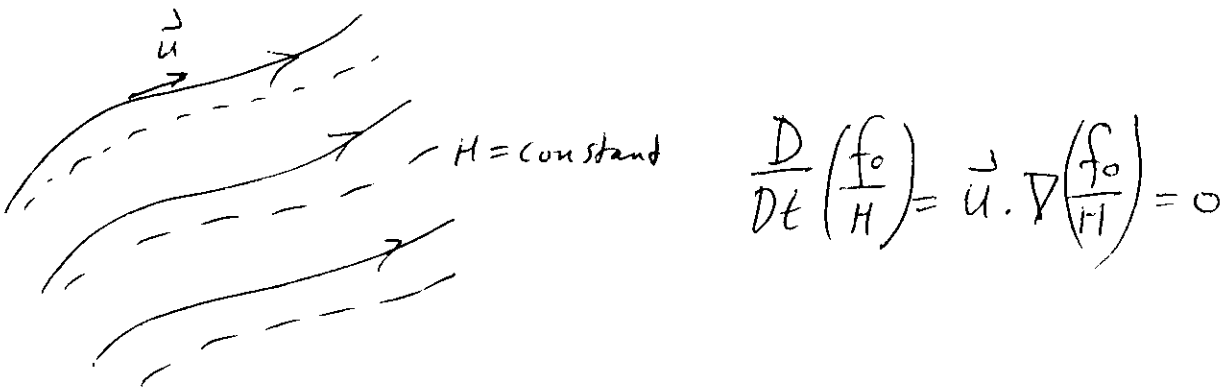
$$\vec{u} \cdot \nabla\eta = \frac{g}{f}(\hat{k} \times \nabla\eta) \cdot (\nabla\eta) \equiv 0 \tag{2.138}$$

or the horizontal velocity is everywhere parallel to the isolines of the surface elevation η, $\vec{u} \cdot \nabla\eta = 0$.



$$\vec{u} \cdot \nabla\eta = 0$$

On the other hand, (2.134) implies that on an f-plane ($\beta = 0$, $f = f_0$) any material element must move along isolines of H (total depth), since $\frac{D}{Dt}(f_0/H) = f_0\vec{u}\cdot\nabla(1/H) = 0$



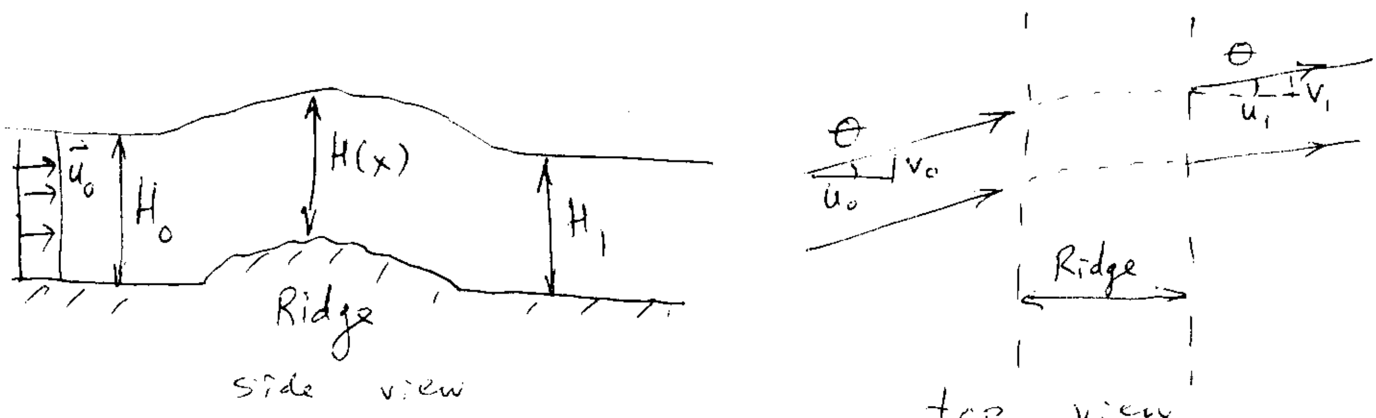
Approximately speaking, $H = h+\eta \simeq h$ ($\eta \ll h$), so that the above is equivalent to the requirement that fluid columns more along bottom contours. The combination of the above two requirements than imply that the isolines the surface displacement η and depth h are parallel to each other. In fact, this can be shown exactly, by multiplying (2.138) by the factor f_0/H^2 and subtracting from (2.133):

$$\frac{f_0}{H^2}\vec{u}\cdot\nabla(H-\eta) = \frac{f_0}{H^2}\vec{u}\cdot\nabla h =, \tag{2.139}$$

which requires that velocity is parallel to bathymetric contours.

2.4.2. Flow over a topographic ridge (f-plane)

As a second application consider the steady (but obviously not geostrophic) flow over a ridge shaped topographic barrier:



Consider the flow to be independent of the y-coordinate (along the ridge), and let the incoming velocity be also independent of y, but assume it comes at an angle to the ridge with components (u_0, v_0) such that the initial vorticity $\zeta_0 = 0$:

Let the velocity components be (u_1, v_1) and the total depth be H_1 after the ridge. Let the total depth on any point on the ridge be H, the velocity components be (u, v) and vorticity be ζ. Then potential vorticity conservation requires (for an f-plane)

$$\frac{f_0 + \zeta(x)}{H(x)} = \frac{f_0}{H_0} = constant \tag{2.140}$$

Since velocity components should be independent of y everywhere

$$\zeta(x) = \frac{dv(x)}{dx} \tag{2.141}$$

so that from(2.140)

$$\zeta = \frac{dv}{dx} = f_0(\frac{H}{H_0} - 1) \tag{2.142}$$

which is integrated to yield

$$v = v_0 + f_0 \int_{-\infty}^{x} \frac{H - H_0}{h_0} dx \tag{2.143}$$

The x-component of velocity is determined by the continuity equation (2.90), which is

$$\frac{\partial}{\partial x}(uH) = 0 \tag{2.144}$$

implying that

$$uH = u_0 H_0 \tag{2.145}$$

Then, on the downstream side of the topographic barrier and sufficiently far from it (2.143) and (2.145) give

$$v_1 = v_0 - \frac{f_0}{H_0} A \tag{2.146.a}$$

$$u_1 = u_0 \frac{H_0}{H_1} = u_0 \tag{2.146.b}$$

where

$$A = \int_{-\infty}^{\infty} (h_0 - H) dx. \tag{2.146.c}$$

$(H_0 = H_1$ by continuity). Note that if the surface displacement ζ is neglected, (2.146.c) is

$$A \simeq \int_{-\infty}^{\infty} (h_0 - H)dx \tag{2.147}$$

which is approximately the cross-sectional area of the ridge.

The angle of incidence θ_0, and the angle of transmission θ_1, are given by

$$\tan\theta_0 = \frac{u_0}{v_0}, \tag{2.148.a}$$

$$\tan\theta_1 = \frac{v_0 - \frac{fA}{H_0}}{u_0}, \tag{2.148.b}$$

so that the angle through which the velocity vector turns upon passing the ridge ($\alpha = \theta_0 - \theta_1$) can be calculated from trigonometry:

$$\tan\alpha = \tan(\theta_0 - \theta_1) = \frac{\tan\theta_0 - \tan\theta_1}{1 + \tan\theta_0 \tan\theta_1} = \frac{u_0(fA/h_0)}{u_0^2 + v_0^2 - v_0(fA/H_0)} \tag{2.149}$$

Therefore, the flow will be deflected in a clockwise sense upon passing the ridge. Immediately over the ridge (2.142) can be approximated as

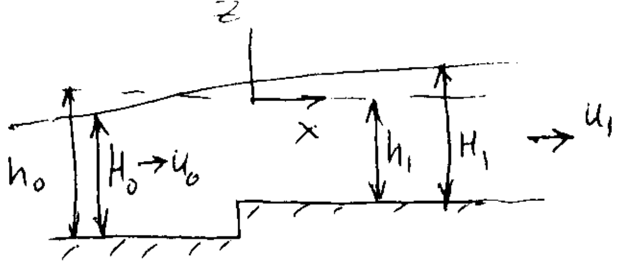
$$\zeta \simeq f_0\left(\frac{h}{h_0} - 1\right) \tag{2.150}$$

and since $h < h_0$, it is seen that a negative vorticity ($\zeta < 0$) is imparted on the fluid by the ridge:

Since negative (anticyclonic) vorticity is often associated with high pressure centers (as in the case of atmospheric highs), we expect a pressure excess (*i.e.*, raised surface elevation) a top the ridge. Excess pressures are often observed on top of maintains in the atmosphere.

2.4.3. Flow over a depth discontinuity (β−plane)

We consider a flow approaching a depth discontinuity in the zonal direction as shown:



We consider the flow in a β-plane, and assume that the y-direction is aligned towards the north (northern hemisphere). We assume that the surface displacement η is small compared to the total depth ($\eta \ll H_0$), consistent with a *rigid lid approximation.* Therefore, the depths H_0, H_1 on two sides are approximated by h_0, h_1 respectively. In the β-plane potential vorticity conservation requires that

$$\frac{f_0 + \beta y + \zeta}{h_1} = \frac{f_0 + \beta y_0}{h_0} \tag{2.151}$$

where it has been assumed that the approaching flow has no vorticity ($\zeta = 0$ for $x < 0$), and y_0 represents the y-position of any approaching fluid particle. The velocity (for $x > 0$) is no longer uniform in the y-direction by virtue of (2.151), and since the depth is constant in $x > 0$, the velocity components satisfy

$$\nabla \cdot \vec{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.152}$$

At the depth discontinuity, we must have

$$u_1 = u_0 h_0 / h_1 \tag{2.153}$$

required by continuity, because the flow is independent of y in the region $x \leq 0$. By virtue of (2.152) we can define a stream function ψ such that

$$\vec{u} = \hat{k} \times \nabla \psi \tag{2.154}$$

which readily satisfies the continuity equation. On the other hand, vorticity can be expressed as

$$\zeta = \hat{k} \cdot \nabla \times \vec{u} = \hat{k} \cdot \nabla \times \hat{k} \times \nabla \psi = \nabla^2 \psi. \tag{2.155}$$

Substituting in (2.151) yields

$$\nabla^2 \psi + \beta y + f_0 = \frac{h_1}{h_0}(f_0 + \beta y_0). \tag{2.156}$$

The stream function immediately near $x = 0$, approaching from $x > 0$ is given by

$$u_1 = -\frac{\partial \psi}{\partial y} \text{ at } x = 0 \tag{2.157.a}$$

so that

$$\psi = -u_1 y = -u_1 y_0 \tag{2.157.b}$$

since $y = y_0$ at $x = 0$. Now, substituting (2.157) into (2.156) we have

$$\nabla^2\psi + p^2\psi = -\beta y - (\frac{h_0 - h_1}{h_0}) f_0 \tag{2.158}$$

where

$$p^2 = \beta h_1/u_1 h_0 \tag{2.159}$$

The solution of the linear equation (2.158) can be written as

$$\psi = -(y + \frac{f_0}{\beta})F(x) - \frac{1}{p^2}[f_0(\frac{h_0 - h_1}{h_0}) + \beta y] \tag{2.160}$$

where the second term represents a particular solution. Substituting the above form into (2.158), we observe that $F(x)$ should satisfy

$$\frac{d^2F}{dx^2} + p^2 F = 0 \tag{2.161.a}$$

The boundary conditions are obtained by requiring (2.157.a) at x=0

$$u = -\frac{\partial\psi}{\partial y}|_{x=0} = -F(0) - \frac{\beta}{p^2} = u_1 \tag{2.161.b}$$

and requiring that the y-component of velocity should vanish at x=0

$$v|_{x=0} = \frac{\partial\psi}{\partial x}|_{x=0} = (y + \frac{f_0}{\beta})\frac{dF}{dx}|_{x=0} = 0. \tag{2.161.c}$$

Noting that $\beta/p^2 = u_0$, and $u_1 = (h_0/h_1)u_0$, and denoting $\Delta h = h_0 - h_1$, the above equations are simply

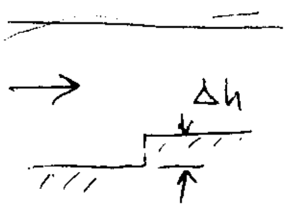
$$\begin{aligned} &\frac{d^2F}{dx^2} + \frac{\beta}{u_0}F = 0 \\ &F(0) = \frac{\Delta h}{h_1}u_0 \\ &F'(0) = 0 \end{aligned} \tag{2.162.a - c}$$

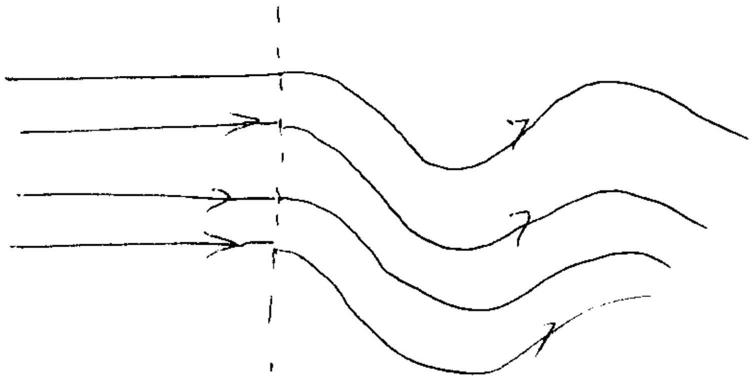
Integration of (2.162.a) with boundary conditions (2.162.a,b) yields

$$F(x) = \frac{\Delta h}{h_1}u_0 \cos\sqrt{\frac{\beta x}{u_0}}, \tag{2.163.a}$$

and, by substitution into (2.160), the full solution is obtained as follows

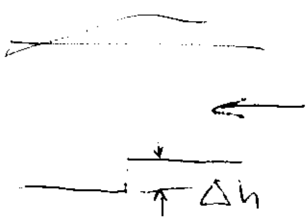
$$\psi = -\frac{u_0}{\beta}\frac{\Delta h}{h_1}(f_0 + \beta y)\cos\sqrt{\frac{\beta x}{u_0}} - \frac{u_0}{\beta}(f_0\frac{\Delta h}{h_1} + \beta y). \tag{2.163.b}$$

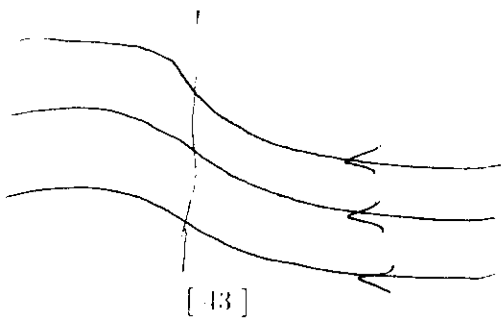




The solution is sketched above. For real valued p, (*i.e.*, if $u_0 > 0$ in the northern hemisphere) the solution is oscillatory in $x > 0$, with a wave length of $2\pi/p = 2\pi\sqrt{u_0/\beta}$. These waves, resulting from restoring forces in a β−plane, are called *Rossby waves*. Note thet the amplitude of the oscillatory part of (2.163) increases with latitude y. Note also that if we were to calculate an average position of a streamline for $y > 0$, by averaging the solution over one wavelength, we observe that the streamline is displaced to the south by a net distance $\Delta y = -\frac{f_0}{\beta}\frac{\Delta h}{h_1}$. At the step, the jet is deflected anticyclonically to the south by vorticity conservation, and as it moves south, gains vorticity by compensation of the decrease in planetary vorticity (β effect), and eventually has to turn back north when the initially negative fluid vorticity becomes positive. If the depth increased rather than decreased across the step, i.e. $\Delta h < 0$, then the jet would deflect north by an equal amount, gain positive vorticity which would decrease by compensation against inreasing planetary vorticity, and again form a wave motion with mean position of streamlines displaced to the north.

Note that if $p^2 < 0$ (if $u_1 < 0$ or if $u_0 < 0$, as in a westward flow, the solution would be exponentially decaying after the step, and since the first derivatives would have to be matched at the step, the curvature of the solution would have to continue before the step, and therefore the approaching flow would feel the step beforehand, as shown below.





The nature of the westward and eastward flows approaching a step are vastly different; which points us to the basic assymetry in geophysical flows. All of these new features are direct results of the *β-effect.*

2.4.4. Planetary waves

We have seen in the last Section that Rossby waves are generated by a uniform flow impinging on a north-south aligned depth discontinuity. This waves are a direct result of the variation of the coriolis parameters with latitude.

It is easy to show, by setting $\beta = 0$ in the solutions of Section 2.4.3, that the wave motion ceases to exist. Since only linear variations of f on a β-plane were considered to the lowest order, and since the f-variation actually occurs on a planetary scale, these waves are alternatively referred to as *planetary waves.* These are not the only the waves that can be sustained in rotating flow, but planetary waves are those waves resulting directly from latitude variation of the Coriolis effect.

The case of constant total depth, $H = H_0$, is the simplest configuration to show the existence of planetary waves. We shall also assume that the total depth can be approximated by the still-water depth h_0, *i.e.*, $H_0 = h_0 + \eta \simeq h_0$ since $\eta \ll h$ (the rigid-lid approximation). We can then use (2.154) and (2.155), *i.e.*

$$\vec{u} = \hat{k} \times \nabla\psi \tag{2.154}$$

$$\zeta = \nabla^2\psi, \tag{2.155}$$

to express potential vorticity conservation via equation (2.127):

$$\begin{aligned}
\frac{D}{Dt}\left(\frac{f_0 + \beta y + \zeta}{H_0}\right) &\simeq \frac{1}{h_0}\frac{D}{Dt}(\beta y + \nabla^2\psi) \\
&= \frac{1}{h_0}\{\frac{\partial}{\partial t}\nabla^2\psi + \vec{u}\cdot\nabla(\beta y + \nabla^2\psi)\} \\
&= \frac{1}{h_0}\{\frac{\partial}{\partial t}\nabla^2\psi + (\hat{k}\times\nabla\psi)\cdot\nabla(\beta y + \nabla^2\psi)\} \\
&= \frac{1}{h_0}\{\frac{\partial}{\partial t}\nabla^2\psi + \beta\frac{\partial\psi}{\partial x} + \hat{k}\times\nabla\psi\cdot\nabla(\nabla^2\psi)\} = 0
\end{aligned} \tag{2.164}$$

Here, we may introduce the definition of a *Jacobian:*

$$J(A,B) = \frac{\partial A}{\partial x}\frac{\partial B}{\partial y} - \frac{\partial A}{\partial y}\frac{\partial B}{\partial x} = (\hat{k}\times\nabla A)\cdot\nabla B = \hat{k}\cdot\nabla A\times\nabla B. \tag{2.165}$$

Setting $A = \psi$, $B = \nabla^2\psi$ in (2.165) and comparing with (2.164), the vorticity conservation equation becomes:

$$\frac{\partial}{\partial t}\nabla^2\psi + \beta\frac{\partial\psi}{\partial x} + J(\psi, \nabla^2\psi) = 0. \tag{2.166}$$

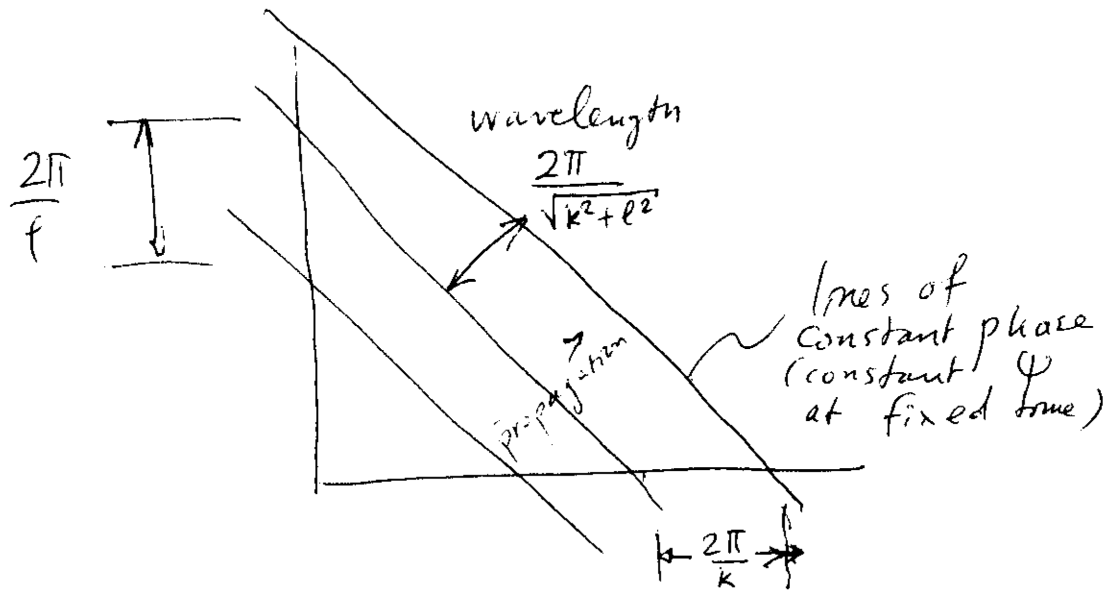
Note that this is a nonlinear equation since the Jacobian (2.165) is nonlinear by definition. If this equation is somehow linearized (assuming the Jacobian term is much smaller than the other terms), it would read

$$\frac{\partial}{\partial t}\nabla^2\psi + \beta\frac{\partial\psi}{\partial x} = 0 \tag{2.167}$$

Next we investigate the existence of a *plane-wave solution* of the form

$$\psi = \psi_0 e^{i(kx+ly-\omega t)} = \psi_0 e^{i\theta(x,y,t)}. \tag{2.168}$$

where k, l represent *wave number* in the x, y directions respectively, and ω represents the *angular frequency* of the motion. $\theta = (kx+ly-\omega t)$ is called the *phase* of the wave motion; when θ =constant it follows that ψ =constant. We sketch the wave motion at a fixed time, as follows:



Now, in substituting (2.168) into (2.167) we first note that

$$\nabla^2\psi = -\psi_0(k^2+l^2)e^{i(kx+ly-\omega t)} = (k^2+l^2)\psi$$

and therefore the Jacobian term would identically vanish, although we have not made an *a priori* assumption of linearity. It can also be verified to be true by virtue of an important property of the Jacobian: that if A and B are *linearly dependent*, *i.e.*, if $B = f(A)$, then (2.165) yields

$$J(A,B) = J(A,f(A)) = \frac{\partial A}{\partial x}\frac{\partial f}{\partial A}\frac{\partial A}{\partial y} - \frac{\partial A}{\partial y}\frac{\partial f}{\partial A}\frac{\partial A}{\partial x} \equiv 0 \tag{2.169}$$

Therefore, plane wave solutions of the form (2.168) do not contribute to the nonlinear terms, and the nonlinear equation (2.166) is equivalent to (2.167) in this special case.

Substituting (2.168) into (2.167) yields

$$-i\omega(-k^2 - l^2) + \beta ik = 0$$

or

$$\omega = -\frac{\beta k}{k^2 + l^2} \tag{2.170}$$

which is the ***dispersion relation*** for planetary waves. The *phase speed* in the x and y directions are respectively calculated as

$$C_x = \frac{\omega}{k} = -\frac{\beta}{k^2 + l^2} \tag{2.171.a}$$

$$C_y = \frac{\omega}{l} = -\frac{\beta}{k^2 + l^2}\frac{k}{l} \tag{2.171.b}$$

Note that $C_x < 0$ for all possible values of k, l, the wave propagation is always in the negative x-direction, *i.e. towards the west.* We also note that these waves can only exist if $\beta \neq 0$, *i.e.*, they arise as a result of the β-effect.

We can also define a *wave number vector* $\vec{\kappa}$ with components $\vec{\kappa} = (k,\ l)$. The phase speed in the direction of the wavenumber vector (or in the direction of phase propagation) is

$$C = \frac{\omega}{|\vec{\kappa}|} = \frac{\omega}{\sqrt{k^2 + l^2}} = -\frac{\beta k}{(k^2 + l^2)^{3/2}} = -\beta\frac{k}{|\vec{\kappa}|^3} \tag{2.172}$$

Note that the phase-speed does not satisfy vector decomposition (!) since the resultant of (2.171.a,b) does not satisfy (2.172). This means that the direction of wave propagation is not normal to the *wave fronts*. It is also seen that the phase speed is maximum for westward propagating waves ($|\vec{\kappa}| = k$ and zero for waves propagating in the north-south direction, *i.e.* wave patterns with north-south orientation do not propagate.

Note that, although the nonlinear terms vanish for any single plane-wave solution, the superposition of a number of plane-waves does not necessarily constitute a solution for the nonlinear equation (2.166).

The *group velocity* is the velocity of propagation of wave energy (*i.e.*, of wave packets), with components defined as

$$C_{gx} = \frac{\partial\omega}{\partial k}, \quad C_{gy} = \frac{\partial\omega}{\partial l} \tag{2.173.a, b}$$

These can be evaluated from (2.170) to be:

$$C_{gx} = \frac{\beta(k^2 - l^2)}{(k^2 + l^2)^2} \tag{2.174.a}$$

$$C_{gy} = \frac{2\beta kl}{(k^2 + l^2)^2} \tag{2.174.b}$$

It can be observed that although the phase velocity in the x-direction is always negative (westward phase propagation), the group velocity can be positive ($k > l$) or negative ($k < l$), *i.e.* while the wave is propagating west it can transfer energy in either of the horizontal directions. It is useful to construct a ***group velocity vector*** by using (2.173 a.b):

$$\vec{C}_g = \nabla_\kappa \omega = \hat{i}\frac{\partial \omega}{\partial k} + \hat{j}\frac{\partial \omega}{\partial l} \tag{2.175}$$

where ∇_κ denotes gradient in the direction of the wavenumber vector $\vec{\kappa}$.

Through definitions (2.172), the above can be modified to read as

$$\vec{C}_g = \nabla_\kappa |\vec{\kappa}| C = |\vec{\kappa}| \nabla_\kappa C + C \nabla_\kappa |\vec{\kappa}| = |\vec{\kappa}| \nabla_\kappa C + 2\frac{\vec{\kappa}}{|\vec{\kappa}|} C, \tag{2.176}$$

where $\vec{C}$ denotes phase speed (vector) in the direction of the wave number $\vec{\kappa}$. Since C depends on $\vec{\kappa}$ by virtue of (2.172), $\vec{C}_g$ and $\vec{C}$ differ in magnitude and direction. Such waves are called dispersive waves. [Nondispersive waves are those for which phase speed is independent of wave number, or for which group and phase velocities are equal].

Note that a Rossby wave is that it is a *transverse wave*. By making the rigid-lid assumption $\eta = 0$, and considering a constant depth case, the continuity equation (2.90) becomes

$$\nabla \cdot \vec{u} = 0. \tag{2.177}$$

By substituting the plane-wave solution

$$\vec{u} = \vec{u}_0 e^{i(kx+ly-\omega t)} = \vec{u}_0 e^{i(\vec{\kappa}\cdot\vec{x}-\omega t)} \tag{2.178}$$

into (2.177), it can be shown that

$$\vec{\kappa} \cdot \vec{u}_0 = 0, \tag{2.179}$$

i.e., the fluid velocity $\vec{u}$ is always perpendicular to the wave propagation direction $\vec{\kappa}$. This is a very important result, since it explains why the nonlinear terms $\vec{u} \cdot \nabla \vec{u}$ do not make any contribution for this type of wave, although we have not specificly made the linearity assumption.

In order to interpret the physical reason for the westward propagation of Rossby waves, consider the vorticity equation (2.167), which alternatively could be written as

$$\frac{\partial \zeta}{\partial t} + \beta v = 0 \tag{2.170}$$

According to this equation, a fluid particle with positive velocity in the y-direction ($v > 0$) will gain negative vorticity, and a particle with $v < 0$ will gain positive vorticity. Therefore, if we consider an initially straight line of fluid particles having zero initial vorticity, and displace this line to give it a sinusoidal form, the resulting vorticity distribution could be sketched as follows:

The train of vortices generated by the displacements will have a self induced velocity field carrying the pattern towards west. By virtue of (2.171.a,b), the solutions would be stationary for an observer moving west with speed C_x. Similarly, if the waves were superposed on an easterly current exactly opposing C_x, the resulting pattern would be stationary. This is the case for the north-south step problem discussed earlier in Section 2.4.3, where the wave adjusts its structure (*i.e.*, wavelength) to a stationary pattern in order match the boundary conditions at the step.

In the more general case of superposition of an easterly flowing current and planetary waves, whether the waves would appear to be propagating to the east or west depends on whether or not the speed of the current overcomes the westerly phase speed of the waves. For instance in the tropics, the mean flow in the atmosphere is in a westerly direction ("easterlies") so that atmospheric systems which typically have higher phase speeds most often travel west. In the mid-latitudes there are strong "westerlies" (*i.e.*, easterly flowing mean currents) which often overcome the phase speed of planetary waves, and therefore mid-latitude weather systems are often observed moving east.

2.5. Small Amplitude Motions with a Free Surface

In the foregoing sections, we have used the rigid-lid assumption ($\eta = 0$). To see the effects of surface displacement, we reconsider the shallow water equations (2.89) and (2.90) simplified by neglecting the forcing terms on the right hand side of (2.89). Furthermore, small amplitude motions will be considered by neglecting the nonlinear terms. For example, we assume

$$\frac{\partial \vec{u}}{\partial t} \gg \vec{u} \cdot \nabla \vec{u}$$

in equation (2.89), and in the second term of (2.90) expressed as

$$\nabla \cdot H\vec{u} = \nabla \cdot \eta\vec{u} + \nabla \cdot h\vec{u},$$

assuming that the nonlinear first term on the right hand side is much smaller than the second term ($\eta \ll h$). Then equations (2.89) and (2.90) are simplified to give

$$\frac{\partial \vec{u}}{\partial t} + f\hat{k} \times \vec{u} = -g\nabla\eta, \tag{2.181.a}$$

$$\frac{\partial \eta}{\partial t} + \nabla \cdot h\vec{u} = 0. \tag{2.181.b}$$

These are linear equations from which the unknowns $\vec{u}$ and η can be solved. We also assume that $f = f_0$=constant. In order to eliminate one of the unknowns, say $\vec{u}$, from the equations, we can multiply (2.178.a) by h, and take first the divergence, and then, the curl of this equation, yielding

$$\frac{\partial}{\partial t}\nabla \cdot h\vec{u} - f\hat{k} \cdot \nabla \times h\vec{u} = -g\nabla \cdot h\nabla\eta \tag{2.182.a}$$

$$\frac{\partial}{\partial t}\nabla \times h\vec{u} + f\hat{k}\nabla \cdot h\vec{u} = -g\nabla h \times \nabla\eta \tag{2.182.b}$$

Note that use has been made of the vector identities [DOI-1.27.b,c,d]. Then, $\nabla \times h\vec{u}$ is eliminated from (2.181.a,b), yielding

$$\left(\frac{\partial^2}{\partial t^2} + f^2\right) \nabla \cdot h\vec{u} = -g\{\frac{\partial}{\partial t}\nabla \cdot h\nabla\eta + f\hat{k} \cdot \nabla h \times \nabla\eta\}. \tag{2.183}$$

Utilization of (2.178.b) eliminates $\nabla \cdot h\vec{u}$ from the above equations:

$$\frac{\partial}{\partial t}\{\left(\frac{\partial^2}{\partial t^2} + f^2\right)\eta - g\nabla \cdot h\nabla\eta\} - gfJ(h,\eta) = 0, \tag{2.184}$$

where the Jacobian $J(h,\eta)$ is defined through (2.165). The last equation is essentially a wave equation for η and is much similar to the wave equation

$$\frac{\partial^2 \eta}{\partial t^2} = C_0^2\nabla^2\eta \tag{2.185}$$

(which occurs only if $f = 0$, $h = h_0$=constant), where

$$C_0^2 = gh_0, \tag{2.186}$$

C_0 being the phase speed of wave solutions that can be obtained for the simple case (2.185). Once η is obtained as solution of the equation (2.184), the velocity field can be obtained from (2.181.a), or following some manipulation, from the following equation:

$$\left(\frac{\partial^2}{\partial t^2} + f^2\right) \vec{u} = -g\left(\frac{\partial}{\partial t}\nabla\eta - f\hat{k} \times \nabla\eta\right). \tag{2.187}$$

2.5.1. Plane Waves for Constant Depth

Consider free oscillations in the case of a constant depth $h = h_0$. Further, assume plane wave solutions for (2.184), of the form

$$\eta = \Re\{\eta_0 e^{i(kx+ly-\omega t)}\} = \Re\{\eta_0 e^{i(\vec{\kappa}\cdot\vec{x}-\omega t)}\}. \tag{2.188}$$

Substitution of (2.188) into (2.184) gives

$$i\omega\eta_0[f^2 - \omega^2 + gh_0(k^2 + l^2)] = 0 \tag{2.189}$$

which is a dispersion relation for these waves. Letting $\kappa = |\vec{\kappa}|$ be the magnitude of the wave number vector $\vec{\kappa} = (k, l)$, the above equation is simplified to

$$\omega = \pm\sqrt{f^2 + C_0^2\kappa^2} \tag{2.190}$$

where C_0 is given by (2.186). Since the wave frequency ω depends on κ, the waves are *dispersive*. For each value of the wave number κ there are two waves with opposite phase speeds of

$$C = \frac{\omega}{\kappa} = \pm\sqrt{C_0^2 + \frac{f^2}{\kappa^2}} \tag{2.191}$$

In the non-rotating case of $f = 0$, $C = C_0 = \pm\sqrt{gh_0}$, and the waves propagate with the classical shallow water wave speed, in a *non-dispersive* mode. Therefore, we can see that rotation introduces dispersion, as well as increasing the wave speed. Also by virtue of (2.190), it is clear that all possible wave frequencies $|\omega|$ must exceed f, corresponding to *super-inertial* frequencies ($|\omega| > f$). The velocity field can be obtained by substituting

$$\vec{u} = \Re\{\vec{u}_0 e^{i(\vec{\kappa}\cdot\vec{x}-\omega t)}\} \tag{2.192}$$

and (2.188) into (2.187), yielding

$$(f^2 - \omega^2)\vec{u}_0 = -g(\omega\vec{\kappa} - if\hat{k}\times\vec{\kappa})\eta_0. \tag{2.193}$$

The right hand side has two terms: one parallel and the other perpendicular to the $\vec{\kappa}$ vector. For $\omega \gg \kappa$ (*gravity waves*) the above equation reduces to

$$\vec{u}_0 \simeq \frac{g}{\omega}\eta_0\vec{\kappa} \tag{2.194}$$

and the particle velocity is parallel to the wavenumber vector, *i.e.* the motion is a *longitudinal wave*. In the general case, equation (2.193) indicates that the velocity vector makes an angle with the propagation direction. This more general case of motion is called an *inertia-gravity wave*, characterized by a mixture of longitudinal and transverse modes. The vector $\vec{u}_0$ can be decomposed into two components $\vec{u}_0 = (u_{0_\parallel},\ v_{0_\perp})$ such that (using 2.193),

$$u_{0_\parallel} = \vec{u}_0 \cdot \frac{\vec{\kappa}}{|\vec{\kappa}|} = -\frac{g\omega|\vec{\kappa}|}{f^2-\omega^2}\eta_0, \tag{2.195.a}$$

$$u_{0_\perp} = \vec{u}_0 \cdot \frac{\hat{k}\times\vec{\kappa}}{|\vec{\kappa}|} = i\frac{gf|\vec{\kappa}|}{f^2-\omega^2}\eta_0. \tag{2.195.b}$$

Substituting from (2.190), and using $\omega/\kappa = C_0 = \sqrt{gh_0}$, one obtains

$$u_{0_\parallel} = \frac{g\omega\kappa\eta_0}{C_0^2\kappa^2} = -\frac{C_0}{h_0}\eta_0 \tag{2.196.a}$$

$$u_{0_\perp} = \frac{gf\kappa\eta_0}{C_0^2\kappa^2} = i\frac{C_0}{\eta_0}\frac{f}{\omega}\eta_0 \tag{2.196.b}$$

Note that, in general, η_0, $u_{0_\parallel}$, $u_{0_\perp}$ are complex numbers. Velocity components can be calculated from (2.192), by similarly decomposing the velocity vector as $\vec{u} = (u_\parallel, u_\perp)$,

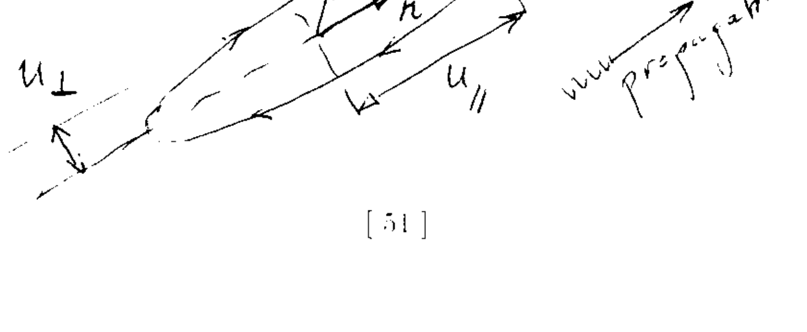
$$\begin{aligned} u_\parallel &= \Re\{-\frac{C_0}{h_0}\eta_0 e^{i(\vec{\kappa}\cdot\vec{x}-\omega t)}\} \\ &= -\frac{C_0}{h_0}|\eta_0|\cos(kx+ly-\omega t+\phi) \end{aligned} \tag{2.197.a}$$

$$\begin{aligned} u_\perp &= \Re\{i\frac{C_0}{h_0}\frac{f}{\omega}\eta_0 e^{i(\vec{\kappa}\cdot\vec{x}-\omega t)}\} \\ &= -\frac{C_0}{h_0}\frac{f}{\omega}|\eta_0|\sin(kx+ly-\omega t+\phi) \end{aligned} \tag{2.197.b}$$

where ϕ is the phase of η_0, such that $\eta_0 = |\eta_0|e^{i\phi}$. It can thus be shown, by combining (2.197.a,b), that

$$u_\parallel^2 + u_\perp^2\left(\frac{\omega}{f}\right)^2 = \left(\frac{C_0|\eta_0|}{h_0}\right)^2 \tag{2.198}$$

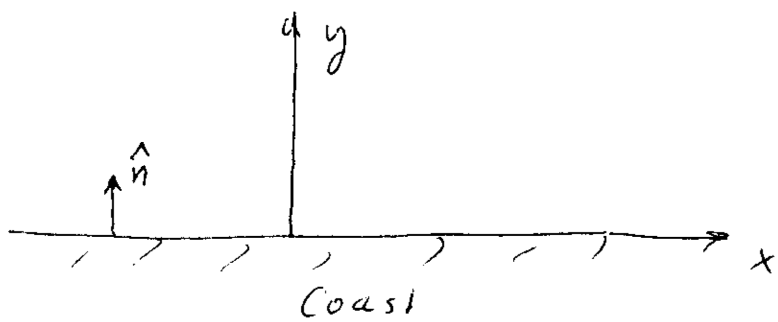
which is the equation for an ellipse alongated in the propagation direction.



Note that by virtue of (2.197.a) and (2.197.b) the current vector is rotating in the clockwise direction, because of the clockwise deflection of the Coriolis effect. For $\omega >> f$ (***gravity waves***), the ellipse becomes narrower and becomes aligned in the direction of propagation ($u_{0_\perp}$ becomes vanishingly small by virtue of 2.196.b). On the other hand, as $\omega \to f$ (*inertial motion*), both components of velocity become equal and the ellipse becomes a circle. In the general case of ***inertia-gravity waves*** ($\omega > f$) the current ellipse is aligned in the propagation direction. While it is not directly applicable here, it is worth noting that for the case $\omega < f$, the ellipse is oriented perpendicular to the propagation, and only in the limit $\omega \to 0$ (*geostrophic motion*) the current becomes perpendicular to the isolines of surface elevation (isobars). Therefore, it can be seen that, in general, inertia-gravity waves are far from being geostrophic. Certainly they are another class of waves much different from Rossby waves.

2.5.2. Poincare and Kelvin Waves

Consider now a coast aligned with the x-axis:



Let the depth be constant, $h = h_0$. Then equation (2.184) reduces to

$$\frac{\partial}{\partial t}\left\{\left(\frac{\partial^2}{\partial t^2} + f^2\right)\eta - C_0^2\nabla^2\eta\right\} = 0, \tag{2.199}$$

where $C_0^2 = gh_0$ as in (2.186). The boundary condition at the coast is

$$\vec{u}\cdot\hat{n} = \vec{u}\cdot\hat{j} = 0 \tag{2.200}$$

i.e., the normal velocity at the coast must vanish. this boundary condition can be conveniently expressed, using (2.187) as

$$\left(\frac{\partial}{\partial t}\nabla\eta - f\hat{k}\times\nabla\eta\right)\cdot\hat{j} = -\frac{1}{g}\left(\frac{\partial^2}{\partial t} + f^2\right)\vec{u}\cdot\hat{n} = 0 \quad \text{on } y = 0 \tag{2.201}$$

or as :

$$\frac{\partial^2 \eta}{\partial t \partial y} - f\frac{\partial \eta}{\partial x} = 0 \quad \text{on } y = 0. \tag{2.202}$$

Another boundary condition is needed at the open boundary, which will be the requirement that

$$\eta \rightarrow \text{ bounded, as } y \rightarrow \infty. \tag{2.203}$$

We will consider solutions which are periodic along the coast and with respect to time:

$$\eta = \Re\{N(y)e^{i(kx-\omega t)}\} \tag{2.204}$$

which, upon substitution into (2.199), (2.202) and (2.203), gives the set

$$\frac{d^2N}{dy^2} + \{\frac{\omega^2 - f^2}{C_0^2} - k^2\}N = 0 \tag{2.205}$$

$$\frac{dN}{dy} + f\frac{k}{\omega}N = 0 \quad \text{on } y = 0 \tag{2.206}$$

$$N \rightarrow \text{ bounded, as } y \rightarrow \infty. \tag{2.207}$$

For simplicity, let

$$l^2 = \frac{\omega^2 - f^2}{C_0^2} - k^2, \tag{2.208}$$

which is the same expression as the dispersion relation (2.189) or (2.190).

Now the solution has different character for different values of l, as shown in the following cases.

`2.5.2.1. Solution I - Poincare Waves` ($l^2 \geq 0$)

For $l^2 \geq 0$ (l is a real number), the solution to (2.205) is

$$N(y) = Ae^{ily} + Be^{-ily} \tag{2.209}$$

upon substituting in boundary condition (2.206), this gives

$$\left(il + f\frac{k}{\omega}\right)A + \left(-il + f\frac{k}{\omega}\right)B = 0. \tag{2.210}$$

The boundary condition (2.207) is readily satisfied by the form of the solution. The solution (2.204) is then:

$$\eta = \Re\{Ae^{i(kx+ly-\omega t)} + Be^{i(kx-ly-\omega t)}\} \tag{2.211}$$

This solution, then, represents an incident wave with amplitude B and a reflected wave with amplitude A, whose ratio is given by (2.210)

$$\frac{A}{B} = \frac{is-1}{is+1} = -\frac{(is-1)^2}{1+s^2} = -\frac{1-s^2-2is}{1+s^2} \tag{2.212}$$

where

$$s = \frac{l}{k}\frac{\omega}{f}. \tag{2.213}$$

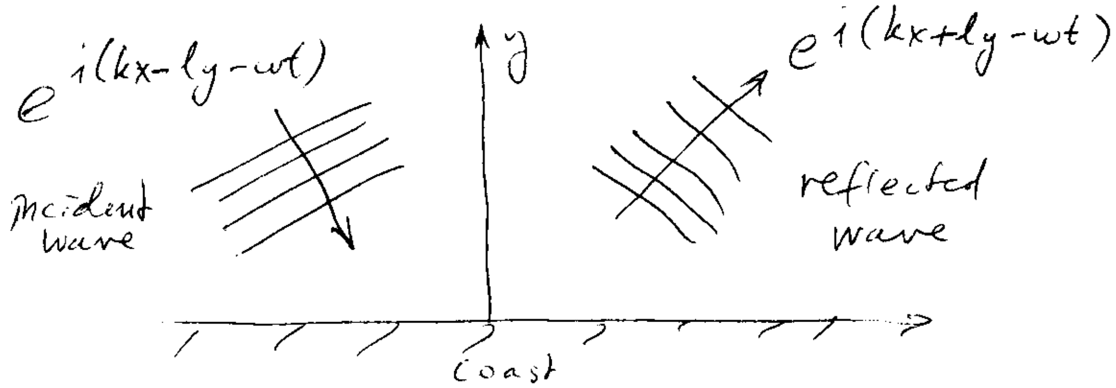
Substituting

$$B = |B|e^{i\delta} \tag{2.214}$$

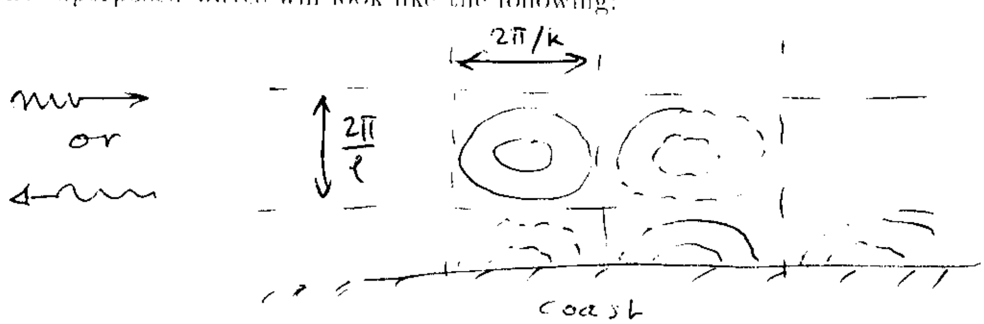
yields

$$\eta = |B|\Re\{e^{i(kx-ly-\omega t+\delta)} - \frac{1-s^2-2is}{1+s^2}e^{i(kx+ly-\omega t+\delta)}\} \tag{2.215}$$

as the solution.



The superposed waves will look like the following:



Note that the coefficient of the reflected wave can be written as

$$\begin{aligned} -\frac{1-s^2-2is}{1+s^2} &= -\frac{\sqrt{(1-s^2)^2+(2s)^2}}{1+s^2}e^{-i\ \tan^{-1}(\frac{2s}{1-s^2})} \\ &= -\frac{\sqrt{1+s^4-2s^2+4s^2}}{1+s^2}e^{-i\ \tan^{-1}(\frac{2s}{1-s^2})} \\ &= -1e^{-i\ \tan^{-1}(\frac{2s}{1-s^2})} = e^{i\theta} \end{aligned} \tag{2.216}$$

with θ replacing

$$\theta = \pi - \tan^{-1}(\frac{2s}{1-s^2}),$$

so that (2.215) becomes

$$\eta = |B|\{\cos(kx - ly - \omega t + \delta) + \cos(kx + ly - \omega t + \delta + \theta)\} \tag{2.217}$$

i.e. the only change in the reflected wave is that it merely suffers a phase shift.

These waves are called *Poincaré waves*. A special case occurs when $l = 0$, (*i.e.* waves propagating along the coast) in which

$$\omega^2 = \omega_0^2 = f^2 + C_0^2 k^2 \tag{2.218}$$

gives the minimum frequency ω_0 possible for Poincaré waves. By virtue of (2.208), Poincaré waves with $l \neq 0$ always have

$$\omega > \omega_0. \tag{2.219}$$

Note also that for all possible cases, $\omega > f$ (period of motion is less than the inertial period)

2.5.2.2. Solution II - Kelvin Wave ($l^2 < 0$):

$$l^2 = -\alpha^2 < 0 \tag{2.220}$$

For $l^2 < 0$ (l is an imaginary number), we can interchange l with α, such that $\alpha^2 = -l^2 > 0$ (α is real). The solution to (2.205) is expressed as

$$N = Ae^{+\alpha y} + Be^{-\alpha y}. \tag{2.221}$$

Since the first term can not satisfy the boundary condition (2.207), we must have

$$A = 0. \tag{2.222}$$

The boundary condition (2.206) requires

$$\left(-\alpha + f\frac{k}{\omega}\right) B = 0 \tag{2.223}$$

and using (2.208) and (2.220) in (2.223),

$$\alpha^2 = k^2 - \frac{\omega^2 - f^2}{C_0^2} = \frac{f^2 k^2}{\omega^2}, \tag{2.224}$$

and this reduces to

$$\frac{\omega^2 - f^2}{C_0^2} = \left(1 - \frac{f^2}{\omega^2}\right) k^2 = \frac{\omega^2 - f^2}{\omega^2} k^2,$$

or simply

$$\left(\frac{\omega}{k}\right)^2 = C_0^2 \tag{2.225}$$

This equation has both positive and negative roots $\omega/k = \pm C_0$. On the other hand, since $\alpha > 0$, equation (2.223) requires that only a positive root can be accepted:

$$\frac{\omega}{k} = +C_0 = \sqrt{gh_0}. \tag{2.226}$$

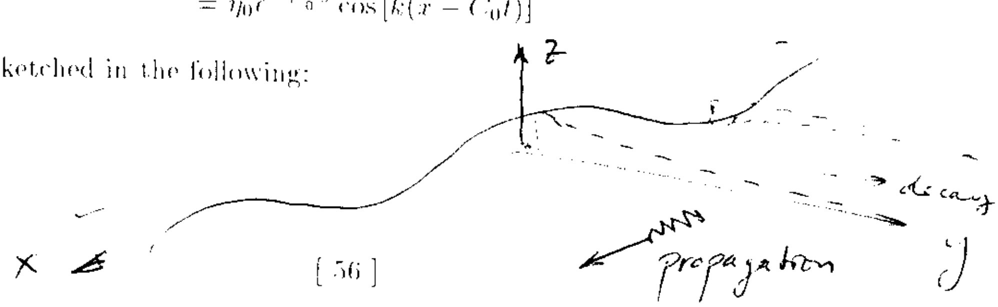
The choice of only one of the roots imposes a preferential direction to the motion. Furthermore, this simple result in itself is remarkable, yielding a *non-dispersive wave solution*, propagating with the classical shallow water phase speed C_0, (for a non-rotating fluid), despite the fact that the wave motion owes its existence to rotation. The decay parameter alpha is calculated from (2.224) and (2.226) as

$$\alpha = \frac{f}{C_0} = \frac{f}{\sqrt{gh_0}}. \tag{2.227}$$

Substituting (2.227) and (2.226), and letting $B = \eta_0$ (a real number) without loss of generality, the solution (2.204) takes the following form:

$$\begin{aligned} \eta &= \eta_0 \Re\{e^{-\frac{f}{C_0}y} e^{ik(x - C_0 t)}\} \\ &= \eta_0 e^{-\frac{f}{C_0}y} \cos[k(x - C_0 t)] \end{aligned} \tag{2.228}$$

The wave motion is sketched in the following:



Since $C_0 > 0$, (2.228) indicates that the *Kelvin wave* always travels in the positive x-direction for increasing t, *i.e. it takes the coast to its right.* The wave motion has a maximum amplitude of η_0 at the coast and decays offshore within an e-folding distance of

$$R = \frac{C_0}{f} = \frac{\sqrt{gh_0}}{f}, \tag{2.229}$$

which is the ***Rossby radius of deformation.*** An estimate, with $h_0 = 1000m$, $f = 10^{-4} rad/sec$, for the ocean gives $R \simeq 1000km$; *i.e.* the wave amplitude decays within several thousands of kilometers from the coast (!). This solution does not seem to be of physical significance in the ocean, but its analogue in the atmosphere is feasible. In the ocean, a similar pattern of motion, the internal Kelvin wave, occurs only in a stratified fluid. The velocity $\vec{u}$ is obtained from (2.187), by assuming a form of

$$\vec{u} = \Re\{\vec{u}_0 e^{-\frac{f}{C_0}y} e^{ik(x-C_0t)}\} \tag{2.230}$$

and by substituting this in (2.187):

$$\begin{aligned}(f^2 - k^2C_0^2)\vec{u}_0 &= -g\{-ikC_0\left(ik\hat{\imath} - \frac{f}{C_0}\hat{\jmath}\right) + f\left(-\frac{f}{C_0}\hat{\imath} - ik\hat{\jmath}\right)\}\eta_0 \\ &= -g\{\left(k^2C_0 - \frac{f^2}{C_0}\right)\hat{\imath} + i(fk - fk)\hat{\jmath}\}\eta_0 \\ &= g\frac{\eta_0}{C_0}(f^2 - k^2C_0^2)\hat{\imath}.\end{aligned} \tag{2.231}$$

This result shows that the velocity field has only a u component. We have

$$u_0 = \vec{u}_0 \cdot \hat{\imath} = \frac{g\eta_0}{C_0} = \frac{\eta_0}{h_0}C_0, \qquad v_0 = \vec{u}_0 \cdot \hat{\jmath} = 0. \tag{2.232.a, b}$$

and by (2.228) and (2.230), we obtain for the velocity components

$$u = \vec{u} \cdot \hat{\imath} = \frac{\eta_0}{h_0}C_0 e^{-\frac{f}{C_0}y}\cos k(x - C_0t), \qquad v = \vec{u} \cdot \hat{\jmath} = 0. \tag{2.233.a, b}$$

Since the v-velocity vanishes everywhere, the corresponding terms in the shallow water equations (2.181.a,b) vanish, to yield

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} \tag{2.234.a}$$

$$fu = -g\frac{\partial \eta}{\partial y} \tag{2.234.b}$$

$$\frac{\partial \eta}{\partial t} + h_0\frac{\partial u}{\partial x} = 0 \tag{2.234.c}$$

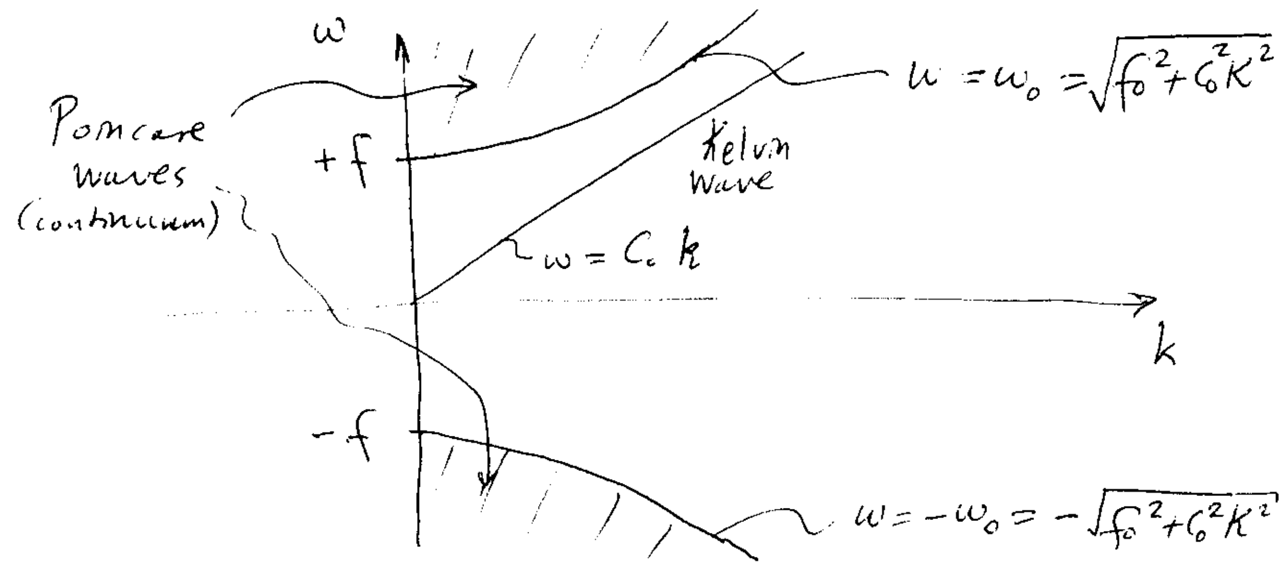
The equations with the remaining terms indicate an interesting balance. While the alongshore acceleration is driven by the alongshore pressure gradient, the velocity is in geostrophic balance with the cross-shore pressure gradient. Elimination of u from (2.234.a), (2.234.c) gives

$$\frac{\partial^2 \eta}{\partial t^2} = C_0^2 \frac{\partial^2 \eta}{\partial x^2} \tag{2.235}$$

i.e., a classic wave equation for the propagation (with speed C_0) in the x-direction. Of the two possible wave solutions with phase speeds $\pm C_0$,only the positive valued solution can satisfy the boundary conditions.

2.5.2.3. Dispersion diagrams

For both of the above solutions, *i.e.* the Poincaŕe and Kelvin waves, a dispersion diagram can be constructed:



The Poincaré waves occupy the shaded regions in which all frequencies that satisfy the dispersion relation (2.208) are allowable. For the Kelvin wave, there is only one wave number satisfying the dispersion relation (2.226) at each frequency.

2.5.3. Topographic Rossby Waves

2.5.3.1. One Dimensional Depth Variations

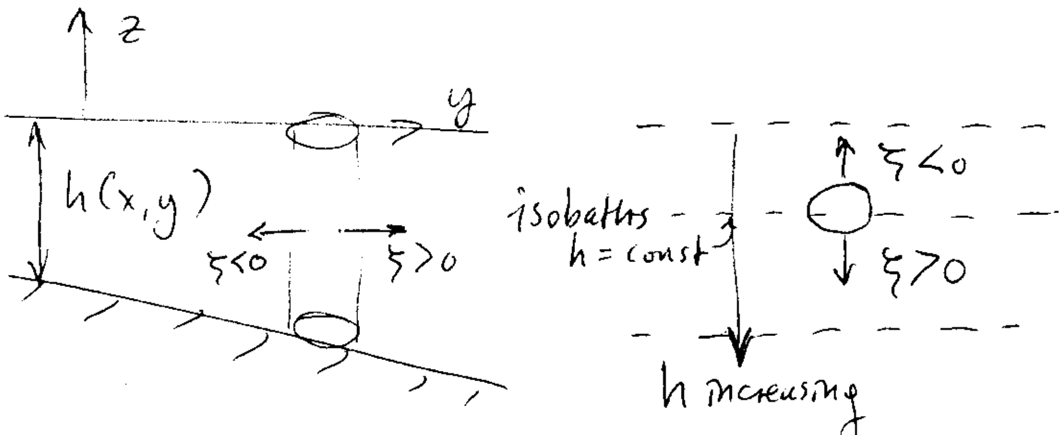
Let us now consider the effects of depth variation. The governing equation (2.184) can be written as

$$\frac{\partial}{\partial t}\{\left(\frac{\partial_2}{\partial t^2} + f^2\right)\eta - g\left(\frac{\partial}{\partial x}h\frac{\partial \eta}{\partial x} + \frac{\partial}{\partial y}h\frac{\partial \eta}{\partial y}\right)\} - fg\left(\frac{\partial h}{\partial x}\frac{\partial \eta}{\partial y} - \frac{\partial h}{\partial y}\frac{\partial \eta}{\partial x}\right) = 0. \tag{2.236}$$

Without loss of generality, consider depth variations in the y-direction, $h = h(y)$. Then (2.236) reduces to

$$\frac{\partial}{\partial t}\{\left(\frac{\partial^2}{\partial t^2}+f^2\right)\eta-g\left(h\frac{\partial^2\eta}{\partial x^2}+\frac{\partial}{\partial y}h\frac{\partial\eta}{\partial y}\right)\}+fg\frac{\partial h}{\partial y}\frac{\partial\eta}{\partial x}=0. \tag{2.237}$$

The depth variations induce a restoring effect, as seen through the potential vorticity conservation (2.109), in a much similar way to the β-effect. Motion across isobaths will cause stretching or extension of fluid columns and thus generate vorticity.



Therefore we may expect topographic Rossby waves of the form

$$\eta=\Re\{N(y)e^{i(kx-\omega t)}\} \tag{2.238}$$

which reduces (2.237) to

$$-i\omega\{(f^2-\omega^2)N-g\left(-k^2hN+h\frac{d^2N}{dy^2}+\frac{dh}{dy}\frac{dN}{dy}\right)\}+fg\left(-ikN\frac{dh}{dy}\right)=0$$

or

$$N''+\left(\frac{h'}{h}\right)N'-\left(\frac{f^2-\omega^2}{gh}+k^2-f\frac{k}{\omega}\frac{h'}{h}\right)N=0, \tag{2.239}$$

where primes denote differentiation with respect to y. We can further denote

$$F(y)=\frac{1}{h}\frac{dh}{dy}=\frac{h'}{h} \tag{2.240}$$

and substitute

$$N(y)=M(y)e^{-\frac{1}{2}\int F(y)dy} \tag{2.241}$$

to transform the variables into a new form:

$$
\begin{aligned}
N &= M e^{-\frac{1}{2}\int F dy} \\
N' &= (M' - \frac{1}{2} F M) e^{-\frac{1}{2}\int F dy} \\
N'' &= \left[M'' - F M' + \left(\frac{1}{4} F^2 - \frac{1}{2} F' \right) M \right] e^{-\frac{1}{2}\int F dy}
\end{aligned}
\qquad (2.242.a-c)
$$

Then, using these substitutions in (2.239) reduces it to

$$
M'' - \left[\frac{f^2 - \omega^2}{gh} + k^2 - \frac{f}{\omega}\frac{h'}{h} k + \frac{1}{4}\left(\frac{h'}{h}\right)^2 + \frac{1}{2}\left(\frac{h'}{h}\right)' \right] M = 0. \qquad (2.243)
$$

Consider the flow on a sloping bottom adjacent to a coast:

At the coast ($y = 0$), we must require the normal velocity to vanish $\vec{u} \cdot \hat{\jmath} = 0$, so that boundary condition (2.206) applies. With the substitutions (2.242 a-b), (2.206) reads

$$
M' + \left(\frac{f}{\omega} k - \frac{1}{2}\frac{h'}{h} \right) M = 0 \quad \text{on } y = 0. \qquad (2.244)
$$

Far away from the coast, the motion must vanish, so that boundary condition (2.207) applies. But if N is bounded as $y \to \infty$, we have by virtue of (2.241),

$$
M \to \text{ bounded, as } y \to \infty. \qquad (2.225)
$$

A solution can then be obtained for (2.243), with boundary conditions (2.244) and (2.245). By virtue of (2.240) and (2.241), we have

$$
\begin{aligned}
N &= M(y) e^{-\frac{1}{2}\int \frac{1}{h}\frac{dh}{dy} dy} \\
&= M(y) e^{-\frac{1}{2}\int \frac{d \ln h}{dy} dy} \\
&= M(y) [e^{\ln h}]^{\frac{1}{2}} \\
&= h^{-\frac{1}{2}(y)M(y)},
\end{aligned}
\qquad (2.246)
$$

so that the assumed solution (2.238) can be written as

$$\eta = \Re\{h^{-\frac{1}{2}(y)} M(y) e^{i(kx-\omega t)}\} \tag{2.247}$$

where $M(y)$ is determined by solving (2.243) with boundary conditions of the type (2.244) and (2.245) imposed.

Note that (2.243) is a differential equation with non-constant coefficients. Furthermore equation (2.243) with boundary conditions of type (2.244) and (2.245) constitute an eigenvalue problem, which must yield eigenvalues which relate ω to k. However the terms in square brackets of (2.243) are extremely complex. For example, assuming an eigenvalue is determined, these terms are cubic in frequency ω and quadratic in wave number k. Therefore analytic solutions can be obtained only in simple cases where a certain simple topography is assumed and with further possible simplifications.

A further simplification can often be made in (2.243) with respect to the first term in the brackets. In fact, let L be the horizontal y-scale of the motion, and let h_0 be a depth scale; then comparing the first and second terms of (2.243), we have

$$\frac{(\frac{f^2-\omega^2}{gh}M)}{(\frac{d^2M}{dy^2})} = \frac{O(\frac{f^2M}{gh_0})}{O(\frac{M}{L^2})} = O\left(\frac{f^2L^2}{gh_0}\right) = O(\delta) \tag{2.248}$$

where

$$\delta \equiv \frac{f^2L^2}{gh_0} \tag{2.249}$$

is defined as the *divergence parameter*. Note that this parameter is equal to the ratio

$$\delta = \left(\frac{L}{R}\right)^2 \tag{2.250}$$

where R is the Rossby radius of deformation defined in (2.229). The scale L can be thought of as the horizontal scale in which depth variations occur (*i.e.*, the shelf width).

For typical values of $f = 10^4\ rad/sec$, $L = 10km$, $g = 10m/sec^2$, $h_0 = 100m$; δ is calculated to be $\delta \simeq 10^{-3} \ll 1$, and therefore the second term of (2.243) is much smaller than the first. With this approximation ($\delta \ll 1$) in place, (2.243) reads

$$M'' - \left[k^2 - f\frac{k}{\omega}\left(\frac{h'}{h}\right) + \frac{1}{4}\frac{h'^2}{h} + \frac{1}{2}\left(\frac{h'}{h}\right)'\right] M = 0. \tag{2.251}$$

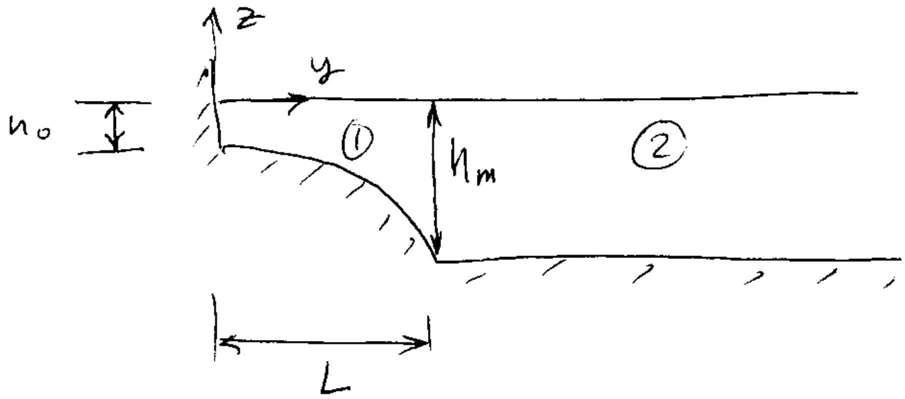
The approximation for the horizontal scale being much less than the Rossby radius ($\delta \ll 1$) is in fact equivalent to the rigid-lid approximation, and amounts to neglecting the $\frac{\partial\eta}{\partial t}$ term in the continuity equation (2.181.b). This can simply be verified by making the appropriate substitution in equation (2.183), *i.e.* $\nabla \cdot h\vec{u} = 0$, and similarly causes the first term of (2.243) to vanish.

2.5.3.2. Shelf Waves - Exponential Shelf

As an example of simple case with an analytical solution, consider the coast adjoined by the following shelf topography

$$h(y) = \begin{cases} h_0 e^{2by}, & 0 < y < l \\ h_0 e^{2bL} = h_m, & L < y < \infty, \end{cases} \tag{2.252}$$

visualised in the following sketch:



Region 1 and 2 are the continental shelf and deep ocean regions respectively. In region 1 (the shelf), (2.240) is a constant

$$F(y) = \frac{h'}{h} = \frac{2bh_0 e^{2by}}{h_0 e^{2by}} = 2b \tag{2.253}$$

and in the deep ocean region

$$F(y) = 0, \tag{2.254}$$

so that the corresponding equations (2.243) become:

region 1:

$$M_1'' - [\frac{f^2 - w^2}{gh_0} e^{-2by} + k^2 - \frac{f}{\omega} 2bk + b^2] M_1 = 0 \tag{2.255.a}$$

region 2:

$$M_2'' - [\frac{f^2 - w^2}{gh_m} + k^2] M_2 = 0 \tag{2.255.b}$$

The second term of (2.255.a), which gives rise to non-constant coefficients, can be neglected with the rigid-lid approximation, $\delta \ll 1$, yielding:

region 1:

$$M_1'' - [k^2 - \frac{f}{\omega}2bk + b^2]M_1 = 0 \qquad (2.256.a)$$

region 2:

$$M_2'' - k^2 M_2 = 0 \qquad (2.256.b)$$

We define

$$\gamma^2 = k^2 - \frac{f}{\omega}2bk + b^2 \qquad (2.257)$$

to replace the constant coefficient in the first equation.

The boundary condition at the coast is obtained by substituting from (2.253),

$$M_1' + [f\frac{k}{\omega} - b]M_1 = 0 \text{ at } y = 0 \qquad (2.258.a)$$

We will be interested in waves that are trapped in the shelf region, so that far from the coast we not only want the solution be bounded, but also to vanish:

$$M_2 \to 0 \text{ as } y \to \infty \qquad (2.258.b)$$

Finally we need jump conditions at $x = L$ where the shelf joins the deep ocean. For this we require that η and $\vec{u} \cdot \hat{\jmath}$ be continuous at the the junction,yielding:

$$M_1 = M_2 \text{ at } y = L \qquad (2.258.c)$$

$$M_1' + \left(\frac{f}{\omega}k - b\right) M_1 = M_2' + \left(\frac{f}{\omega}k\right) M_2 \text{ at } y = L \qquad (2.258.d)$$

The solution to (2.256.a,b) can be written as

$$M_1 = Ae^{\gamma(y-L)} + Be^{-\gamma(y-L)} \qquad (2.259.a)$$

$$M_2 = Ce^{k(y-L)} + De^{-k(y-L)} \qquad (2.259.b)$$

Boundary condition (2.258.b) then requires that

$$C = 0,$$

while boundary conditions (2.258.a,c,d)require:

$$(\gamma + \frac{f}{\omega}k - b)e^{-\gamma L}A + (-\gamma + \frac{f}{\omega}k - b)e^{\gamma L}B = 0 \quad (2.260.a)$$

$$A + B = D \quad (2.260.b)$$

$$\left(\gamma + \frac{f}{\omega}k - b\right)A + \left(-\gamma + \frac{f}{\omega}k - b\right)B = (-k + sk)D \quad (2.260.c)$$

and eliminating D from 2.260.b,c we have

$$(\gamma + k - b)\,A + (-\gamma + k - b)\,B = 0 \quad (2.261)$$

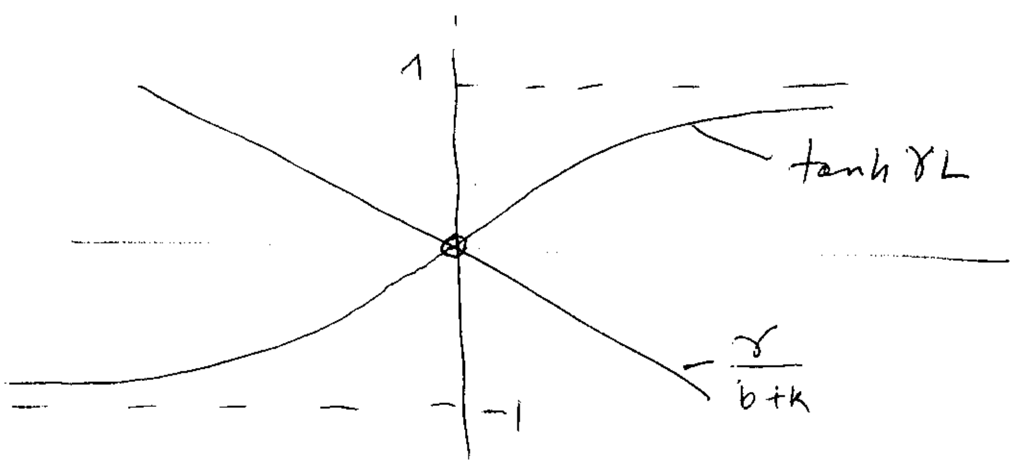
For non trivial values of A and B, (2.260.a) and (2.261) then yield

$$\left(\gamma + \frac{f}{\omega}k - b\right)(-\gamma + k - b)e^{-\gamma L} - \left(-\gamma + \frac{f}{\omega}k - b\right)(\gamma + k - b)e^{\gamma L} = 0, \quad (2.262)$$

or, rearranging terms, and utilizing (2.257)

$$\tanh \gamma L = -\frac{\gamma}{b + k}. \quad (2.263)$$

This is a transcendental equation for which the roots γ_n must be obtained. Graphically the right and left hand sides can be sketched as follows:



The only possible root is $\gamma = 0$ which is trivial. In (2.257) we have imperatively assumed that $\gamma^2 > 0$ (γ is real), but we now find that non-trivial solutions are not possible. However, if we let $\gamma^2 < 0$ (γ is imaginary),

$$\gamma^2 = -\mu^2 < 0 \tag{2.264}$$

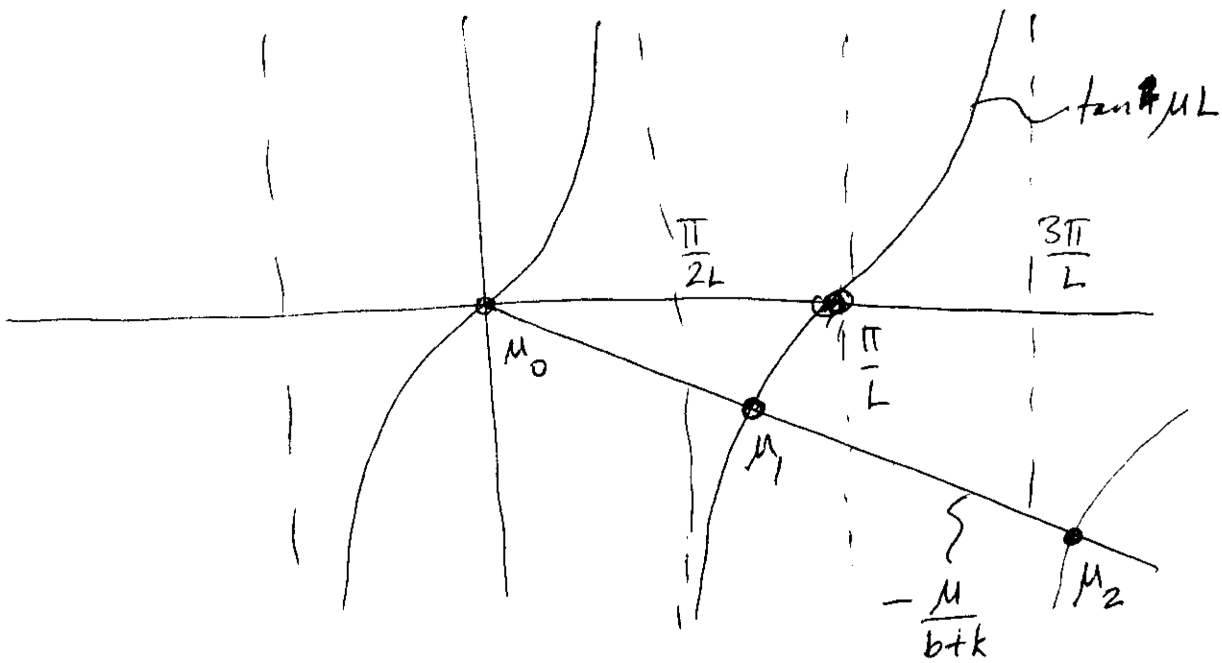
($\mu^2 > 0$, μ is real), then (2.257) becomes

$$\mu^2 = \frac{f}{\omega} 2bk - b^2 - k^2 \tag{2.265}$$

and (2.263) becomes (with $\gamma = i\mu$),

$$\tanh i\mu L = i \tan \mu L = -i \frac{\mu}{b+k}. \tag{2.266}$$

This equation possesses an infinite number of discrete roots $\mu_n = \mu_1, \mu_2, \ldots (n > 0)$



and by virtue of (2.265) an infinite number of discrete frequencies

$$\frac{\omega}{f} = \frac{2bk}{b^2 + k^2 + \mu_n^2}. \tag{2.267}$$

where μ_n are the roots of (2.266) for $n > 0$. The solutions (2.259.a,b) become

$$M_1 = A_n e^{i\mu_n(y-L)i} + B_n e^{-i\mu_n(y-L)} \tag{2.268.a}$$

$$M_2 = D_n e^{-k(y-L)} \tag{2.268.b}$$

and the relations between the complex-valued coefficients A_n, B_n and D_n are given by (2.260.b) and (2.261). By making use of (2.266),

$$\begin{aligned}\frac{B_n}{A_n} &= -\frac{k-b+i\mu_n}{k-b-i\mu} \\ &= -\frac{(k-b)-(k+b)\tanh i\mu_n L}{(k-b)+(k+b)\tanh i\mu_n L} \\ &= -\frac{k(1-\tanh i\mu_n L)-b(1+\tanh i\mu_n L)}{k(1+\tanh i\mu_n L)-b(1-\tanh i\mu_n L)} \\ &= -\frac{ke^{-i\mu_n L}-be^{i\mu_n L}}{ke^{i\mu_n L}-be^{-i\mu_n L}}\end{aligned} \tag{2.269.a}$$

$$\begin{aligned}\frac{D_n}{A_n} = 1+\frac{B_n}{A_n} &= -\frac{2i\mu_n}{k-b-i\mu_n} \\ &= \frac{2(k+b)\tanh i\mu_n L}{(k-b)+(k+b)\tanh i\mu_n L} \\ &= \frac{(k+b)(e^{i\mu_n L}-e^{-i\mu_n L})}{ke^{i\mu_n L}-be^{-i\mu_n L}} \\ &= \frac{(k+b)2i\sin\mu_n L}{ke^{i\mu_n L}-be^{i\mu_n L}}.\end{aligned} \tag{2.269.b}$$

Letting

$$A_n = \frac{1}{2i}\bar{A}_n(ke^{i\mu_n L}-be^{-i\mu_n L}) \tag{2.270}$$

The solutions (2.268.a,b) become

$$\begin{aligned}M_1 &= \frac{\bar{A}_n}{2i}\left\{[ke^{i\mu_n L}-be^{-i\mu_n L}]e^{i\mu_n(y-L)}-[ke^{-i\mu_n L}-be^{i\mu_n L}]e^{i\mu_n(y-L)}\right\} \\ &= \frac{\bar{A}_n}{2i}\left\{k[e^{i\mu_n y}-e^{-i\mu_n y}]-b[e^{i\mu_n(y-2L)}-e^{-i\mu_n(y-2L)}]\right\} \\ &= \bar{A}_n\{k\sin\mu_n y-b\sin\mu_n(y-2L)\}\end{aligned} \tag{2.271}$$

and

$$M_2 = \bar{A}_n(k+b)\sin\mu_n Le^{-k(y-L)}. \tag{2.272}$$

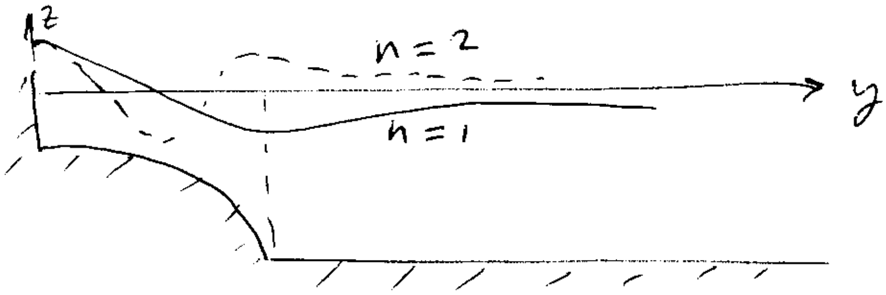
Substituting

$$\eta_0 = b(\sin 2\mu_n L)h_0^{-1/2}\bar{A}_n \tag{2.273}$$

and utilizing (2.247),(2.252),(2.271) and (2.272), the full solution can be written as

$$\eta = \begin{cases} \frac{\eta_0}{b \sin 2\mu_n L} \left[k \sin \mu_n y - b \sin \mu_n (y - 2b) \right] e^{-by} \cos(kx - \omega_n t), & 0 < y < L, \\ \frac{\eta_0 \sin \mu_n L}{b \sin 2\mu_n L} (k + b) e^{-bL} e^{-k(y-L)} \cos(kx - w_n t), & L < y < \infty. \end{cases} \tag{2.274}$$

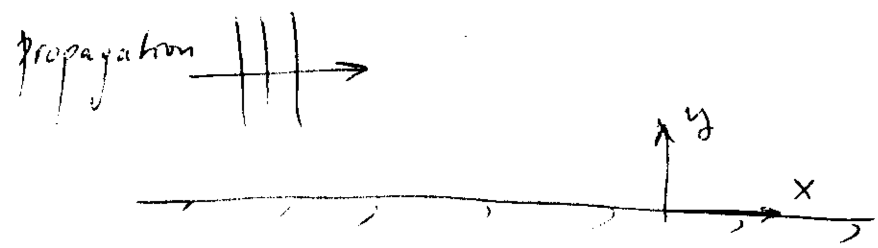
where η_0 is the amplitude of the wave at the coast. The amplitude of these waves with offshore distance can be sketched as follows



The wave motion is thus trapped in the continental shelf region. Note that by virtue of (2.267), the frequency ω_n is positive for any positive k, so that

$$\cos(kx - \omega_n t)$$

always represents a wave moving in the positive x-direction:



The waves preferentially take the coast to their right (similar to the case for Kelvin waves).

The dispersion relation (2.267) yields an infinite number of frequencies ω_n corresponding to the roots μ_n. Since μ_n are ordered in an increasing sequence for $n = 1, 2, \ldots$ higher modes have smaller frequencies, *i.e.* for

$$\mu_1 < \mu_2 < \mu_3 \ldots$$

the frequencies are ordered as:

$$\omega_1 > \omega_2 > \omega_3 \ldots$$

Note that (2.267) has a maximum when

$$k^2 = k_{\max}^2 = b^2 + \mu_n^2, \tag{2.275}$$

which corresponds to a maximum frequency of

$$\left(\frac{\omega_n}{f}\right)_{\max} = \frac{b}{\sqrt{b^2 + \mu_n^2}} = \frac{b}{k_{\max}} \tag{2.276}$$

for each mode. Note that for any possible mode, $\omega_{n,\max} < f$ (since $\mu_n^2 > 0$), *i.e.* possible frequencies are always smaller than the *inertial frequency*, or the Coriolis parameter f.

Furthermore, note that for a fixed frequency ω_n of any single mode, (2.267) yields two possible wavenumbers, k_n^+ and k_n^-, since (2.267) can be written as

$$k^2 - \left(2b\frac{f}{\omega_n}\right) k + b^2 + \mu_n^2 = 0 \tag{2.277}$$

and it follows that

$$\begin{aligned} k_n^+,\ k_n^- &= b\left(\frac{f}{\omega_n}\right) \pm \sqrt{\left(b\frac{f}{\omega_n}\right)^2 - (b^2 + \mu_n^2)} \\ &= b\left(\frac{f}{\omega_n}\right)\left[1 \pm \sqrt{1 - \frac{b^2 + \mu_n^2)}{(bf/\omega_n)^2}}\right] \\ &= \left(\frac{\omega_{n,\max}}{\omega_n}\right) k_{\max} \left\{1 \pm \sqrt{1 - \left(\frac{\omega_n}{\omega_{n,\max}}\right)^2}\right\} \end{aligned} \tag{2.278}$$

where (2.275) and (2.276) have been utilized.

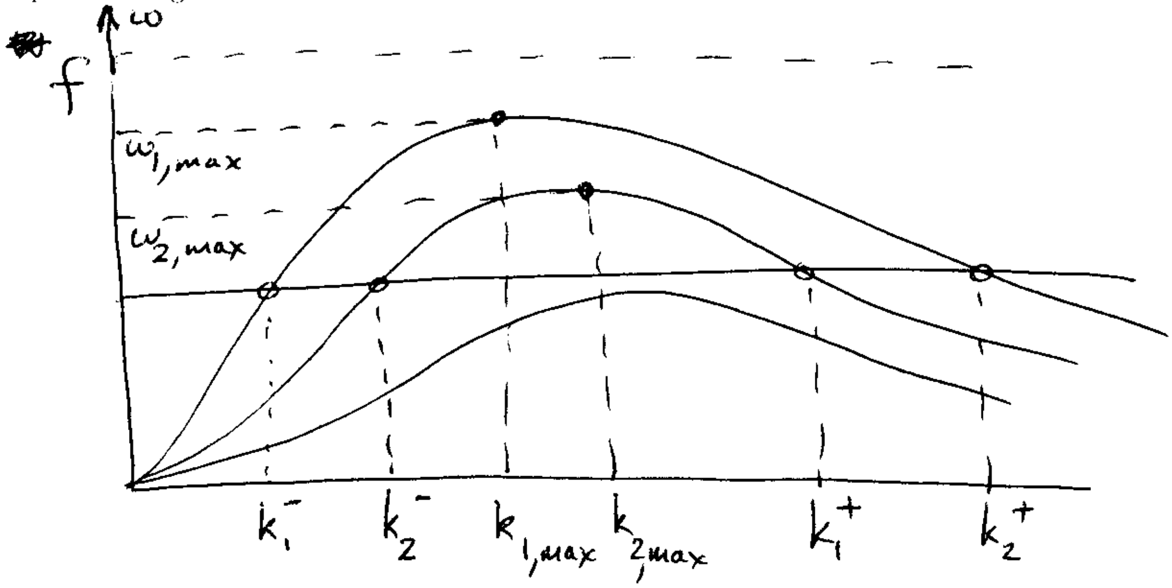
The phase speed of shelf waves is always in the positive x-direction since these waves always propagate with the coast on the right hand side.

$$C_x = \frac{\omega}{k} = \frac{2b}{(b^2 + k^2 + \mu_n^2)} > 0 \tag{2.279}$$

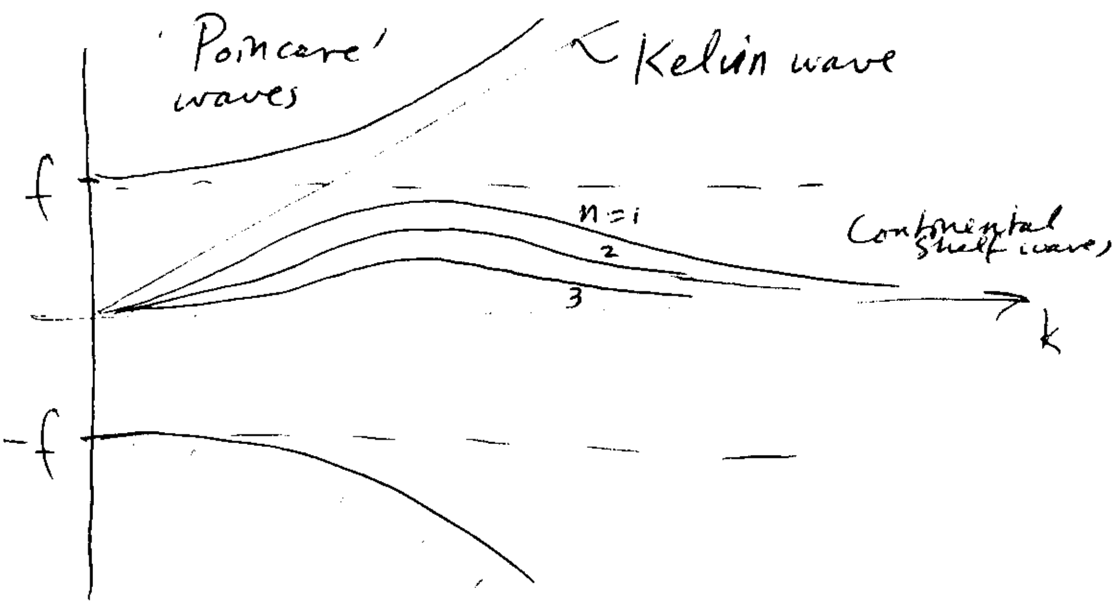
A calculation of the group velocity can be made as follows:

$$C_{gx} = \frac{\partial \omega}{\partial k} = 2bf \frac{b^2 + \mu_n^2 - k^2}{(b^2 + \mu_n^2 + k^2)^2} \tag{2.280}$$

Now, note that $C_{gx} < 0$ if $k > k_{max}$. Therefore it can be verified that for a fixed frequency ω_n of any mode the wave with wavenumber k^- carries energy in the same direction as phase propagation, while the wave with wavenumber k^+ carries energy in the opposite direction. Also note that at $k = k_{\max}$ the group velocity vanishes, implying that no energy can be transmitted by such waves. The dispersion diagram can be sketched as follows



In the above analytical solution, we have we have excluded the Poincare and Kelvin waves because we have used the rigid-lid approximation. However, it is often found that the dispersion characteristics of these waves are only slightly modified by the presence of bottom topography. The general form of the dispersion diagram is sketched below:



CHAPTER 3

QUASIGEOSTROPHIC THEORY

3.1. An Overview and Derivation of of Quasi-Geostrophic Equations

In Chapter 2 we derived the shallow water equations, and studied possible solutions to these equations. In general terms, we have always seeked to represent the complex geophysical flows by simplified equations which could be used to better understand the possible motions. We have done this in two ways: (i) in Section 2.4, we have made the rigid lid approximation and utilized the potential vorticity conservation to investigate low frequency motions such as Rossby waves, (ii) in Section 2.5, we have allowed surface displacement, but in order to simplify the equations, we have neglected nonlinear terms, upon which we discovered new types of motions which were mainly of high frequency.

In this Section, we will develop the approximate theory for the first type of motions considered above. We have seen in Section 2.4.1 that the motion becomes geostrophic in the steady limit. We have also shown that the low frequency motion (Rossby waves) carry many features of the geostrophic case, such as the particle motions being transverse to the surface elevation gradient and feeling topographic steering effects. The motion is than essentially close to being geostrophic (which we have shown to be a degenerate case), and hence called *quasi-geostrophic*, because of the set of simplifications consistently approximating the primitive equations.

Let us re-write the shallow water equations (2.89) and (2.90) excluding the barometric pressure and frictional effects:

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} + f\hat{k} \times \vec{u} = -g\nabla\eta \tag{3.1}$$

$$\frac{\partial \eta}{\partial t} + \nabla \cdot H\vec{u} = 0 \tag{3.2}$$

where

$$H = \eta + h \tag{3.3}$$

is the total depth and

$$f = f_0 + \beta y \tag{3.4}$$

is the Coriolis parameter, as it was approximated earlier in Section 2.3.

We now want to focus our attention on motions with time scales much larger than the (internal period) f_0^{-1}. Therefore we want to scale the equations accordingly, choosing the following scales:

$$\vec{x} \sim L, \; t \sim T$$
$$H \sim H_0, \; \beta \sim \frac{U}{L^2}, \; f \sim f_0 \tag{3.5}$$
$$\eta \sim \frac{fUL}{g}, \; \vec{u} \sim U$$

upon which the non-dimensional forms of the equations (3.1) and (3.2) become

$$\epsilon_T \frac{\partial \vec{u}}{\partial t} + \epsilon(\vec{u} \cdot \nabla \vec{u} + \beta y \hat{k} \times \vec{u}) + \hat{k} \times \vec{u} = -\nabla \eta \tag{3.6}$$

$$\delta \epsilon_T \frac{\partial \eta}{\partial t} + \nabla \cdot H \vec{u} = 0 \tag{3.7}$$

where

$$\epsilon = \frac{U}{f_0} L \tag{3.8.a}$$
$$\epsilon_T = \frac{1}{f_0} T \tag{3.8.b}$$
$$\delta = \frac{f_0^2 L^2}{g H_0} \tag{3.8.c}$$

The first of these nondimensional numbers, ϵ, is the Rossby number, the second one ϵ_T measures the ratio of the inertial time scale f_0^{-1} to the time scale T of the motion. For the type of motion considered, we insist that the rotational effects are important, $\epsilon \ll 1$, and that the time scale of the motion is much grater than the inertial time scale, *i.e.* the motion is of low frequency so that $\epsilon_T \ll 1$. While both of these numbers are small, their ratio

$$\frac{\epsilon}{\epsilon_T} = \frac{UT}{L} \tag{3.9.a}$$

determines the relative importance of the nonlinear terms in equations (3.6) and (3.7). When the ratio (3.9.a) is small the equations can be linearized. Since we want to keep the nonlinearity, we must also insist that the ratio is unity, or

$$\epsilon_T = \epsilon \tag{3.9.b}$$

Finally the third parameter δ in (3.8.c) is called the *divergence parameter*.

We have seen earlier that (cf.equation 2.250)

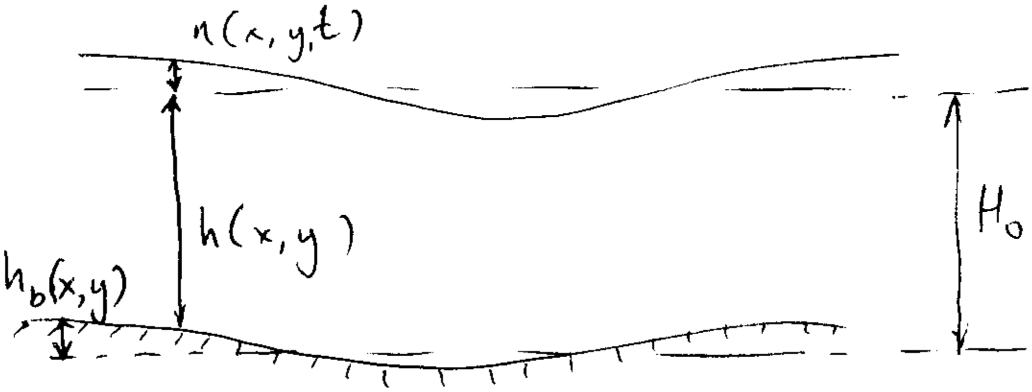
$$\delta = \left(\frac{L}{R}\right)^2 \tag{3.10}$$

where R is the Rossby Radius of deformation. Its order of magnitude depends on the application. For example, in the context of shelf waves, we have seen that $\delta \ll 1$, and such motions are called *quasi-nondivergent* if corresponding terms have a small contribution, or *nondivergent* if these terms are completely neglected. The nondivergent case is also known as the rigid-lid approximation. Here, we will assume that

$$\delta = O(1) \tag{3.11}$$

and keep the corresponding terms. Therefore, the only small parameter in the equations is the Rossby number, ϵ. We can now make a perturbation analysis in terms of the small parameter ϵ.

Before carrying out this analysis, let us first consider (3.3) and make further approximations about the depth variations:



Let us assume a mean depth H_0 (which was used as a scale earlier) and call the bottom topographic deviations from this mean depth h_b(x,y). Equation (3.3) becomes

$$H = \eta + h = \eta + H_0 - h_b, \tag{3.12.a}$$

and dividing by H_0,

$$\frac{H}{H_0} = \frac{H_0 + \eta - h_b}{H_0} = 1 + \frac{\eta}{H_0} - \frac{h_b}{H_0}.$$

Then, utilizing the scales (3.5), the nondimensional form of (3.12.a) becomes

$$H = 1 + \epsilon\delta\eta - \xi, \tag{3.12.b}$$

where since $\epsilon\delta = fUL/gH_0$ and ξ is defined as

$$\xi = \frac{h_b}{H_0} \tag{3.13}$$

Upon substitution of (3.12.b), equation (3.7) becomes

$$\delta\epsilon\partial\eta\partial t + H\nabla\cdot\vec{u} + \vec{u}\cdot\nabla H = \delta\epsilon\frac{\partial\eta}{\partial t} + (1 + \epsilon\delta\eta - \xi)\nabla\cdot\vec{u} + \vec{u}\cdot\nabla(\epsilon\delta\eta - \xi) \tag{3.14}$$

A formal perturbation expansion of the unknown variables in the small parameter ϵ proceeds as

$$\vec{u}(\vec{x}, t, \epsilon) = \vec{u}^{(0)}(\vec{x}, t) + \epsilon(\vec{u}^{(1)}(\vec{x}, t) + \epsilon^2\vec{u}^{(2)}(\vec{x}, t) + \ldots \tag{3.15.a}$$

$$\vec{\eta}(\vec{x}, t, \epsilon) = \eta^{(0)}(\vec{x}, t) + \epsilon\eta^{(1)}(\vec{x}, t) + \epsilon^2\eta^{(2)}(\vec{x}, t) + \ldots \tag{3.15.b}$$

and substituting the above expressions in equations (3.6) and (3.14), and collecting terms with respect to the powers of ϵ yields:

$$\left[\hat{k}\times\vec{u}^{(0)} + \nabla\eta^{(0)}\right] + \epsilon\left[\frac{\partial\vec{u}^{(0)}}{\partial t} + \vec{u}^{(0)}\cdot\nabla\vec{u}^{(0)} + \beta y\hat{k}\times\vec{u}^{(0)} + \hat{k}\times\vec{u}^{(1)} + \nabla\eta^{(1)}\right] + \epsilon^2[\ldots] + \ldots = 0 \tag{3.16.a}$$

$$\left[(1-\xi)\nabla\cdot\vec{u}^{(0)} - \vec{u}^{(0)}\cdot\nabla\xi\right] + \epsilon\left[\delta\left(\frac{\partial\eta^{(0)}}{\partial t} + \eta^{(0)}\nabla\cdot\vec{u}^{(0)} + \vec{u}^{(0)}\cdot\nabla\eta^{(0)}\right) + \nabla\cdot\vec{u}^{(1)}\right] + \epsilon^2[\ldots] + \ldots = 0 \tag{3.16.b}$$

The coefficients for each term in powers of ϵ must vanish, since the equations must be valid for arbitrary values of ϵ and for all x, y, t. Therefore, we get separate equations setting each of the concerned brackets in equations (3.16.a,b) to zero. For example, the first order equations (terms of order $\epsilon^0 = 1$) are:

$O(\epsilon^0)$:

$$\hat{k}\times\vec{u}^{(0)} = -\nabla\eta^{(0)} \tag{3.17.a}$$

$$(1-\xi)\nabla\cdot\vec{u}^{(0)} - \vec{u}^{(0)}\cdot\nabla\xi = 0 \tag{3.17.b}$$

We are mainly interested in the first order flow, since ϵ is small. However, if we only keep the $O(\epsilon^0)$ terms, equations (3.17.a) and (3.17.b) would be the same as the geostrophic equations (1.21 and 1.22), if we were to drop the bottom topography term ξ. In fact, using vector identities and taking the curl of (3.17.a) yields

$$\nabla\times\hat{k}\times\vec{u}^{(0)} = -\nabla\times\nabla\eta^{(0)} = 0$$

or

$$\nabla \cdot \vec{u}^{(0)} = 0. \tag{3.18}$$

Substituting into (3.17.b) gives

$$\vec{u}^{(0)} \cdot \nabla \xi = 0. \tag{3.19.a}$$

In dimensional terms, this is

$$\vec{u}^{(0)} \cdot \nabla h_b = 0 \tag{3.19.b}$$

i.e. the first order (pure geostrophic) motion must follow the isobaths. Since H_0 is constant, (3.19.b) can also be written in the same form as equation (2.139):

$$\vec{u}^{(0)} \cdot \nabla (H_0 - h_b) = \vec{u}^{(0)} \cdot \nabla h = 0. \tag{3.19.c}$$

The requirement for the flow to follow the isobaths is a very strong constraint, and would fail when $H = H_0$=constant. At this point, we have another choice: perhaps we should limit the effects of bottom topography, by requiring that the bottom variations are of the same order as the Rossby number, *i.e.*

$$\xi = \frac{h_b(x,y)}{H_0} = \epsilon \eta_b(x,y) = O(\epsilon) \ll 1 \tag{3.20.a}$$

such that

$$h_b = O(\epsilon H_0) \ll H_0. \tag{3.20.b}$$

With this choice of orders, it is assumed that the bottom topography variations (deviations from the mean depth) are much smaller than the mean depth.

With approximation (3.20) imposed, the terms proportional to ξ in (3.16.b) are carried over from order one terms to order ϵ terms, and the first order equations are reduced to

$O(\epsilon^0)$:

$$\hat{k} \times \vec{u}^{(0)} = -\nabla \eta^{(0)} \tag{3.21.a}$$

$$\nabla \cdot \vec{u}^{(0)} = 0 \tag{3.21.b}$$

i.e. the geostrophic equations. As we have seen earlier, the geostrophic equations are degenerate (linearly dependent), since manipulation of (3.21.a) directly yields (3.18), making one of the two equations redundant. It is evident that the description of the first order flow will only be possible by obtaining corrections from the second order. Substituting (3.20) in (3.16 a,b) yields the second order equations

$O(\epsilon^1)$:

$$\hat{k} \times \vec{u}^{(1)} + \nabla \eta^{(1)} + \frac{\partial \vec{u}^{(0)}}{\partial t} + \vec{u}^{(0)} \cdot \nabla \vec{u}^{(0)} + \beta y \hat{k} \times \vec{u}^{(0)} = 0 \tag{3.22.a}$$

$$\nabla \cdot \vec{u}^{(1)} + \delta \frac{\partial \eta^{(0)}}{\partial t} + (\delta \eta^{(0)} - \eta_b) \nabla \cdot \vec{u}^{(0)} + \vec{u}^{(0)} \nabla (\delta \eta^{(0)} - \eta_b) = 0 \tag{3.22.b}$$

Note that the third term of (3.22.b) vanishes by virtue of (3.21.b). We then take the curl of (3.22.a), first using vector identities to write

$$\begin{aligned} \vec{u}^{(0)} \cdot \nabla \vec{u}^{(0)} &= \frac{1}{2} \nabla (\vec{u}^{(0)} \cdot \vec{u}^{(0)}) + (\nabla \times \vec{u}^{(0)}) \times \vec{u}^{(0)} \\ &= \frac{1}{2} \nabla (\vec{u}^{(0)} \cdot \vec{u}^{(0)}) + \xi^{(0)} \hat{k} \times \vec{u}^{(0)} \end{aligned}$$

and

$$\begin{aligned} \nabla \times (\vec{u}^{(0)} \cdot \nabla \vec{u}^{(0)}) &= \frac{1}{2} \nabla \times \nabla (\vec{u}^{(0)} \cdot \vec{u}^{(0)}) + \nabla \times (\xi^{(0)} \hat{k} \times \vec{u}^{(0)}) \\ &= \vec{u}^{(0)} \cdot \nabla \xi^{(0)} \hat{k} + \xi^{(0)} \hat{k} \nabla \cdot \vec{u}^{(0)} \\ &= \hat{k} \vec{u}^{(0)} \cdot \nabla \xi^{(0)} \end{aligned}$$

by virtue of (3.21.b). Note that some terms have diasppeared by vector identities, and the definition of vorticity (2.104) have been used in the above, where

$$\vec{\omega}^{(0)} = \xi^{(0)} \hat{k} = \nabla \times \vec{u}^{(0)}.$$

Similarly,

$$\begin{aligned} \nabla \times \beta y \hat{k} \times \vec{u}^{(0)} &= \vec{u}^{(0)} \cdot \nabla \beta y \hat{k} + \beta y \hat{k} \nabla \cdot \vec{u}^{(0)} \\ &= \hat{k} \vec{u}^{(0)} \cdot \nabla \beta y \\ &= \hat{k} \beta v^{(0)} \end{aligned}$$

and

$$\nabla \times \hat{k} \times \vec{u}^{(1)} = \hat{k} \nabla \cdot \vec{u}^{(1)}.$$

With the above substitution, the curl of equation (3.22.b) becomes

$$\nabla \cdot \vec{u}^{(1)} + \frac{\partial \xi^{(0)}}{\partial t} + \vec{u}^{(0)} \cdot \nabla \xi^{(0)} + \beta v^{(0)} = 0 \tag{3.23}$$

and eliminating $\nabla u^{(1)}$ between (3.23) and (3.22.b) then yields:

$$\frac{\partial \xi^{(0)}}{\partial t} + \vec{u}^{(0)} \cdot \nabla \xi^{(0)} + \beta v^{(0)} - \delta \frac{\partial \eta^{(0)}}{\partial t} - \vec{u}^{(0)} \cdot \nabla(\delta \eta^{(0)} - \eta_b) = 0, \tag{3.24}$$

or,

$$\frac{D}{Dt}\left(\xi^{(0)} + \eta_b - \delta \eta^{(0)}\right) + \beta v^{(0)} = 0 \tag{3.25}$$

or using $\beta v^{(0)} = \vec{u}^{(0)} \cdot \nabla \beta y$,

$$\frac{D}{Dt}\left(\xi^{(0)} + \eta_b - \delta \eta^{(0)} + \beta y\right) = 0. \tag{3.26}$$

We essentially have derived the approximate form of the potential vorticity equation, *i.e.* the ***quasi-geostrophic vorticity equation.***

In fact, the same equation can be obtained through the exact form of the nondimensional form of the potential vorticity equation (2.127)

$$\frac{D}{Dt}\left(\frac{\epsilon\xi + 1 + \epsilon\beta y}{H}\right) = 0 \tag{3.27}$$

To see this, we first expand

$$\begin{aligned} \xi = \hat{k} \cdot \times \vec{u} &= \hat{k} \cdot \times(\vec{u}^{(0)} + \epsilon \vec{u}^{(1)} + \ldots) \\ &= \xi^{(0)} + \epsilon \xi^{(1)} + \ldots \end{aligned} \tag{3.28}$$

and using (3.12.b) and (3.20.a), write

$$\frac{1}{H} = \frac{1}{1 + \epsilon(\delta\eta - \eta_b)} = \frac{1}{1 + \epsilon(\delta\eta^{(0)} - \eta_b) + \epsilon^2 \delta \eta^{(1)} + \ldots} \tag{3.29}$$

Since $\epsilon \ll 1$, (3.29) can be approximated as

$$\frac{1}{H} = 1 - \epsilon(\delta\eta^{(0)} - \eta_b) + \epsilon^2(\ldots) + \ldots \tag{3.30}$$

So that (3.27) can equivalently be approximated as

$$\frac{D}{Dt}\{1 + \epsilon[\beta y + \xi^{(0)} + \delta\eta^{(0)} - \eta_b] + \epsilon^2[\ldots] + \ldots\} = 0 \tag{3.31}$$

which then, to first order yields (3.26).

The finishing touch to equation (3.25) or (3.26) comes from the definition or vorticity. Reorganizing (3.21.a) and taking curl:

$$\vec{u}^{(0)} = \hat{k} \times \nabla \eta^{(0)} \tag{3.32}$$

that one obtains:

$$\begin{aligned} \xi^{(0)} = \hat{k} \cdot \nabla \times \vec{u}^{(0)} &= \hat{k} \cdot \nabla \times \hat{k} \times \nabla \eta^{(0)} \\ &= \nabla \cdot \nabla \eta^{(0)} = \nabla^2 \eta^{(0)} \end{aligned} \tag{3.33}$$

Alternatively, a stream function

$$\psi = \eta^{(0)} \tag{3.34}$$

can be defined since (3.32) is in the form of the definition of this stream function which readily satisfies continuity equation (3.21) . Therefore, substituting (3.34) and

$$\xi^{(0)} = \nabla^2 \psi \tag{3.35}$$

in (3.26) gives

$$\frac{D}{Dt}\left(\nabla^2 \psi - \delta \psi + \beta y + \eta_b\right) = 0 \tag{3.36.a}$$

We also note:

$$\begin{aligned} \frac{D}{Dt} &= \frac{\partial}{\partial t} + \vec{u}^{(0)} \cdot \nabla \\ &= \frac{\partial}{\partial t} + (\hat{k} \times \nabla \psi) \cdot \nabla \end{aligned}$$

so that (3.36) can be written as

$$\left[\frac{\partial}{\partial t} + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\right]\left[\nabla^2 \psi - \delta \psi + \beta y + \eta_b\right] = 0 \tag{3.36.b}$$

or alternatively

$$\frac{\partial}{\partial t}[\nabla^2 \psi - \delta \psi] + J[\psi, (\nabla^2 \psi - \delta \psi + \beta y + \eta_b)] = 0 \tag{3.36.c}$$

and equivalently as

$$\frac{\partial}{\partial t}\left[\nabla^2 \psi - \delta \psi\right] + \beta \frac{\partial \psi}{\partial x} + J[\psi, (\nabla^2 \psi - \delta \psi + \eta_b)] = 0 \tag{3.36.d}$$

Note that when $\delta = 0$ (rigid-lid) and $\eta_b = 0$ (constant depth) the above equation reduces to (2.166), derived earlier in Section (2.4.4).

There are several points to note in the quasi-geostrophic theory:

(*i*) While the flow is in geostrophic balance by virtue of equations (3.21), the actual dynamics is governed by a simple equation (3.36) in the stream function $\psi = \eta^{(0)}$ which equivalently represents the dynamic pressure. This equation essentially incorporates the corrections to geostrophy by the surface elevation, β-effect, small bathymetric influences, and nonlinearity.

(*ii*) Once the stream function (or pressure) is determined from (3.36) then the velocity field can be obtained geostrophically from (3.32).

(*iii*) We have assumed earlier that the depth variations, are small as compared to the total depth, $(h_b/H_0) = \epsilon\eta_b = O(\epsilon)$, where $\eta_b = O(1)$. If we relax this assumption by insisting that $(h_b/H_0) = O(1)$, then $\eta_b = O(1/\epsilon)$ becomes large as compared to the other terms in (3.36.a) so that

$$\frac{D\eta_b}{Dt} = \vec{u}^{(0)} \cdot \nabla\eta_b = 0 \qquad (3.37)$$

which is the same as the constraint (3.19) obtained without making the assumption of small depth variations. Therefore, that possibility is readily contained in the present theory.

(*iv*) We have noted that equation (3.26) is a statement for conservation of *quasi-geostrophic potential vorticity* defined as

$$\Pi_g = \xi^{(0)} + \eta_b - \delta\eta^{(0)} + \beta y \qquad (3.38.a)$$

in the one dimensional variables. In dimensional variables this is equivalent to the following:

$$\Pi_g = \xi^{(0)} + \left(\frac{f_0}{H_0}\right)(h_b - \eta^{(0)}) + \beta y. \qquad (3.38.b)$$

The above definitions of quasi-geostrophic potential vorticity differs from the exact definition by a fixed constant which is of no relevance, and is the first order approximate form of the latter. Note that relative vorticity $\xi^{(0)}$ and planetary vorticity βy contribute to the *quasi-geostrophic potential vorticity* as well as the bottom topography and surface elevation. It is also worth mentioning that a positive bottom topography makes a positive contribution, whereas a positive surface elevation makes a negative contribution to the quasi-geostrophic potential vorticity. The first and the third terms are related to the flow and the sum is therefore total fluid vorticity. The second and third terms are independent of the flow and are therefore called *ambient potential vorticity*.