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INTERNATIONAL ATOMIC ENERGY AGENCY
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
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H4.SMR/994-16

SPRING COLLEGES IN
COMPUTATIONAL PHYSICS

19 May - 27 June 1997

FLUX VORTICES IN SUPERCONDUCTORS - COSMIC STRINGS

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Flux vortices in superconductors – cosmic strings.

Matter field φ, complex, scalar:
φ describes charged particles of 0 spin (bosons)
$\Rightarrow$ superconducting Cooper pairs, Higgs field.

Degrees of freedom:
- real field, expand into normal modes, associate a quantized oscillator to each mode → → particle interpretation;
- complex field, real + imaginary parts give origin to 2 sets of oscillators → two sets of particles of positive + negative charge.

NB: the identification is <u>not</u>

positive, negative $\Leftrightarrow$ real, imaginary,

rather

positive, negative $\Leftrightarrow$ real $\pm$ i imaginary.

Equivalently (not necessarily obvious, though) field has 2 degrees of freedom, modulus + phase:

$$\varphi = e^{i\vartheta}\,\Phi \quad , \quad (\Phi \text{ real}).$$

Generator of phase rotations, $-i\frac{\partial}{\partial\vartheta}$, is like component of angular momentum (e.g. L_z) and has eigenvalues

$$m = 0 \quad \pm 1 \quad \pm 2 \quad \ldots \quad (\text{corresponding to } e^{im\vartheta}),$$

which we can associate with charge.

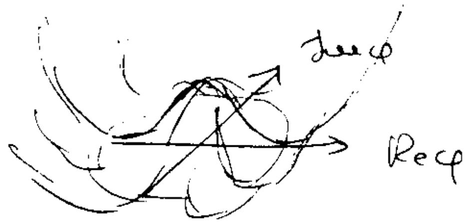
Energy of system.

For normal systems, $E = 0$ for $\varphi = 0$;
for superconductors (+ cosmic strings)

$$E = 0 \qquad \text{for} \quad |\varphi| = \varphi_0 .$$

Energy density can be approximated by

$$\frac{\lambda}{4}\left(|\varphi|^2 - \varphi_0^2\right)^2$$

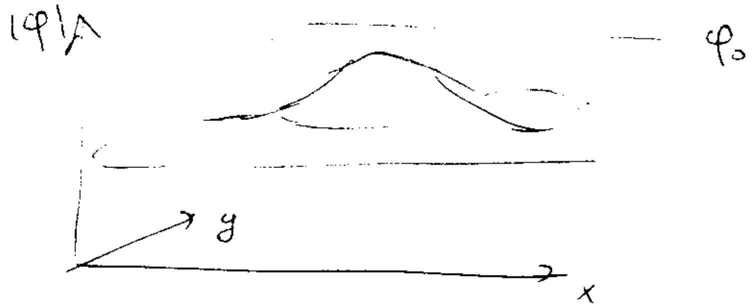


Vacuum, i.e. state with lowest energy, corresponds to

$$\varphi = e^{i\chi} \varphi_0 :$$

$\exists$ arbitrary phase $\Rightarrow$ multiplicity of vacua.

If φ is perturbed away from $|\varphi| = \varphi_0$



it will decay to $\varphi = \varphi_0$, but consider $\varphi(x,y)$ behaving as

$$\varphi(x,y) \xrightarrow[r=\sqrt{x^2+y^2}\to\infty]{} \frac{x+iy}{r}\varphi_0 ,$$

i.e., the phase of φ rotates by $360°$ as one goes around very large circle in xy plane, then continuity demands $|\varphi| = 0$ (and $\neq \varphi_0$)

some-where in the plane.

Consider configurations with translational symmetry along z axis and energy per unit length ($\equiv E$) along z.

E will contain term

$$\int dx\,dy \;\frac{\lambda}{4}\;(|\varphi|^2 - \varphi_0^2)^2,$$

but also term imposing smoothness

$$\int dx\,dy \quad |\vec{\nabla}\varphi|^2 \equiv$$

$$\equiv \int dx\,dy \;(\partial^i \bar{\varphi})(\partial_i \varphi)\,, \qquad \left(\partial_i \equiv \frac{\partial}{\partial x^i}\right.\quad\left.\vphantom{\frac{\partial}{\partial x^i}}\right)$$

$x^1 = x \;\; x^2 = y$.

We use units with $\hbar = c = 1$ and arbitrarily normalize φ, E so that coeff. of this second term is 1.

Setting

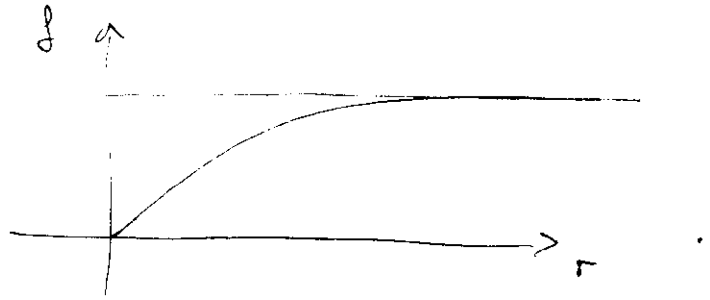
$$\varphi = \frac{x+iy}{r} f(r)\, \varphi_0 ,$$

$$f(0)=0 \qquad f(\infty) = 1 ,$$

we may substitute in E to find

$$E = 2\pi \int r\,dr \left\{ \left[\left(\frac{df}{dr}\right)^2 + \frac{f^2}{r^2} \right] \varphi_0^2 + \frac{\lambda}{4} \varphi_0^4 \left(f^2-1\right)^2 \right\} ,$$

and minimize to find profile of stable static vortex solution.



This is O.K. if we cut off $\int$ at $r = R_{max}$,

but with $R_{max} = \infty$ E is logarithmically divergent

$$\int^{R_{max}} r\,dr\, \frac{1}{r^2} \propto \ln R_{max} .$$

This is because phase rotation contributes to $\vec{\nabla}\varphi$ a term $\propto \frac{1}{r}$.

Introduce e.m. field (indeed, in this static situation only m. field) via potential

$$\vec{A} \quad ; \quad \vec{B} = \vec{\nabla} \times \vec{A} .$$

NB : since $A_z = 0$, $A_x \, A_y = A_{x,y}$ (x,y only)

$$B_x = \partial_y A_z - \partial_z A_y = 0 \, , \quad B_y = 0 \, ,$$

and only non vanishing component is

$$B_z \equiv B = \partial_x A_y - \partial_y A_x \equiv \partial_1 A_2 - \partial_2 A_1 .$$

$\vec{A}$ is introduced in E by making the derivative gauge covariant.

$\partial_i \varphi$ is not invariant for

$$\varphi \Rightarrow e^{ie\chi(x,y)} \varphi \qquad \text{(gauge transf.)}$$

Because

$$\partial_i \varphi \Rightarrow e^{i\chi}\partial_i \varphi + ie(\partial_i\chi)\, e^{i\chi}\varphi .$$

But, with $A_i \Rightarrow A_i + \partial_i \chi$ (gauge transf.)

$$(\partial_i - ieA_i)\varphi \Rightarrow e^{i\chi}\partial_i\varphi + ie(\partial_i\chi)e^{i\chi}\varphi -$$

$$- ieA_i e^{i\chi}\varphi - ie(\partial_i\chi)e^{i\chi}\varphi =$$

$$= e^{i\chi}(\partial_i - ieA_i)\varphi ,$$

i.e. $(\partial_i - ieA_i)\varphi$ transforms in a gauge covariant manner, like φ itself, and, in particular $|(\partial_i - ieA_i)\varphi|^2$ is gauge invariant.

Thus, adding also energy of magnetic field,

$$E = \int dx\,dy \left\{ \overline{(\partial_i - ieA_i)\varphi}\,(\partial^i - ieA^i)\varphi + \right.$$

$$\left. + \frac{\lambda}{4}\left(|\varphi|^2 - \varphi_0^2\right)^2 + \frac{1}{2}B^2 \right\}$$

is gauge invariant.

Now phase of φ can be arbitrarily changed, provided a corresponding A is introduced to compensate. Equivalently, a rotation of the phase, with $|\varphi| = \varphi_0$, will not introduce a contribution to the energy if there is a suitable A.

E.g. take
$$\varphi = \frac{x+iy}{r}\varphi_0 ,$$

$$A_x = -\frac{1}{e}\frac{y}{r^2} ,$$

$$A_y = \frac{1}{e}\frac{x}{r^2} ,$$

and verify

$$(\partial_x - ieA_x)\varphi = \frac{\varphi_0}{r} - \frac{x+iy}{r^2}\frac{x}{r}\varphi_0$$
$$+ \frac{iy}{r^2}\left(\frac{x+iy}{r}\right)\varphi_0 = \frac{\varphi_0}{r^3}\left(x^2+y^2-x^2-ixy+ixy-y^2\right)$$
$$= 0 ,$$

similarly $(\partial_y - ieA_y)\varphi = 0$, and also

$$B = \partial_x A_y - \partial_y A_x = \frac{1}{e}\left(\frac{1}{r^2} - \frac{2x}{r^3}\frac{x}{r} - \right.$$

$$\left. + \frac{1}{r^2} - \frac{2y}{r^3}\frac{y}{r} \right) = \frac{1}{er^4}\left(2x^2 + 2y^2 - 2x^2 - 2y^2 \right) = 0$$

Locally, for very large r, the system is identical to the vacuum; however, the global phase rotation by 360° cannot be undone, and the topologically non trivial vacuum configuration for large r must enclose region where the energy density is $\neq 0$.

Problem: find the profiles of the φ and A fields which minimize E, i.e. the static vortex solution.

Rescale φ_0 and e out of the eqs.:

dim φ = mass = [length]$^{-1}$; use units of mass

such that $\varphi_0 = 1$.

Then

$$E = \int d^2x \left\{ |(\partial_x - ieA)\varphi|^2 + \frac{\lambda}{4}(|\varphi|^2-1)^2 + \frac{1}{2}(\partial_x \times A)^2 \right\}.$$

Also, set

$$x = \frac{\hat{x}}{e} \quad \Rightarrow \quad \partial_x = e\,\partial_{\hat{x}}, \text{ then}$$

$$E = \frac{1}{e^2}\int d^2\hat{x} \left\{ e^2 |(\partial_{\hat{x}} - iA)\varphi|^2 + \frac{\lambda}{4}(|\varphi|^2-1)^2 + \frac{e^2}{2}(\partial_{\hat{x}} \times A)^2 \right.$$

$$= \int d^2\hat{x} \left\{ |(\partial_{\hat{x}} - iA)\varphi|^2 + \frac{\lambda}{4e^2}(|\varphi|^2-1) + (\partial_{\hat{x}} \times A)^2 \right\}$$

By defining $\frac{\lambda}{e^2} = \hat{\lambda}$ every reference to e

disappear ; set $e = 1$.

Ansatz

$$\varphi(x,y) = \frac{x+iy}{r} f(r),$$

$$A_x(x,y) = -\frac{y}{r^2} b(r),$$

$$A_y(x,y) = \frac{x}{r^2} b(r),$$

with

$$f(r) \to 1, \quad b(r) \to 1, \quad r \to \infty,$$

$$f \sim r, \quad b \sim r^2, \quad r \to 0.$$

This gives

$$E = 2\pi \int_0^\infty r\,dr \left\{ \left(\frac{df}{dr}\right)^2 + \frac{1}{2r^2}\left(\frac{db}{dr}\right)^2 + \right.$$

$$\left. + f^2 \frac{(b-1)^2}{r^2} + \frac{\lambda}{4}\left(f^2-1\right)^2 \right\} :$$

find $f(r)$ and $b(r)$ which minimize E.

2^{nd} order coupled non-linear diff eqs for f and b can be derived. We will solve, however, by parametrizing $f + b$ and finding coefficients which minimize $\frac{E}{2\pi}$.

Expected asymptotic behaviors:

$$f = 1 - c_1 e^{-\sqrt{\lambda}\, r},$$

$$b = 1 - c_2 e^{-\sqrt{2}\, r}.$$

We will therefore parametrize by expanding into basis of orthogonal functions with exponential fall-offs.

Orthogonal polynomials:

$$\int \rho(x)\, P_m(x)\, P_n(x)\, dx = c_m \delta_{mn}$$

$$\Rightarrow \quad \int \rho(x)\, dx .. = \int_0^\infty e^{-x}\, dx .. \Rightarrow$$

$\Rightarrow$ P_n = Laguerre polynomials
call them $L_n(x)$.

$L_n(x)$ satisfy: $L_0 = 1$,

$L_1 = 1 - x$,

$$x \frac{d}{dx} L_n(x) = n L_n(x) - n L_{n-1}(x),$$

$$(n+1) L_{n+1}(x) = (2n+1-x) L_n(x) - n L_{n-1}(x),$$

$$\int_0^\infty e^{-x} L_m(x) L_n(x) = \delta_{mn}.$$

Therefore the set of functions

$$\ell_m(x) = e^{-\frac{x}{2}} L_m(x)$$ form an orthonormal set in $0 \to \infty$.

$\ell_m(\alpha x)$ has an asymptotic behavior

$$e^{-\frac{\alpha x}{2}} \times \text{polynomial}.$$

We will expand

$$f = 1 - \sum_{m=0}^{L_{max}} f_m e^{-\sqrt{\lambda} r} L_m(2\sqrt{\lambda} r),$$

note negative sign

$$b = 1 - \sum_{m=0}^{L_{max}} b_m e^{-\sqrt{2} r} L_m(2\sqrt{2} r),$$

We shall minimize E taking

$f_1 \ldots f_{L_{max}}$ $b_2 \ldots b_{L_{max}}$ as free

parameters, and constraing f_0 b_0 b_1

so that $f(0) = 0$ $b(0) = b'(0) = 0$.

L_m's satisfy $L_m(0) = 1$ $L'_m(0) = -m$,

hence we shall impose

$$f_0 + f_1 + \ldots f_{L_{max}} = 1,$$

$$b_0 + b_1 + \ldots b_{L_{max}} = 1,$$

$$b_0 + b_1 + \ldots b_{L_{max}} + 2(b_1 + 2b_2 + \ldots L_{max} b_{L_{max}}) = 0.$$