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A MATHEMATICAL MODEL OF POPULATION DYNAMICS, INVOLVING MIGRATION AND RESOURCES

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A MATHEMATICAL MODEL OF POPULATION DYNAMICS, INVOLVING MIGRATION AND RESOURCES

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The problem of population dynamics depending on ages and involving migration has been considered by many authors (see e.g. [1][2][6]). Our aim is to approach the same problem, by supposing, in addition, the dependence of the population variation not only on age and time, but also on a vector of resources; this one is conceived in a general manner, in order to simplify the exposition more precisely, if a component of this vector is an obstacle at the development of the population, it will be interpreted as a pollution; so that we will not impose sign restrictions on this vector.

Obviously, we must introduce a new equation in the system, beside the usual ones, namely the dependence of this vector on the population; we will suppose the time variation of the resources depending linearly on the population of various ages.

§1. <u>Statement of the problem</u>. In what follows, we will denote

i) $u(t,a,x)$ the number of individuals of age a, at the moment t, and the point x of the space;

ii) $Du = \lim\limits_{h\to o} \dfrac{u(t+h,\ a+h,x) - u(t,x)}{h}$ (see f.i. [3], [5]);

iii) A-the migration coefficient, which we will suppose constant

iv) $r(t,x)$ - an n-dimensional vector, depending on time <u>t</u> and point x of the space, representing the resources, and $r_o(x)$ the initial repartition of these resources;

v) λ the death coefficient, depending on time, age and resources, i.e. $\lambda = \lambda(t,a,r(t,x))$;

vi) μ the birth coefficient, depending on time, age and on the point x, $\mu = \mu(t,a,x)$;

vii) φ the initial repartition of the population $\varphi = \varphi(a,x)$;

viii) B the number of offsprings $B = B(t,x)$, at moment t, and point x ;

ix) $S=S(t,a,x)$ an n-vector $R_+ \times R_+ \times R \to R^n$,

x) $g=g(t,x)$ also such a vector, this two characterizing the variation of the resources vector.

The space in which migration takes place will be R. The equations of our model will be

$$(1.1)\qquad Du - A^2 \frac{\partial^2 u}{\partial x^2} = \lambda(t,a,r(t,x))u,$$

$$(1.2)\qquad B(t,x) = \int_o^\infty \mu(t,\alpha,x)u(t,\alpha,x)d\alpha.$$

$$(1.3)\qquad \frac{\partial r}{\partial t} = \int_o^\infty S(t,\alpha,x)u(t,\alpha,x)d\alpha + g(t,x),$$

$$(1.4)\qquad u(o,a,x) = \varphi(a,x),\quad r(o,x) = r_o(x),$$

$$(1.5)\qquad \varphi(o,x) = \int_o^\infty \mu(o,\alpha,x)\varphi(\alpha,x)d\alpha$$ - an obvious compatibility condition.

<u>Remarks</u>. 1. If A=o, i.e. if there does not exist migration, then λ-the death rate, must be negative. In our case, this condition is no more valid, since the contribution of the migratio can lead to the fact that in some points x, $\lambda \geq o$.

2. The equation (1.2) is the same that can be found in [3],[5]

3. The third equation expresses the fact that the variation speed of the resources depends linearly on the population, with coefficients depending on ages, on time and on points; . . it

follows that the derivatives of resources, are integrals $\int_0^{\infty} S(t,\alpha,x)u(t,\alpha,x)d\alpha$, S being an n-vector. If a certain component i of S is positive, and $r_i > 0$, then this can mean that the resources are produced by the population, if $r_i <$ • and $S_i < 0$, then thus can mean that the component i of r is a pollution, which grows with the number of individuals in the population.

§2. Hypotheses. We suppose

A_1. λ is continuous on $R_+ \times R_+ \times R^n$ and bounded

$$|\lambda(t,a,r)| \le \lambda_0(t,a) \le \Lambda, \text{ and } \int_0^{\infty}\lambda_0(t-s,a-s)ds \le \Lambda, \quad \Lambda < 1/2,$$

for every $r \in R^n$, $\|r\| < M$.

A_2. λ is lipschitz with respect to the last variables, i.e.

$$|\lambda(t,a,r_1) - \lambda(t,a,r_2)| \le Le^{-ka}\|r_1-r_2\|,$$

where L, k are positive constants, and $\|.\|$ is the norm in R^n.

B_1. μ is continuous in $R_+ \times R_+ \times R$, and satisfies (see also [5])

$$0 \le \mu \le \mu_0 a e^{-ha}, \qquad \mu_0,\ h \text{ positive constants.}$$

C. The vector S is continuous and bounded on $R_+ \times R_+ \times R$ and g is continuous on $R \times R$, and

$$\|S\| \le S_0(t)e^{-sa} \qquad s = \text{const.} \quad \int_0^{\infty}S_0(t)dt = S_1 = \text{const.}$$

$$\|g(t,x)\| \le g_1 \quad \text{and} \quad \int_0^t \|g(t,x)\| \le G = \text{const.}$$

D. The function φ is continuous on $R_+ \times R$ and bounded

$$0 \le \varphi \le \varphi_0 .$$

E. The function r_0 is continuous on R and bounded

$$\|r_0\| \le R .$$

F. The above constants satisfy

$$R + \left(\frac{1}{s} + 1 + G\right)\frac{\varphi_0 S_1}{s} < M, \quad S_1 Q/h \le 1.$$

§3. Transformation of the system (1.1)-(1.5). As usual, we consider first a pair of arbitrary positive, but fixed numbers $(t_0,a_0) \in R_+ \times R_+$, and denote

$$(3.1)\qquad \begin{aligned} u(t_0+\tau,\ a_0+\tau,x) &= \bar{u}(\tau,x),\\ r(t_0+\tau,x) &= \bar{r}(\tau,x),\\ \lambda(t_0+\tau,\ a_0+\tau,\bar{r}(\tau,x)) &= \bar{\lambda}(\tau,r),\\ \mu(t_0+\tau,a_0+\tau,x) &= \bar{\mu}(\tau,x),\\ \varphi(a_0+\tau,x) &= \bar{\varphi}(\tau,x). \end{aligned}$$

Then, one sees easily that

$$(3.2)\qquad Du(t_0+\tau,\ a_0+\tau,x) = \frac{\partial\bar{u}}{\partial\tau}(\tau,x),$$

and (1.1) becomes

$$(3.3)\qquad \frac{\partial\bar{u}}{\partial t} - A^2\frac{\partial^2\bar{u}}{\partial x^2} = \bar{\lambda}(\tau,r)\bar{u}.$$

Taking into account the thermic potentials, (3.3) can be written under the form

$$(3.4)\quad \bar{u}(\tau,x) = \int_0^{\tau}d\sigma\int_{-\infty}^{\infty}E(\tau,x;\sigma,\xi)\bar{\lambda}(\sigma,\bar{r}(\sigma,\xi))\bar{u}(\sigma,\xi)d\xi + \int_{-\infty}^{\infty}E(\tau,x;0,\xi)\bar{u}(0,\xi)d\xi,$$

where, obviously

$$E(t,x;\sigma,\xi) = \frac{1}{2A\sqrt{\pi(t-\sigma)}}\exp\left(-\frac{(x-\xi)^2}{4A^2(t-\sigma)}\right).$$

Supposing now $\bar{u}(0,\xi)$ a known function, and denoting

$$(3.5)\qquad v(\tau,x) = \int_{-\infty}^{\infty}E(\tau,x;0,\xi)\,\bar{u}(0,\xi)d\xi,$$

the equation (3.3) is a linear integral equation with kernel $K(\tau,x;\sigma,\xi)\bar{\lambda}(\sigma,\bar{r}(\sigma,\xi))$ where $\bar{\lambda}$, as a consequence from

A_1, satisfies

$$\int_0^{\infty} |\bar{\lambda}(\sigma, \bar{r}(\sigma, x))| \, d\sigma \le \Lambda \quad . \tag{3.6}$$

Taking into account the identity

$$\frac{(x-x_1)^2}{\tau-\tau_1} + \frac{(x_1-x_2)^2}{\tau_1-\tau_2} = \frac{(x-x_2)^2}{\tau-\tau_2} + \frac{(x_1-\tilde{x})^2}{\dfrac{(\tau-\tau_1)(\tau_1-\tau_2)}{\tau-\tau_2}}, \tag{3.7}$$

with

$$\tilde{x} = \frac{(\tau_1-\tau_2)x_1 + (\tau-\tau_1)x_2}{\tau-\tau_2}$$

and, supposing r and v known-functions, we have, for the resolvent kernel

$$\mathcal{R}(\tau, x; \sigma, \xi) \le \frac{1}{1-\Lambda} \frac{1}{2A\sqrt{\pi(\tau-\sigma)}} \left|\bar{\lambda}(\sigma, \bar{r}(\sigma, \xi))\right| \cdot \exp\left(-\frac{(x-\xi)^2}{4A^2(\tau-\sigma)}\right), \tag{3.8}$$

and so

$$\bar{u}(\tau, x) = \int_0^{\tau} d\sigma \int_{-\infty}^{\infty} \mathcal{R}(\tau, x; \sigma, \xi) v(\sigma, \xi) d\xi + v(\tau, x),$$

from which we deduce

$$\frac{1-2\Lambda}{1-\Lambda} v(\tau, x) \le \bar{u}(\tau, x) \le \frac{1}{1-\Lambda} v(\tau, x), \tag{3.9}$$

which proves the positiveness of $\bar{u}(\tau, x)$.

§4. Taking now, for $a < t$

$$a_0 = 0; \quad t_0 = t-a, \quad \tau = a,$$

(3.4) becomes

$$u(t,a,x) = \int_0^a d\sigma \int_{-\infty}^{\infty} E(a,x;\sigma,\xi)\, \lambda(t-a+\sigma, \sigma, r(t-a+\sigma, \xi))\, u(t-a+\sigma, \sigma, \xi) d\xi + v(a,x), \tag{4.1}$$

with

$$v(a,x) = \int_{-\infty}^{\infty} \frac{1}{2A\sqrt{\pi a}} \exp\left(-\frac{(x-\xi)^2}{4A^2 a}\right) B(t-a, \xi) d\xi, \tag{4.2}$$

the solution of which is

$$u(t,a,x) = \int_0^a d\sigma \int_{-\infty}^{\infty} \mathcal{R}(a,x;\sigma,\xi) v(\sigma,\xi) d\xi + v(a,x),$$

and is bounded by

$$0 \le u(t,a,x) \le \frac{1}{1-\Lambda} \int_0^a d\sigma \int_{-\infty}^{\infty} \frac{\bar{\lambda}(t-a+\sigma, \sigma, r(t-a+\sigma, \xi))}{2A\sqrt{\pi(a-\sigma)}} \exp\left(-\frac{(x-\xi)^2}{4A^2(a-\sigma)}\right) \left(\int_{-\infty}^{\infty} \frac{1}{2A\sqrt{\pi\sigma}} \exp\left(-\frac{(\xi-\eta)^2}{4A^2\sigma}\right) B(t-a, \eta) d\eta\right) d\xi .$$

Using (3.7), we obtain

$$u(t,a,x) \le \frac{1}{1-\Lambda} \int_{-\infty}^{\infty} \frac{1}{2A\sqrt{\pi a}} \exp\left(-\frac{(x-\xi)^2}{4A^2 a}\right) B(t-a, \xi) d\xi \le \frac{1}{1-\Lambda} \max_{\xi \in \mathbb{R}} \left(B(t-a,\xi)\right), \tag{4.3}$$

$$u(t,a,x) \ge \frac{1-2\Lambda}{1-\Lambda} \int_{-\infty}^{\infty} \frac{1}{2A\sqrt{\pi a}} \exp\left(-\frac{(x-\xi)^2}{4A^2 a}\right) B(t-a,\xi) d\xi$$

For $a > t$, we take:

$$a_0 = a-t, \quad t_0 = 0, \quad \tau = t,$$

and obtain in the same manner:

$$u(t,a,x) \le \frac{1}{1-\Lambda} \int_{-\infty}^{\infty} \frac{1}{2A\sqrt{\pi t}} \exp\left(-\frac{(x-\xi)^2}{4A^2 t}\right) \cdot \varphi(a-t, \xi) d\xi \le \frac{1}{1-\Lambda} \varphi_0, \tag{4.4}$$

$$u(t,a,x) \ge \frac{1-2\Lambda}{1-\Lambda} \int_{-\infty}^{\infty} \frac{1}{2A\sqrt{\pi t}} \exp\left(-\frac{(x-\xi)^2}{4A^2 t}\right) \cdot \varphi(a-t, \xi) d\xi .$$

§5. Write now (1.2) under the form:

$$B(t,x) = \int_0^t \mu(t,\alpha,x) u(t,\alpha,x) d\alpha + \int_t^{\infty} \mu(t,\alpha,x) u(t,\alpha,x) d\alpha , \tag{5.1}$$

Using (4.3), (4.5), and (3.5), we get

$$(5.2)\quad B(t,x) = \int_0^t d\alpha \int_{-\infty}^{\infty} H(t,x;\alpha,\xi) B(\alpha,\xi)\, d\xi + F(t,x),$$

where

$$(5.3)\quad H(t,x;\alpha,\xi) = \mu(t, t-\alpha, x)\Big\{\int_0^{t-\alpha} d\sigma \int_{-\infty}^{\infty} \frac{1}{2A\sqrt{\pi\sigma}}$$

$$\mathcal{R}(t-\alpha,x;\sigma,\eta)\exp\Big(-\frac{(\eta-\xi)^2}{4A^2\sigma}\Big)d\eta + \frac{1}{2A\sqrt{\pi(t-\alpha)}}\exp\Big(-\frac{(x-\xi)^2}{4A^2(t-\alpha}$$

and

$$(5.4)\quad F(t,x) = \int_0^{\infty} \mu(t,t+\alpha,x)\,d\alpha \Big\{\int_0^t d\sigma \int_{-\infty}^{\infty} \frac{1}{2A\sqrt{\pi\sigma}} \mathcal{R}(t,x;\sigma,\eta)$$

$$\exp\Big(-\frac{(\eta-\xi)^2}{4A^2\sigma}\Big)d\eta + \frac{1}{2A\sqrt{\pi t}}\exp\Big(-\frac{(x-\xi)^2}{4A^2 t}\Big)\Big\}\varphi(\alpha,\xi)\,d\xi.$$

From hypothesis B_1, the bound (3.8), and the identity (3.7), it results

$$|H(t,x;\alpha,\xi)| \le \frac{\mu_0(t-\alpha)e^{-h(t-\alpha)}}{(1-\Lambda)\, 2A\sqrt{\pi(t-\alpha)}}\exp\Big(-\frac{(x-\xi)^2}{4A^2(t-\alpha)}\Big)$$

and for the resolvent kernel:

$$0 \le \mathcal{H}(t,x;\alpha,\xi) \le \frac{1}{2A\sqrt{\pi(t-\alpha)}}e^{-h(t-\alpha)}$$

$$\sum_1^{\infty}\frac{\mu_0^n}{(1-\Lambda)^n}\,\frac{(t-\alpha)^{2n-1}}{(2n-1)!}\exp\Big(-\frac{(x-\xi)^2}{4A^2(t-\alpha)}\Big).$$

For $F(t,x)$, using the same relations as before, we obtain

$$0 \le F(t,x) \le \frac{\mu_0}{(1-\Lambda)h^2}(ht+1)e^{-ht}\int_{-\infty}^{\infty}\frac{1}{2A\sqrt{\pi t}}\exp\Big(-\frac{(x-\xi)^2}{4A^2 t}\Big)\varphi_0\, d\xi;$$

and then, denoting

$$(5.5)\qquad \nu^2 = \frac{\mu_0}{1-\Lambda},$$

we obtain

$$(5.6)\qquad B(t,x) \le \nu\rho\frac{\varphi_0}{h^2}\exp((\nu-h)t)\ ,\quad \rho = \max(h,\nu).$$

Suppose now the supplementary hypothesis

$$(B_2)\qquad \mu(t,a,x) \ge \mu_1 a\, e^{-ha} \qquad\qquad \mu_1 < \mu_0;$$

Denoting

$$\nu_1^2 = \frac{(1-2\Lambda)}{1-\Lambda},$$

$$\varphi_1(t,x) = \int_0^{\infty} d\beta \int_{-\infty}^{\infty}\beta\, e^{-h\beta}\frac{1}{2A\sqrt{\pi t}}\exp\Big(-\frac{(x-\xi)^2}{4A^2 t}\Big)\varphi(\beta,\xi)\,d\xi$$

$$\varphi_2(t,x) = \int_0^{\infty} d\beta \int e^{-h\beta}\frac{1}{2A\sqrt{\pi t}}\exp\Big(-\frac{(x-\xi)^2}{4A^2 t}\Big)\varphi(\beta,\xi)\,d\xi,$$

and

$$\tilde{\varphi}(t,x) = \min(\nu,\ \varphi_1(t,x),\ \varphi_2(t,x)),$$

it follows

$$(5.7)\qquad B(t,x) \ge \nu_1\tilde{\varphi}(t,x)\exp[(-h+\nu_1)t].$$

Remark. 1 Taking into account that, after hypotheses, $\tilde{\varphi}$ is bounded, it follows from (5.6) and (5.7) if $\nu < h$, that the offspring tends to zero as t tends to infinity; and if $\nu_1 > h$, then the offspring tends to infinity as t tends to infinity, in each compact of R.

2. If $\nu < h$, then $\rho = h$, and

$$|B(t,x)| \le \frac{\nu\varphi_0}{h} \le \varphi_0,$$

and from (4.3) we obtain, when $a < t$:

$$(5.8)\qquad u(t,a,x) \le \frac{1}{1-\Lambda}\varphi_0,$$

so that, this inequality remains valid for all a,t.

§6. To study the equation (1.3) we will assume in the following $\nu \leq h$; then let r_1, r_2 be two vectors in R^n, depending on t and x; for $\bar{u}_1$, $\bar{u}_2$ from (3.4), we get

$$|\bar{u}_1(\tau,x)-\bar{u}_2(\tau,x)| \leq \int_0^\tau d\sigma \int_{-\infty}^{\infty} E(\tau,x;\sigma,\xi)\Big\{|\bar{\lambda}(\sigma,\bar{r}_1(\sigma,\xi)) - \bar{\lambda}(\sigma,\bar{r}_2(\sigma,\xi))|\,\bar{u}_1(\sigma,\xi) + |\bar{\lambda}(\sigma,\bar{r}_2(\sigma,\xi))|\,|\bar{u}_1(\sigma,\xi)-\bar{u}_2(\sigma,\xi)|\Big\}d\xi + \int_{-\infty}^{\infty} E(\tau,x;o,\xi)\,|\bar{u}_1(o,\xi)-\bar{u}_2(o,\xi)|\,d\xi$$

and then, after hypothesis A_2:

$$(1-\Lambda)\max|\bar{u}_1(\tau,x)-\bar{u}_2(\tau,x)| \leq \frac{\Lambda L}{(1-\Lambda)k}\max\|r_1-r_2\|\int_{-\infty}^{\infty}E(\tau,x;o,\xi)\cdot\bar{u}(o,\xi)d\xi + \|\bar{u}_1(o,\xi)-\bar{u}_2(o,\xi)\|,$$

from which, for $a < t$, we deduce, with regard to (3.9) and (4.2):

$$(6.1)\quad \max|u_1(t,a,x)-u_2(t,a,x)| \leq \frac{\Lambda L}{(1-\Lambda)^2 k}\max\|r_1-r_2\| \int_{-\infty}^{\infty}\frac{1}{2A\sqrt{\pi a}}\exp\Big(-\frac{(x-\xi)^2}{4A^2 a}\Big)B(t-a,\xi)d\xi + \frac{1}{1-\Lambda}\max_{\substack{t\geq a\\ x\in R_+}}\|B_1-B_2\|,$$

and, with (5.5)

$$\max|u_1(t,a,x)-u_2(t,a,x)| \leq \frac{\Lambda L}{(1-\Lambda)k}\varphi_o\max\|\bar{r}_1-r_2\|\cdot\exp(\nu-h)(t-a) + \frac{1}{1-\Lambda}\max|B_1(t-a,\xi)-B_2(t-a,\xi)|$$

and for $a > t$:

$$(6.2)\quad \max|u_1(t,a,x)-u_2(t,a,x)| \leq \frac{\Lambda L}{(1-\Lambda)^2 k}\max\|r_1-r_2\|.$$

Further, from (5.1) we have

$$|B_1(t,x)-B_2(t,x)| \leq \int_0^t \mu(t,\alpha,x)|u_1(t,\alpha,x)-u_2(t,\alpha,x)|d\alpha + \int_t^{\infty}\mu(t,\alpha,x)\,|u_1(t,\alpha,x)-u_2(t,\alpha,x)|d\alpha$$

and taking into account (6.1) and (6.2), and the hypothesis B_1, we have

$$|B_1(t,x)-B_2(t,x)| \leq \frac{\Lambda L}{(1-\Lambda)^2 k}\varphi_o\mu_o\|r_1-r_2\|\Big[\frac{1}{\nu^2}e^{\nu t} + \frac{1}{\nu^2}(th+1)\Big]e^{-ht} + \frac{1}{(1-\Lambda)h^2}\max|B_1-B_2|.$$

Supposing in addition $(1-\Lambda)h^2 > 1$, we deduce:

$$(6.3)\quad \max\|B_1-B_2\| \leq P\max\|r_1-r_2\|, \text{ with } P = \frac{3\Lambda L\varphi_o h^2}{[(1-\Lambda)h^2-1]\,k}.$$

Coming back to (6.1) and (6.2), we obtain

$$(6.4)\quad \max|u_1(t,a,x)-u_2(t,a,x)| \leq Q\max\|r_1-r_2\|,$$

where

$$Q = \frac{\Lambda L}{(1-\Lambda)^2 k}\varphi_o + \frac{1}{1-\Lambda}P \leq 4\,\frac{\Lambda L\varphi_o}{(1-\Lambda)^2[(1-\Lambda)h^2-1]}$$

§7. We come now to equation (1.3), which we write under the form

$$(7.1)\quad r(t,x)=r_o(x)+\int_0^t d\tau\int_0^{\infty}S(\tau,\alpha,x)u(\tau,\alpha,x)\,d\alpha + \int_0^t g(\tau,x)d\tau.$$

For this equation we must prove that it has a continuous bounded solution $R^n \to R$.

To this end, we consider the operator $\mathcal{A}$, defined by

$$\tilde{r}(t,x) = (\mathcal{A}r)(t,x) = r_o(x) + \int_0^t d\tau\int_0^{\infty}S(t,\alpha,x)u(t,\alpha,x)d\alpha + \int_0^t g(t,x)$$

First we see that, taking into account (4.3), (4.5) and (5.6) with $\nu < h$, and the hypothesis E, we have:

$$\|\tilde{r}(t,x)\| \leq R + S_o\frac{\rho\nu\varphi_o}{h^2}\int_0^t\Big\{\frac{1}{(h-\rho+s)}(1-\exp e^{-\rho\tau})\exp(-s\tau) + \varphi_o S_o\frac{1}{s}\exp(-s\tau)\Big\}d + G.$$

$$\leq R + \frac{\varphi_o S_o}{s}\Big(\frac{\rho\nu}{h-\rho+s}+1\Big)+G,$$

which after hypothesis is smaller than M, i.e. the space of vector functions (7.2) is transformed by the operator defined in the left-hand side of (7.2), into itself.

Concerning the contraction property of this operator we see that if r_1 and r_2 are two vector-functions satisfying (7.2) and $\tilde{r}_1$ and $\tilde{r}_2$ their transforms through $\mathcal{A}$, then, taking into account (6.4), we have:

$$\|\tilde{r}_1-\tilde{r}_2\| \le \int_0^t d\tau \int_0^\infty S(\tau,\alpha,x)Q\,\|r_1-r_2\|\,d\alpha \le$$

$$\int_0^t d\tau \int_0^\infty S_0 e^{s\alpha}\, Q\,\|r_1-r_2\|\,d\alpha \le \frac{S_1}{s} Q\,\|r_1-r_2\|$$

and as by hypothesis, $S_1Q/h<1$, the existence follows.

§8. <u>The influence of the migration</u>. To study the influence of the migration, we must see the dependence of the solution on the coefficient A. To this end, we add to the hypotheses in §2, the following ones:

A_3. λ has bounded derivatives with respect to the components of the third variable

$$\left|\frac{\partial\lambda}{\partial r}\right| \le \Lambda_1 .$$

B_2. μ has a bounded derivative with respect to the third variable

$$\left|\frac{\partial\mu}{\partial s}\right| \le \bar{\mu}_0\, e^{-pa}, \qquad \bar{\mu}_0,\text{p-constants and } \mu_0, p>0 .$$

C_2. The components of the vectors S are differentiable and satisfy

$$\int_0^\infty \left\|\frac{\partial S}{\partial x}\right\| d\alpha \;\; \bar{S}_1 e^{mt}, \qquad \left\|\frac{\partial \rho}{\partial x}\right\| \le \gamma.$$

E_2. r_0 is differentiable and

$$\left\|\frac{\partial r_0}{\partial x}\right\| \le R_1 ,$$

F_2. $$1-\Lambda-\frac{L\,\varphi_0}{(1-\Lambda)k}-\frac{\mu_0}{h^2} > 0 .$$

We obtain so

$$\|u\| \le \frac{1}{1-\Lambda}\varphi_0$$

(8.1) $$\left|\frac{\partial u}{\partial x}\right| \le \begin{cases} k\dfrac{2\varphi_0}{A}\left(\sqrt{a}+\dfrac{1}{\sqrt{a}}\right), & \text{if } a<t \\ k\dfrac{2\varphi_0}{A}\left(\sqrt{t}+\dfrac{1}{\sqrt{t}}\right), & \text{if } a>t . \end{cases}$$

(8.2) $$\left|\frac{\partial B}{\partial x}\right| \le M_1+\frac{1}{A}M_2\,\phi_1(t),$$

(8.3) $$\left\|\frac{\partial r}{\partial x}\right\| \le Q_1+\frac{1}{A}Q_2\Phi_2(t),$$

ϕ_1, ϕ_2 being two increasing on t functions, and M_1, M_2, Q_1, Q_2 constants.

Then, taking into account the differentiability with respect to x of the functions λ, μ, S, we have, from (3.4):

(8.4) $$\begin{aligned}\bar{u}(\tau,x) = {}& \frac{1}{\sqrt{\pi}}\int_0^\tau d\sigma\int_{-\infty}^{\infty} e^{-\xi^2}\,\bar{\lambda}(\sigma,\bar{r}(\sigma,x))\bar{u}(\sigma,x)\,d\xi \;+ \\ &+\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\xi^2}\,\bar{u}(0,x)\,d\xi+\frac{1}{\sqrt{\pi}}\int_0^\tau d\sigma\int_0 e^{-\xi^2}\Big\{\frac{\partial\bar{\lambda}}{\partial \bar{r}}(\sigma,\bar{r}(\sigma,x))\cdot \\ &\cdot\frac{\partial \bar{r}}{\partial x}(\sigma,x)\bar{u}(\sigma,x^*)\;+ \\ &+\bar{\lambda}(\sigma,r(\sigma,x^*))\frac{\partial\bar{u}}{\partial x}(\sigma,x^*)\Big\}2A\sqrt{\tau-\sigma}\,d\xi \\ &+\frac{1}{\sqrt{\pi}}\frac{\partial\bar{u}}{\partial x}(\sigma,x^*)2A\sqrt{\tau},\end{aligned}$$

where x^* is a value between x and $x+2A\sqrt{\tau-\sigma}\cdot\eta$ and, after considering the above estimates we find

$$\left|\bar{u}(\tau,x)-\int_0^\tau \bar{\lambda}(\sigma,\bar{r}(\sigma,x))\bar{u}(\sigma,x)\,d\sigma+\bar{u}(0,x)\right| \le C_1A\sqrt{\tau^3}+C_2 \ldots$$

where C_1, C_2 are constants depending on the constants involved in the hypotheses, independent on A, and F is an increasing on t functions, also independent on A;

Consider now the system (1.1) - (1.5) with A=o, and denote by $(v(t,x), b(t,x), \mathcal{R}(t,x))$ its solution. By analogous notations and operations, we obtain

$$(8.5)\quad \begin{aligned} \bar{v}(\tau,x) &= \int_0^{\tau} \bar{\lambda}(\sigma, \mathcal{R}(\sigma,x))\bar{v}(\sigma,x) + \bar{v}(o,x), \\ b(t,x) &= \int_0^{\infty} \bar{\mu}(t,\alpha,x)v(t,\alpha,x)d\alpha, \\ \frac{\partial R}{\partial t} &= \int_0^{\infty} S(t,\alpha,x)v(t,\alpha,x)d\alpha + G(t,x), \\ v(o,a,x) &= \varphi(a,x), \quad \mathcal{R}(o,x) = r_0(x). \end{aligned}$$

Comparing (8.4), with the first equation (8.5), we obtain:

$$|\bar{u}(\tau,x) = \bar{v}(\tau,x)| \leq \int_0^{\tau} Le^{-k(a_0+\sigma)} \|r - R\| \frac{\varphi_0}{1-\Lambda} d\sigma +$$

$$+ \max^{\Lambda} |\bar{u}(\beta,x) - \bar{v}(\sigma,x)|\, d\sigma + |\bar{u}(o,x) - \bar{v}(o,x)| +$$

$$+ C_1 A\sqrt{\tau^3} + C_2 F(t),$$

which means:

<u>for a >t</u>

$$(8.6)\quad (1-\Lambda) \max_{a<t} |u(t,a,x) - v(t,a,x)| \leq \frac{L\varphi_0}{(1-\Lambda)k} \max \|r - R\| +$$

$$+ \max_{a\leq t} |B(t-a,x) - b(t-a,x)| + C_1 A\sqrt{t^3} + C_2 F(t)$$

and <u>for a >t</u>

$$(8.7)\quad (1-\Lambda) \max |u(t,a,x) - v(t,a,x)|\ \frac{L\varphi_0}{(1-\Lambda)k} \|r - R\| + C_1 A\sqrt{t^3} + C_2 F$$

From the other relations we obtain

$$\|r - R\| \leq \frac{\bar{S}_1}{\Delta} \max \|u - v\|, \quad |B - b| \leq \frac{\mu_0}{h^2} \max |u - v|$$

So that, finally we have

$$\max |u-v| \leq c_1 A \sqrt{t^3} + c_2 F(t).$$

where

$$c_1 = C_1 / [1 - \Lambda - \frac{L\varphi_0}{(1-\Lambda)k} - \frac{\mu_0}{h^2}.$$

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