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SUMMER SCHOOL IN HIGH ENERGY PHYSICS AND COSMOLOGY

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LECTURES ON D-BRANES, GAUGE THEORY AND M(ATRICES)

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Lectures on D-branes, Gauge theory, and M(atrices)

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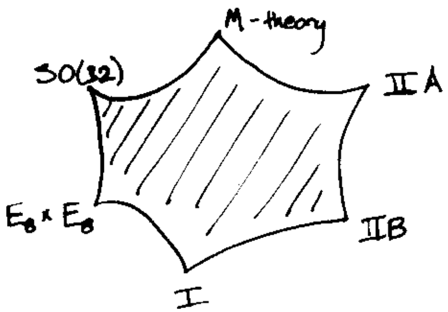
Overview

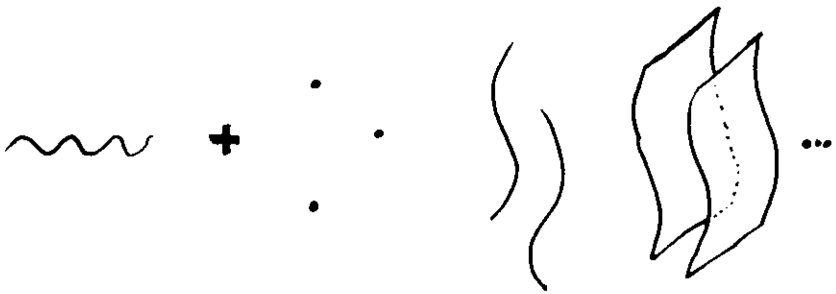
I. Introduction

In the last several years there has been a revolution in string theory.

1. All five string theories (+ 11D Sugr.) are aspects of one underlying theory.



2. String theories contain Dirichlet p-branes.



D-branes are dynamical objects, with low-energy physics described by supersymmetric gauge theory.

Philosophy of these lectures:

Describe D-brane physics using Super-Yang-Mills as the starting point.

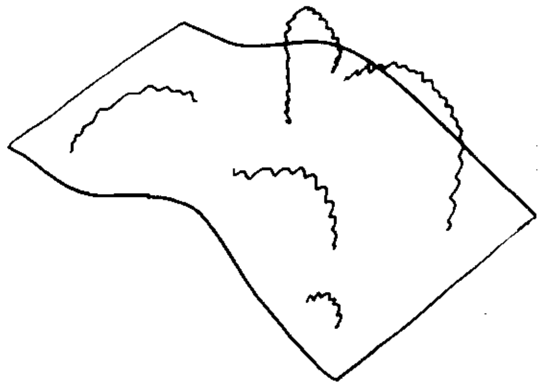
Motivation:

(1) SYM describes much interesting D-brane physics, without needing complicated stringy technology.

(2) M(atrix) conjectures say that SYM is *everything*.

Heuristic discussion: D-branes from string theory

In string theory, D-branes define open string backgrounds.



String ends live on D-branes.

Strings produce D-brane dynamics.

String spectra (from world-sheet a la GSW)

Consider open and closed strings

Bosonic **fields** X^μ and fermionic fields ψ.

Closed **string is like** two copies of open string (left/right)

Boundary **conditions** give different fermion sectors
Open: NS/R, Closed: NS-NS/R-R/NS-R/R-NS

There is a tachyon, removed by GSO projection.

Left/right GSO projection same (different) → **IIB (IIA)**.

Massless fields characterized by SO(8) rep.:

Open string: $8_v \rightarrow A_\mu$ (NS), $8_s \rightarrow \psi_\mu$ (R).

Closed:

	NS-NS	R-R
IIA	$g^{\mu\nu}, \phi, B^{\mu\nu}$	$A^{(1)}_\mu, A^{(3)}_{\mu\nu\rho}$
IIB	$g^{\mu\nu}, \phi, B^{\mu\nu}$	$A^{(0)}, A^{(2)}_{\mu\nu}, A^{(4)}_{\mu\nu\rho\sigma}$

where $A^{(4)}$ is self-dual.

D-branes couple to R-R fields, $\int_{p+1} A^{(p+1)}$ (Polchinski).

Type IIA: $p = 0, 2, 4, 6, 8$; type IIB: $p = -1, 1, 3, 5, 7, 9$

In a D-brane background

open string vector field A^μ decomposes into:
- $p + 1$-dimensional U(1) gauge field A^α
- $9 - p$ transverse coordinates X^a

D-brane action for massless string fields is (Leigh)

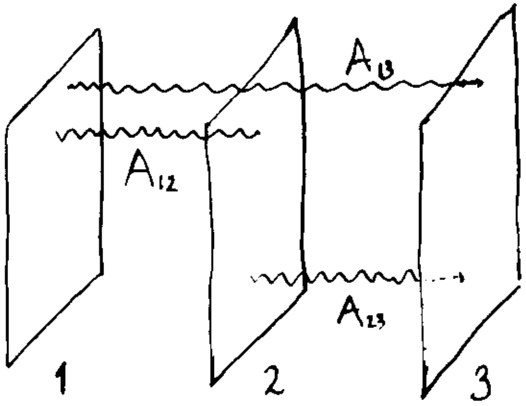
$$S = -\tau_p \int d^{p+1}\xi \ \det{}^{1/2}(G + B + 2\pi\alpha' F) + \text{ferm.} + \text{CS}$$

$$\tau_p = \frac{T_p}{g} = \frac{1}{g\sqrt{\alpha'}} \frac{1}{(2\pi\sqrt{\alpha'})^p}$$

For weak field in flat space, gives U(1) SYM

$$S \sim \int d^{p+1}\xi(-F_{\alpha\beta}F^{\alpha\beta} - (D_\alpha X^a)^2 - \text{ferm.})$$

Strings stretching between D-branes have mass ~ length.



Fields A_{ij}^{α} become massless as branes i, j approach.

N Parallel D-branes acquire $U(N)$ gauge theory upon coincidence. (Witten)

Low-energy action is 10D SYM action

$$S \sim \int d^{10}\xi \; - \text{Tr}\, F_{\mu\nu}F^{\mu\nu} + i\text{Tr}\, \bar{\psi}\Gamma^{\mu}D_{\mu}\psi$$

reduced to $p+1$ dimensions

This is the starting point for most of the material in these lectures.

II. D-branes and Super-Yang-Mills

Starting point

Low-energy physics of N coincident Dirichlet p-branes is described by $(p+1)$-dimensional Super-Yang-Mills, given by dimensional reduction of $\mathcal{N} = 1$ SYM in 10D.

10D Super-Yang-Mills

Ten dimensional $U(N)$ Super-Yang-Mills has an action

$$S = \int d^{10}\xi \left(-\frac{1}{4}\text{Tr}\, F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\text{Tr}\, \bar{\psi}\Gamma^{\mu}D_{\mu}\psi \right)$$

The field strength

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} - ig_{YM}[A_{\mu}, A_{\nu}]$$

Is the curvature of a $U(N)$ gauge field A_{μ}.

A_{μ} and ψ both live in the adjoint of $u(N)$ and carry a (suppressed) adjoint index.

The covariant derivative D_{μ} of ψ is

$$D_{\mu}\psi = \partial_{\mu}\psi - ig_{YM}[A_{\mu}, \psi]$$

ψ is a Majorana-Weyl spinor of $SO(1,9)$, acted on by 32×32 Γ matrices.

The action is invariant under the supersymmetry

$$\delta A_{\mu} = \frac{i}{2}\bar{\epsilon}\Gamma_{\mu}\psi$$

$$\delta\psi = -\frac{1}{4}F_{\mu\nu}\Gamma^{\mu\nu}\epsilon$$

10D SYM has 8 bosonic degrees of freedom and 8 fermionic degrees of freedom.

There are 16 supercharges.

10D SYM is a well-defined classical field theory but is anomalous and problematic quantum mechanically.

After dimensional reduction, theory is better behaved.

Convention: absorb g_{YM} in A so action is

$$S = \frac{1}{4g_{YM}^2} \int d^{10}\xi \ \left(-\mathrm{Tr}\ F_{\mu\nu}F^{\mu\nu} + 2i\mathrm{Tr}\ \bar{\psi}\Gamma^\mu D_\mu \psi\right)$$

and covariant derivative is

$$D_\mu = \partial_\mu - iA_\mu$$

Dimensional reduction

Dimensionally reduce to $p+1$ dimensions by making fields independent of $9-p$ coordinates.

Fields in $p+1$ dimensions:
$(p+1)$-dimensional gauge field A_α
$9-p$ adjoint fields X^i (transverse positions)
associated fermions

Action:

$$S \sim \int d^{p+1}\xi \mathrm{Tr}\ (-F_{\alpha\beta}F^{\alpha\beta} - (D_\alpha X^i)^2 + [X^i, X^j]^2 + \text{ferm.})$$

Classical vacua:
F, fermions vanish
X^i constant and $[X^i, X^j] = 0$

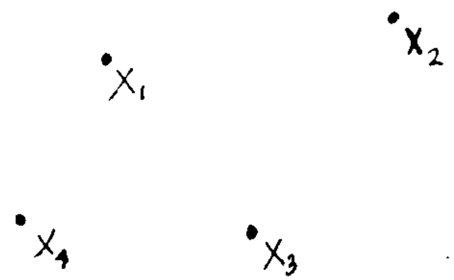
When X^i commute, simultaneously diagonalizable.

$$X^i = \begin{pmatrix} x_1^i & 0 & 0 & \ddots \\ 0 & x_2^i & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ \ddots & 0 & 0 & x_N^i \end{pmatrix}$$

x_k^i are coordinates of kth D-brane.

Configuration space is

$$\frac{(\mathbf{R}^{9-p})^k}{S_k}$$



Permutation symmetry is residual Weyl symmetry.

Example: 3-branes

Dimensional reduction of $\mathcal{N}=1$ 10D SYM to **4D**

$A_\mu \to$ 6 transverse scalars X^a, 4D connection A_α

Gives $\mathcal{N}=4$ SYM in 4D.

In $\mathcal{N}=1$ language, this theory has chiral superfields ϕ, B, C

with superpotential

$$W = \text{Tr}\ \phi[B,C].$$

Superfields have six real bosonic components, corresponding to transverse X fields.

$$\phi = X^4 + iX^5, B = X^6 + iX^7, C = X^8 + iX^9$$

potential is

$$\sum |\frac{\partial W}{\partial \phi^i}|^2 = \text{Tr}\ \left(|[B,C]|^2 + |[\phi,B]|^2 + |[\phi,C]|^2\right)$$

$$\text{Tr}\ (D^a)^2 = \frac{1}{4}\text{Tr}\ (|[\phi,\phi^\dagger]|^2 + |[B,B^\dagger]|^2 + |[C,C^\dagger]|^2)$$

gives

$$V = -\sum_{i<j}[X^i, X^j]^2$$

Example: 0-branes

Dimensionally reduced SYM becomes $\mathcal{N}=16$ SUSY matrix quantum mechanics.

In units with $2\pi\alpha' = 1$, in gauge $A_0 = 0$

$$\mathcal{L} = \frac{1}{2g}\left[\dot{X}^a\dot{X}_a + \sum_{a<b}\text{Tr}\ [X^a, X^b]^2 + 2\theta^T(\dot{\theta} + \Gamma_a[X^a,\theta])\right]$$

There are nine adjoint scalar matrices X^a.

Superpartners are 16-component spinors θ

Transform under $SO(9)$ Clifford algebra.

Classical static solutions have $[X^a, X^b] = 0$.

Diagonal elements are 0-brane positions.

Classical configuration space

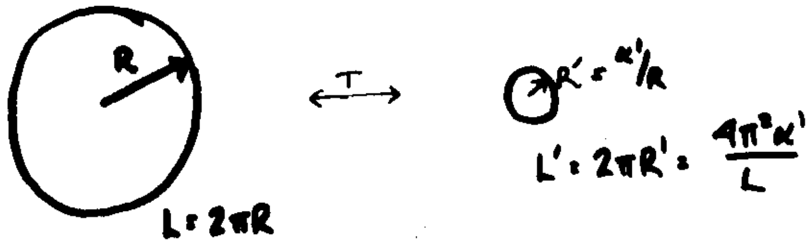
$$\frac{(\mathbf{R}^9)^N}{S_N}$$

Off-diagonal elements give noncommutative geometry

III. D-branes and Duality

T-duality

Compactify on a circle of radius R, giving $\mathbf{R}^9 \times S^1$.



T-duality maps $\frac{\text{IIA}}{S^1} \leftrightarrow \frac{\text{IIB}}{S^1}$, $R \leftrightarrow \frac{\alpha'}{R}$.

D-branes first understood from T-duality in string theory

For *fixed* b.c., T-duality maps Neumann $\leftrightarrow$ Dirichlet.

Thus, T-duality maps

$$p - \text{branes} \leftrightarrow (p \pm 1) - \text{branes}$$

Expect that T-duality is a symmetry of string theory.

Investigate in context of U(N) D-brane gauge theory.

Prove T-duality from Super-Yang-Mills.

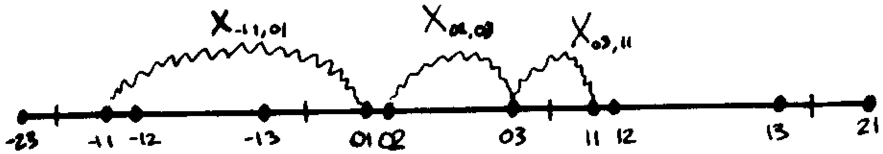
Set $2\pi\alpha' = 1$ for now, will restore later.

0-branes in flat space described by super QM

0-branes on R^9/Γ where Γ is discrete:
Consider R^9 with $|\Gamma|$ copies, impose symmetry.

Example: ALE spaces (Douglas/Moore)

Study 0-branes on S^1 by looking at covering space $\mathbf{R}$.



Index copies by $n \in \mathbf{Z}$ since $S^1 = \mathbf{R}/\mathbf{Z}$.

$U(\infty)$ Quantum mechanics with constraints.

Fields $X^a_{mi,nj}$; write as $N \times N$ matrices X^a_{mn}

Translation invariance says

$$\begin{aligned} X^a_{mn} &= X^a_{(m-1)(n-1)}, \quad a > 1 \\ X^1_{mn} &= X^1_{(m-1)(n-1)}, \quad m \neq n \\ X^1_{nn} &= 2\pi R I + X^1_{(n-1)(n-1)} \end{aligned}$$

So X^1 in the compact direction looks like

$$\begin{pmatrix} \ddots & X_1 & X_2 & X_3 & \ddots \\ X_{-1} & X_0 - 2\pi RI & X_1 & X_2 & X_3 \\ X_{-2} & X_{-1} & X_0 & X_1 & X_2 \\ X_{-3} & X_{-2} & X_{-1} & X_0 + 2\pi RI & X_1 \\ \ddots & X_{-3} & X_{-2} & X_{-1} & \ddots \end{pmatrix}$$

where $X_k = X^1_{0k}$

This is a matrix representation of

$$X^1 = i\hat{\partial} + A(\hat{x})$$

on a Fourier decomposition of functions

$$\phi(\hat{x}) = \sum_n \hat{\phi}_n e^{in\hat{x}/R'}$$

with $R' = \alpha'/R = 1/(2\pi R)$.

$$i\partial = \text{diag}(\ldots, -4\pi RI, -2\pi RI, 0, 2\pi RI, 4\pi RI, \ldots)$$

and decomposing

$$A = \sum_n e^{in\hat{x}/R'} X_n$$

identifies $X_n = X^1_{0n}$ with modes of A.

We can therefore identify

$$X^a \sim i\partial^a + A^a$$

under T-duality in the compact direction.

Gives an explicit SYM derivation of T-duality.

0-brane and 1-brane SYM actions on S^1 are equivalent.

After replacement,

$$\text{Tr } [X^1, X^a]^2$$

becomes

$$-\left(\frac{1}{2\pi R'}\right) \int dx \text{ Tr } (\partial_1 X^a - i[A^1, X^a])^2$$

$$= -\left(\frac{1}{2\pi R'}\right) \int dx \text{ Tr } (D_1 X^a)^2$$

Factor of $R = 1/(2\pi R')$ needed to extract zero mode.

Similarly,

$$-\text{Tr } (D_0 X^1)^2 \to -\left(\frac{1}{2\pi R'}\right) \int dx \text{ Tr } F^2_{01}$$

Resulting action is 2D SYM with 16 supercharges.

Generalizes to arbitrary dimensions:

p-brane & q-brane SYM actions on $T^{|p-q|}$ are equivalent.

After T-duality on two compact directions

$$([X^a, X^b])^2 \to -\left(\frac{1}{4\pi^2 R'_a R'_b}\right)\int dx^a dx^b \, (F^{ab})^2$$

Including constants, transformation is

$$X^a \leftrightarrow (2\pi\alpha')(i\partial^a + A^a)$$

We will use this relation to study many D-brane configurations and their transformations under T-duality.

Essential feature:

T-duality exchanges string winding and momentum, as expected from string theory.

We can generalize to twisted sectors.

Constraints can be expressed abstractly as

$$UX^aU^{-1} = X^a + \delta^{a1}2\pi RI.$$

This is satisfied formally by

$$X^1 = i\partial^1 + A^1, \qquad U = e^{2\pi i\hat{x}^1 R}$$

More generally, can include a gauge transformation

$$UX^aU^{-1} = g(X^a + \delta^{a1}2\pi RI)g^{-1}.$$

This is satisfied formally by

$$X^1 = i\partial^1 + A^1, \qquad U = g \cdot e^{2\pi i\hat{x}^1 R}$$

Corresponds to a bundle with nontrivial BC.

For example, if g is a permutation it switches labels of 0-branes on each sheet of covering space.

S-duality

IIB string appears to have an S-duality symmetry.

Symmetry group is $SL(2,Z)$.

Exchanges strong and weak coupling.

Exchanges strings and 1-branes.

Exchanges 5-branes and NS 5-branes.

Leaves 3-branes unchanged.

On 3-branes, appears to act as SYM $\mathcal{N}=4$ S-duality.

SYM S-duality conjectured by Montonen and Olive.

Generalization of Maxwell duality

$$\begin{array}{ccc} F & \leftrightarrow & *F \\ E & \leftrightarrow & B \\ e & \leftrightarrow & g \end{array}$$

For $U(1)$ theory this is a symmetry.

For non-abelian theories, only seems to be true with $\mathcal{N}=4$ supersymmetry.

Still unproven, but much evidence.

S-duality maps group G to dual group $\widehat{G}$.

Yang-Mills coupling and theta angle can be packaged in

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$$

$SL(2,Z)$ acts on coupling by

$$\tau \to \frac{a\tau + b}{c\tau + d}$$

where $a, b, c, d \in \mathbf{Z}$ with $ad - bc = 1$.

In particular,

$$\tau \to \tau + 1$$

is periodicity of θ.

And

$$\tau \to -1/\tau$$

inverts coupling and corresponds to strong-weak duality.

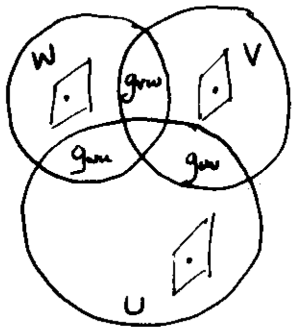
Dyon masses (BPS) agree with duality.

Partition function checked in some cases on compact manifolds in twisted theory by Vafa and Witten.

IV. Branes and Bundles

Review of vector bundles

A vector bundle with structure group G looks locally like $U \times R^k$.



Transition functions between patches g_{uv} identify (x, f) and (x', f') where $f' = g_{uv} f$.

Transition functions satisfy cocycle conditions

$$g_{uv} g_{vw} g_{wu} = 1.$$

A Yang-Mills connection is associated with a principal bundle with fiber G. Transforms as

$$A' = g \cdot A \cdot g^{-1} - i\, dg \cdot g^{-1}$$

Matter fields are sections of associated vector bundles.

Over compact manifolds like T^{2n}, nontrivial bundles are characterized by topological invariants like Chern classes

$$c_1 = \frac{1}{2\pi} \text{Tr } F$$

$$c_2 = \frac{1}{8\pi^2} (\text{Tr } F \wedge F - (\text{Tr } F) \wedge (\text{Tr } F)) \ldots$$

c_i are integral cohomology classes.

On D-branes, topological invariants carry brane charges

$\int F$ carries $(p-2)$-brane charge.

$\int F \wedge F$ carries $(p-4)$-brane charge.

$\int F \wedge F \wedge F$ carries $(p-6)$-brane charge ...

Originally understood using string methods

Term in D-brane action

$$\text{Tr} \int_{\Sigma_{p+1}} C \wedge e^F$$

where C is sum of R-R fields.

For example, on a 4-brane $F \wedge F$ couples to $A^{(1)}$, giving 0-brane charge.

$$\text{Tr} \int_{\Sigma_5} A^{(1)} \wedge F \wedge F$$

Fluxes and T-duality

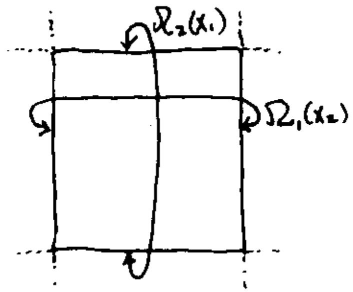
Compactify on T^2, with radii

$$R_1 = L_1/(2\pi),\ R_2 = L_2/(2\pi)$$

Wrapping N 2-branes gives $U(N)$ SYM on T^2.

Nontrivial $U(N)$ bundles given by transition functions

$$\Omega_1(x_2),\ \Omega_2(x_1)$$



Fundamental matter fields obey

$$\begin{aligned} \phi(x_1 + L_1, x_2) &= \Omega_1(x_2)\phi(x_1, x_2) \\ \phi(x_1, x_2 + L_2) &= \Omega_2(x_1)\phi(x_1, x_2). \end{aligned}$$

Connection satisfies

$$\begin{aligned} A_2(x_1 + L_1, x_2) &= \Omega_1(x_2)A_2(x_1, x_2)\Omega_1^{-1}(x_2) \\ &\quad -i\,(\partial_2\Omega_1(x_2))\cdot\Omega_1^{-1}(x_2) \end{aligned}$$

etc.

Cocycle condition for a well-defined bundle is

$$\Omega_2^{-1}(L_1)\Omega_1^{-1}(0)\Omega_2(0)\Omega_1(L_2) = 1.$$

For compactification on T^n story is same.

Transition functions Ω_i satisfy cocycle condition for all pairs i, j

Example: T^2

$U(N)$ bundles over T^2 are classified by $C_1 = \int c_1$.

Curvature is

$$\frac{1}{2\pi}\int \mathrm{Tr}\ F = k \in \mathbf{Z}$$

Decomposing

$$U(N) = (U(1) \times SU(N))/\mathbf{Z}_N$$

This corresponds to $U(1)$ flux $F = kI/N$

and an $SU(N)$ bundle with 't Hooft twist $e^{2\pi ik/N}$.

Twisted bundles satisfy

$$\Omega_2^{-1}(L_1)\Omega_1^{-1}(0)\Omega_2(0)\Omega_1(L_2) = Z$$

where $Z = e^{2\pi ik/N}I$ is central in $SU(N)$.

Consider a $U(N)$ theory on T^2 with flux $\int F = 2\pi$.

We can choose boundary conditions

$$\begin{aligned} \Omega_1(x_2) &= e^{2\pi i (x_2/L_2) T} V \\ \Omega_2(x_1) &= 1 \end{aligned}$$

where

$$V = \begin{pmatrix} & 1 & & \\ & & 1 & \\ & & & \ddots \\ 1 & & & \end{pmatrix}$$

And $T = (0, 0, \ldots, 0, 1/N)$.

These boundary conditions admit a linear connection with constant curvature

$$\begin{aligned} A_1 &= 0 \\ A_2 &= F x_1 + \frac{2\pi}{L_2} \text{Diag}(0, 1/N, \ldots, (N-1)/N) \end{aligned}$$

with

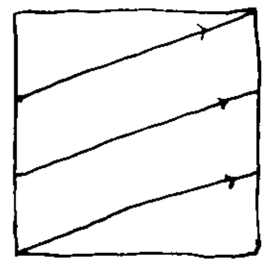
$$F = \frac{2\pi}{N L_1 L_2} I.$$

$\Omega_2 = 1$ so T-dualize using $X^2 = (2\pi\alpha')(i\partial_2 + A_2)$.

This gives

$$X^2 = \left(\frac{4\pi^2\alpha'}{L_2}\right) \cdot \left(\frac{1}{N}\right) \cdot \begin{pmatrix} \frac{x_1}{L_1} & 0 & 0 & \ddots \\ 0 & \frac{x_1}{L_1} + 1 & 0 & \ddots \\ 0 & \ddots & \ddots & 0 \\ \ddots & 0 & 0 & \frac{x_1}{L_1} + (N-1) \end{pmatrix}$$

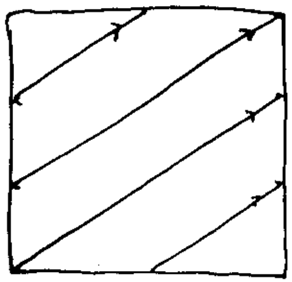
Corresponding to a 1-brane wrapped once around X^2. (Dual radius is $R_2' = \alpha'/R_2 = 2\pi\alpha'/L_2$.)



Since wrapped 1-branes are T-dual to 0-branes, the original flux on the 2-brane corresponds to a 0-brane.

Notes:

- Similar construction possible for k 0-branes.



(3,2) wrapping

- Chose $\Omega_2 = 1$ for convenience.

Could have used more standard ('t Hooft)

$$\begin{aligned}\Omega_1(x_2) &= e^{2\pi i (x_2/L_2)(1/N) I} U \\ \Omega_2(x_1) &= V\end{aligned}$$

$$U = \begin{pmatrix} 1 & & & \\ & e^{\frac{2\pi i}{N}} & & \\ & & \ddots & \\ & & & e^{\frac{2\pi i (N-1)}{N}} \end{pmatrix}$$

Now translation in covering space gives rotation by V.

Example: T^4

Now let's consider an instanton on T^4 with sides L.

No $U(N)$ Yang-Mills solution with $k = 1$ unless instanton is pointlike.

Consider $k = 2, N = 2$.

Take transition functions

$$\begin{aligned}\Omega_2 = \Omega_4 &= 1 \\ \Omega_1 &= e^{2\pi i (x_2/L)\tau_3} \\ \Omega_3 &= e^{2\pi i (x_4/L)\tau_3}\end{aligned}$$

where

$$\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Admits connection

$$\begin{aligned}A_1 = A_3 &= 0 \\ A_2 &= \frac{2\pi x_1}{L^2}\tau_3 \\ A_4 &= \frac{2\pi x_3}{L^2}\tau_3\end{aligned}$$

with curvature

$$F_{12} = F_{34} = \frac{2\pi}{L^2}\tau_3$$

Since Tr $F = 0$ there is no 2-brane charge.

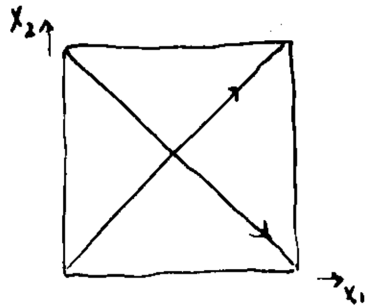
Instanton number is

$$C_2 = \frac{1}{8\pi^2} \int d^4x \; F \wedge F = 2$$

which we expect corresponds to two 0-branes.

Under T-duality in directions 2, 4 we get two 2-branes

$$\begin{aligned} X_2(x_1, x_3) &= \pm 4\pi^2 \alpha' x_1 / L^2 \\ X_4(x_1, x_3) &= \pm 4\pi^2 \alpha' x_3 / L^2 \end{aligned}$$



2-brane charges correspond to homology 2-cycles.

brane 1 $\rightarrow$ $(13) + (14) + (23) + (24)$
brane 2 $\rightarrow$ $(13) - (14) - (23) + (24)$

Total is $2(13) + 2(24)$.

T-dual to two 4-branes and two 0-branes.

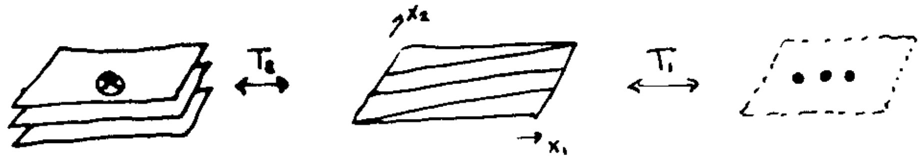
Branes from smaller branes

So far we've built $(p-2k)$-branes from fluxes.

Now, through T-duality, we'll build $(p+2k)$-branes.

Consider again the diagonal $(N, 1)$ 1-brane on T^2

(Set $L_i = 1$ for the moment)



Satisfies

$$[(\partial_1 - iA_1), X^2] = \frac{4\pi^2\alpha'}{N} I$$

T-dual on X^2 to N 2-branes with unit flux

$$[(\partial_1 - iA_1), (\partial_2 - iA_2)] = -iF = \frac{-2\pi i}{N} I$$

T-dual on X^1 to N 0-branes satisfying

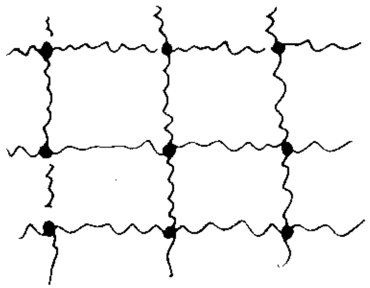
$$[X^1, X^2] = \frac{8\pi^3 \alpha'^2 i}{N} I$$

Normalizing radii of T^2 to 1,

N 0-branes on T^2 with matrices X satisfying

$$\mathrm{Tr}\,[X^1, X^2] = 2\pi i$$

carry a unit of 2-brane charge.



Note: impossible for finite N without compactification.

Generalizes naturally to higher dimensions.

N 0-branes on T^4 with

$$\mathrm{Tr}\,\epsilon_{ijkl} X^i X^j X^k X^l = 8\pi^2$$

carry a unit of 4-brane charge, etc.

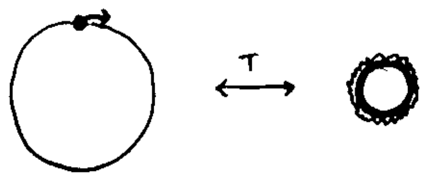
Conditions can be mixed. 2-branes wrapped on $1,2$ with

$$\mathrm{Tr}\,\left(F_{12}[X^3, X^4] - (D_1 X^3)(D_2 X^4) + (D_1 X^4)(D_2 X^3)\right) = 8\pi^2$$

has a unit of 4-brane charge on dimensions (1234).

Strings and electric fields

We can give a 0-brane on S^1 momentum.



Momentum proportional to $\dot{X}^1$.

Under T-duality,

$$\dot{X}^1 \rightarrow \int (2\pi\alpha')\dot{A}^1$$

Thus, momentum $\leftrightarrow$ electric flux.

This corresponds to string momentum $\leftrightarrow$ winding.

Momentum and T-dual string winding are quantized through QM.

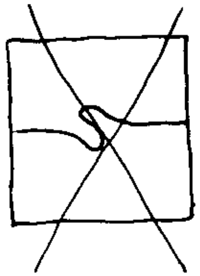
Unlike fluxes which are topologically quantized.

V. Born-Infeld Theory

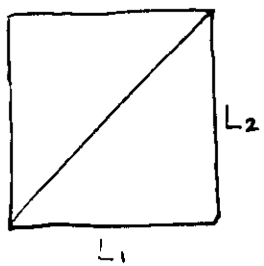
For M(atrix) theory, SYM is starting point.

Inadequate for describing details of type II D-branes.

- Static gauge



- Small field, slope



$$E_{\text{string}} = \tau_1 L = \tau_1 \sqrt{L_1^2 + L_2^2}$$

$$E_{\text{YM}} = \tau_2 L_1 L_2 (F^2) = \text{constant} \cdot \tau_1 \left(\frac{L_2^2}{2L_1} \right)$$

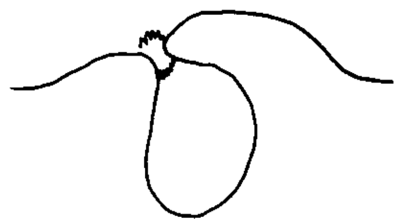
Yang-Mills energy is leading correction to $\tau_1 L_1$.

Complete description of D-brane physics requires Born-Infeld

$$S = -\tau_p \int d^{p+1}\xi \ \det{}^{1/2}(G + B + 2\pi\alpha' F) + \text{ferm.} + \text{CS}$$

However, there are complications:

- Difficult to supersymmetrize (κ-symmetry)
- Correct non-abelian Born-Infeld action unknown
- Nonlocal interactions



Simplified version: static gauge, abelian, flat background

Dimensionally reduce

$$S = -\int d^{10}x \sqrt{-\det\left(\eta_{\mu\nu} + F_{\mu\nu} - 2\bar{\lambda}\Gamma_\mu \partial_\nu \lambda + \bar{\lambda}\Gamma^\rho \partial_\mu \lambda \bar{\lambda}\Gamma_\rho \partial_\nu\right.}$$

to $p+1$ dimensions (Aganagic/Popescu/Schwarz)

After dimensional reduction,

$$S = -\tau_p \int d^{p+1}\xi \ \det{}^{1/2}(\eta_{\alpha\beta}+\partial_\alpha X^a \partial_\beta X^a + 2\pi\alpha' F_{\alpha\beta}) + \text{ferm.}$$

Generally, difficult to extend to non-abelian case.

Possible when F and ∂X^a are constant and commute.

Can use to describe simple aspects of multi-brane configurations:

- Energy
- Fluctuation spectra
- Scattering amplitudes

In particular, consider configurations of p-branes with

1. $X^a = 0$ for all transverse a, $\lambda = 0$
2. $F_{\alpha\beta}$ constant for all α, β
3. $[F_{\alpha\beta}, F_{\gamma\delta}] = 0$ for all $\alpha, \beta, \gamma, \delta$.

After simultaneous diagonalization,

$$S = -\tau_p \int d^{p+1}\xi \text{Tr} \sqrt{-\det(\eta_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta})}$$

Energies

First, fix normalization constants.

$$\tau_p = \frac{T_p}{g} = \frac{1}{g\sqrt{\alpha'}} \frac{1}{(4\pi^2\alpha')^{p/2}}$$

where

$$g = e^\phi$$

is the string coupling, related to the dilaton field.

From T-duality, we need

$$\tau_p = \tau'_{p+1}(2\pi R')$$

This is in accord with

$$g = g'\sqrt{\alpha'}/R'$$

which comes from invariance of

$$g_9 = g\sqrt{\alpha'/R_9}.$$

Let's now use Born-Infeld to calculate the energies of some simple systems.

N 2-branes with q units of flux have

$$F_{12} = \frac{2\pi q}{L_1 L_2 N} I.$$

$$\det\left(\delta_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta}\right) = 1 + 4\pi^2\alpha'^2 F_{12}^2$$

Energy is

$$\begin{aligned} E &= N\tau_2 L_1 L_2 \sqrt{1 + 4\pi^2\alpha'^2 F_{12}^2} \\ &= \sqrt{(N\tau_2 L_1 L_2)^2 + (q\tau_0)^2} \end{aligned}$$

Energy for a completely separated system of N 2-branes and q 0-branes would be

$$N\tau_2 L_1 L_2 + q\tau_0$$

After T-dualizing in direction 2,

$$\begin{aligned} E &= \tau_1' \sqrt{(NL_1)^2 + (4\pi^2\alpha' q)^2 / L_2^2} \\ &= \tau_1' \sqrt{(NL_1)^2 + (qL_2')^2} \end{aligned}$$

as desired.

Now consider 2 4-branes with instanton number $k = 2$ on T^4 with volume $V_4 = L_1 L_2 L_3 L_4$

As before, we can choose a linear connection

$$\begin{aligned} A_1 = A_3 &= 0 \\ A_2 &= \frac{2\pi x_1}{L_1 L_2}\tau_3 \\ A_4 &= \frac{2\pi x_3}{L_3 L_4}\tau_3 \end{aligned}$$

with curvature

$$\begin{aligned} F_{12} &= \frac{2\pi}{L_1 L_2}\tau_3 \\ F_{34} &= \frac{2\pi}{L_3 L_4}\tau_3 \end{aligned}$$

Conditions 1-3 are satisfied. Energy is

$$\begin{aligned} E &= \tau_4 V_4 \mathrm{Tr}\, \sqrt{\det\left(\delta_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta}\right)} \\ &= 2\tau_4 V_4 \sqrt{(1 + 4\pi^2\alpha'^2 F_{12}^2)(1 + 4\pi^2\alpha'^2 F_{34}^2)} \end{aligned}$$

When $L_1 L_2 = L_3 L_4$ configuration is self-dual and

$$E = 2\tau_4 V_4 + 2\tau_0$$

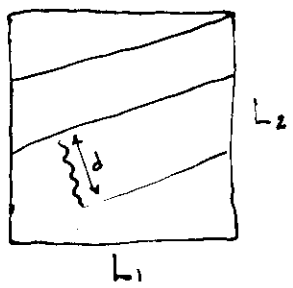
which is precisely the energy of the separated branes.

Exercise: check that E is τ_2' times the total volume of the two 2-branes after T-dualizing in directions 2, 4.

VI. D-brane Fluctuations

Fractional Momenta

Consider a 1-brane wrapped around $(N, 1)$ on T^2.



We expect momentum modes with

$$E = \frac{2\pi}{L} n$$

where

$$L = \sqrt{N^2 L_1^2 + L_2^2}$$

is the length of the 1-brane

and winding modes where

$$E = \frac{d}{2\pi\alpha'} m$$

where

$$d = \frac{L_1 L_2}{\sqrt{N^2 L_1^2 + L_2^2}}$$

is the minimal length of a winding string.

Dual Yang-Mills picture is N 2-branes with 1 flux quantum.

Boundary conditions

$$\begin{aligned} \Omega_1(x_2) &= e^{2\pi i (x_2/L_2) T} V \\ \Omega_2(x_1) &= 1 \end{aligned}$$

or

$$\begin{aligned} \Omega_1(x_2) &= e^{2\pi i (x_2/L_2)(1/N) I} U \\ \Omega_2(x_1) &= V \end{aligned}$$

give fractional momenta.

More transparent with latter BC's.

$$\delta A_i(x_1, x_2 + L_2) = V \cdot \delta A_i(x_1, x_2) \cdot V^{-1}$$

makes both components periodic in x_2 with period NL_2.

$$\delta A_i(x_1 + L_1, x_2) = U \cdot \delta A_i(x_1, x_2) \cdot U^{-1}$$

makes the $j - k$ diagonal transform by $e^{2\pi i (j-k)/N}$ which gives momenta $n, n + 1/N, \ldots, n + (N-1)/N$.

Thus, momentum in both directions is quantized in units of $1/N$.

N^2-fold degeneracy is raised.

Same results for any BC's for same bundle.

So for Yang-Mills, flux causes fractional quantization.

Corresponds to dual string excitations only in limit

$$qL'_2 = 4\pi^2\alpha' q/L_2 \ll NL_1$$

For exact correspondence, need Born-Infeld.

When $F \sim I$ commutes with everything, computation simplifies

Fluctuations in F_{12} rescaled by $(1 + (2\pi\alpha' F)^2)^{-2}$

Fluctuations in F_{0i} rescaled by $(1 + (2\pi\alpha' F)^2)^{-1}$

Causes overall rescaling of fluctuation spectra by

$$\frac{1}{1 + (2\pi\alpha' F)^2} = \frac{1}{1 + \left(\frac{4\pi^2\alpha'}{NL_1L_2}\right)^2}$$

$$E^2 = \frac{1}{N^2 + \left(\frac{4\pi^2\alpha'}{L_1L_2}\right)^2}\left(\left(\frac{2\pi}{L_1}n_1\right)^2 + \left(\frac{2\pi}{L_2}n_2\right)^2\right)$$

in agreement with string theory.

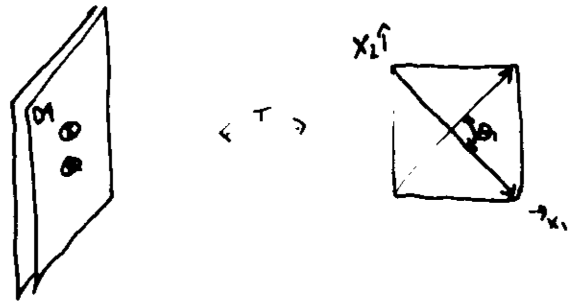
Fluctuation spectra provide a check of non-abelian Born-Infeld.

One suggestion for NBI was made by Tseytlin:

$$\mathcal{L} = \mathrm{STr}\sqrt{-\det(\eta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})}$$

where trace is symmetrized after formal expansion.

This can be checked in a background of intersecting branes dual to a gauge theory background.



Consider two 4-branes on T^4 with $k = 2$ 0-branes.

$$\begin{aligned} \tan(\theta_1/2) &= 2\pi\alpha' F_{12} \\ \tan(\theta_2/2) &= 2\pi\alpha' F_{34} \end{aligned}$$

relates angles of T-dual 2-branes to fluxes.

Compare string spectrum in 2-brane background with Yang-Mills and Born-Infeld spectrum in 4 + 0 background.

String calculation of fluctuation spectra around two 4-branes at angles θ_1, θ_2 (Berkooz, Douglas, Leigh)

$$M = \theta_1/\pi \cdot (n_1 + k_1/2) + \theta_2/\pi \cdot (n_2 + k_2/2)$$

for certain combinations k_1, k_2 and integer $n \geq 0$.

Yang-Mills fluctuation spectra on T^4 (van Baal)

$$M = 2\tan(\theta_1/2)/\pi \cdot (n_1 + k_1/2) + 2\tan(\theta_2/2)/\pi \cdot (n_2 + k_2/2)$$

When $\theta_1 \neq \theta_2$ both spectra have tachyons from

$$(k_1, k_2) = (1, -1) \text{ or } (-1, 1)$$

This corresponds to unstable background.

Signs of spectra are same, but exact values are different.

Symmetrized trace prescription doesn't correct completely.

In special case $\theta_1 = \theta_2$, n terms agree, not k terms.

Discrepancy probably due to $[F, F]$ or DF terms ignored in Tseytlin's analysis.

Formulation of complete non-abelian Born-Infeld theory is an open problem.

VII. Bound states of D-branes

We've discussed aspects of D-brane geometry, energy and fluctuations.

Now consider interactions, beginning with bound states.

Several systems of interest:

- $p - p'$ bound states
- $p - p$ bound states
- brane-string bound states

Bound states originally understood from supergravity and using duality from perturbative string spectrum.

1-brane/string bound states described by Witten using gauge theory.

Studying bound states has been a major area of interest.

We'll review some important developments which are easily accessible from gauge theory.

BPS states

Certain extended SUSY algebras contain central terms

$$\{Q, Q\} \sim P + Z$$

For example, in $D = 4$, $\mathcal{N} = 2$ $U(2)$ SYM,

$$\{Q_{\alpha i}, \bar{Q}_{\beta j}\} = \delta_{ij}\gamma^{\mu}_{\alpha\beta}P_{\mu} + \epsilon_{ij}\left(\delta_{\alpha\beta}U + (\gamma_5)_{\alpha\beta}V\right)$$

where

$$U = \langle\phi\rangle e \quad V = \langle\phi\rangle g$$

are related to electric and magnetic charges after spontaneous breaking to $U(1)$.

Since $\{Q_{\alpha i}, \bar{Q}_{\beta j}\}$ is positive definite it follows that

$$M^2 \geq U^2 + V^2$$

so

$$M \geq \langle\phi\rangle\sqrt{e^2 + g^2}$$

Inequality saturated when $\{Q_{\alpha i}, \bar{Q}_{\beta j}\}$ has vanishing eigenvalues.

This implies $Q|\text{state}\rangle = 0$ for some Q

In this case, "short" SUSY multiplet protects mass.

Similar BPS states appear in string theory.

Central terms correspond to R-R charges (D-branes).

States which preserve some SUSY are BPS saturated.

Many ways of describing BPS states:

- **Through duality** from perturbative string states
  Allows counting of states
- **Through space-time** SUSY algebra
  Connects to supergravity solutions
- In Yang-Mills/Born-Infeld
  Gives explicit description of bound states
  But story is more complicated (κ symmetry, NBI)

From duality and SUSY algebra, can predict

$p - p$ BPS systems are marginally bound ($E = NE_p$).

$p - (p+4)$ BPS systems are marginally bound.

$p-(p+2)$ BPS systems are truly bound ($E = \sqrt{E_p^2 + E_{p+2}^2}$).

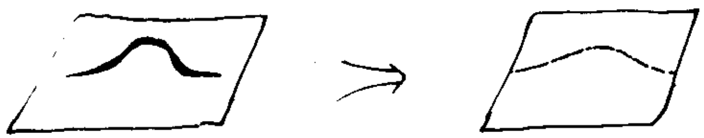
1-brane/string BPS systems are truly bound.

Binding 0-branes and 2-branes

Start with classical Yang-Mills in 2 + 1 dimensions.

Attach a 0-brane to an infinite 2-brane.

Tends to spread out



0-brane charge $\sim \int F$

Energy $\sim \int F^2$.

Scaling $F'(x) = \rho^2 F(\rho x)$ leaves flux fixed

but gives $E' = \rho^2 E$.

Taking $\rho \rightarrow 0$ reduces energy, spreads out flux.

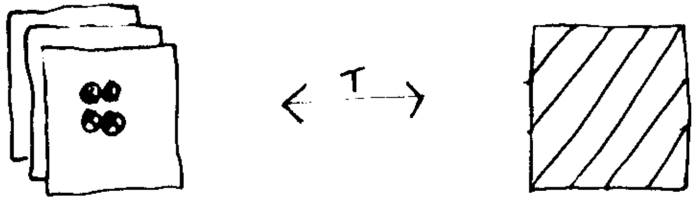
On a compact space, energy minimized when flux is uniform.

Return to T^2. Already considered uniform flux.

Using Born-Infeld energy,

N 2-branes on T^2 with q flux quanta (0-branes)

$$E = \sqrt{(N\tau_2 L_1 L_2)^2 + (q\tau_0)^2}$$



T-dual to diagonal 1-brane with

$$E = \tau_1' \sqrt{(NL_1)^2 + (qL_2')^2}$$

Classical bound states $\sim$ constant flux/diagonal 1-brane.

Classical moduli space $S^1 \times S^1/\mathbf{Z}_N$.

From positions of 1-brane, $U(1)$ holonomy on 1-brane,

or $U(N)$ holonomy (flat connection) on T^2.

0 + 2 bound states saturate BPS bound.

Can try to check SUSY from SYM

$$\delta\psi^a = -\frac{1}{4}F^a_{\mu\nu}\Gamma^{\mu\nu}\epsilon$$

Since

$$(\Gamma^{12})^2 = -1$$

$\delta\psi$ cannot vanish when $F_{12} \sim I$

So the bound state appears to break SUSY.

Catch: IIA has 32 supersymmetries.
Half are broken by a D-brane.

Remaining supersymmetries were originally broken.

State is actually 1/2 supersymmetric.

Careful analysis includes nonlinearly realized SUSY's.
(Green/Gutperle)

Can be extended to Born-Infeld with κ-symmetry
(Bergshoeff et al.)

Still somewhat unclear in non-abelian case.

Unbroken SUSY of D-branes

Type II string theory has $\mathcal{N} = 2$ SUSY in 10D.

Γ^μ are 32 x 32 matrices.

Admit Majorana and Weyl spinors.

Unbroken supersymmetry of a D-brane satisfies

$$\tilde{\epsilon} = \prod_\mu \Gamma^\mu \epsilon$$

with product over orthonormal basis on world-volume.

N parallel p-branes have 1/2 supersymmetry unbroken.

Thus, 0 + 2 bound state is 1/2 supersymmetric.

Separate 0-brane and 2-brane would have

$$\Gamma^0\epsilon = \Gamma^0\Gamma^1\Gamma^2\epsilon$$

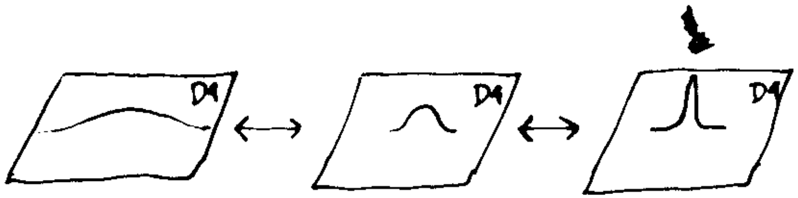
but this is impossible since $(\Gamma^{12})^2 = -1$.

Agrees with BPS saturation of bound state.

0 + 2 system is truly bound when N, q are relatively prime.

Binding 0-branes and 4-branes

In classical Yang-Mills, a 0-brane (instanton) scales **freely**



0-brane charge $\sim \int F \wedge F$

Energy $\sim \int F^2$.

Scaling $F'(x) = \rho^2 F(\rho x)$ leaves $\int F \wedge F$ fixed

and gives $E' = E$.

Changing ρ has no effect on energy.

Yang-Mills solutions with $\int F \wedge F$ nonzero are instantons.

Classical moduli space with size, orientation parameters

On T^4 considered a system of 2 4-branes and 2 0-branes

Born-Infeld energy was

$$E = 2\tau_4 V_4 \sqrt{(1 + 4\pi^2 \alpha'^2 F_{12}^2)(1 + 4\pi^2 \alpha'^2 F_{34}^2)}$$

This satisfies

$$E \geq 2\tau_4 V_4 + 2\tau_0$$

with equality iff F is self-dual

T-dual 2-brane configuration preserves some supersymmetry iff angles θ_1, θ_2 are equal (Berkooz/Douglas/Leigh).

Thus, the minimal energy configuration is indeed BPS.

In Yang-Mills theory, minimal energy instantons are always self-dual or anti-self-dual.

Follows from

$$(F \pm *F)^2 = 2(F^2 \pm F \wedge F) \geq 0$$

Instantons

(Anti)-self-dual instantons should be BPS for **Born-Infeld**

To prove this, non-abelian Born-Infeld is necessary.

This may give a hint for formulating NBI.

On any compact 4-manifold exists an (A)SD moduli space of irreducible instantons.

Important for studying topology of 4-manifolds.

Over S^4 dimension of moduli space is

$$4Nk - N^2 + 1$$

for $U(N)$ k-instantons.

Over T^4 dimension is

$$4[Nk - N^2 + 1]$$

If k is too small, no irreducible connections.

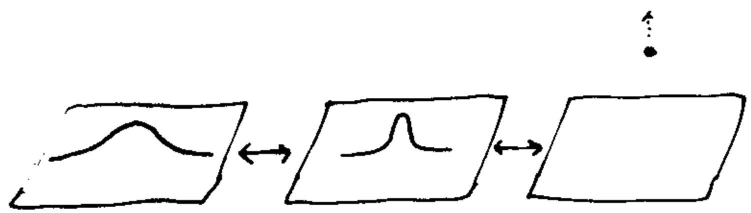
Dimension of moduli space can be determined by counting 0-modes in a brane background.

Dimension calculable through index theorem.

Roughly speaking, classical moduli space of vacua for 0-4 system is moduli space of instantons.

Several complications:

- Shrinking instantons



When an instanton shrinks to a point, the 0-brane can leave the 4-brane.

This cannot be described in gauge theory.

Need a more general description.

One possibility: use two gauge groups $U(N) \times U(k)$.

Expand around vacuum on intersection manifold.

Dropping $U(N)$ of 4-branes leaves a $U(k)$ gauge theory with extra hypermultiplets χ corresponding to 0-4 strings.

$U(k)$ gauge theory on 0-brane world-volume is dimensional reduction of $\mathcal{N}=2$ in 4D with following scalars:

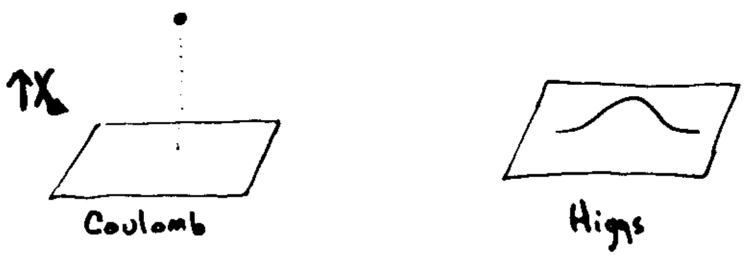
ϕ: adjoint (in gauge multiplet)
B, C: adjoint (from $\mathcal{N}=4$ gauge multiplet)
$\chi, \tilde{\chi}$: in $\overline{U(N)} \times U(k)$ and $U(N) \times \overline{U(k)}$ (fund. and anti.).

In 4D language superpotential is

$$\mathrm{Tr}\ \phi[B,C] + \mathrm{Tr}\ \tilde{\chi}\phi\chi$$

ϕ and A_μ give transverse 0-brane coordinates X^a.

Moduli space of vacua has two branches:



Coulomb branch: $X^a \neq 0$, $\chi, \tilde{\chi} = 0$

Higgs branch: $X^a = 0$, $\chi, \tilde{\chi} \neq 0$.

Shown by Witten (in 5-9 context) that Higgs branch is precisely the moduli space of instantons on $\mathbf{R}^4$, through ADHM hyperkaehler quotient construction.

Discussed by Douglas for arbitrary p-$(p+4)$.

Further evidence through probes (Douglas).

- No instantons for some N, k.

For example, with $N=2, k=1$, instantons shrink to a point.

Need sheaves (Douglas/Moore).

Arguments for using sheaves:

Nahm-Mukai: (N,k) moduli space and (k,N) moduli space are equivalent.

Agrees with T-duality. (Note: doesn't extend to field theory)

Only valid for arbitrary k, N in sheaf context.

Sheaves related to infinite-dimensional symmetry algebras, connected to duality symmetries.

Counting BPS states involves cohomology of moduli space (Vafa)

Precise counting requires resolution of singularities.

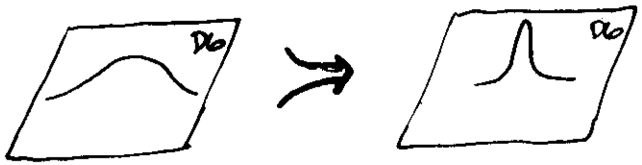
0-branes and 6-branes

Try to attach a 0-brane to a 6-brane

In classical Yang-Mills, 0-brane charge $\sim \int F \wedge F \wedge F$

Energy $\sim F^2$

From scaling argument, 0-brane tends to contract



Scaling $F'(x) = \rho^2 F(\rho x)$ leaves $\int F \wedge F \wedge F$ fixed

but gives $E' = E/\rho^2$.

Taking $\rho \to \infty$ reduces energy.

So generically 0-brane will shrink and be pushed off.

Yang-Mills energy $\sim \tau_6 (2\pi\alpha' F)^2$

0-brane energy $\sim \tau_6 (4\pi^2\alpha')^3$

So for small α', $E_6 + E_{YM} > E_{\text{separate}}$ when $E_{YM} > 0$.

Strangely, there are classical SYM 0 + 6 solutions.

For $N = 4$, with 0-brane charge 4, on "square" T^6

can construct solution with

$$F_{12} = 2\pi\mu_1 \quad F_{34} = 2\pi\mu_2 \quad F_{56} = 2\pi\mu_3$$

where

$$\mu_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \mu_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mu_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Classical solution is quadratically stable.

But solution breaks all supersymmetry.

T-dual to 4 3-branes intersecting pairwise on lines

$$\begin{array}{lll} X_2^{(1)} = x_1 & X_4^{(1)} = x_3 & X_6^{(1)} = x_5 \\ X_2^{(2)} = x_1 & X_4^{(2)} = -x_3 & X_6^{(2)} = -x_5 \\ X_2^{(3)} = -x_1 & X_4^{(3)} = -x_3 & X_6^{(3)} = x_5 \\ X_2^{(4)} = -x_1 & X_4^{(4)} = x_3 & X_6^{(4)} = -x_5 \end{array}$$

Corresponds to metastable non-SUSY black hole.
(Khuri/Ortin,Sheinblatt)

Similar construction for 8 0-branes & 8 8-branes.

Binding p-branes to p-branes

World-volume theory of N p-branes is $U(N)$ $\mathcal{N}=1$ SYM in 10D dimensionally reduced to $p+1$ dimensions.

Boson fields are A_μ and X^a.

Moduli space of classical vacua for N p-branes is

$$\frac{(\mathbf{R}^{9-p})^N}{S_N}$$

where S_N arises from Weyl symmetry.

Threshold bound states can arise quantum mechanically but issues are quite subtle.

Consider "simplest" case: 0-0 binding
(Sethi/Stern)

Related to gravitons in 11D supergravity

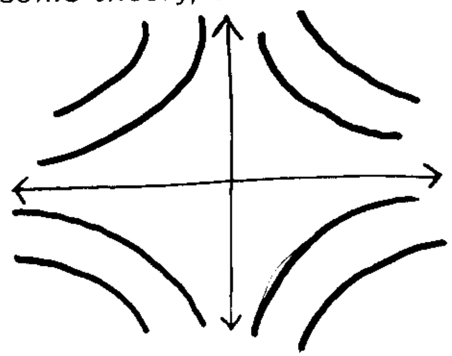
Should be a bound state of N 0-branes $\forall$ N.

Supersymmetric quantum mechanics problem

$$H=\frac{1}{2N}\mathrm{Tr}\,(P^i)^2-\frac{1}{4N}\mathrm{Tr}\,([X^i,X^j]^2)-\frac{1}{2N}\mathrm{Tr}\,(\psi\gamma^i[X^i,\psi])$$

where γ represent $SO(9)$ Clifford algebra (16D)

In purely bosonic theory, there are flat directions



but the spectrum is discrete.

Masses scale as X^2, giving increasing zero point energy.

In supersymmetric theory, fermions give opposite zero point energy, allowing possibility of zero energy ground states.

Proof of existence is rather subtle.

In theory with 8 supercharges (reduction from $\mathcal{N}=1$, $D=6$) there are no zero energy ground states.

Agrees with results of Strominger on conifolds.

Similarly, in dimensional reduction of $\mathcal{N}=1$ 4D SYM there is no bound state.

Number of bound states can in principle be determined from an index theory calculation.

Simple index calculation gives a fractional number.

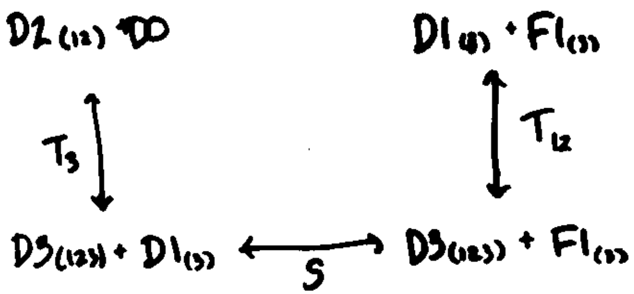
Correct result requires study of long-distance propagator, and calculation of an associated boundary term.

Binding 1-branes and strings

(N, q) states of 1-branes and strings form an SL(2,Z) multiplet under S-duality in IIB.

String winding number proportional to electric flux on 1-brane.

Dual to 0-branes and 2-branes.



Expect truly bound states when N, q relatively prime.

Analyzed by Witten in gauge theory; bound states found.

In gauge theory, string quantization is a quantum effect.

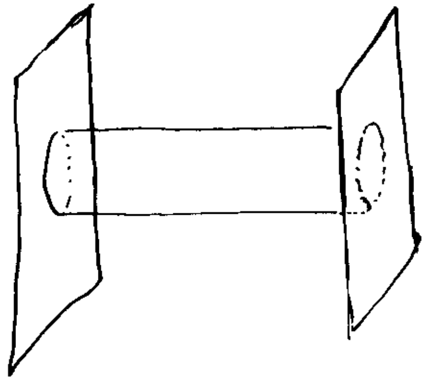
Flux quantum $e \sim 1/N$ (momentum dual to $\dot{A}$).

Born-Infeld energy

$$E \sim \mathrm{Tr}\sqrt{1 + q^2 e^2} \sim \sqrt{N^2 + q^2}$$

VIII. D-brane Interactions

Now consider interactions of separated D-branes.



Long-distance (supergravity) interactions from massless closed strings.

Short-distance (gauge theory) interactions from massless open strings.

Different truncations of full string spectrum.

No a priori reason gauge theory should correctly describe long distance physics.

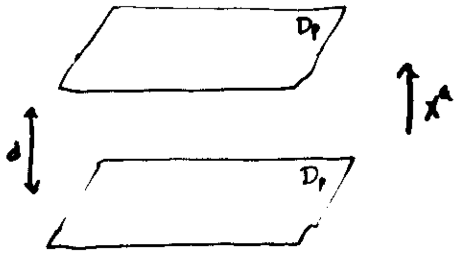
Sometimes may agree because protected by SUSY.

Original computations of D-brane forces from string theory (Polchinski, Bachas, Lifschytz)

Study here from gauge theory.

p-p potential

First consider a pair of parallel p-branes.



In Yang-Mills or Born-Infeld theory this is a $U(2)$ gauge theory with a nonzero scalar VEV

$$\langle X^a \rangle = d(\tau_3 + I)/2 = \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}$$

BPS configuration with Born-Infeld energy 2 E_p.

Therefore there is no force between the branes.

Agrees with string calculation by Polchinski.

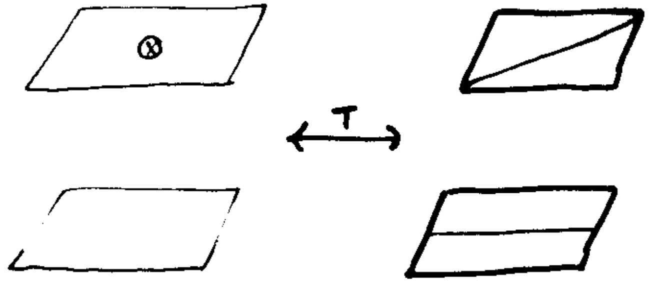
In string calculation, delicate balance between NS-NS and R-R exchanges.

Note that in bosonic theory, branes attract as in 0-0 bosonic system. (Quantum effect.)

0-2 potential

Can't describe separate 2/0 system in gauge theory.

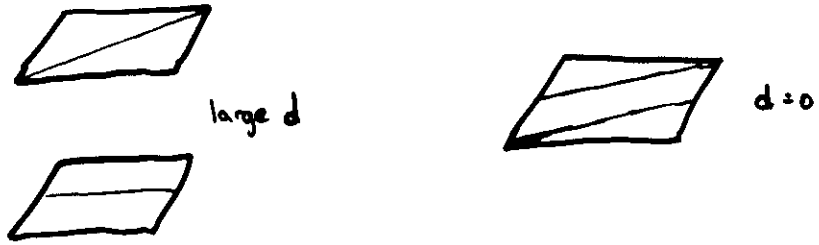
But can describe separated 2/(2 + 0) system on T^2.



T-dual to separated 1-branes at an angle.

When $d = 0$, BPS bound state with

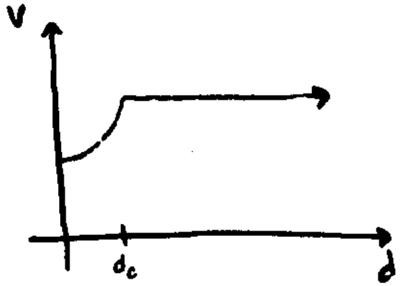
$$E_{\text{bound}} = \sqrt{(2\tau_2 L^2)^2 + \tau_0^2}$$



When d large, flux must live on one brane.

$$E_{\text{separate}} = \tau_2 L^2 + \sqrt{(\tau_2 L^2)^2 + \tau_0^2}$$

Classical potential unstable at close distances



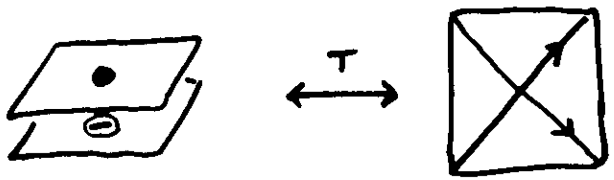
smoothed by quantum effects.

Attraction agrees with string calculation.

Brane-anti-brane forces

Exhibit tachyonic instability (Banks-Susskind)

Consider two 2-branes with fluxes $\pm 2\pi$



T-dual to perpendicular 1-branes.

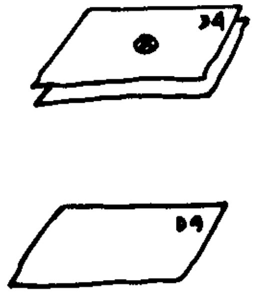
Yang-Mills solution with tachyonic instability.

BC's mean fluctuations are theta functions.

Can analyze instability precisely.
(Hashimoto/WT,Gava/Narain/Sarmadi)

0-4 potential

Consider 3 separate 4-branes, with an instanton on two.



BPS state in moduli space of $U(3)$ instantons

$$E = 3\tau_4 V_4 + \tau_0$$

So no potential between 0-brane and 4-branes

Agrees with string theory calculation.

0-6 potential

Takes extra energy to combine 0-branes and 6-branes.

Repulsive interaction at long and short distances.

0-8 potential

Similar story would naively imply 0-8 repulsion.

Extra complication related to R-R fields.

String creation (Hanany/Witten)

0-brane produces "half string", giving charge density on 8-brane volume.

D-brane scattering

Consider 0-brane/0-brane scattering

For static 0-branes, configuration space is flat

$$\frac{(\mathbf{R}^9)^2}{\mathbf{Z}_2}$$

Protected by supersymmetry.

When relative velocity v is nonzero, SUSY broken.

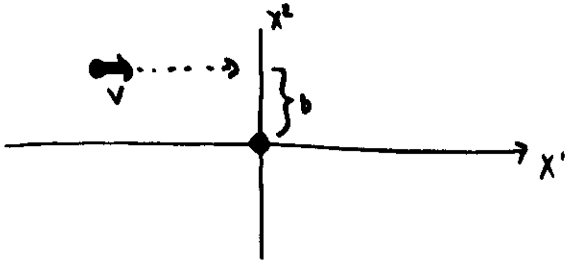
Scattering calculation gives v-dependent potential.

Quadratic term in v would correct metric.

Consider Yang-Mills with background

$$X^2(t) = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$$

$$X^1(t) = \begin{pmatrix} vt & 0 \\ 0 & 0 \end{pmatrix}$$



Integrating out massive off-diagonal fields, get

$$V(b) \sim v^4/b^7$$

No correction to metric.

IX. M(atrix) theory

M-theory

M-theory is a conjectured 11D theory

Low energy limit is 11D supergravity.

11D supergravity has fields:

e^a_μ: vielbein (44 components)
ψ_μ: Majorana fermion gravitino (128 components)
$A_{\mu\nu\rho}$: 3-form potential (84 components)

M-theory contains 2D supermembrane and 5-brane.

M-theory is strong coupling limit of IIA string.

M-theory/S^1 $\sim$ IIA string

$$R = g^{2/3} l_p = g l_s$$

KK photon ($g_{\mu 11}$) $\sim$ RR gauge field A_μ

p_{11} supergraviton $\sim$ 0-brane

wrapped membrane $\sim$ IIA string

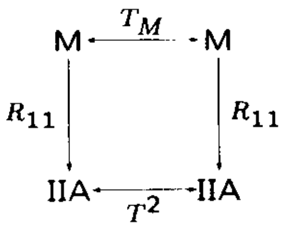
unwrapped membrane $\sim$ IIA RR 2-brane

wrapped 5-brane $\sim$ IIA RR 4-brane

unwrapped 5-brane $\sim$ IIA NS 5-brane

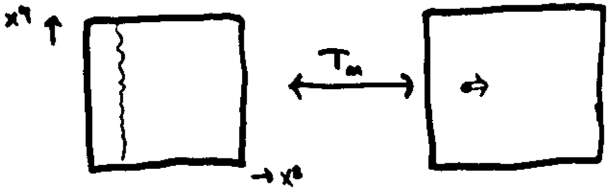
T-duality in M-theory (Schwarz, Sen, Aharony)

Label dimensions $0, 1, \ldots, 9, 11$.

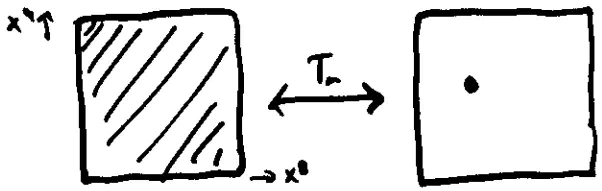


T-dualize **IIA** on 8, 9 $\rightarrow$ T_M acting on 8, 9, 11.

- T_M is symmetric on 8, 9, 11 (with 8 $\leftrightarrow$ 9 exchange).
- $T_M : V = R_8 R_9 R_{11} \leftrightarrow 1/V$.
- T_M exchanges membrane wrapping and momentum



Example: membrane on 9, 11 $\leftrightarrow$ 8-momentum



Example: membrane on 8, 9 $\leftrightarrow$ 11-momentum

Infinite momentum frame

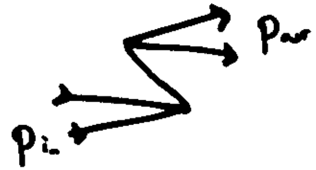
Infinite momentum frame $\sim$ light-cone frame

Choose a longitudinal direction (X^{11} in this case)

Boost system until everything has large p_{11}.

Can integrate out anything with negative or vanishing p_{11}, giving simplified theory.

General idea:



If all in and out states have large p_{11} then intermediate states without large p_{11} are very suppressed.

Zero momentum degrees of freedom can cause complications, particularly moduli.

IMF system has Galilean invariance

$$p_\perp \rightarrow p_\perp + p_{11} v_\perp$$

Massless particle has Galilean energy

$$E = \frac{p_\perp^2}{2p_{11}}$$

If X^{11} is compact, $p_{11} = N/R$ is quantized.

Note: boosting in a compact direction is tricky.

M(atrix) conjecture

The following conjecture was made by Banks, Fischler, Shenker and Susskind:

M-theory in the IMF is exactly described by the $N \to \infty$ limit of 0-brane quantum mechanics

$$\mathcal{L} = \frac{1}{2g}\left[\dot{X}^a \dot{X}_a + \sum_{a<b} \mathrm{Tr}\, [X^a, X^b]^2 + 2\theta^T(\dot{\theta} + \Gamma_a[X^a, \theta])\right]$$

Original evidence included:

- Only 0-branes carry p_{11}
- 10D Super-Galilean invariance
- Correct graviton scattering amplitudes
- Supermembrane arises from 0-branes.

Supporting evidence has continued to appear.

We'll discuss several classes of evidence:

- Construction of M-theory objects from 0-branes
- Appearance of M-theory symmetries in 0-brane QM
- Reproduction of IMF M-theory interactions

M(atrix) compactification

Several ways to think of compactifying M(atrix) theory

Can simply consider M(atrix) theory on an arbitrary manifold to be the large N limit of 0-branes on that manifold.

So far, 0-brane quantum mechanics only really understood on tori and orbifolds.

Some progress on arbitrary manifolds
(Douglas/Ooguri/Shenker, etc.)

Another approach to compactification:

Consider a superselection sector of M(atrix) theory

For example, take infinite matrices satisfying

$$UX^aU^{-1} = X^a + \delta^{a1} 2\pi R I.$$

for some "translation" operator U

Corresponds to S^1 compactification of M(atrix) theory

Also corresponds to SYM in $1+1$ dimensions.

Thus, M(atrix) theory "contains" SYM for all dimensions $d \leq 10$.

Supergraviton

0-branes live in a supermultiplet of 256 states

Corresponds to KK modes of graviton, 3-form, gravitino

$$256 = 44 + 84 + 128$$

In 11D massless states.

In 10D, mass is $1/R$, BPS states

Supergravitons with $p_{11} = N/R$ correspond to bound states of N 0-branes.

These bound states are BPS and exist (Sethi/Stern)

In IMF, we want to take $N \to \infty$

Decoupling of negative $p_{11} \leftrightarrow$ decoupling of anti-0-branes

To get M-theory, we need $R \to \infty$

0-branes are "partons" of holomorphic principle

Second quantization is *automatic* in M(atrix) theory.

$$\begin{pmatrix} M_1^i & 0 & 0 & \ddots \\ 0 & M_2^i & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ \ddots & 0 & 0 & M_k^i \end{pmatrix}$$

Describes a state of k supergravitons.

If M_k are $N_k \times N_k$ matrices, kth p_{11} is N_k/R

Supermembrane

One way to see supermembrane from 0-branes:

After compactification on T^2 with radii R_1, R_2,

A 0-brane configuration with

$$\text{Tr } [X^1, X^2] = 2\pi(R_1 R_2)i$$

carries a charge corresponding to a 2-brane.

The **2-brane of IIA** is the transverse membrane from M-theory.

Scaling

$$\begin{aligned} X^1 &= R_1 Y^1 \\ X^2 &= R_2 Y^2 \end{aligned}$$

we can take the large R limit, and get matrices satisfying

$$[Y^1, Y^2] = \frac{2\pi i}{N}$$

which describe a membrane in the decompactified limit.

Extra energy of this membrane scales as $1/N$.

Agrees with membrane tension.

This is automatic from earlier discussion of $0 + 2$ bound states, and Yang-Mills/Born-Infeld relation.

Historical development

Supermembrane connection to 0-brane QM originally found by de Wit, Hoppe and Nicolai.

They quantized supermembrane.

In light-cone gauge, supermembrane has residual invariance under area-preserving diffeomorphisms.

Area-preserving diffeomorphism group of sphere is a limit as $N \to \infty$ of $SU(N)$. (Generalizes to higher genus)

Hamiltonian of supermembrane can be regularized and becomes precisely 0-brane quantum mechanics model.

Term

$$\{X^i, X^j\}^2$$

becomes

$$[X^i, X^j]^2$$

in matrix system.

0-brane quantum mechanics was first considered several years earlier, by Claudson/Halpern, Flume, Baake et al.

de Wit, Luscher and Nicolai noted continuous spectrum.

Apparent pathology corresponds to second quantization.

5-branes in M(atrix) theory

Two kinds of 5-branes can appear:

Longitudinal 5-branes, appearing as 4-branes in IIA

Transverse 5-branes, appearing as NS 5-branes in IIA

First, we discuss longitudinal 5-branes.

Longitudinal 5-branes first incorporated by Berkooz and Douglas

Described in terms of hypermultiplets in 0-brane theory

Not a dynamical description of 5-brane

But showed correct Berry's phase on membrane.

Intrinsic longitudinal 5-brane analogous to membrane.

4-brane charge on T^4 described by

$$\frac{1}{8\pi^2} \mathrm{Tr}\, \epsilon_{ijkl} X^i X^j X^k X^l$$

Can decompactify and get noncompact 4-brane.

Explicit construction: Banks-Casher instanton

$$X^a = i\partial^a + A^a$$

for an instanton on S^4.

Another way to understand membrane and 5-brane charges in M(atrix) theory is to consider the supersymmetry algebra.

The 11 dimensional SUSY algebra is

$$\{Q, Q\} \sim P^{\mu} + Z^{\mu_1 \mu_2} + Z^{\mu_1 \cdots \mu_5}$$

Central terms correspond to 2-brane and 5-brane charges.

Supersymmetry algebra of M(atrix) theory explicitly computed by Banks, Seiberg and Shenker.

Previous related calculations dropped Tr $[X^i, X^j]$ since they vanish for finite N

BSS find

$$\{Q, Q\} \sim P^{\mu} + z^{a} + z^{ab} + z^{abcd}$$

The charge

$$z^{ab} \sim [X^a, X^b]$$

corresponds to membrane charge.

The charge

$$z^{abcd} \sim X^{[a} X^b X^c X^{d]}$$

corresponds to longitudinal 5-brane charge, as above.

The charge

$$z^{a} \sim \{P^b, [X^a, X^b]\}$$

corresponds to longitudinal membranes (strings)
Dual to Poynting vector $F^{ab} E^b$.

Apparently no charge for transverse 5-brane.

T-duality in M(atrix) theory

M-theory T-duality inverts a 3-torus.

How can we see this in M(atrix) theory?

M(atrix) theory on T^3 equivalent to $\mathcal{N} = 4$ SYM on T^3.

T_M duality on dimensions 7, 8, 9 is SYM S-duality.

Evidence:

- Coupling constant $\tau \rightarrow -1/\tau$

SYM coupling constant is

$$\tau = \frac{4\pi i}{g^2} = iV_{789}$$

- Flux exchange

Membranes $\sim$ magnetic flux

$$\text{Tr } [X^a, X^b] \sim \int iB^{ab}$$

Momentum $\sim$ electric flux

$$\text{Tr } \Pi^a = \text{Tr } \dot{X}^a \sim \int \text{Tr } \dot{A}^a = \int \text{Tr } E^a$$

- Magnetic energy $\sim$ membrane tension

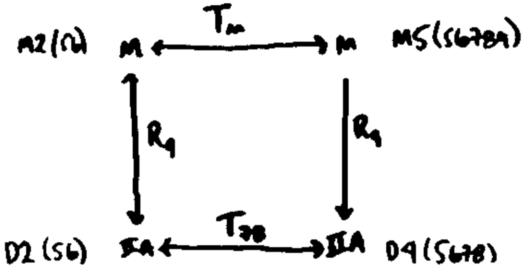
Transverse 5-brane

Given the realization of M-theory T-duality, we can construct a transverse 5-brane.

Compactify M-theory on dimensions 789.

Wrap a membrane on dimensions 56

T-duality on 789 takes the membrane to a 5-brane.



Thus, to construct a transverse 5-brane in M(atrix) theory, we begin with a membrane in dimensions 56,

$$[X^5, X^6] \sim 2\pi i$$

where X^5, X^6 are adjoint fields in $\mathcal{N} = 4$ SYM on $T^3 \times \mathbf{R}$.

Performing SYM S-duality gives a transverse 5-brane.

Puzzles:

How to make construction of state explicit?

Why isn't charge in SUSY algebra?