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I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

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Lecture II

SUMMER SCHOOL IN HIGH ENERGY PHYSICS AND COSMOLOGY

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THE STANDARD MODEL

H. HABER
Department of Physics
University of California, Santa Cruz
Santa Cruz, CA 95064
USA

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MAIN BUILDING STRADA COSTIERA, 11 TEL. 2240111 TELEFAX 224163 TELEX 460392 ADRIATICO GUEST HOUSE VIA GRIGNANO, 9 TEL.224241 TELEFAX 224531 TELEX 460449
MICROPROCESSOR LAB. VIA BEIRUT, 31 TEL. 2249911 TELEFAX 224600 TELEX 460392 GALILEO GUEST HOUSE VIA BEIRUT, 7 TEL.2240311 TELEFAX 2240310 TELEX 460392

Fermions in Quantum Field Theory

Representations of the Poincaré group (Lorentz group + acetime translations) correspond to particle states of definite ass and spin.

Here, we focus on spin-1/2 fields. Under a Lorentz transformation $x'_\mu = L_\mu{}^\nu x_\nu$,

$$\phi'(x') = \phi(x) \qquad \text{spin } 0$$
$$A'_\mu(x') = L_\mu{}^\nu A_\nu(x) \qquad \text{spin } 1$$

$$\Psi'_\alpha(x') = \exp\left(-\frac{i}{2}\Theta^{\mu\nu}S_{\mu\nu}\right)_\alpha{}^\beta \Psi_\beta(x) \qquad \text{general case}$$

[where $S_{\mu\nu} = -S_{\nu\mu}$]

$$[S_{\mu\nu}, S_{\rho\sigma}] = i(g_{\nu\rho}S_{\mu\sigma} - g_{\mu\rho}S_{\nu\sigma} - g_{\nu\sigma}S_{\mu\rho} + g_{\mu\sigma}S_{\nu\rho})$$

re the commutation relations for $SO(3,1) \simeq SL(2,C)$ Lie algebra.

Finite dimensional representations of $SL(2,C)$ correspond to rticles of different spin.

Define:

$$J_i = \frac{1}{2}\varepsilon_{ijk} S^{jk} \qquad \text{generates rotations}$$
$$K_i = S^{0i} \qquad \text{generates boosts}$$

this is a non-compact symmetry. Thus $e^{-\frac{i}{2}\Theta^{\mu\nu}S_{\mu\nu}}$ is **not** unitary.

Then:

$$[J_i, J_j] = i\varepsilon_{ijk} J_k$$
$$[J_i, K_j] = i\varepsilon_{ijk} K_k$$
$$[K_i, K_j] = -i\varepsilon_{ijk} J_k$$

Define: $\vec{J}_+ \equiv \frac{1}{2}(\vec{J} + i\vec{K})$

$$\vec{J}_- \equiv \frac{1}{2}(\vec{J} - i\vec{K})$$

Then,

$$[J_{+i}, J_{+j}] = i\varepsilon_{ijk} J_{+k}$$
$$[J_{-i}, J_{-j}] = i\varepsilon_{ijk} J_{-k}$$
$$[J_{+i}, J_{-j}] = 0$$

Thus, irreducible representations of the Lorentz group correspond to (j_+, j_-), where the eigenvalues of $J_\pm^2$ are $j_\pm(j_\pm+1)$, respectively. The dimension of (j_+, j_-) is $(2j_+ + 1)(2j_- + 1)$.

Infinitesimally,

$$\exp\left(-\frac{i}{2}\theta_{\mu\nu}S^{\mu\nu}\right) \simeq I - i\vec{\theta}\cdot\vec{J} - i\vec{\beta}\cdot\vec{K}$$

where:

$$\theta_i \equiv \frac{1}{2}\varepsilon_{ijk}\theta_{jk}$$
$$\beta_i = \theta_{0i} = -\theta_{i0}$$

Clearly, $(0,0)$ corresponds to a scalar. The next simplest irreducible representations are two-dimensional:

$$(\tfrac{1}{2}, 0) \qquad \left.\begin{aligned} \vec{J}_+ &= \tfrac{1}{2}(\vec{J} + i\vec{K}) = \frac{\vec{\sigma}}{2} \\ \vec{J}_- &= \tfrac{1}{2}(\vec{J} - i\vec{K}) = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} \vec{J} &= \frac{\vec{\sigma}}{2} \\ \vec{K} &= \frac{-i\vec{\sigma}}{2} \end{aligned}$$

$$(0, \tfrac{1}{2}) \qquad \left.\begin{aligned} \vec{J}_+ &= \tfrac{1}{2}(\vec{J} + i\vec{K}) = 0 \\ \vec{J}_- &= \tfrac{1}{2}(\vec{J} - i\vec{K}) = \frac{\vec{\sigma}}{2} \end{aligned}\right\} \Rightarrow \begin{aligned} \vec{J} &= \frac{\vec{\sigma}}{2} \\ \vec{K} &= \frac{i\vec{\sigma}}{2} \end{aligned}$$

For the $(\frac{1}{2},0)$ representation, introduce the two-component field ξ_α $(\alpha=1,2)$ which transforms under Lorentz transformations as:

$$\xi_\alpha \longrightarrow \xi'_\alpha = M_\alpha{}^\beta \xi_\beta \qquad \alpha,\beta=1,2$$

where $M \simeq I - \frac{i\vec{\theta}\cdot\vec{\sigma}}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}$ is a two-dimensional representation of $SL(2,C)$.

In quantum field theory, ξ_α is an anticommuting two-component fermion field.

<u>Aside</u>: If M is a matrix representation of $SL(n,C)$, then, M^*, $(M^{-1})^T$, and $(M^{-1})^\dagger$ are also matrix representations of the same dimension. For $n>2$, all four representations are inequivalent. For $SL(2,C)$, there are only (at most) two distinct matrix representations corresponding to a given dimension: (j_1, j_2) and (j_2, j_1).

It is a simple matter to check that:

$$(M^{-1})^T = i\sigma_2\, M\, (i\sigma_2)^T \qquad i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which follows from:

$$\sigma_2 \vec{\sigma}^T \sigma_2^T = \vec{\sigma}^T$$

Introduce the <u>contragradient</u> representation $(M^{-1})^T$:

$$\xi^\alpha \longrightarrow \xi'^\alpha = (M^{-1})^{T\,\alpha}{}_\beta\, \xi^\beta$$

$$= \left[i\sigma_2 M (i\sigma_2)^T \right]^\alpha{}_\beta\, \xi^\beta$$

which motivates the definition:

$$\varepsilon^{\alpha\beta} \equiv i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

i.e. $\varepsilon^{12} = -\varepsilon^{21} = 1$. Then,

$$\boxed{\xi^\alpha = \varepsilon^{\alpha\beta}\xi_\beta}$$

The matrices M and $(M^{-1})^T$ are related by a similarity transformation, or equivalently by a change in basis; hence the corresponding representations are equivalent.

This is equivalent to the statement that in $SU(2)$, the $\underset{\sim}{2}$ and $\underset{\sim}{2}^*$ representations are equivalent.

Hence, either ξ_α or ξ^α are equally good candidates to describe the $(\frac{1}{2}, 0)$ representation.

For the $(0,\frac{1}{2})$ representation, introduce the "dotted" spinor indices:

$$\bar{\eta}^{\dot{\alpha}} \longrightarrow \bar{\eta}^{\dot{\alpha}\,\prime} = (M^{-1})^{\dagger}{}^{\dot{\alpha}}{}_{\dot{\beta}}\, \bar{\eta}^{\dot{\beta}} \qquad \dot{\alpha},\dot{\beta} = \dot{1},\dot{2}$$

where

$$(M^{-1})^{\dagger} \simeq I - \frac{i\vec{\theta}\cdot\vec{\sigma}}{2} + \frac{\vec{\beta}\cdot\vec{\sigma}}{2}\,.$$

An equivalent description is via the <u>conjugate</u> representation M^*:

$$\bar{\eta}_{\dot{\alpha}} \longrightarrow \bar{\eta}_{\dot{\alpha}}' = (M^*)_{\dot{\alpha}}{}^{\dot{\beta}}\, \bar{\eta}_{\dot{\beta}}$$

where

$$\boxed{\bar{\eta}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\, \bar{\eta}_{\dot{\beta}}}$$

and:

$$\epsilon^{\dot{\alpha}\dot{\beta}} = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \epsilon_{\dot{\alpha}\dot{\beta}} = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

So, $\epsilon^{\dot{\alpha}\dot{\beta}} = \epsilon^{\alpha\beta}$, etc.

Note that $\bar{\eta}_{\dot{\alpha}}$ and η_{α}^* have the same transformation law, so we may equate them:

$$\bar{\eta}_{\dot{\alpha}} = \eta_{\alpha}^*$$

Similarly,

$$\bar{\eta}^{\dot{\alpha}} = \eta^{\alpha\,*}.$$

Thus, ξ_α and $\bar{\eta}^{\dot\alpha}$ are the fundamental building blocks for onstructing spin-½ quantum fields. To construct a field theory, e need to be able to construct Lorentz invariant scalar mbinations of ξ and $\bar{\eta}$ in order to construct the Lagrangian.

basic property of the Lorentz invariant matrix M is that:

$$\varepsilon^{\alpha\beta} M_\beta{}^\rho M_\alpha{}^\sigma = \varepsilon^{\rho\sigma}$$

roof: use $M \simeq I - \frac{i\vec{\theta}\cdot\vec{\sigma}}{2} - \frac{\vec{\beta}\cdot\vec{\sigma}}{2}$ and compute explicitly.

t then follows that under

$$\xi_\alpha \to M_\alpha{}^\beta \xi_\beta$$
$$\chi_\alpha \to M_\alpha{}^\beta \chi_\beta$$

$$\chi\xi \equiv \chi^\alpha \xi_\alpha = \varepsilon^{\alpha\beta}\chi_\beta\xi_\alpha$$

invariant under Lorentz transformations. Similarly,

$$\bar{\chi}\bar{\xi} \equiv \bar{\chi}_{\dot\alpha}\bar{\xi}^{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\bar{\chi}^{\dot\beta}\bar{\xi}^{\dot\alpha}$$

s invariant.

Note carefully the placement of the indices.

<u>otes</u>:

1. $\chi\xi = \xi\chi$, using $\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}$ and anti-commuting properties of the two-component fermion fields.
2. $\bar{\chi}\bar{\xi} = \bar{\xi}\bar{\chi}$
3. $(\chi\xi)^\dagger = (\chi^\alpha\xi_\alpha)^\dagger = \bar{\xi}_{\dot\alpha}\bar{\chi}^{\dot\alpha} = \bar{\xi}\bar{\chi} = \bar{\chi}\bar{\xi}$

Hermitian conjugation reverses the order

Conclusion:

$$\chi\xi + \bar{\chi}\bar{\xi}$$

is Lorentz invariant and Hermitian. This is a candidate for a term in the Lagrangian.

We still need a candidate for a kinetic energy term.
Introduce:

$$\sigma^\mu = (I; \vec{\sigma})$$

$$\bar{\sigma}^\mu = (I, -\vec{\sigma})$$

Note that:

$$p_\mu \sigma^\mu = p_0 I - \vec{p}\cdot\vec{\sigma} = \begin{pmatrix} p_0 - p_3 & -p_1 + ip_2 \\ -p_1 - ip_2 & p_0 + p_3 \end{pmatrix}$$

is a Hermitian 2×2 matrix. So is $M p_\mu \sigma^\mu M^\dagger$. Thus, there exists a p'_μ such that:

$$\boxed{p'_\mu \sigma^\mu = M p_\mu \sigma^\mu M^\dagger}$$

Exercise: Using $\det(p_\mu\sigma^\mu) = p_0^2 - |\vec{p}|^2$ and $\det M = 1$, show that $p_0'^2 - |\vec{p}'| = p_0^2 - |\vec{p}|^2$ and conclude that $p_\mu \to p'_\mu$ under the Lorentz transformation M.

The spinor index structure of the boxed equation above is:

$$p'_\mu \sigma^\mu_{\alpha\dot{\alpha}} = M_\alpha{}^\beta (M^*)_{\dot{\alpha}}{}^{\dot{\beta}} p_\mu \sigma^\mu_{\beta\dot{\beta}}$$

Thus, we have deduced the spinor index structure of σ^μ:

$$\sigma^\mu_{\alpha\dot{\alpha}}$$

which immediately allows one to construct another Lorentz invariant quantity:

$$i\chi^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \bar{\chi}^{\dot{\alpha}} \equiv i\chi\sigma^\mu\partial_\mu\bar{\chi}$$

The factor of i is inserted since $\frac{i}{2}\chi\sigma^\mu\overleftrightarrow{\partial}_\mu\bar{\chi}$ (which differs from $\chi\sigma^\mu\partial_\mu\bar{\chi}$ by a total divergence) is hermitian and thus a candidate for a kinetic energy term in the Lagrangian.

<u>exercise</u>: Show that $(\chi\sigma^\mu\bar{\xi})^\dagger = \xi\sigma^\mu\bar{\chi}$.

Similarly, the index structure of $\bar{\sigma}^\mu$ is:

$$\bar{\sigma}^{\mu\dot{\alpha}\alpha} = \varepsilon^{\dot{\alpha}\dot{\beta}}\varepsilon^{\alpha\beta}\sigma^\mu_{\beta\dot{\beta}}$$

<u>exercise</u>: Show that $\chi\sigma^\mu\bar{\xi} = -\bar{\xi}\bar{\sigma}^\mu\chi$.

That is, $\bar{\sigma}^\mu$ does not lead to an independent Lorentz invariant quantity.

Lorentz transformations in two component notation

$$\sigma^{\mu\nu}{}_{\alpha}{}^{\beta} = \frac{1}{4}\left(\sigma^{\mu}_{\alpha\dot{\alpha}}\,\bar{\sigma}^{\nu\dot{\alpha}\beta} - \sigma^{\nu}_{\alpha\dot{\alpha}}\,\bar{\sigma}^{\mu\dot{\alpha}\beta}\right)$$

$$\bar{\sigma}^{\mu\nu\dot{\alpha}}{}_{\dot{\beta}} = \frac{1}{4}\left(\bar{\sigma}^{\mu\dot{\alpha}\alpha}\sigma^{\nu}{}_{\alpha\dot{\beta}} - \bar{\sigma}^{\nu\dot{\alpha}\alpha}\sigma^{\mu}{}_{\alpha\dot{\beta}}\right)$$

Explicitly,

$$\sigma^{ij} = -i\epsilon^{ijk}\frac{1}{2}\sigma^{k} = \bar{\sigma}^{ij}$$

$$\sigma^{i0} = -\sigma^{0i} = \frac{1}{2}\sigma^{i} = -\bar{\sigma}^{i0} = \bar{\sigma}^{0i}$$

Comparing with

$$\exp -\frac{i}{2}\theta_{\mu\nu}S^{\mu\nu} \simeq I - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}$$

$$\theta_i = \frac{1}{2}\varepsilon_{ijk}\theta_{jk}$$
$$\beta_i = \theta_{0i} = -\theta_{i0}$$

we deduce that

$$\boxed{S^{\mu\nu} = i\sigma^{\mu\nu}}$$ for the $(\frac{1}{2},0)$ representation

Similarly,

$$\boxed{S^{\mu\nu} = i\bar{\sigma}^{\mu\nu}}$$ for the $(0,\frac{1}{2})$ representation.

Four-component notation

$$\psi = \begin{pmatrix} \xi_\alpha \\ \bar{\eta}^{\dot\alpha} \end{pmatrix}$$

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_{\mu\alpha\dot\beta} \\ \bar{\sigma}_\mu^{\dot\alpha\beta} & 0 \end{pmatrix}$$

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

$$\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu] = 2i \begin{pmatrix} \sigma^{\mu\nu}{}_\alpha{}^\beta & 0 \\ 0 & \bar{\sigma}^{\mu\nu\dot\alpha}{}_{\dot\beta} \end{pmatrix}$$

Note: $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$ $\qquad g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$

Projection operators

$$P_L = \frac{1}{2}(1-\gamma_5) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

$$P_R = \frac{1}{2}(1+\gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$$

$$\psi_L \equiv P_L\psi = \begin{pmatrix} \xi_\alpha \\ 0 \end{pmatrix}$$

$$\psi_R \equiv P_R\psi = \begin{pmatrix} 0 \\ \bar{\eta}^{\dot\alpha} \end{pmatrix}$$

Charge conjugation matrix

$$C = i\gamma^0\gamma^2 = \begin{pmatrix} \epsilon_{\beta\alpha} & 0 \\ 0 & \epsilon^{\dot\beta\dot\alpha} \end{pmatrix} = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}$$

Note:

$$\psi^c \equiv C\overline{\Psi}^T = C(\psi^\dagger\gamma^0)^T = \begin{pmatrix} \epsilon_{\beta\alpha} & 0 \\ 0 & \epsilon^{\dot\beta\dot\alpha} \end{pmatrix}\begin{pmatrix} \eta^\alpha \\ \bar\xi_{\dot\alpha} \end{pmatrix} = \begin{pmatrix} \eta_\beta \\ \bar\xi^{\dot\beta} \end{pmatrix}$$

In particular,

$$\psi_L^c \equiv P_L\psi^c = \begin{pmatrix} \eta_\beta \\ 0 \end{pmatrix}$$

$$\psi_R^c \equiv P_R\psi^c = \begin{pmatrix} 0 \\ \bar\xi^{\dot\beta} \end{pmatrix}$$

equivalent descriptions:

(a) ξ, η

(b) ψ_L, ψ_R

(c) ψ_L, ψ_L^c

Four-component Majorana spinor:

Set $\eta = \xi$. Then,

$$\psi_M = \begin{pmatrix} \xi_\alpha \\ \bar\xi^{\dot\alpha} \end{pmatrix} = \begin{pmatrix} \xi \\ i\sigma^2\xi^* \end{pmatrix}$$

which satisfies $\psi_M^c = \psi_M$.

TRANSLATION TABLE

$$\Psi_1 = \begin{pmatrix} \xi_1 \\ \bar\eta_1 \end{pmatrix} \qquad \Psi_2 = \begin{pmatrix} \xi_2 \\ \bar\eta_2 \end{pmatrix}$$

$$\boxed{\begin{aligned} \overline{\Psi}_1 P_L \Psi_2 &= \eta_1 \xi_2 \\ \overline{\Psi}_1 P_R \Psi_2 &= \bar\eta_2 \bar\xi_1 \end{aligned}}$$

$$\overline{\Psi}{}_1^c P_L \Psi_2 = \xi_1 \xi_2$$

$$\overline{\Psi}_1 P_R \Psi_2^c = \bar\xi_1 \bar\xi_2$$

$$\boxed{\overline{\Psi}_1 \gamma^\mu P_L \Psi_2 = \bar\xi_1 \bar\sigma^\mu \xi_2}$$

$$\overline{\Psi}{}_1^c \gamma^\mu P_R \Psi_2^c = -\bar\xi_2 \bar\sigma^\mu \xi_1$$

$$\boxed{\begin{aligned} -\tfrac{i}{2}\overline{\Psi}_1 \sigma^{\mu\nu} P_L \Psi_2 &= \eta_1 \sigma^{\mu\nu} \xi_2 \\ -\tfrac{i}{2}\overline{\Psi}_1 \sigma^{\mu\nu} P_R \Psi_2 &= \bar\xi_1 \bar\sigma^{\mu\nu} \bar\eta_2 \end{aligned}}$$

$$\overline{\Psi}{}_1^c P_L \Psi_2^c = \xi_1 \eta_2$$

$$\overline{\Psi}{}_1^c P_R \Psi_2^c = \bar\xi_2 \bar\eta_2$$

$$\overline{\Psi}_1 P_L \Psi_2^c = \eta_1 \eta_2$$

$$\overline{\Psi}{}_1^c P_R \Psi_2 = \bar\eta_1 \bar\eta_2$$

$$\overline{\Psi}{}_1^c \gamma^\mu P_L \Psi_2^c = \bar\eta_1 \bar\sigma^\mu \eta_2$$

$$\boxed{\overline{\Psi}_1 \gamma^\mu P_R \Psi_2 = -\bar\eta_2 \bar\sigma^\mu \eta_1}$$

$$-\tfrac{i}{2}\overline{\Psi}{}_1^c \sigma^{\mu\nu} P_L \Psi_2^c = \xi_1 \sigma^{\mu\nu} \eta_2$$

$$-\tfrac{i}{2}\overline{\Psi}{}_1^c \sigma^{\mu\nu} P_R \Psi_2^c = \bar\eta_1 \bar\sigma^{\mu\nu} \bar\xi_2$$

It follows that:

$$\overline{\Psi}_1 \Psi_2 = \eta_1 \xi_2 + \bar\eta_2 \bar\xi_1$$

$$\overline{\Psi}_1 \gamma_5 \Psi_2 = -\eta_1 \xi_2 + \bar\eta_2 \bar\xi_1$$

$$\overline{\Psi}_1 \gamma^\mu \Psi_2 = \bar\xi_1 \bar\sigma^\mu \xi_2 - \bar\eta_2 \bar\sigma^\mu \eta_1$$

$$\overline{\Psi}_1 \gamma^\mu \gamma_5 \Psi_2 = -\bar\xi_1 \bar\sigma^\mu \xi_2 - \bar\eta_2 \bar\sigma^\mu \eta_1$$

$$-\tfrac{i}{2}\overline{\Psi}_1 \sigma^{\mu\nu} \Psi_2 = \eta_1 \sigma^{\mu\nu} \xi_2 + \bar\xi_1 \bar\sigma^{\mu\nu} \bar\eta_2$$

Examples:

① Majorana field theory

$$\mathcal{L} = i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi - \frac{1}{2}m(\psi\psi + \bar{\psi}\bar{\psi})$$

$$= \frac{i}{2}\bar{\psi}\bar{\sigma}^\mu\overleftrightarrow{\partial}_\mu\psi - \frac{1}{2}m(\psi\psi + \bar{\psi}\bar{\psi}) + \text{ total divergence}$$

where ψ is a two component spinor.

Define:

$$\psi_M = \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$$

Note that

$$\bar{\psi}_M\gamma^\mu\partial_\mu\psi_M = \bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi - (\partial_\mu\bar{\psi})\bar{\sigma}^\mu\psi$$

$$\equiv \bar{\psi}\bar{\sigma}^\mu\overleftrightarrow{\partial}_\mu\psi$$

Hence,

$$\mathcal{L} = \frac{i}{2}\bar{\psi}_M\gamma^\mu\partial_\mu\psi_M - \frac{1}{2}m\bar{\psi}_M\psi_M \qquad \text{4-component Lagrangian}$$

② Dirac field theory

$$\mathcal{L} = i(\bar{\psi}_1\bar{\sigma}^\mu\partial_\mu\psi_1 + \bar{\psi}_2\bar{\sigma}^\mu\partial_\mu\psi_2) - \frac{1}{2}m_{ij}\psi_i\psi_j - \frac{1}{2}m_{ij}^*\bar{\psi}_i\bar{\psi}_j,$$

$$\text{where } m_{ij} \equiv \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}$$

Define

$$\psi_D = \begin{pmatrix} \psi_1 \\ \bar{\psi}_2 \end{pmatrix}$$

$\overline{\psi}_1 \bar{\sigma}^\mu \partial_\mu \psi_1 + \overline{\psi}_2 \bar{\sigma}^\mu \partial_\mu \psi_2$

$$= \overline{\psi}_1 \sigma^\mu \partial_\mu \psi_1 - (\partial_\mu \overline{\psi}_2) \bar{\sigma}^\mu \psi_2 + \text{total divergence}$$

$$= \overline{\Psi}_D \gamma^\mu \partial_\mu \Psi_D$$

$\psi_1 \psi_2 + \overline{\psi}_1 \overline{\psi}_2 = \overline{\Psi}_D \Psi_D .$

Thus,

$$\boxed{\mathcal{L} = i \overline{\Psi}_D \gamma^\mu \partial_\mu \Psi_D - m \overline{\Psi}_D \Psi_D}$$

Alternatively, we could have first diagonalized the mass matrix $M_{ij} = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}$. The corresponding eigenvalues are $\pm m$

Note: The sign of the mass eigenvalue is proportional to the CP-quantum number of the corresponding mass eigenstate.

Let

$$\psi_a = \frac{\psi_1 + \psi_2}{\sqrt{2}} \qquad \text{or} \qquad \psi_1 = \frac{\psi_a + i\psi_b}{\sqrt{2}}$$

$$i\psi_b = \frac{\psi_1 - \psi_2}{\sqrt{2}} \qquad \qquad \psi_2 = \frac{\psi_a - i\psi_b}{\sqrt{2}}$$

this factor removes the negative sign from the negative mass.

$$\mathcal{L} = i(\bar{\psi}_1 \bar{\sigma}^\mu \partial_\mu \psi_1 + \bar{\psi}_2 \bar{\sigma}^\mu \partial_\mu \psi_2) - \frac{1}{2} m (\psi_1 \psi_2 + \bar{\psi}_1 \bar{\psi}_2)$$

$$= i(\bar{\psi}_a \bar{\sigma}^\mu \partial_\mu \psi_a + \bar{\psi}_b \bar{\sigma}^\mu \partial_\mu \psi_b) - \frac{m}{2}(\psi_a \psi_a + \bar{\psi}_a \bar{\psi}_a + \psi_b \psi_b + \bar{\psi}_b \bar{\psi}_b)$$

onclusion

A Dirac fermion is equivalent to two mass-degenerate Majorana fermions with opposite CP-quantum numbers.

onversely, we can always combine two mass-degenerate Majorana fermions (whose mass eigenvalues are equal n magnitude but opposite in sign) into one Dirac fermion.

③ The see-saw mechanism

$$\mathcal{L} = i(\bar{\psi}_1 \bar{\sigma}^\mu \partial_\mu \psi_1 + \bar{\psi}_2 \bar{\sigma}^\mu \partial_\mu \psi_2) - \frac{1}{2} m_{ij} \psi_i \psi_j - \frac{1}{2} m_{ij}^* \bar{\psi}_i \bar{\psi}_j$$

where $m_{ij} = \begin{pmatrix} 0 & m_D \\ m_D & M \end{pmatrix}$ and $m_D \ll M$.

Eigenvalues of m_{ij}: $\frac{1}{2} M\left[1 \pm \sqrt{1 + \frac{4m_D^2}{M^2}}\right]$

i.e. for $m_D \ll M$, the two mass eigenvalues are $M, \frac{-m_D^2}{M}$.

Eigenstates:

$$i\psi_a \simeq \psi_1 - \frac{m_D}{M}\psi_2$$

$$\psi_b \simeq \psi_2 + \frac{m_D}{M}\psi_1$$

Then,

$$\frac{1}{2} m_D (\psi_1 \psi_2 + \psi_2 \psi_1) + \frac{1}{2} M \psi_2 \psi_2 + h.c.$$

$$\simeq \frac{1}{2}\left[\frac{m_D^2}{M} \psi_a \psi_a + M \psi_b \psi_b + h.c.\right] + O\left(\frac{m_D^3}{M^2}\right)$$

which corresponds to a theory of two Majorana fermions, one very light and one very heavy (the seesaw).

Fermions in gauge theories

- The basic irreducible parts consist of two-component fermion fields. Each field has a particular set of gauge quantum numbers.

- Compute the masses and interactions of the two-component fermions with the gauge and Higgs bosons. Look for mass-degenerate pairs to assemble into Dirac fermions.

- Convert all interaction terms into four-component notation and derive the Feynman rules.

One generation of quarks

two-component fermion fields	$SU(2)_L$	Y	T_3	$Q = T_3 + \frac{1}{2}Y$
$\begin{pmatrix} \psi_{Q_1} \\ \psi_{Q_2} \end{pmatrix}$	doublet	$1/3$	$1/2$ $-1/2$	$2/3$ $-1/3$
ψ_U	singlet	$-4/3$	0	$-2/3$
ψ_D	singlet	$2/3$	0	$1/3$

four-component fields [after $SU(2)_L \times U(1)_Y$ breaking]

$$u = \begin{pmatrix} \psi_{Q_1} \\ \overline{\psi}_U \end{pmatrix} \qquad d = \begin{pmatrix} \psi_{Q_2} \\ \overline{\psi}_D \end{pmatrix}$$

note: the conserved quantum number Q of the upper and lower components match.

$$\mathcal{L}_{KE} = i\bar{\psi}_i \bar{\sigma}^\mu \partial_\mu \psi_i$$ summed over all two-component fermion species

is invariant under the global symmetry:

$$\psi_i \longrightarrow U_{ij}\psi_j$$
$$\bar{\psi}_i \longrightarrow U_{ij}^* \bar{\psi}_j$$
(U is unitary)

but not under the local symmetry. To accomplish the latter, replace ∂_μ with the covariant derivative D_μ:

$$(D_\mu)_{ij} = \delta_{ij}\partial_\mu + ig_a A_\mu^a T^a$$

Then, we have:

$$\boxed{\mathcal{L} = i\bar{\psi}_i \bar{\sigma}^\mu D_\mu \psi_i}$$

which contains the interaction of the fermions with the gauge fields.

<u>Application to the electroweak Standard Model</u>

$$ig_a A_\mu^a T^a = \frac{ig}{\sqrt{2}}\left(T^+ W_\mu^+ + T^- W_\mu^-\right)$$
$$+ \frac{ig}{\cos\theta_W}\left(T^3 - Q\sin^2\theta_W\right) Z_\mu + ieQA_\mu$$

<u>Note</u>: this is applied to the ψ_i which are complex two-component fields. Hence, we will <u>not</u> double the size of the matrix generators as we previously did when we employed the <u>real</u> representation.

The ψ_i are either $SU(2)_L$ singlets or doublets. When acting on the doublet fields, LSc

$$T^+ = \frac{\sigma_1 + i\sigma_2}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$T^- = \frac{\sigma_1 - i\sigma_2}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$T^3 = \frac{\sigma_3}{2} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and we find:

$$\begin{aligned}
\mathcal{L} = {} & i\left(\bar\psi_{Q_1}\bar\sigma^\mu\partial_\mu\psi_{Q_1} + \bar\psi_{Q_2}\bar\sigma^\mu\partial_\mu\psi_{Q_2} + \bar\psi_U\bar\sigma^\mu\partial_\mu\psi_U + \bar\psi_D\bar\sigma^\mu\partial_\mu\psi_D\right) \\
& - \frac{g}{\sqrt{2}}\left(\bar\psi_{Q_1}\bar\sigma^\mu\psi_{Q_2} W^+_\mu + \bar\psi_{Q_2}\bar\sigma^\mu\psi_{Q_1} W^-_\mu\right) \\
& - \frac{g}{\cos\theta_W}\Big[\left(\tfrac{1}{2} - e_u\sin^2\theta_W\right)\bar\psi_{Q_1}\bar\sigma^\mu\psi_{Q_1} \\
& \qquad + \left(-\tfrac{1}{2} - e_D\sin^2\theta_W\right)\bar\psi_{Q_2}\bar\sigma^\mu\psi_{Q_2} \\
& \qquad + e_u\sin^2\theta_W\,\bar\psi_U\bar\sigma^\mu\psi_U + e_D\sin^2\theta_W\,\bar\psi_D\bar\sigma^\mu\psi_D\Big] Z_\mu \\
& - e\Big[e_u\left(\bar\psi_{Q_1}\bar\sigma^\mu\psi_{Q_1} - \bar\psi_U\bar\sigma^\mu\psi_U\right) \\
& \qquad + e_D\left(\bar\psi_{Q_2}\bar\sigma^\mu\psi_{Q_2} - \bar\psi_D\bar\sigma^\mu\psi_D\right)\Big] A_\mu
\end{aligned}$$

where $e_u = +2/3$

$e_D = -1/3$

Logically, the next step is to compute masses for $\psi_{Q_1}, \psi_{Q_2}, \psi_U$ and ψ_D. We will find that, indeed, one can combine into Dirac spinors:

$$u = \begin{pmatrix} \psi_{Q_1} \\ \overline{\psi}_U \end{pmatrix}, \qquad d = \begin{pmatrix} \psi_{Q_2} \\ \overline{\psi}_D \end{pmatrix}$$

Thus, we can rewrite $\mathcal{L}$ completely in terms of four-component Dirac spinors:

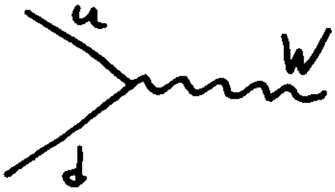
$$\begin{aligned}
\mathcal{L} = \; & i(\bar{u}\gamma^\mu \partial_\mu u + \bar{d}\gamma^\mu \partial_\mu d) \\
& - \frac{g}{\sqrt{2}}\left(\bar{u}\gamma^\mu P_L d\, W_\mu^+ + \bar{d}\gamma^\mu P_L u\, W_\mu^-\right) \\
& - \frac{g}{\cos\theta_W} Z_\mu \Big[\bar{u}\gamma^\mu [(\tfrac{1}{2} - e_u \sin^2\theta_W) P_L - e_u \sin^2\theta_W P_R] u \\
& \qquad + \bar{d}\gamma^\mu [(-\tfrac{1}{2} - e_D \sin^2\theta_W) P_L - e_D \sin^2\theta_W P_R] d \Big] \\
& - eA_\mu [e_u \bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d]
\end{aligned}$$

note sign

<u>A detail:</u>

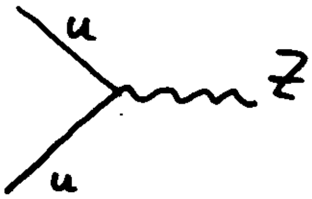
$$\begin{aligned}
& \overline{\psi}_{Q_1} \bar{\sigma}^\mu \partial_\mu \psi_{Q_1} + \overline{\psi}_U \bar{\sigma}^\mu \partial_\mu \psi_U \\
& = \overline{\psi}_{Q_1} \sigma^\mu \partial_\mu \psi_{Q_1} - (\partial_\mu \overline{\psi}_U) \bar{\sigma}^\mu \psi_U + \cancel{\text{total divergence}}^{\text{ drop}} \\
& = \bar{u}\gamma^\mu \partial_\mu u
\end{aligned}$$

Feynman rules



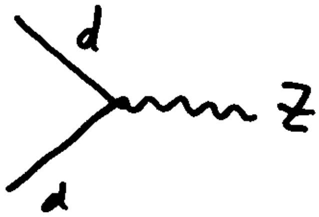
$$\frac{-ig}{2\sqrt{2}}\gamma^{\mu}(1-\gamma_5)$$

pure
left-handed
interaction



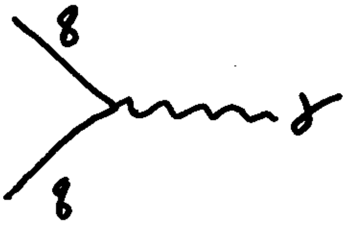
$$\frac{-ig\gamma^{\mu}}{2\cos\theta_W}\left[(T_{3U}-e_u\sin^2\theta_W)(1-\gamma_5) - e_u\sin^2\theta_W(1+\gamma_5)\right]$$

$T_{3U}=\frac{1}{2}$, $e_u = 2/3$



$$\frac{-ig\gamma^{\mu}}{2\cos\theta_W}\left[(T_{3D}-e_D\sin^2\theta_W)(1-\gamma_5) - e_D\sin^2\theta_W(1+\gamma_5)\right]$$

$T_{3D}=-\frac{1}{2}$, $e_D = -1/3$



$$-iee_q\gamma^{\mu}$$

pure
vector
coupling

Fermion masses

The most general form for mass terms is:

$$-\mathcal{L}_{mass} = m_{ij}\, \psi_i \psi_j + m_{ij}^* \bar{\psi}_i \bar{\psi}_j$$

In the electroweak Standard Model, $m_{ij} = 0$ due to gauge invariance!

example:

$$-\mathcal{L}_m = m_u (\psi_{Q_1} \psi_u + \bar{\psi}_{Q_1} \bar{\psi}_u)$$

$$= m_u \bar{u} u$$

but this term conserves neither $SU(2)_L$ nor $U(1)_Y$.

However, we have yet to consider the interactions of the fermions with the Higgs scalars.

The electroweak quantum numbers — a reprise

field	$SU(2)_L$	Y	T_3	$Q = T_3 + \frac{1}{2}Y$	
$\begin{pmatrix}\psi_{Q_1}\\ \psi_{Q_2}\end{pmatrix}$	doublet	$1/3$	$+1/2$ $-1/2$	$+2/3$ $-1/3$	two-component quark fields
ψ_U	singlet	$-4/3$	0	$-2/3$	
ψ_D	singlet	$2/3$	0	$+1/3$	
$\begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix}$	doublet	$+1$	$1/2$ $-1/2$	$+1$ 0	Higgs scalars
$\begin{pmatrix}\tilde\phi^0\\ \tilde\phi^-\end{pmatrix}$	doublet	-1	$1/2$ $-1/2$	0 -1	

Remark: Since $\Phi \equiv \begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix}$ is a complex doublet, it represents both a $Y=+1$ and $Y=-1$ doublet of states. Formally,

$$\tilde\Phi = i\sigma_2\Phi^* = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}(\phi^+)^*\\ (\phi^0)^*\end{pmatrix} = \begin{pmatrix}\phi^{0*}\\ -\phi^-\end{pmatrix}$$

or equivalently,

$$\tilde\phi^i = \epsilon^{ij}\phi^{*j}$$

Higgs-fermion interactions

$$\mathcal{L}_{int} = -h_U(\tilde{\Phi}^\dagger \psi_Q)\psi_U - h_D(\Phi^\dagger \psi_Q)\psi_D + h.c.$$

Check gauge invariance:

$$\psi_Q = \begin{pmatrix} \psi_{Q_1} \\ \psi_{Q_2} \end{pmatrix}$$

Note that if U is an $SU(2)$ transformation, then

$$\psi_Q \to U\psi_Q$$
$$\Phi \to U\Phi$$
$$\tilde{\Phi} \to U\tilde{\Phi}$$

← this follows from $i\sigma_2 U (i\sigma_2)^T = U^*$

so $(\tilde{\Phi}^\dagger \psi_Q)$ and $(\Phi^\dagger \psi_Q)$ are $SU(2)$-singlets.

Check the $U(1)_Y$ quantum numbers:

$\tilde{\Phi}^\dagger$	$+1$	$\Phi^\dagger$	-1
ψ_Q	$+1/3$	ψ_Q	$+1/3$
ψ_U	$-4/3$	ψ_D	$+2/3$
	0 ✓		0 ✓

$$\mathcal{L}_{int} = -h_U(\phi^0 - \phi^+)\begin{pmatrix}\psi_{Q_1} \\ \psi_{Q_2}\end{pmatrix}\psi_U - h_D(\phi^- \ \phi^{0*})\begin{pmatrix}\psi_{Q_1} \\ \psi_{Q_2}\end{pmatrix}\psi_D + h.$$

$$= -h_U(\phi^0\psi_{Q_1}\psi_U - \phi^+\psi_{Q_2}\psi_U)$$
$$-h_D(\phi^-\psi_{Q_1}\psi_D + \phi^{0*}\psi_{Q_2}\psi_D) + h.c.$$

Go to the unitary gauge:

$$\phi^+ = \phi^- = \mathrm{Im}\,\phi^0 = 0$$
$$\mathrm{Re}\,\phi^0 = \frac{1}{\sqrt{2}}(v + H^0)$$

Then,

$$\mathcal{L}_{int} = -\frac{h_U v}{\sqrt{2}}(\psi_{Q_1}\psi_U + \overline{\psi}_{Q_1}\overline{\psi}_U)\left(1 + \frac{H^0}{v}\right)$$
$$-\frac{h_D v}{\sqrt{2}}(\psi_{Q_2}\psi_D + \overline{\psi}_{Q_2}\overline{\psi}_D)\left(1 + \frac{H^0}{v}\right)$$

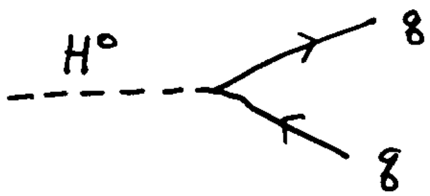
The fermions acquire Dirac masses via spontaneous symmetry breaking!!

In four component notation, this reads:

$$\mathcal{L}_{int} = -m_u\bar{u}u\left(1 + \frac{H^0}{v}\right) - m_d\bar{d}d\left(1 + \frac{H^0}{v}\right),$$

$$\boxed{m_u = \frac{h_U v}{\sqrt{2}}, \quad m_d = \frac{h_D v}{\sqrt{2}}}$$

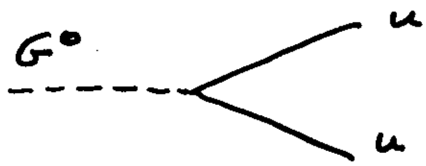
Higgs-fermion interactions: Feynman rules



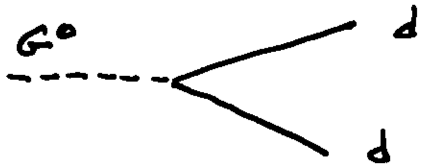
$$\frac{-i m_q}{v} = \frac{-i g m_q}{2 m_W}$$

The Higgs boson couplings are proportional to mass.

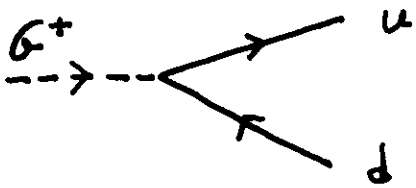
<u>Aside</u>: In the R_ξ-gauge, one must also include the interactions with the Goldstone fields. Here they are:



$$-\frac{g m_u}{2 m_W} \gamma_5$$



$$+\frac{g m_d}{2 m_W} \gamma_5$$



$$\frac{-i g}{2\sqrt{2}\, m_W} \left[m_d (1+\gamma_5) - m_u (1-\gamma_5) \right]$$

<u>Note</u>:

H^0 is a CP-even scalar

G^0 is a CP-odd scalar, with pseudoscalar couplings to fermions

dding one generation of leptons

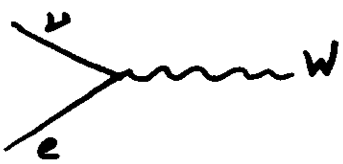
o-component pton fields	$SU(2)_L$	Y	T_3	$Q = T_3 + \frac{1}{2}Y$
$\begin{pmatrix} \psi_{L_1} \\ \psi_{L_2} \end{pmatrix}$	doublet	-1	$+\frac{1}{2}$ $-\frac{1}{2}$	0 -1
ψ_E	singlet	0	$+2$	$+1$
$\boxed{\psi_N}$	singlet	0	0	0

not (yet) a confirmed member of the electroweak Standard Model

epton interactions with gauge fields

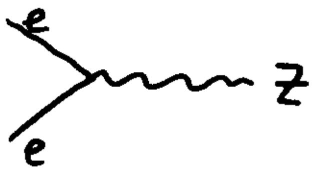
Replace $e_U \to e_\nu = 0$

$e_D \to e_e = -1$



$$\frac{-ig}{2\sqrt{2}}\gamma^\mu(1-\gamma_5)$$

$$\frac{-ig}{4\cos\theta_W}\gamma^\mu(1-\gamma_5)$$



$$\frac{ig}{4\cos\theta_W}\gamma^\mu(1-4\sin^2\theta_W-\gamma_5)$$

$$ie\gamma^\mu$$

neutrinos only exhibit left-handed interactions

Lepton interactions with the Higgs fields

$$\mathcal{L}_{int} = -h_E(\Phi^\dagger \psi_L)\psi_E + h.c.$$

hypercharge gauge invariance:

$\Phi^\dagger$	-1
ψ_L	-1
ψ_E	2
	0 ✓

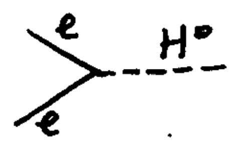
So,

$$\mathcal{L}_{int} = -h_E(\phi^- \psi_{L_1}\psi_E + \phi^{0*}\psi_{L_2}\psi_E) + h.c.$$

In unitary gauge, going over to four-component fermions:

$$\mathcal{L}_{int} = -m_e \bar{e}e\left(1+\frac{H^0}{v}\right)$$

$$e = \begin{pmatrix}\psi_{L_2}\\ \bar{\psi}_E\end{pmatrix} \qquad m_e \equiv \frac{h_E v}{\sqrt{2}}$$



$$\frac{i g\, m_e}{2m_W}$$

The neutrino remains massless, since it has nothing to pair up with.

$$m_\nu = 0 \qquad \nu_M = \begin{pmatrix}\psi_{L_1}\\ \bar{\psi}_{L_1}\end{pmatrix}$$

No $H^0 \nu\bar{\nu}$ vertex.

More traditionally, one writes:

$$\nu \equiv \nu_L = P_L \nu_M = \begin{pmatrix}\psi_{L_1}\\ 0\end{pmatrix}$$

$$\bar{\nu} \equiv \bar{\nu}_R = P_R \nu_M = \begin{pmatrix}0\\ \bar{\psi}_{L_1}\end{pmatrix}$$

Right-handed neutrinos?

If ψ_N exists, then a more general form for $\mathcal{L}_{int}$ exists consistent with gauge invariance:

$$\mathcal{L}_{int} = -h_E(\Phi^\dagger\psi_L)\psi_E - h_N(\tilde{\Phi}^\dagger\psi_L)\psi_N - \tfrac{1}{2}M\psi_N\psi_N + h.c$$

Since ψ_N is completely neutral under $SU(2)_L \times U(1)_Y$, there is no reason why M has anything to do with the electroweak scale. In fact, there are good arguments to suggest that $M \gg v$.

We can easily compute masses by setting $\langle\phi^0\rangle = \frac{v}{\sqrt{2}}$. Then,

$$\begin{aligned} -\mathcal{L}_{mass} = {} & \frac{h_E v}{\sqrt{2}}(\psi_{L_2}\psi_E + \overline{\psi}_{L_2}\overline{\psi}_E) \leftarrow \text{Dirac term for e.} \\ & + \frac{h_N v}{\sqrt{2}}(\psi_{L_1}\psi_N + \overline{\psi}_{L_1}\overline{\psi}_N) \\ & + \tfrac{1}{2}M(\psi_N\psi_N + \overline{\psi}_N\overline{\psi}_N) \end{aligned}$$

Denoting $m_D \equiv \frac{h_N v}{\sqrt{2}}$, we find a neutrino mass matrix:

$$-\mathcal{L}_{\nu\text{-mass}} = \tfrac{1}{2}(\psi_{L_1}\ \psi_N)\begin{pmatrix} 0 & m_D \\ m_D & M \end{pmatrix}\begin{pmatrix} \psi_{L_1} \\ \psi_N \end{pmatrix} + h.c.$$

which is precisely the see-saw mechanism exhibited earlier.

Cabibbo-Kobayashi-Maskawa mixing

In order to describe three generations of quarks, we must provide generation labels for our fermion fields.

Return to the interaction of the Higgs fields with the two-component quark fields:

$$\mathcal{L}_{int} = -h_{U ij}(\tilde{\Phi}^{\dagger}\psi_{Qi})\psi_{Uj} - h_{D ij}(\Phi^{\dagger}\psi_{Qi})\psi_{Dj} + h.c.$$

$i,j = 1,2,3$ (generation labels)

To determine the physical states of the theory, we must study the quadratic terms in the Lagrangian.
Replacing $\phi^0 \longrightarrow \langle\phi^0\rangle = \frac{v}{\sqrt{2}}$,

$$-\mathcal{L}_{mass} = \frac{h_{U ij} v}{\sqrt{2}}\psi_{Q_1 i}\psi_{Uj} + \frac{h_{D ij} v}{\sqrt{2}}\psi_{Q_2 i}\psi_{Dj} + h.c.$$

<u>Notation:</u>

$$X_{1ij} = \frac{v}{\sqrt{2}} h_{U ij}$$
$$X_{2ij} = \frac{v}{\sqrt{2}} h_{D ij}$$

$$\left.\begin{aligned} \psi_{Q_k i} &= (\psi_{Q_1 i}, \psi_{Q_2 i}) \\ \psi_{R_k i} &= (\underset{\substack{\uparrow \\ k=1}}{\psi_{Ui}}, \underset{\substack{\uparrow \\ k=2}}{\psi_{Di}}) \end{aligned}\right\}\ \text{generation label: } i = 1,2,3.$$

$$-\mathcal{L}_{mass} = \sum_{k=1}^{2} \psi_{Qki}(X_k)_{ij}\, \psi_{Rkj}$$

implicit sum over $i,j=1,2,3$

As per previous instructions, we must diagonalize this mass matrix.

ψ_{Qki}, ψ_{Rki} : "interaction" eigenstates

Introduce:

ξ_{ki}, η_{ki} : "physical" or "mass" eigenstates

$$\xi_{ki} = (V_k)_{ij}\, \psi_{Qkj}$$

$$\eta_{ki} = (U_k)_{ij}\, \psi_{Rkj}$$

$k=1,2$

U_k, V_k are unitary matrices

such that:

$$-\mathcal{L}_{mass} = \sum_{k=1}^{2} \eta_{ki}(M_k)_{ij}\, \xi_{kj} + h.c.$$

$$(M_1)_{ij} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}$$

$$(M_2)_{ij} = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}$$

In order to accomplish this result, we must have:

$$M_1 = V_1^* X_1 U_1^{-1}$$

$$M_2 = V_2^* X_2 U_2^{-1}$$

Now, X_1 and X_2 are known matrices. To deduce the V_k and U_k, simply note that:

$$M_k^\dagger M_k = U_k (X_k^\dagger X_k) U_k^{-1}$$

$$M_k M_k^\dagger = V_k^* (X_k X_k^\dagger) V_k^{*-1}$$

so all we need to do is to diagonalize $X_k^\dagger X_k$ and $X_k X_k^\dagger$ by standard techniques.

The physical four-component quark states are:

$$u_i = \begin{pmatrix} \xi_{1i} \\ \overline{\eta}_{1i} \end{pmatrix}, \qquad d_i = \begin{pmatrix} \xi_{2i} \\ \overline{\eta}_{2i} \end{pmatrix} \qquad i = 1, 2, 3$$

with

$$-\mathcal{L}_{mass} = \sum_i m_{q_i} \overline{q}_i q_i$$

We now revisit the coupling of quarks to gauge bosons.
These involve the "interaction" eigenstates, which arise from:

$$\mathcal{L} = i\,\overline{\psi}_i\,\bar{\sigma}^\mu\left[\delta_{ij}\,\partial_\mu + i g_a A^a_\mu T^a_{ij}\right]\psi_j$$

This does not mix up the generations. Thus, e.g.

$$\mathcal{L}_{Wqq'} = -\frac{g}{\sqrt{2}}\sum_i\Big[\overline{\psi}_{Q_1 i}\,\bar{\sigma}^\mu\,\psi_{Q_2 i}\,W^+_\mu$$
$$+\ \overline{\psi}_{Q_2 i}\,\bar{\sigma}^\mu\,\psi_{Q_1 i}\,W^-_\mu\Big]$$

xpress this in terms of physical mass eigenstates:

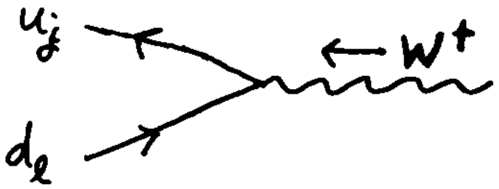
$$\overline{\psi}_{Q_1 i}\,\bar{\sigma}^\mu\,\psi_{Q_2 i} = V_{1ji}\,V^{-1}_{2i\ell}\;\overline{\xi}_{1j}\,\bar{\sigma}^\mu\,\xi_{2\ell}$$
$$= K_{j\ell}\;\overline{\xi}_{1j}\,\bar{\sigma}^\mu\,\xi_{2\ell}$$

$K = V_1 V_2^{-1}$ is the Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix

<u>note</u>: $K^\dagger = K^{-1}$!

onverting to four-component notation:

$$\mathcal{L}_{Wqq'} = -\frac{g}{\sqrt{2}}\sum_{j\ell}\left(K_{j\ell}\,\bar{u}_j\,\gamma^\mu P_L\, d_\ell\, W^+_\mu + \text{h.c.}\right)$$



$$\frac{-ig}{2\sqrt{2}}\,\gamma^\mu(1-\gamma_5)\,K_{j\ell}$$

Neutral current couplings to Z° and γ

These involve terms of the form

$$\bar{\psi}_{Q k i}\,\bar{\sigma}^\mu\,\psi_{Q k i} \quad \text{or} \quad \bar{\psi}_{R k i}\,\bar{\sigma}^\mu\,\psi_{R k i}$$

i.e. they are diagonal in the interaction basis. But,

$$\bar{\psi}_{Q k i}\,\bar{\sigma}^\mu\,\psi_{Q k i} = (V_k V_k^{-1})_{j\ell}\;\bar{\xi}_{k j}\,\bar{\sigma}^\mu\,\xi_{k\ell}$$
$$= \bar{\xi}_{k j}\,\bar{\sigma}^\mu\,\xi_{k j}$$

$$\bar{\psi}_{R k i}\,\bar{\sigma}^\mu\,\psi_{R k i} = (U_k U_k^{-1})_{j\ell}\;\bar{\eta}_{k j}\,\bar{\sigma}^\mu\,\eta_{k\ell}$$
$$= \bar{\eta}_{k j}\,\bar{\sigma}^\mu\,\eta_{k j}$$

since V_k and U_k are unitary matrices.

Thus, the neutral current couplings are also diagonal in the physical mass basis!

This is the famous GIM-mechanism.

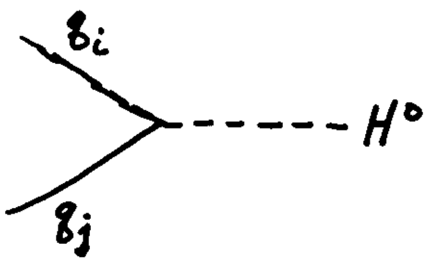
No flavor-changing neutral currents (FCNC's) at tree-level!

The GIM mechanism also operates in the Higgs-quark couplings. Repeating the previous derivation of the quark masses, we see that one simply replaces the quark mass matrices as follows:

$$-\mathcal{L}_{int} = \sum_{k=1}^{2} \psi_{Qki}(X_k)_{ij}\,\psi_{Rkj}\left(1+\frac{H}{v}\right)$$

$$= \sum_{k=1}^{2} \eta_{ki}(M_k)_{ij}\,\xi_{kj}\left(1+\frac{H}{v}\right) + h.c.$$

$$= \sum_{i} m_{q_i}\,\bar{q}_i q_i\left(1+\frac{H}{v}\right)$$



$$\frac{igm_{q_i}}{2m_W}\,\delta_{ij}$$

The $H^0 q_i \bar{q}_j$ interaction is also diagonal.

<u>Aside:</u> In the R_ξ-gauge, the $G^0 q_i \bar{q}_j$ interaction is diagonal, while the $G^+ q_i q_j'$ involves K_{ij}.

Absence of tree-level FCNC's

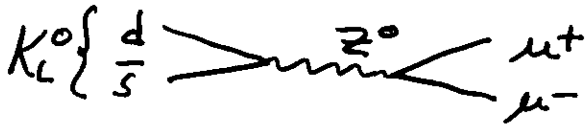
The extreme rarity of FCNC's was a crucial clue in the development of the Standard Model. It also provides strong constraints on any model building that attempts to introduce new physics beyond the Standard Model.

One such example is the rare decay $K_L^0 \to \mu^+\mu^-$. Experimental observations yield:

$$BR\,(K_L^0 \to \mu^+\mu^-) = (7.2 \pm 0.5) \times 10^{-9}.$$

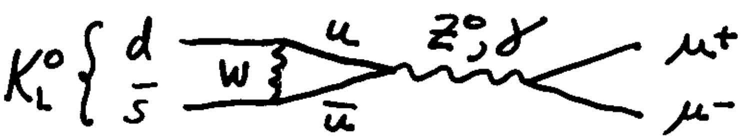
If tree-level FCNC's were present, one would expect to find

suppressed typically by a mixing angle the size of a CKM mixing angle. But then, one would predict:



at a rate far greater than the observed number quoted above.

In the Standard Model, $K_L^0 \to \mu^+\mu^-$ can take place via a one-loop process, such as:



Explicit theoretical calculations yield results that are consistent with the observed rate.

By contrast, the charged current interactions at tree-level are generation-changing and governed in strength by K_{ij}.

How many free parameters?

We started off with four unitary 3×3 matrices. We noted that only one combination: $K = V_1 V_2^{-1}$ was physical.

A priori, an arbitrary 3×3 unitary matrix contains 3 real angles and 6 real phases. But even some of these are not physical.

Recall that

$$\xi_{ki} = (V_k)_{ij}\, \psi_{Q\,kj}$$
$$\eta_{ki} = (U_k)_{ij}\, \psi_{R\,kj}$$

with $M_k = V_k^* X_k U_k^{-1}$ and

$$-\mathcal{L}_{mass} = \sum_{k=1}^{2} \eta_{ki} (M_k)_{ij}\, \xi_{kj} + h.c.$$

It is clear that

$$\xi_{ki} \longrightarrow e^{i\theta_{ki}}\, \xi_{ki}$$
$$\eta_{ki} \longrightarrow e^{-i\theta_{ki}}\, \eta_{ki}$$

leaves $-\mathcal{L}_{mass}$ invariant.

In particular, this means that

$$(V_k)_{ij} \longrightarrow e^{i\theta_{ki}}(V_k)_{ij}$$

has no effect on the physics. Using $K = V_1 V_2^{-1}$, it follows that

$$K_{ij} \longrightarrow e^{i(\theta_{1i} - \theta_{2j})} K_{ij}$$

has no effect on the physics. For $i,j = 1,2,3$ there are five phases $\theta_{1i} - \theta_{2j}$ which can be removed from K.

<u>End result</u>: Physical parameters of K

3 real angles

$6-5 = 1$ real phase $\qquad (\Rightarrow$ CP-violation!)

There are many ways to parametrize the CKM mixing matrix. Here's one:

$$K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & -s_2 \\ 0 & s_2 & c_2 \end{pmatrix} \begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_3 & s_3 \\ 0 & -s_3 & c_3 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 & s_1 c_3 & s_1 s_3 \\ -s_1 c_2 & c_1 c_2 c_3 + s_2 s_3 e^{i\delta} & c_1 c_2 s_3 - s_2 c_3 e^{i\delta} \\ -s_1 s_2 & c_1 s_2 c_3 - c_2 s_3 e^{i\delta} & c_1 s_2 s_3 + c_2 c_3 e^{i\delta} \end{pmatrix}$$

$c_i \equiv \cos\theta_i$

$s_i \equiv \sin\theta_i$

From the Particle Data Group (1996)

$$K = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

The 90% CL limits on the magnitude of the CKM elements:

$$\begin{pmatrix} 0.9745 \text{ to } 0.9757 & 0.219 \text{ to } 0.224 & 0.002 \text{ to } 0.005 \\ 0.218 \text{ to } 0.224 & 0.9736 \text{ to } 0.9750 & 0.036 \text{ to } 0.046 \\ 0.004 \text{ to } 0.014 & 0.034 \text{ to } 0.046 & 0.9989 \text{ to } 0.9993 \end{pmatrix}$$

with an assumption that only 3 generations exist (so that the unitarity of K can be used to constrain some of the mixing elements.

Counting parameters for n-generations

An $n \times n$ unitary matrix can be written as:

$$U = Q \exp iS \qquad n^2 \text{ real parameters}$$

where:

Q is a real orthogonal $n \times n$ matrix : $\frac{1}{2}n(n-1)$ real angles

S is a real symmetric $n \times n$ matrix : $\frac{1}{2}n(n+1)$ real phases

Physics is unaffected by

$$K_{ij} \to e^{i(\theta_{1i} - \theta_{2j})} K_{ij}$$

which removes $2n-1$ real phases.

End result:

$$\tfrac{1}{2}n(n-1) \qquad \text{real angles}$$

$$\tfrac{1}{2}n(n+1) - (2n-1) = \tfrac{1}{2}(n-1)(n-2) \qquad \text{real phases}$$

Thus, for at least one CP-violating phase, one needs at least $n=3$.

For $n=2$, $K = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix}$ $\qquad \theta_c =$ Cabibbo angle

$$\sin\theta_c \simeq 0.22$$

Three generations of leptons

Following the analysis for quarks, there are again no FCNC's. At first glance, there are generation changing charged current interactions governed by a leptonic-CKM matrix $\tilde{K}$:

$$\mathcal{L}_{We\nu} = \frac{-g}{\sqrt{2}} \sum_{j\ell} \left(\tilde{K}_{j\ell}\, \overline{\nu_j}\, \gamma^\mu P_L\, \ell_\ell\, W_\mu^+ + h.c. \right)$$

But in the Standard Model, with no right-handed neutrinos, all the neutrinos are massless. For this argument, it is sufficient that all neutrinos are mass-degenerate. In this case, we are free to rotate the neutrino eigenstates with no penalty.

In particular, simply replace

$$\tilde{K}^\dagger_{j\ell}\, \nu_\ell \longrightarrow \nu_j$$

and associate this with ℓ_j. Then,

$$\mathcal{L}_{We\nu} = \frac{-g}{\sqrt{2}} \sum_{i} \left(\overline{\nu_j}\, \gamma^\mu P_L\, \ell_j\, W_\mu^+ + h.c. \right)$$

which is diagonal in generation space. Thus, there are no generation-changing processes involving leptons (to all orders in perturbation theory!)

This implies that there are three separate discrete lepton number symmetries: L_e, L_μ, and L_τ.

The electroweak Standard Model parameter count

parameters	number	
g, g'	2	
λ (or m_H)	1	
v (or m_W)	1	
m_q	6	13
m_ℓ	3	
K_{ij}	4	
	17	

Another method to arrive at the 13 parameters that govern fermion masses and mixing.

Start with the Higgs-fermion Yukawa matrices:

$$h_{U ij},\ h_{D ij},\ h_{E ij}$$

Each h_{ij} is a complex 3×3 matrix which yields 9 real parameters and 9 phases.

total: 27 real parameters
27 phases

(The four 3×3 unitary matrices U_1, U_2, V_1, V_2 plus two counterparts from the lepton sector yield $6\times9 = 54$ parameters. This matches the counting above.)

If I put all the $h_{ij} = 0$, then the Standard Model exhibits a global $U(3)^5$ symmetry

3: number of generations

5: number of $SU(2) \times U(1)$ multiplets:

$$\begin{pmatrix} \psi_{Q_1} \\ \psi_{Q_2} \end{pmatrix}, \ \psi_U, \psi_D, \ \begin{pmatrix} \psi_{L_1} \\ \psi_{L_2} \end{pmatrix}, \ \psi_E$$

A $U(3)$ matrix has 3 angles and 6 phases. Thus, when $h_{ij} \neq 0$, I can make $U(3)^5$ rotations with no effect on the fermion-gauge boson interactions. I can then use these rotations to remove some of the 27 real parameters + 27 phases of h_{Uij}, h_{Dij}, h_{Eij} ...

... almost, but not quite. Sitting among the $U(3)^5$ rotations are four $U(1)$ global rotations which have <u>no effect</u> on the physics: these are $\underset{\substack{\uparrow \\ \text{baryon} \\ \text{number}}}{B}$, $\underbrace{L_e, L_\mu, L_\tau}_{\text{lepton numbers}}$.

Thus, we can use the $U(3)^5$ rotations to remove:

$5 \times 3 = 15$ angles

$(5 \times 6) - 4 = 26$ phases

<u>End</u> <u>result</u>: We are left with:

$$\left.\begin{array}{l} 27 - 15 = 12 \text{ real parameters} \\ 27 - 26 = 1 \text{ phase} \end{array}\right\} 13.$$