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AUTUMN COURSE ON MATHEMATICAL ECOLOGY

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STOCHASTIC POPULATION THEORY
DIFFUSION PROCESSES I

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Continuous approximation to population size.

Stochastic process

$\mathcal{E}$: random experiments specified by

i) its outcomes ξ forming a space S

ii) certain subsets of S, i.e. the "events" and by their probabilities

To each outcome ξ we associate a function of time $X(t,\xi)$ (real or complex). $X(t,\xi)$, or $X(t)$ for brevity, is a stochastic process. In general, $X(t)$ can be:

- discrete in space and in time (simple random walk)
- space-discrete, time-continuous (birth and death simple processes)
- space continuous - time discrete (general random walks)
- continuous both in space and time (diffusions, for instance)

From now on, we shall consider mainly the last case.

Description

n fixed instants $t_1, t_2, \dots, t_n$. Then $\{X(t_i)\}$ $(i=1 \dots n)$ is a family of r.v.'s whose p.d.f.

$$f_n\,[X(t_1), X(t_2) \dots, X(t_n)]$$

is known if $X(t)$ is specified. Conversely, to specify $X(t)$ we need to know for all n and t_i the functions f_n.

Chosing t_i's sufficiently close to one another we can replace $X(t)$ by a sequence of r.v.'s $X(t_i)$ with a practically sufficient accuracy. $X(t)$ can thus be characterised by

an infinite sequence of p.d.f.'s of increasing order:

$$f_1\,[X(t_1)],\ f_2\,[X(t_1); X(t_2)],\ \dots,\ f_n\,[X(t_1); X(t_2) \dots; X(t_n)],\ \dots$$

$$\int dX(t_j)\, f_n\,[X(t_1); X(t_2); \dots; X(t_n)] = f_{n-1}\,[X(t_1); \dots \widehat{X(t_j)}; \dots; X(t_n)],\ \forall\, i \leq j \leq n$$

Markov processes

Assume that for all n and $t_1 < t_2 < \dots < t_n$ one has:

$$f[X(t_n) \mid X(t_{n-1}), X(t_{n-2}), \dots, X(t_1)] = f[X(t_n) \mid X(t_{n-1})]$$

where:

$$f[X(t_n) \mid X(t_{n-1}), X(t_{n-2}), \dots, X(t_1)] = \frac{f_n\,[X(t_1); X(t_2); \dots; X(t_n)]}{f_{n-1}\,[X(t_1); X(t_2); \dots; X(t_{n-1})]}$$

f : transition p.d.f. Depends on $x_n \equiv X(t_n)$, $x_{n-1} \equiv X(t_{n-1})$, t_n, t_{n-1}.

$$f_n\,[X(t_1); X(t_2); \dots; X(t_n)] = f[X(t_n) \mid X(t_{n-1})]\, f[X(t_{n-1}) \mid X(t_{n-2})] \dots$$
$$\dots f[X(t_2) \mid X(t_1)]\ f_1\,[X(t_1)]$$

(Gaussian processes: f_2).

How does one get f ?

Let

$$t_0 < \tau < t$$

be *arbitrary* instants and set:

$$x_0 = X(t_0)\ ,\ y = X(\tau),\ x = X(t)$$

Then:

$$f_2(x,t\,;x_0,t_0) = \int dy\ f_3(x,t\,;\,y,\tau\,;\,x_0,t_0)$$

$\therefore$

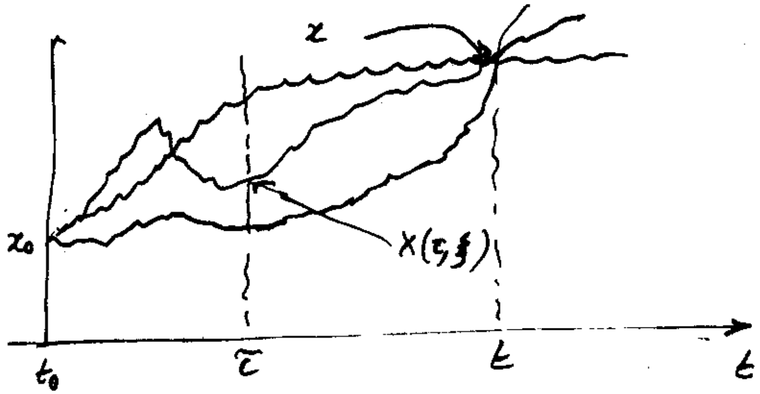
$$f(x,t\,|x_0,t_0)\ \cancel{f_1(x_0,t_0)} = \int dy\ f(x,t\,|y,\tau;x_0,t_0)\cdot f_2(y,\tau\,|x_0,t_0)\,\cancel{f_1(x_0,t_0)}$$

$\therefore$ (due to Markov assumption)

$$f(x,t\,|x_0,t_0) = \int dy\ f(x,t\,|y,\tau)\ f(y,\tau\,|x_0,t_0)$$

(Smoluchowski or Chapman-Kolmogorov eqn.)

Initial condition: $f(x,t\,|x_0,t_0) \to$ delta function as $t \to t_0$



<u>Differential expansions of C.K. eqn.</u>

$$f(x,t\,|x_0,t_0) = \int dy\ f(x,t\,|y,\tau)\ f(y,\tau\,|x_0,t_0)$$

$t \to t+\Delta t$

$\tau \to t$

$\therefore$

$$f(x,t+\Delta t\,|x_0,t_0) - f(x,t\,|x_0,t_0) = \int dy\ f(x,t+\Delta t\,|y,t)\,f(y,t\,|x_0,t_0) - f(x,t\,|x_0,t_0)$$

$R(x)$: $\quad R^{(n)}(x) \to 0\quad$ as $\quad x \to \pm\infty,\ \forall\, n \in \mathbb{N}$

$$\int dx\ R(x)\,\frac{f(x,t+\Delta t\,|x_0,t_0) - f(x,t\,|x_0,t_0)}{\Delta t} = \frac{1}{\Delta t}\int dx\ R(x)\int dy\ f(x,t+\Delta t\,|y,t)\,f(y,t\,|x_0,t_0) - \frac{1}{\Delta t}\int dx\ R(x)\ f(x,t\,|x_0,t_0)$$

$$\begin{cases} R(x) = R(y) + \displaystyle\sum_{n=1}^{\infty} \frac{d^n R(y)}{dy^n}\,\frac{(x-y)^n}{n!} \\ \Delta t \to 0 \end{cases}$$

$\therefore$

$$\int dx\ R(x)\,\frac{\partial f}{\partial t} = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int dy\, R(y) f(y,t\,|x_0,t_0)\overbrace{\int dx\ f(x,t+\Delta t\,|y,t)}^{=1}$$

$$+ \sum_{n=1}^{\infty}\frac{1}{n!}\int dy\ \frac{d^n R(y)}{dy^n} f(y,t\,|x_0,t_0)\cdot \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int dx\,(x-y)^n f(x,t+\Delta t\,|y,t)$$

$$- \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int dx\ R(x)\ f(x,t\,|x_0,t_0)$$

$\therefore$

$$\int dx\, R(x) \frac{\partial f}{\partial t} = \sum_{n=1}^{\infty} \frac{1}{n!} \int dx \frac{d^n R(x)}{dx^n} f(x,t|x_0,t_0)\, A_n(x,t)$$

where:

$$A_n(x,t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int dy\, (y-x)^n f(y,t+\Delta t|x,t) \qquad (n=1,2,\dots)$$

Integrating by parts:

$$\int dx\, R(x) \left\{ \frac{\partial f}{\partial t} - \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x^n} \left[A_n(x,t) f(x,t|x_0,t_0) \right] \right\} = 0$$

$\therefore$

$$\frac{\partial f}{\partial t} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x^n} \left[A_n(x,t) f \right] \qquad \text{kinetic equation}$$

A_n: infinitesimal moments (time independent for homogeneous process)

$$A_n(x,t) \cdot \Delta t \simeq \langle (\Delta x)^n \,|\, X(t)=x \rangle$$

$\therefore$

$$\langle \Delta x \,|\, x \rangle \simeq A_1 \cdot \Delta t$$
$$\langle (\Delta x)^2 \,|\, x \rangle \simeq A_2 \cdot \Delta t$$

- - - - - -

$$\sigma^2\{\Delta x \,|\, x\} = A_2 \cdot \Delta t - A_1^2 (\Delta t)^2$$

$\therefore$

$$A_2 \simeq \frac{\sigma^2}{\Delta t}$$

Backward eqn.

Fix present time and present state.

$$f(x,t|x_0,t_0) = \int dy\, f(x,t|y,\tau)\, f(y,\tau|x_0,t_0)$$

$$t_0 \to t_0 - \Delta t$$
$$\tau \to t_0$$

$$f(x,t|x_0,t_0-\Delta t) = \int dy\, f(x,t|y,t_0)\, f(y,t_0|x_0,t_0-\Delta t)$$

$\therefore$

$$f(x,t|x_0,t_0-\Delta t) - f(x,t|x_0,t_0) = \int dy\, f(y,t_0|x_0,t_0-\Delta t) \underbrace{\left[f(x,t|y,t_0) - f(x,t|x_0,t_0) \right]}_{\sum_{n=1}^{\infty} \frac{\partial^n f(x,t|x_0,t_0)}{\partial x_0^n} \frac{(y-x_0)^n}{n!}}$$

Hence, dividing by $(-\Delta t)$, when $\Delta t \to 0$ we get:

$$\frac{\partial f(x,t|x_0,t_0)}{\partial t_0} = -\sum_{n=1}^{\infty} \frac{A_n(x_0,t_0)}{n!} \frac{\partial^n f(x,t|x_0,t_0)}{\partial x_0^n}$$

Diffusion equations

In principle, infinitesimal moments $A_1, A_2, \dots$ can be arbitrary (some zero, some $\neq 0$). Actually, this is not the case:

Pawula, R.F. (1967). Generalizations and extensions of the Fokker-Planck-Kolmogorov equations. *IEEE Trans. Information Theory* IT 13, 33-41

Pawula theorem

If $\forall n$ $A_n(x,t)$ exist, then the vanishing of any even-order moment implies

$$A_n(x,t)=0 \quad , \; n \geq 3$$

Corollary. Whenever the kinetic equation contains a finite number of derivatives, it is of the second order at most.

i) $A_2 = A_3 = \cdots = 0$

Deterministic process

ii) $A_2 \neq 0, \; A_3 = A_4 = \cdots = 0$

Diffusion processes

In case ii) we have:

$$\begin{cases} \dfrac{\partial f}{\partial t} = -\dfrac{\partial}{\partial x}[A_1(x,t)f] + \dfrac{1}{2}\dfrac{\partial^2}{\partial x^2}[A_2(x,t)f] \\ \dfrac{\partial f}{\partial t} = A_1(x_0,t_0)\dfrac{\partial f}{\partial x_0} + \dfrac{1}{2}A_2(x_0,t_0)\dfrac{\partial^2 f}{\partial x_0^2} \\ \lim\limits_{t\to t_0} f(x,t|x_0,t_0) = \delta(x-x_0) \end{cases}$$

Time homogeneous case: infinitesimal moments are time-indepdt.

How does one calculate $A_1(x), B_2(x)$ for specific models?

- directly from definitions
- by writing fluctuation equations.

Exponential growth with random harvesting and immigration.

Assume that a population undergoes exponential growth:

$$x(t) = x_0 e^{\lambda t}$$

$\therefore$

$$dx = \lambda x \, dt$$

Now, assume the population is also subject to Poisson immigration with parameter α_e and to Poisson harvesting with parameter α_i (recall that population size is taken as a continuous variable).

Then:

$$A_n(x,t) \equiv A_n(x) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int (y-x)^n f(y,t+\Delta t|x,t)\,dy$$

where:

$$f(y,t+\Delta t|x,t) \equiv f(y,\Delta t|x)$$

Immigration and harvesting occur in bunches in the sense that e.g. the number of immigrations in unit time is Poisson and each immigration phenomenon instantaneously adds e individuals to the population. Similarly, i individuals disappear at each harvesting:

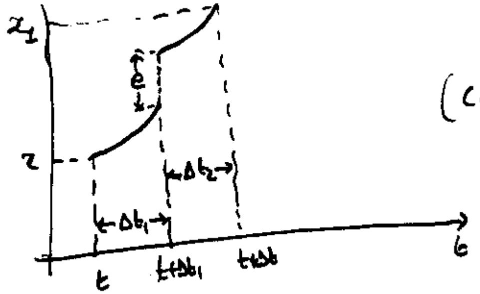
$$x \to x+e \quad (e>0)$$
$$x \to x+i \quad (i<0)$$

Therefore, starting from x individuals at time t, after time Δt the number of individuals will be as follows:

i) $x_0 = x + x\lambda\Delta t$

ii) $x_1 = x + x\lambda\Delta t_1 + e + (x + x\lambda\Delta t_1 + e)\lambda\Delta t_2 \simeq x + x\lambda\Delta t + e + (x+e)\lambda\Delta t_2$

iii) $x_2 = x + x\lambda\Delta t_1 + i + (x + x\lambda\Delta t_1 + i)\lambda\Delta t_2 \simeq x + x\lambda\Delta t + i + (x+i)\lambda\Delta t_2$



(case ii))

Hence:

$$f(y, t+\Delta t \mid x, t) = [1-(\alpha_e+\alpha_i)\Delta t]\,\delta(y-x_0) + \alpha_e\Delta t\,\delta(y-x_1) + \alpha_i\Delta t\,\delta(y-x_2)$$

$\therefore$

$$A_1 = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\Big\{[1-(\alpha_e+\alpha_i)\Delta t]\int (y-x)\,\delta(y-x_0)\,dy + \alpha_e\Delta t\int (y-x)\,\delta(y-x_1)\,dy + \alpha_i\Delta t\int (y-x)\,\delta(y-x_2)\,dy\Big\}$$

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\Big\{[1-(\alpha_e+\alpha_i)\Delta t](x_0-x) + \alpha_e\Delta t(x_1-x) + \alpha_i\Delta t(x_2-x)\Big\}$$

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\Big\{[1-(\alpha_e+\alpha_i)\Delta t]\,x\lambda\Delta t + \alpha_e\Delta t\,[x\lambda\Delta t + e + (x+e)\lambda\Delta t_1] + \alpha_i\Delta t\,[x\lambda\Delta t + i + (x+i)\lambda\Delta t_1]\Big\}$$

$\therefore$

$A_1(x) = \lambda x + \alpha_e e + \alpha_i i$

Similarly:

$A_n(x) \equiv A_n = \alpha_e e^n + \alpha_i i^n \qquad (n = 2, 3, \dots)$

Diffusion approximation:

$$\alpha_e \to \infty,\ \alpha_i \to \infty \qquad : \quad \alpha_e e = -\alpha_i i$$
$$e \to 0,\ i \to 0 \qquad\quad \alpha_e e^2 + \alpha_i i^2 = \sigma^2 < \infty$$

$\therefore$

$$\begin{cases} \dfrac{\partial f}{\partial t} = -\dfrac{\partial}{\partial x}(\lambda x f) + \dfrac{\sigma^2}{2}\dfrac{\partial^2 f}{\partial x^2} \\ f(x,t \mid x_0, 0) = \delta(x-x_0) \end{cases}$$

$\therefore$

$$f(x,t \mid x_0, 0) = G\Big\{x_0 e^{\lambda t},\ \frac{\sigma^2}{2\lambda}\big(1-e^{2\lambda t}\big)\Big\}$$

Boundary condition expressing the stop of the process when $x = \varepsilon \geq 0$:

$f(\varepsilon, t \mid x_0, 0) = 0.$

Extinction probability is then the probability for the process to reach level ε for the first time:

$T = \inf\{t : X(t) > \varepsilon\}$

$g(\varepsilon, t \mid x_0) = \dfrac{\partial}{\partial t} P\{T \leq t\}$

$$M = \theta\left\{ \frac{1}{2}\left[\sum_{n=1}^{\infty} \frac{y^{2n}}{n(2n-1)!!} - \sum_{n=1}^{\infty} \frac{x^{2n}}{n(2n-1)!!}\right] + \sqrt{\frac{\pi}{2}}\left[x\,\Phi\left(\tfrac{1}{2},\tfrac{3}{2};\tfrac{x^2}{2}\right) - y\,\Phi\left(\tfrac{1}{2},\tfrac{3}{2};\tfrac{y^2}{2}\right)\right]\right\}$$

Set: $\boxed{\eta=0,\ \theta=\frac{1}{\lambda}}$

$$t_2 = \theta^2\left\{\left[\sum_{n=1}^{\infty}\frac{x^{2n}}{n(2n-1)!!} - \sum_{n=1}^{\infty}\frac{y^{2n}}{n(2n-1)!!}\right]\cdot\left[\sqrt{\frac{\pi}{2}}\,y\,\Phi\left(\tfrac{1}{2},\tfrac{3}{2};\tfrac{y^2}{2}\right) - \frac{1}{2}\sum_{n=1}^{\infty}\frac{y^{2n}}{n(2n-1)!!}\right]\right.$$
$$+\left[x\,\Phi\left(\tfrac{1}{2},\tfrac{3}{2};\tfrac{x^2}{2}\right) - y\,\Phi\left(\tfrac{1}{2},\tfrac{3}{2};\tfrac{y^2}{2}\right)\right]\cdot\left[\sqrt{2\pi} - \sqrt{\frac{\pi}{2}}\sum_{n=1}^{\infty}\frac{1}{n(2n+1)} +\right.$$
$$\left.+\sqrt{\frac{\pi}{2}}\sum_{1}^{\infty}{}_n\frac{y^{2n}}{n(2n-1)!!} - \pi y\,\Phi\left(\tfrac{1}{2},\tfrac{3}{2};\tfrac{y^2}{2}\right)\right]$$
$$+\frac{1}{2}\left[\sum_{2}^{\infty}{}_n\frac{x^{2n}}{n(2n-1)!!}\sum_{k=1}^{n-1}\frac{1}{k}\right] - \sum_{2}^{\infty}{}_n\frac{y^{2n}}{n(2n-1)!!}\sum_{k=1}^{n-1}\frac{1}{k}\Big]$$
$$\left.+\sqrt{2\pi}\left[\sum_{n=1}^{\infty}\frac{y^{2n+1}}{n!\,2^n(2n+1)}\sum_{k=1}^{n}\frac{1}{2k-1} - \sum_{n=1}^{\infty}\frac{x^{2n+1}}{n!\,2^n(2n+1)}\sum_{k=1}^{n}\frac{1}{2k-1}\right]\right\}$$

where:

Skewness ??

$$x = \sqrt{\frac{2}{\sigma^2\theta}}\,(\eta\theta - x_0)$$
$$y = \sqrt{\frac{2}{\sigma^2\theta}}\,(\eta\theta - S)$$
$$\Phi(a,c;z) = \sum_{n=1}^{\infty}\frac{a(a+1)\dots(a+n-1)}{c(c+1)\dots(c+n-1)} * 1$$

$$g_\theta(\varepsilon\,|\,x_0) = \int_0^\infty e^{-\theta t} g(\varepsilon, t\,|\,x_0)\,dt$$

$\therefore$

$$g_\theta(\varepsilon\,|\,x_0) = \exp\left[(x_0^2 - \varepsilon^2)\frac{\lambda}{2\sigma^2}\right]\frac{D_{-\frac{\theta}{\lambda}}\left[\left(\frac{2\lambda}{\sigma^2}\right)^{1/2}(-x_0)\right]}{D_{-\frac{\theta}{\lambda}}\left[-\sqrt{\frac{2\lambda}{\sigma^2}}\,S\right]}$$

$$\langle T\rangle = -\left.\frac{d g_\theta(\varepsilon\,|\,x_0)}{d\theta}\right|_{\theta=0}$$

(cf. page -10 bis-)

etc.

Details can be found, mutatis mutandis, in

L. M. Ricciardi, Lecture notes in Biomath. Springer 1977

In the future we shall see how to obtain infinitesimal moments by stochastic differential equations.

Diffusion approximations (time-homogeneous case)

$$\begin{cases} \dfrac{\partial f}{\partial t} = -\dfrac{\partial}{\partial x}\left[A_1(x) f\right] + \dfrac{1}{2}\dfrac{\partial^2}{\partial x^2}\left[A_2(x)\cdot f\right] \\ \dfrac{\partial f}{\partial t} = A_1(x_0)\dfrac{\partial f}{\partial x_0} + \dfrac{1}{2}A_2(x_0)\dfrac{\partial^2 f}{\partial x_0^2} \end{cases}$$

$$\begin{cases} \frac{1}{2}A_2(x_0) = a(x) \\ A_1(x_0) = b(x) \end{cases}$$

$$\begin{cases} \dfrac{\partial f}{\partial t} = \dfrac{\partial^2}{\partial x^2}\left[a(x)\cdot f\right] - \dfrac{\partial}{\partial x}\left[b(x)\cdot f\right] \\[2ex] \dfrac{\partial f}{\partial t} = a(x_0)\dfrac{\partial^2 f}{\partial x_0^2} + b(x_0)\dfrac{\partial f}{\partial x_0} \end{cases}$$

$$\lim_{t\to t_0} f(x,t\,|\,x_0,t_0) = \delta(x-x_0)$$

$$I \equiv (r_1, r_2) \qquad -\infty \le r_1 < r_2 \le \infty$$

$$\begin{cases} a(x),\ a'_x(x),\ b(x) \text{ continuous for } x\in I \\ a(x) > 0\ ,\ x\in(r_1,r_2) \end{cases}$$

$\therefore$

$$r_i \begin{cases} \text{accessible} \begin{cases} \text{regular} \\ \text{exit} \end{cases} \\ \text{inaccessible} \begin{cases} \text{natural} \\ \text{entrance} \end{cases} \end{cases}$$

$$x' :\quad r_1 < x' < r_2$$

$$\begin{cases} f(x) = \exp\left[-\displaystyle\int_{x'}^{x} dz\, \frac{b(z)}{a(z)}\right] \\[2ex] g(x) = \left[a(x)\, f(x)\right]^{-1} \\[1ex] h(x) = f(x) \displaystyle\int_{x'}^{x} dz\, g(z) \\[2ex] k(x) = g(x) \displaystyle\int_{x'}^{x} dz\, f(z) \end{cases}$$

$$\psi(x) \in \mathcal{L}(I_i) \leftrightarrow \int_{I_i} dx\ \psi(x) < \infty$$

$$\psi(x) \geq 0$$

$$I_i \equiv (x', r_i)$$

Then :

r_i is
- **regular**, if $f(x) \in \mathcal{L}(J_i)$ and $g(x) \in \mathcal{L}(J_i)$
- **exit**, if $g(x) \notin \mathcal{L}(J_i)$ and $h(x) \in \mathcal{L}(J_i)$
- **entrance**, if $f(x) \notin \mathcal{L}(J_i)$ and $k(x) \in \mathcal{L}(J_i)$
- **natural**, otherwise

r_i regular:

$$\lim_{x \to r_i} \left[a(x)\, f(x)\, f(x,t \mid x_0, t_0) \right] = 0$$

total absorption at $x = r_i$

$$\longrightarrow$$

$$\frac{\partial f}{\partial t} = \frac{\partial^2}{\partial x^2}(r x f) - \frac{\partial}{\partial x}\left[(p x + q) f \right], \quad r > 0$$

$$I = (0, \infty)$$

$$x = 0 \text{ is } \begin{cases} \text{exit}, & q \le 0 \\ \text{regular}, & 0 < q < r \\ \text{entrance}, & q \ge r \end{cases}$$

transition p.d.f. for case $q \le 0$ is written down explicitly on page 16

$$\frac{\partial f}{\partial \tau} + b(y,\tau) \frac{\partial f}{\partial y} + a(y,\tau) \frac{\partial^2 f}{\partial y^2} = 0$$

$$f(x,t \mid y,\tau)$$

$$T : f(x,t \mid y,\tau) \to f'(x',t' \mid y',\tau')$$

$$\frac{\partial f'}{\partial \tau'} + \frac{\partial^2 f'}{\partial y'^2} = 0$$

Chernasov, I.D. (1957) On the transformation of the diffusion process to a Wiener process. Theory Prob. Appl. **2**, 373-376

$$y' = \psi(y,\tau) \qquad x' = \psi(x,t)$$

$$\tau' = \varphi(\tau) \qquad t' = \varphi(t)$$

$$f(x,t \mid y,\tau) = \left| \frac{\partial \psi(x,t)}{\partial x} \right| f'(x',t' \mid y',\tau')$$

$$b(y,\tau) = \frac{a'_y(y,\tau)}{2} + \frac{\sqrt{a(y,\tau)}}{2} \left\{ c_1(\tau) + \int^y dz\, \frac{c_2(\tau)\, a(z,\tau) + a'_\tau(z,\tau)}{[a(z,\tau)]^{3/2}} \right\}$$

$$\begin{cases} \psi(y,\tau) = \exp\left[-\frac{1}{2} \int^\tau ds\, c_2(s) \right] \int^y dz\, [a(z,\tau)]^{-1/2} - \frac{1}{2} \int^\tau d\theta\, c_1(\theta) \exp\left[-\frac{1}{2} \int^\theta ds\, c_2(s) \right] \\ \varphi(\tau) = \int^\tau d\theta\, \exp\left[-\int^\theta ds\, c_2(s) \right] \end{cases}$$

$$\frac{\partial f}{\partial \tau} + ry\,\frac{\partial f}{\partial y} + (py+q)\,\frac{\partial^2 f}{\partial y^2} = 0$$

$q \le 0$ exit

$0 < q < r$ regular

$q \ge r$ entrance

$q = 0$

$$f(x,t\,|\,y) = \frac{p}{r(e^{pt}-1)} \exp\left\{\frac{-p(x+y\,e^{pt})}{r(e^{pt}-1)}\right\} .$$

$$\cdot \left[e^{-pt}\,\frac{x}{y}\right]^{\frac{q-r}{2r}} \cdot I_{1-q/r}\left\{\frac{2p}{r(1-e^{-pt})}\,\left(e^{-pt}\,xy\right]^{1/2}\right\}$$

$$I_k(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{2n+k}}{n!\;\Gamma(n+k+1)}$$

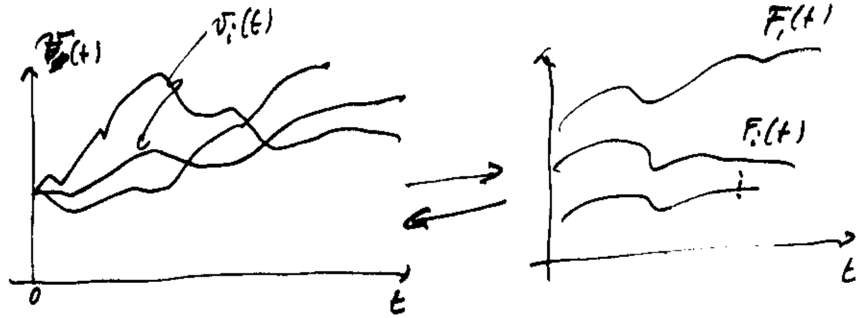
L.M. Ricciardi (1976). On the transformation of diffusion processes into the Wiener process. J. Math. Anal. Appl. 54, 195-199

R.M. Capocelli and L.M. Ricciardi. On the transformation of diffusion processes into the Feller process. Math. Biosciences 29, 219-234 (1976).

Diffusion processes via fluctuation equations.

One dimensional motion of a Brownian particle in a viscous fluid:

$$\begin{cases} m\,\dfrac{dv_i}{dt} = -\beta v_i + F_i(t) \\ v_i(0) = v_i^0 \end{cases}$$



What about if $F(t)$ includes a rapidly fluctuating component? As far as the sample paths of $F(t)$ are smooth, we expect smooth sample paths for $v(t)$ and the above D.E. can be integrated by referring to the sample path functions by using ordinary calculus rules. Whenever such smoothness ceases, the D.E. risks to become meaningless.

Consider an equation such as:

$$\frac{dX(t)}{dt} = f\,[X(t), t, \eta(t)]\,, \quad t \ge 0$$

where f is assigned and $\eta(t)$ is a random disturbance. If f and η are suitably restricted, we can interprete

$\int_{t_0}^{t} f[X(\tau),\tau,\eta(\tau)]\,d\tau$ in the mean-square (m.s.) sense. Then, interpreting also $\frac{dX(t)}{dt}$ in the m.s. sense, we ~~can~~ write:

$$X(t) - X(t_0) = \int_{t_0}^{t} f[X(\tau),\tau,\eta(\tau)]\,d\tau.$$

Important special case:

$$\frac{dX(t)}{dt} = f(X,t) + g(X,t)\Lambda(t) \qquad (*)$$

with $\Lambda(t)$ <u>white noise</u>, i.e. a stationary Gaussian process with zero mean and correlation function of delta-type:

$$E[\Lambda(t)] = 0 \qquad , t \geq 0$$

$$\gamma(t_1,t_2) = E[\Lambda(t_1)\Lambda(t_2)] = \sigma^2\,\delta(t_2 - t_1) \qquad 0 \leq t_1 < t_2 < \infty$$

Since $\Lambda(t)$ is delta correlated, it is not Riemann m.s. integrable (correlation function ~~is not m.~~ $\gamma(t_1,t_2)$ is not Riemann integrable over any square of the type $[0,T]\times[0,T]$). Hence, eq. $(*)$ has no meaning.

Consider a standard Brownian motion or Wiener process $W(t)$. this does not admit of derivative in the usual or m.s. sense (indeed $\frac{\partial^2\gamma(t,\tau)}{\partial t\,\partial\tau}$ does not exist at (t,t)). Hence,

$$\frac{dW(t)}{dt} = \Lambda(t)$$

is not correct. Still, let us ~~assume the relation~~

$$\Lambda(t)\,dt \simeq dW(t)$$

to hold on heuristic grounds. Then $(*)$ can be written as:

$$dX(t) = f(X,t)\,dt + g(X,t)\,dW(t) \qquad (**)$$

This will be taken as equivalent to:

$$X(t) - X(t_0) = \int_{t_0}^{t} f(X,\tau)\,d\tau + \int_{t_0}^{t} g(X,\tau)\,dW(\tau) \qquad (***)$$

where the first integral can be defined as a m.s. Riemann integral or as an ordinary integral for sample functions. The second integral cannot be defined for the sample functions because Brownian motion process has unbounded variations. Hence, we need a new definition of integral (and presumibly a new Calculus) to make $(**)$ and $(***)$ meaningful.

<u>The Ito integral</u>

Set $g(X,\tau) \equiv g[X(\tau),\tau] = g(\tau,\omega) = g_\tau(\omega)$ and $W(\tau) = W_\tau$

so that:

$$\int_a^b g(X,\tau)\,dW(\tau) = \int_a^b g_\tau(\omega)\,dW_\tau$$

$T = [a,b]$, W_t Wiener process with $\gamma(t,\tau) = \sigma^2 \min(t,\tau)$

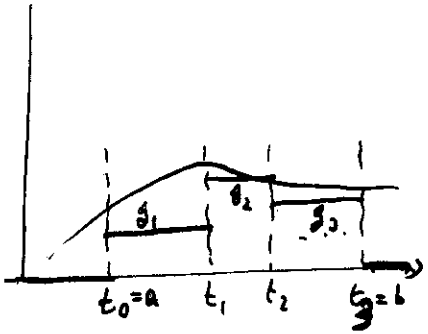
Let:

$a = t_0 < t_1 < t_2 < \ldots < t_n = b$

and set:

$$g_t(\omega) = \begin{cases} 0, & t < t_0 \\ g_i(\omega), & t_i < t < t_{i+1} \\ 0, & t \geq t_n \end{cases}$$

i.e. we consider a step function approximation in $[t_0, t]$



Also, we assume that $g_i(\omega)$'s are independent of

$\{ W_{t_k} - W_{t_\ell} \; : \; t_i \leq t_\ell \leq t_k \leq b \}$

and such that

$$E\left[|g_i(\omega)|^2 \right] < \infty$$

For any such function $g_t(\omega)$ the Ito integral is:

$$\int_T g_t(\omega)\, dW_t \triangleq \sum_{i=0}^{n-1} g_i(\omega) \left[W_{t_{i+1}} - W_{t_i} \right]$$

Due to the independence assumption one easily sees that:

$$E\left\{ \int_T g_t(\omega)\, dW_t \right\} = 0$$

Furthermore, for any two step functions $f_t(\omega)$ and $g_t(\omega)$ of the above type one has:

$$E\left\{ \int_T g_t(\omega)\, dW_t \int_T f_t(\omega)\, dW_t \right\} = \sigma^2 \int_T E[g_t f_t]\, dt \quad (+)$$

Now, let $\{g_t^{(n)}\}$ be a sequence of step-functions converging to the random function $g_t(\omega)$ in the sense that

$$\int_T E\left\{ |g_t - g_t^{(n)}|^2 \right\} dt \to 0 \quad \text{as } n \to \infty \qquad (++)$$

Then, due to $(+)$ we have:

$$E\left\{ \left| \int_T g_t\, dW_t - \int_T g_t^{(n)} dW_t \right|^2 \right\} \equiv E\left\{ \left| \int_T [g_t - g_t^{(n)}] dW_t \right|^2 \right\}$$

$$= \sigma^2 \left\{ E\left[\int_T |g_t - g_t^{(n)}|^2 dt \right] \right\} \to 0 \quad \text{as } n \to \infty$$

i.e.

$$\int_T g_t(\omega)\, dW_t = \underset{n \to \infty}{\text{l.i.m.}} \int_T g_t^{(n)} dW_t \qquad \text{Ito stoch. integral}$$

This integral can be shown to exist for all functions $g_t(\omega)$ such that:

i) $g_t(\omega)$ is independent of $\{W_{t_k} - W_{t_\ell} : t \le t_\ell \le t_k \le b\}$ for all $t \in T$;

ii) $\int_T E\{|g_t(\omega)|^2\} dt < \infty$

since (as shown by Doob) any such function can be approximated by a sequence of step functions in the sense (++).

Theorem Let the random functions $g_t(\omega)$ and $f_t(\omega)$ satisfy i) and ii). Then Ito integrals are well defined and:

$$E\left\{ \int_T g_t(\omega)\, dW_t \right\} = 0$$

$$E\left\{ \int_T g_t(\omega)\, dW_t \int_T f_t(\omega)\, dW_t \right\} = \sigma^2 \int_T E\{g_t f_t\}\, dt$$

Under certain condition Ito integral is the m.s. limit of Riemann-Stieltjes sums:

Theorem

Theorem Let $g_t(\omega)$ be such that

$$E\{|g_t(\omega)|^2\} < \infty \quad , \forall t \in T$$

and let (as before) $g_t(\omega)$ be independent of

$\{W_{t_k} - W_{t_\ell} : t \le t_\ell \le t_k \le b\}$ for all $t \in T$.

Also, suppose $g_t(\omega)$ is m.s. continuous on T. Then, setting

T : $a = t_0 < t_1 < \ldots < t_n = b$

$\rho = \max_i (t_{i+1} - t_i)$

one has:

$$\underbrace{\int_T g_t(\omega)\, dW_t}_{\text{Ito integral}} = \underset{\rho \to 0}{\text{l.i.m.}} \sum_{i=0}^{n-1} g_{t_i}(\omega)\left[W_{t_{i+1}} - W_{t_i}\right]$$

Theorem. Let

$$x_t = \int_a^t g_\tau(\omega)\, dW_\tau \qquad t \in T$$

Then, under conditions of previous theorem x_t is m.s. continuous on T.

How about the practical use of Ito integral?

Fundamental corollary of Ito calculus

Let $\varphi(x)$ be a twice continuously differentiable real scalar function of the real variable x, and let

$$\frac{d\varphi}{dx} \equiv \psi(x)$$

Then $(b > a)$:

$$\int_a^b \psi(W_t)\,dW_t = \varphi(W_b) - \varphi(W_a) - \frac{\sigma^2}{2}\int_a^b \frac{d^2\varphi(W_t)}{dx^2}\,dt$$

Stratonovich "symmetric" integral

Let $g_t(\omega)$ be an explicit function of W_t :

$$g_t(\omega) = g[W_t, t]$$

Then:

$$\oint g(W_t,t)\,dW_t \triangleq \underset{\rho\to 0}{\mathrm{l.i.m.}} \sum_{i=0}^{n-1} g\left(\frac{W_{t_i}+W_{t_{i+1}}}{2}, t_i\right)\left(W_{t_{i+1}} - W_{t_i}\right)$$

where $\rho = \max_i (t_{i+1} - t_i)$ and $t_0 = a < t_1 < \dots < t_n = b$

Theorem If $g(W,t)$ is continuous in t and has a continuous partial derivative $\frac{\partial g}{\partial W} = g_W(W,t)$ and further satisfies $\int_T E[|g(W_t,t)|^2]\,dt < \infty$, then the m.s. limit in the definition of $\oint$ exists and:

$$\oint_T g(W_t,t)\,dW_t = \int_T g(W_t,t)\,dW_t + \frac{\sigma^2}{2}\int_T g_W(W_t,t)\,dt$$

Usual rules of Calculus

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