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Lecture III

SUMMER SCHOOL IN HIGH ENERGY PHYSICS AND COSMOLOGY

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# THE STANDARD MODEL

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## Anomalies – constraints on model building

Not all classical conservation laws hold quantum mechanically. However, if quantum mechanical effects spoil the conservation of a current that couples to a gauge field, then renormalizability and unitarity of the theory is spoiled.

Example of a conservation law that holds both classically and quantum mechanically:

conservation of the electromagnetic current (and corresponding electric charge).

$$\partial_\mu j^\mu_{EM} = 0$$

$$\frac{dQ}{dt} = 0 \qquad \text{where} \quad Q = \int d^3x \, j^0$$

For example, in QED

$$\mathcal{L}_{QED} = \bar{\Psi}(i\partial\!\!\!/ - m + eA\!\!\!/)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Define:

$$j_\mu \equiv \bar{\Psi}\gamma_\mu\Psi$$

$$j_\mu^5 \equiv \bar{\Psi}\gamma_\mu\gamma_5\Psi$$

$$P \equiv \bar{\Psi}\gamma_5\Psi$$

$\mathcal{L}_{QED}$ is invariant under local $U(1)$ gauge transformations:

$$\psi \to e^{ie\Lambda(x)}\psi$$
$$\bar{\psi} \to \bar{\psi} e^{-ie\Lambda(x)}$$
$$A_\mu \to A_\mu + \partial_\mu \Lambda(x)$$

By the Noether procedure, one concludes that $\partial^\mu j_\mu = 0$. Alternatively, one can check that by using the equations of motion:

$$\partial^\mu j_\mu = 0$$
$$\partial^\mu j_\mu^5 = 2imP$$

Gauge invariance imposes constraints on the Green functions of QED. Consider the 1PI Green functions with $n$ external photon lines (and no external fermion lines):

$$i\Gamma_{\mu_1\mu_2\cdots\mu_n}(k_1, k_2, \ldots, k_n) =$$ 

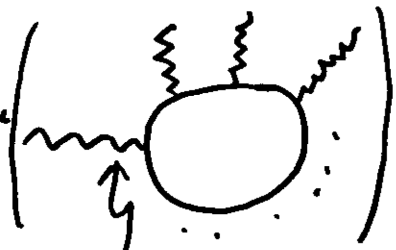

where the $k_i$ are the four momenta of the external (off-shell) photons. [Note: if $n$ is odd, then $\Gamma = 0$ by C-invariance; this is Furry's theorem.]

Then, gauge invariance implies that:

$$\boxed{k_i^{\mu_i}\,\Gamma_{\mu_1\mu_2\cdots\mu_n} = 0}$$ a Ward identity

One consequence of this Ward identity is $q_\mu \Pi^{\mu\nu}(q) = q_\nu \Pi^{\mu\nu}(q) = 0$, a result used in the "proof" that gauge bosons are massless.

One way to prove this Ward identity is by a diagrammatic analysis. The external photon line corresponding to $k_i$ must attach itself to a closed fermion loop, since the Green function under consideration has no external fermions. Thus, we are led to study:

$k_i^\mu$  $= 0$

sum over all possible insertions of this photon line

The Ward identity can be explicitly verified at one-loop, indicating that quantum corrections do not violate the symmetry.

Subtlety: for , the graph is divergent, so the calculation may lead to an ambiguous result. Using dimensional regularization automatically preserves gauge invariance. Using a "bad" regulator could lead to a violation of gauge invariance; but then one is able to add a local counterterm to restore gauge invariance.

Thus, it appears that even if quantum corrections lead to an anomalous violation of the classical symmetry, one could then restore the symmetry by hand. So what's the problem?

Return to QED, but now set $m=0$. Then, there is a new <u>global</u> symmetry present:

$$\psi \to e^{i\beta\gamma_5}\psi$$
$$\bar{\psi} \to \bar{\psi} e^{i\beta\gamma_5}$$

which by the Noether procedure (or by invoking the equations of motion) lead to:

$$\partial^\mu j_\mu^5 = 0$$

In two-component formalism, these classical symmetries are readily apparent, since for massless QED we have:

$$\mathcal{L} = i(\bar{\psi}_1 \bar{\sigma}^\mu D_\mu \psi_1 + \bar{\psi}_2 \bar{\sigma}^\mu D_\mu \psi_2) - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}$$

with the four component spinor given by $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$,

which leads to two conserved currents:

$$j_L^\mu = \bar{\psi}_1 \bar{\sigma}^\mu \psi_1 = \bar{\psi}\gamma^\mu P_L \psi = \frac{1}{2}(j^\mu - j_5^\mu)$$
$$j_R^\mu = -\bar{\psi}_2 \bar{\sigma}^\mu \psi_2 = \bar{\psi}\gamma^\mu P_R \psi = \frac{1}{2}(j^\mu + j_5^\mu)$$

corresponding to the symmetry of phase rotations of $\psi_1$ and $\psi_2$, respectively

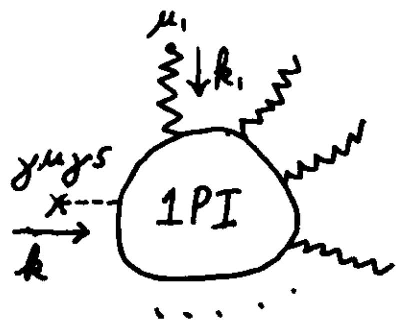
## The axial Ward identity

In analogy to the QED Ward identity, one can formally derive:

$$\boxed{k^{\mu}\Gamma^{5}_{\mu\mu_1\mu_2\cdots\mu_n} = 0}$$

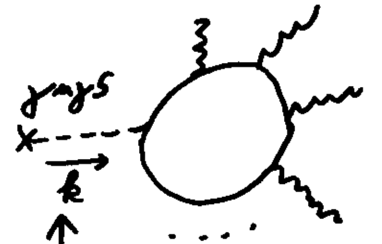
where

$$i\Gamma^{5}_{\mu\mu_1\mu_2\cdots\mu_n} =$$




To check whether the Ward identity holds diagrammatically, compute:

$$k^{\mu}\Bigg( \qquad \Bigg) \stackrel{?}{=} 0$$

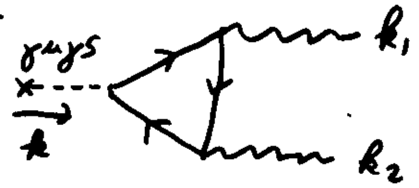


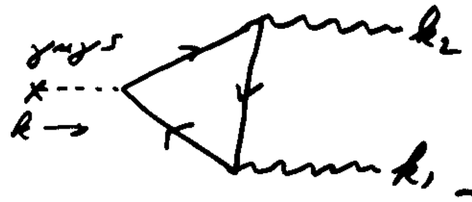

sum over all possible insertions

By C-invariance of QED, one need only consider graphs with an even number of external photon lines.

It is simple to show that the axial Ward identity is satisfied when the number of external photon lines is $n \geq 4$. The case of $n=0$ is trivial, so the only real case of interest is $n=2$.

# The VVA triangle

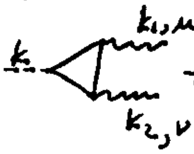
$k^\mu \Big[$ 

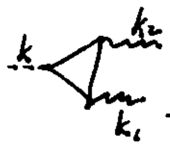
$+$ 

$\Big] \overset{?}{=} 0$

Naive manipulations of these linearly divergent integrals does confirm the above result. But if one is careful to regulate the infinities using a gauge-invariant regulator (like dimensional regularization), then one finds a non-zero result!

Explicitly:

$k^\mu \Big[$ 

$+$ 

$\Big]$

$$= \frac{-ie^2}{8\pi^2} \mathrm{Tr}(\gamma^5 \gamma^\mu \gamma^\nu \not{k}_1 \not{k}_2)$$

$$= \frac{e^2}{2\pi^2} \varepsilon^{\mu\nu\alpha\beta} k_{1\alpha} k_{2\beta}$$

Perhaps we can define this anomalous result away with some local counterterm. But this would upset gauge invariance.

The anomaly results from a clash between the vector and axial vector symmetries, or equivalently between gauge invariance and chiral invariance.

In QED, we choose to regard gauge invariance as sacred. Then it is the axial current that is anomalous, with:

$$\boxed{\partial^\mu j_\mu^5 = \frac{e^2}{16\pi^2} \varepsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}}$$

The axial anomaly in QED is harmless, since $J_\mu^5$ does not couple to a gauge boson (i.e. the axial symmetry is global, not local).

In fact, it is precisely this anomaly that is responsible for $\pi^0 \to \gamma\gamma$ decay, since $\partial^\mu J_\mu^5$ can serve as an interpolating field for the $\pi^0$. Thus nature confirms that the axial anomaly has physical consequences.

But, in electroweak theory, we have seen that $W^\pm$ and $Z^0$ couplings to fermions exhibit both vector and axial vector pieces. Thus, the VVA triangle arises in radiative corrections to electroweak processes. If an anomaly is present, it is disasterous since now the axial vector current does couple to gauge bosons. We demand that the Ward identities for both vector and axial vector currents coupled to $W^\pm, Z, \gamma$ be satisfied, otherwise renormalizability and unitarity of the electroweak theory is destroyed.

Aside:

1. An anomaly also appears in AAA triangles.
2. The Adler-Bardeen theorem states that
$$\partial^\mu J_\mu^5 = \frac{e^2}{16\pi^2} \varepsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}$$
is not renormalized to all orders in perturbation theory. Thus, a one-loop analysis is sufficient
3. Extensions to non-abelian theories yields similar results.

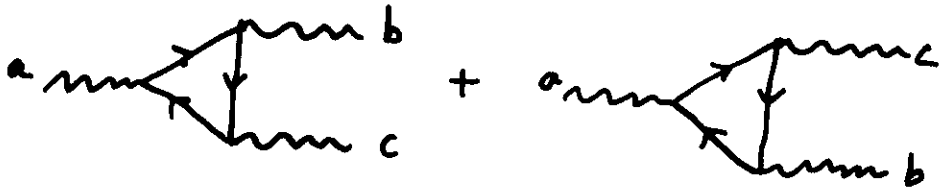
# Cancellation of anomalies in electroweak theory

In two-component formalism, the coupling of gauge bosons to fermions arises from:

$$\mathcal{L} = i\bar{\Psi}_i \bar{\sigma}^\mu (D_\mu)_{ij} \Psi_j$$

where $D_\mu = \partial_\mu + igA^a_\mu T^a$

When we consider triangle diagrams of the form:




because of the factors of $T$ associated with each vertex, I find anomalous behavior of the corresponding Ward identity as before, but now proportional to:

$$\mathrm{Tr}\; T^a(T^bT^c + T^cT^b)$$

The Tr (trace) means to sum over all fermions that can appear in the loop, with corresponding quantum number corresponding to the generator appearing above.

It is convenient to consider the gauge bosons $W^a_\mu$ $(a=1,2,3)$ and $B_\mu$ instead of $W^\pm, Z, \gamma$ since the former couple directly to $T^a$ and $\frac{Y}{2}$.

<u>example</u>: 


$\sim \quad \mathrm{Tr}\, Y^3$

Recall the $SU(2)\times U(1)$ quantum numbers of the two-component fermions:

| | $Y$ | $Y^3$ | degeneracy |
|---|---|---|---|
| $\begin{pmatrix}\psi_{Q_1}\\ \psi_{Q_2}\end{pmatrix}$ | $1/3$ | $1/27$ | $3\times 2$ |
| $\psi_U$ | $-4/3$ | $-64/27$ | $3$ |
| $\psi_D$ | $2/3$ | $8/27$ | $3$ |
| $\begin{pmatrix}\psi_{L_1}\\ \psi_{L_2}\end{pmatrix}$ | $-1$ | $-1$ | $1\times 2$ |
| $\psi_E$ | $2$ | $8$ | $1$ |

3 = color factor
2 = $SU(2)_L$ factor

$$\mathrm{Tr}\, Y^3 = 3\left(\frac{2}{27} - \frac{64}{27} + \frac{8}{27}\right) - 2 + 8 = 0 \qquad !!$$

<u>Thus, there is no hypercharge gauge anomaly</u>!

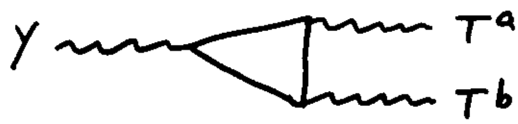
<u>Ingredients of the cancellation</u>

1. Color factor of 3 is crucial.
2. Anomaly does not cancel in quark sector alone; but involves a cancellation between the quark and lepton sectors.
3. Requires that the hypercharge quantum numbers are all integer multiples of a fundamental unit. This is the only place where quantization of a $U(1)$-charge <u>is required</u>!

<u>Aside:</u> Since we are putting the two-component fermions in the loop (equivalent to left handed fields : $\begin{pmatrix} u_L \\ d_L \end{pmatrix}$, $u_L^c, d_L^c$, $\begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$, $e_L^c$ in four-component notation), what we are really computed is the anomaly corresponding two a triangle of left-handed currents; i.e.


which contains both the VVA and AAA anomalies. This is in fact sufficiently general, since one can show that the AAA anomaly is necessarily related (and proportional) to the VVA anomaly.

<u>example</u>:




This is proportional to

$$\mathrm{Tr}\left[Y(T^aT^b+T^bT^a)\right]$$

Thus, only the SU(2)-doublets contribute to the trace.
For SU(2)-doublet representation matrices,

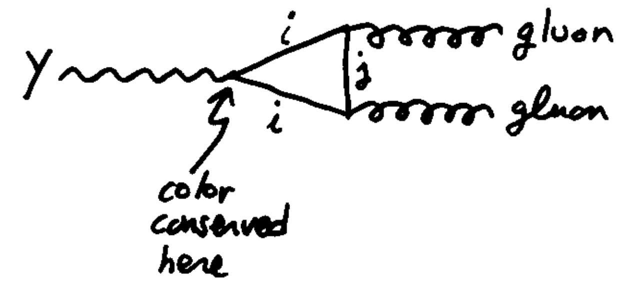
$$T^aT^b+T^bT^a = \frac{1}{2}(\sigma^a\sigma^b+\sigma^b\sigma^a) = \frac{1}{2}\delta^{ab}$$

so, we obtain:

| | $Y$ | degeneracy |
|---|---|---|
| $\begin{pmatrix} \psi_{Q_1} \\ \psi_{Q_2} \end{pmatrix}$ | $1/3$ | 3 |
| $\begin{pmatrix} \psi_{L_1} \\ \psi_{L_2} \end{pmatrix}$ | $-1$ | 1 |

$$\mathrm{Tr}\left[Y(T^aT^b+T^bT^a)\right] = \frac{1}{2}\delta^{ab}\left[3\times\frac{1}{3} - 1\right] = 0 \qquad !!$$

<u>example</u>:




$$\mathrm{Tr}\; Y(T^a_{ij}T^b_{ji} + T^b_{ij}T^b_{ji}) = (2\,\mathrm{Tr}\,T^aT^b)\,\mathrm{Tr}\,Y$$

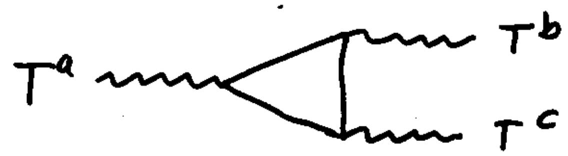
$$= \mathrm{Tr}\,Y \quad \text{summed over colored states only}$$

$$= 2\times\tfrac{1}{3} - \tfrac{4}{3} + \tfrac{2}{3}$$

$$= 0 \quad !!$$

| | $Y$ | degeneracy |
|---|---|---|
| $\begin{pmatrix}\psi_{Q_1}\\ \psi_{Q_2}\end{pmatrix}$ | $1/3$ | 2 |
| $\psi_U$ | $-4/3$ | 1 |
| $\psi_D$ | $2/3$ | 1 |

<u>example</u>:




$$\mathrm{Tr}\; T^a(T^bT^c + T^cT^b) = \tfrac{1}{2}\delta^{bc}\,\mathrm{Tr}\,T^a = 0$$

which in fact holds for any $SU(2)_L$-multiplet.

All other potential anomalies are trivially zero. Thus, we have demonstrated that the Standard Model is anomaly free!

<u>Remark</u>: This looks like magic. But there may be deeper significance. One generation of the Standard Model (plus a right-handed neutrino) fits exactly into the <u>16</u> of SO(10). By simple group theoretical analysis, one can show that SO(10) GUTs is automatically anomaly free.

Aside: automatic cancellation of anomalies for vector-like multiplets.

$SU(2)_L \times U(1)_Y$ is a chiral theory, i.e. the left and right-handed fermions transform differently under $SU(2)_L \times U(1)_Y$ transformation. Hence, anomaly cancellation is not automatic, and requires a special choice for the gauge quantum numbers of the fermions.

In contrast, a vector-like representation (i.e. fermions whose left and right handed components transform the same way under the gauge group) is automatically anomaly free.

In two-component notation, this is easy to see. A vector-like representation satisfies the following property:

- For every two-component fermion of a given set of gauge quantum numbers, there is another fermion that possesses exactly the opposite sign for the corresponding quantum numbers.

[Note: if $f_L$ and $f_L^c$ have opposite sign for all gauge quantum numbers, then $f_L$ and $f_R$ have identical gauge quantum numbers and forms a vector-like representation.]

Consequently,

$$\mathrm{Tr}\left[T^a(T^bT^c + T^cT^b)\right] = 0$$

since for every positive contribution, there is a corresponding negative contribution.

# Electroweak Theory at One-Loop

The renormalization of electroweak theory at one-loop is a straightforward, but arduous task. Without going through all the complexities of renormalization theory, I shall summarize its essence.

Quantum field theory exhibits infinities which must be dealt with in some manner. Suppose I regularize the theory by imposing a large momentum cut-off when performing loop integrals. I then find only a few physical quantities that blow-up if I try to take the cut-off $\Lambda \to \infty$. For example, in QED the only physical quantities of this type are the electron mass and e. Both diverge as $\ln \Lambda$ as $\Lambda \to \infty$. The meaning is clear: m and e are quantities that are extremely sensitive to new physics at very high energy scales.

We know QED is not <u>the</u> fundamental theory. Given an absence of knowledge of the high energy theory, m and e cannot be predicted. So, instead we take a practical viewpoint. Denote the parameters that appear in the QED Lagrangian $m_0$ and $e_0$. Then, the physical masses m and e are functions of $m_0, e_0$ and $\Lambda$:

$$m = f(m_0, e_0, \Lambda)$$
$$e = g(m_0, e_0, \Lambda)$$

Accepting the lack of predictivity of m, e in QED we proceed to compute other QED observables, namely S-matrix elements of scattering processes.

In perturbation theory,

$$S = S(m_0, e_0, \Lambda).$$

But now replace $m_0, e_0 \longrightarrow m, e$ by solving the equations for $m_0$ and $e_0$ implicitly. The result is

$$S = \lim_{\Lambda\to\infty} S(m(m_0,\Lambda), e(e_0,\Lambda), \Lambda).$$

The success of the renormalization program is that this limit produces a finite answer for <u>all</u> observables of the theory, once the basic parameters of the theory (e, m in QED) have been fixed.

Thus, our procedure is:

1. Identify all parameters of the theory that are sensitive to an arbitrary high energy scale.
   - This sensitivity is usually $\sim \ln \Lambda$, but in some cases like squared scalar masses, the sensitivity is quadratic in $\Lambda$.
   - The theory is renormalizable if the number of such parameters is finite
2. Fix these parameters by comparing with some well-defined procedure to measure them in the laboratory
3. Compute observables in terms of the measured parameters.

In the $SU(2)_L \times U(1)_Y$ theory, let us identify the parameters that must be fixed by experiment. There are 17 such parameters, which we have already identified:

$g, g', m_H, m_W$, 6 quark masses, 3 lepton masses,
4 CKM mixing parameters (3 angles + 1 phase).

The choice is not unique. For example, note that $m_Z$ is not listed. This means that given the above parameters, $m_Z$ must be a prediction of the theory. We have already noted the tree-level result:

$$m_Z = \frac{m_W}{\cos\theta_W}\,, \qquad \cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}$$

This is a tree-level result, and is called a "natural relation" of the $SU(2)_L \times U(1)_Y$ model. That is, when radiative corrections are included, we would find:

$$m_Z = \frac{m_W\sqrt{g^2+g'^2}}{g}\left[1 + \underbrace{O(g^2) + O(g'^2) + \ldots}_{\text{finite corrections}}\right].$$

Actually, this is a silly procedure, since $g, g'$ are not know to high accuracy while $m_Z$ is known to four significant figures. Thus, it is better to choose the best measured parameters as input parameters, and predict other observables in terms of the

First, let's take a brief look at a number of tree-level electroweak relations.

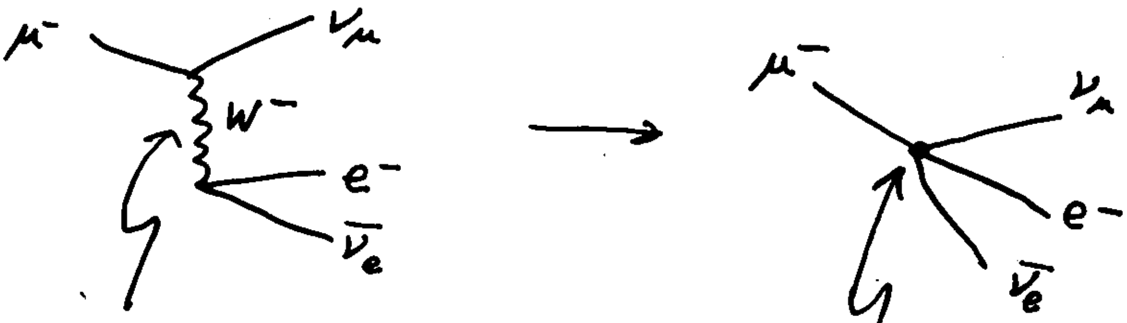
$$m_W^2 = \frac{1}{4} g^2 v^2$$

$$m_Z^2 = \frac{1}{4}(g^2 + g'^2) v^2$$

$$e = \frac{g g'}{\sqrt{g^2 + g'^2}}$$

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8 m_W^2}$$

The last relation is associated with the tree-level calculation of $\mu$-decay. Since $m_W \gg m_\mu$, we can approximate




$$\frac{-i g^{\mu\nu}}{q^2 - m_W^2} \simeq \frac{i g^{\mu\nu}}{m_W^2}$$

$$-\frac{G_F}{\sqrt{2}} \gamma^\mu (1-\gamma_5) \otimes \gamma_\mu (1-\gamma_5)$$

Then,

$$\tau_\mu^{-1} = \frac{G_F^2 m_\mu^5}{192\pi^3} \left(1 + \frac{3}{5}\frac{m_\mu^2}{m_W^2}\right) \left[1 - 8x + 8x^3 - x^4 - 12x^2 \ln x\right]$$

first order correction due to $m_W \neq \infty$.

$$x \equiv \frac{m_e^2}{m_\mu^2}$$

Thus, $G_F$ can be extracted from the measurement of the $\mu^-$ lifetime

The most accurately measured electroweak parameters are:

$$G_F,\ m_Z,\ \alpha \equiv \frac{e^2}{4\pi}$$

Thus, my choice for the 17 electroweak parameters to be fixed by experiment are:

$$\boxed{G_F,\ m_Z,\ \alpha,\ m_H,\ m_t},\ \text{12 other fermion sector parameters}$$

these play the dominant role in the analysis of precision electroweak observables at LEP.

Remark: $m_H$ is not known, so any prediction of precision electroweak observables will be a function of $m_H$.

## QCD corrections

QCD corrections can be important and affect the analysis of precision electroweak observables. This adds one extra parameter — the strong coupling $\alpha_s$ — which must be fixed by experiment. [One other QCD parameter, $\bar{\theta}_{QCD} \lesssim 10^{-9}$ and will play a role only if the electric dipole moment of the neutron or heavy atoms are observed.]

Numerical values of the input parameters:

$$m_Z = 91.1863 \pm 0.0019 \text{ GeV}$$

$$\alpha^{-1} = 137.0359895\,(61)$$

$$G_F = 1.16639(2) \times 10^{-5} \text{ GeV}^{-2} \; *$$

$$\alpha_s(m_Z)_{\overline{MS}} = 0.118 \pm 0.003$$

$$m_t = 175.6 \pm 5.5 \text{ GeV}$$

---

* In practice, $G_F$ is defined from the tree-level formula for $\tau_\mu^{-1}$ augmented by QED corrections to four-fermi theory which are finite. Specifically,

$$\tau_\mu^{-1} = (\tau_\mu^{-1})_{tree}\left[1 + \frac{\alpha(m_\mu)}{2\pi}\left(\frac{25}{4} - \pi^2\right)\right]$$

where $\alpha(m_\mu)^{-1} \simeq 136$.

Note that $m_W$ is not fixed by experiment; it is a prediction of the electroweak theory. At tree level,

$$m_W^2 = \frac{\sqrt{2}\, g^2}{8 G_F}$$

$$\cos\theta_W = \frac{m_W}{m_Z}$$

$$e = g \sin\theta_W = g\left(1 - \frac{m_W^2}{m_Z^2}\right)^{1/2}$$

$$\alpha = \frac{e^2}{4\pi}$$

from which it follows that:

$$m_W^2\left(1 - \frac{m_W^2}{m_Z^2}\right) = \frac{\pi\alpha}{\sqrt{2}\, G_F}$$

which is a quadratic equation whose solution yields $m_W$.

At one loop,

$$m_W^2\left(1 - \frac{m_W^2}{m_Z^2}\right) = \frac{\pi\alpha}{\sqrt{2}\, G_F}(1 + \Delta r)$$

where $\Delta r$ is finite and calculable.

## The electroweak $\rho$-parameter

$$\rho = \frac{m_W^2}{m_Z^2 \cos^2\theta_W} = 1$$

at tree-level. How do we go beyond tree level? This is somewhat ambiguous, since there are a number of observables that depend on $\cos\theta_W$, but each has their own set of radiative corrections.

Thus, one can define a number of different $\cos\theta_W$'s. Here are two examples:

1. $\sin^2\theta_W = 1 - \dfrac{m_W^2}{m_Z^2}$     Sirlin's definition

2. $\sin^2\theta_{eff}^{lept} = \dfrac{1}{4}\left(1 - \dfrac{g_V}{g_A}\right)$

where


$$\frac{-ig}{2\cos\theta_W}\gamma^\mu(g_V - g_A\gamma_5)$$

At tree level, $\quad g_V = 2\sin^2\theta_W - \frac{1}{2}$

$\qquad\qquad\qquad g_A = -\frac{1}{2}$

**Data:**

$$1 - \frac{m_W^2}{m_Z^2} = 0.2232 \pm 0.0016 \quad (\text{using } m_W = 80.37 \pm 0.08\ \text{Ge}$$

$$\sin^2\theta_{eff}^{lept} = 0.23157 \pm 0.00022$$

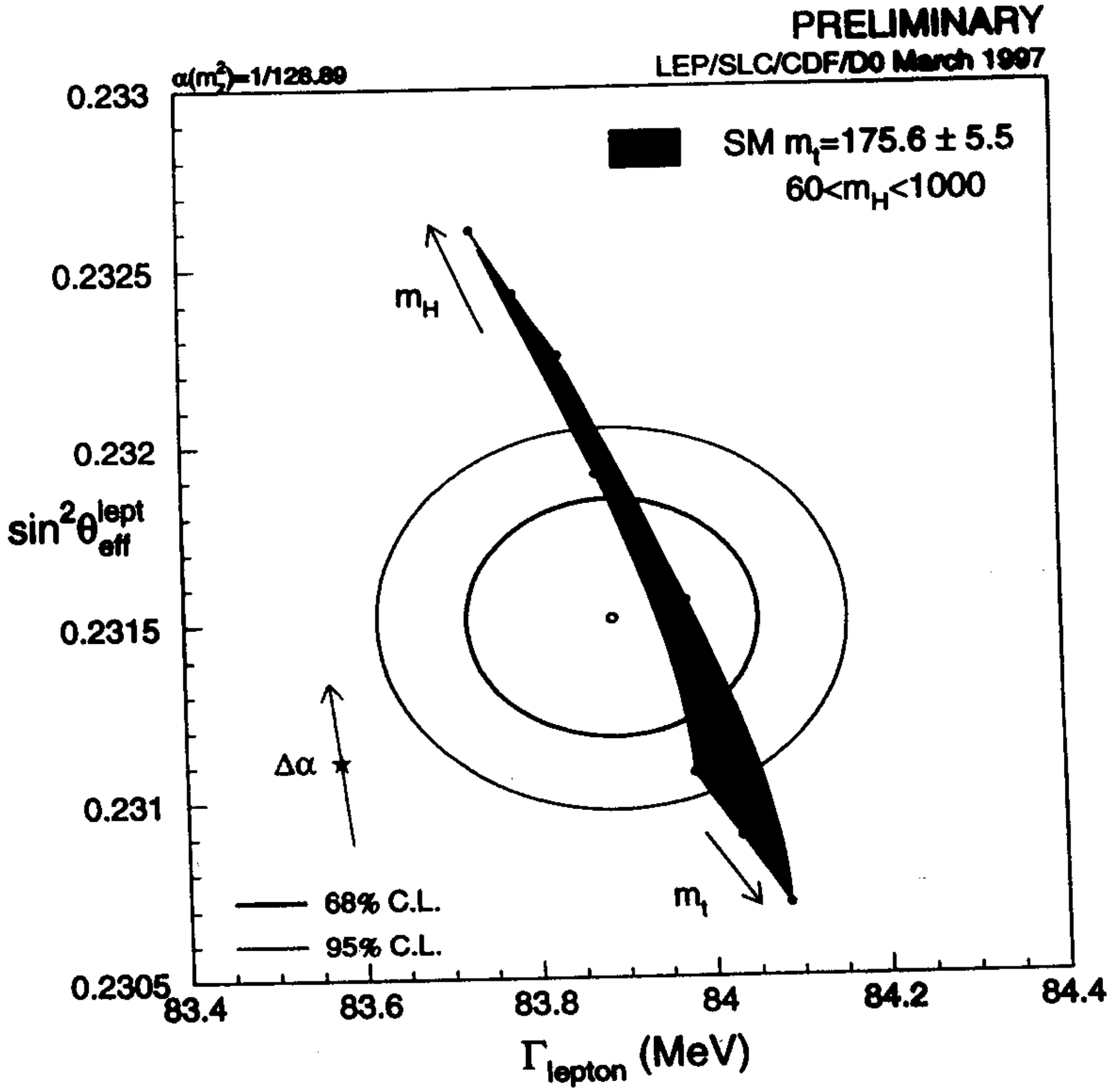
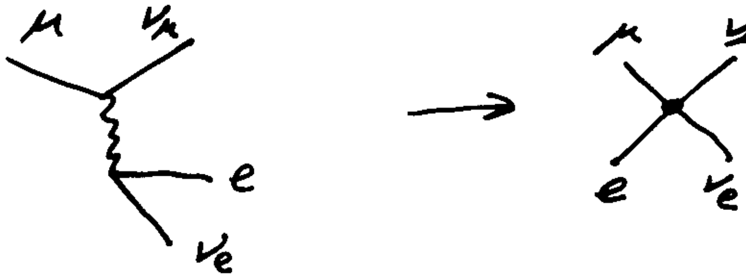



Figure 5: The LEP/SLD measurements of $\sin^2\theta_{\rm eff}^{\rm lept}$ (Table 20) and $\Gamma_{\ell\ell}$ (Table 9) and the Standard Model prediction. The star shows the predictions if among the electroweak radiative corrections only the photon vacuum polarisation is included. The corresponding arrow shows variation of this prediction if $\alpha(m_Z^2)$ is changed by one standard deviation. This variation gives an additional uncertainty to the Standard Model prediction shown in the figure.

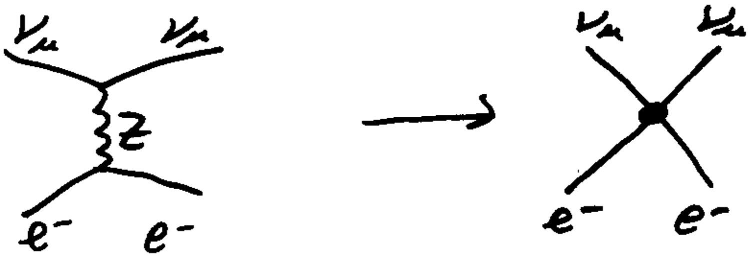
So, one could define

$$\rho_{eff}^{lept} = \frac{m_W^2}{m_Z^2 \cos^2\theta_{eff}^{lept}} = 1.011 \pm 0.002$$

More traditional is to define $\rho_{NC}$ through the measurement of low energy ($q^2 \ll m_Z^2$) neutral current processes. We can mimic the definition of $G_F$ for the neutral current:




$$-\frac{G_F}{\sqrt{2}}\gamma^\mu(1-\gamma_5) \otimes \gamma^\mu(1-\gamma_5)$$




$$-\frac{G_{NC}}{\sqrt{2}} J^\mu_{NC} \otimes J_{\mu\, NC}$$

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8 m_W^2}$$

$$\frac{G_{NC}}{\sqrt{2}} = \frac{g^2}{8 m_Z^2 \cos^2\theta_W^{NC}}$$

$$\boxed{\rho_{NC} \equiv \frac{G_{NC}}{G_F} = \frac{m_W^2}{m_Z^2 \cos^2\theta_W^{NC}}}$$

Data implies that $\rho_{NC} \approx \rho_{eff}^{lept}$.

## The need for radiative corrections

The fact that $(\sin^2\theta_W)_{Sirlin} \neq \sin^2\theta_{eff}^{lept}$ is an indication that radiative corrections are necessary. Another can be seen by computing the tree-level W mass. Using:

$$\frac{\pi\alpha}{\sqrt{2}G_F} = (37.28\,\mathrm{GeV})^2$$

we solve the tree-level expression for $m_W$:

$$m_W^2 = \frac{1}{2} m_Z^2 \left[ 1 + \sqrt{1 - \left(\frac{74.56\,\mathrm{GeV}}{m_Z}\right)^2} \right]$$

$$= (80.94\,\mathrm{GeV})^2$$

to be compared with

$$m_W = 80.37 \pm 0.08\,\mathrm{GeV}.$$

This discrepency indicates the need for electroweak radiative corrections.

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From the LEP Electroweak Working Group (LEWWG) 1997

| | Measurement with Total Error | Systematic Error | Standard Model | Pull |
|---|---|---|---|---|
| $\alpha(m_Z^2)^{-1}$ [50] | $128.896 \pm 0.090$ | 0.083 | 128.909 | −0.2 |
| a) **LEP** | | | | |
| **line-shape and lepton asymmetries:** | | | | |
| $m_Z$ [GeV] | $91.1863 \pm 0.0019$ | [(a)]0.0015 | 91.1862 | 0.1 |
| $\Gamma_Z$ [GeV] | $2.4947 \pm 0.0026$ | [(a)]0.0017 | 2.4966 | −0.7 |
| $\sigma_h^0$ [nb] | $41.489 \pm 0.055$ | 0.054 | 41.464 | 0.5 |
| $R_\ell$ | $20.783 \pm 0.029$ | 0.024 | 20.760 | 0.8 |
| $A_{FB}^{0,\ell}$ | $0.0177 \pm 0.0010$ | 0.007 | 0.0161 | 1.6 |
| + correlation matrix Table 8 | | | | |
| $\tau$ polarisation: | | | | |
| $\mathcal{A}_\tau$ | $0.1401 \pm 0.0067$ | 0.0045 | 0.1467 | −1.1 |
| $\mathcal{A}_e$ | $0.1382 \pm 0.0076$ | 0.0021 | 0.1467 | −1.0 |
| b and c quark results: | | | | |
| $R_b^{0(b)}$ | $0.2179 \pm 0.0011$ | 0.0009 | 0.2158 | 1.9 |
| $R_c^{0(b)}$ | $0.1720 \pm 0.0056$ | 0.0042 | 0.1723 | −0.1 |
| $A_{FB}^{0,b(b)}$ | $0.0985 \pm 0.0022$ | 0.0010 | 0.1028 | −2.0 |
| $A_{FB}^{0,c(b)}$ | $0.0734 \pm 0.0048$ | 0.0026 | 0.0734 | 0.0 |
| + correlation matrix Table 12 | | | | |
| $q\bar{q}$ charge asymmetry: | | | | |
| $\sin^2\theta_{eff}^{lept}$ ($\langle Q_{FB}\rangle$) | $0.2322 \pm 0.0010$ | 0.0008 | 0.23157 | 0.6 |
| $m_W$ [GeV] | $80.38 \pm 0.14$ | 0.05 | 80.366 | 0.1 |
| b) SLD [23] | | | | |
| $\sin^2\theta_{eff}^{lept}$ ($A_{LR}$) | $0.23055 \pm 0.00041$ | 0.00014 | 0.23157 | −2.5 |
| $R_b^{0(b)}$ | $0.2152 \pm 0.0038$ | 0.0016 | 0.2158 | −0.2 |
| $R_c^{0(b)}$ | $0.1756 \pm 0.0181$ | 0.0085 | 0.1723 | 0.2 |
| $\mathcal{A}_b$ | $0.897 \pm 0.047$ | 0.032 | 0.935 | −0.8 |
| $\mathcal{A}_c$ | $0.623 \pm 0.085$ | 0.041 | 0.668 | −0.5 |
| c) $p\bar{p}$ and $\nu$N | | | | |
| $m_W$ [GeV] ($p\bar{p}$ [51]) | $80.37 \pm 0.10$ | 0.09 | 80.366 | 0.0 |
| $1 - m_W^2/m_Z^2$ ($\nu$N [52–54]) | $0.2244 \pm 0.0042$ | 0.0036 | 0.2232 | 0.3 |
| $m_t$ [GeV] ($p\bar{p}$ [55–57]) | $175.6 \pm 5.5$ | 4.2 | 172.7 | 0.5 |

Table 21: Summary of measurements included in the combined analysis of Standard Model parameters. Section a) summarises LEP averages, Section b) SLD results for $\sin^2\theta_{eff}^{lept}$ from the measurement of the left-right polarisation asymmetry, for $R_b$ and for $\mathcal{A}_b$ and $\mathcal{A}_c$ from polarised forward-backward asymmetries and Section c) electroweak precision measurements from $p\bar{p}$ colliders and $\nu$N scattering. The total errors in column 2 include the systematic errors listed in column 3. The determination of the systematic part of each error is approximate. The Standard Model results in column 4 and the pulls (difference between measurement and fit in units of the total measurement error) in column 5 are derived from the Standard Model fit including all data (Table 22, column 3) with the Higgs mass treated as a free parameter.

[(a)]The systematic errors on $m_Z$ and $\Gamma_Z$ contain the errors arising from the uncertainties in the LEP energy only.
[(b)]For fits which combine LEP and SLD heavy flavour measurements we use as input the heavy flavour results given in Equation (10) and their correlation matrix in Table 13 in Section 4 of this note.

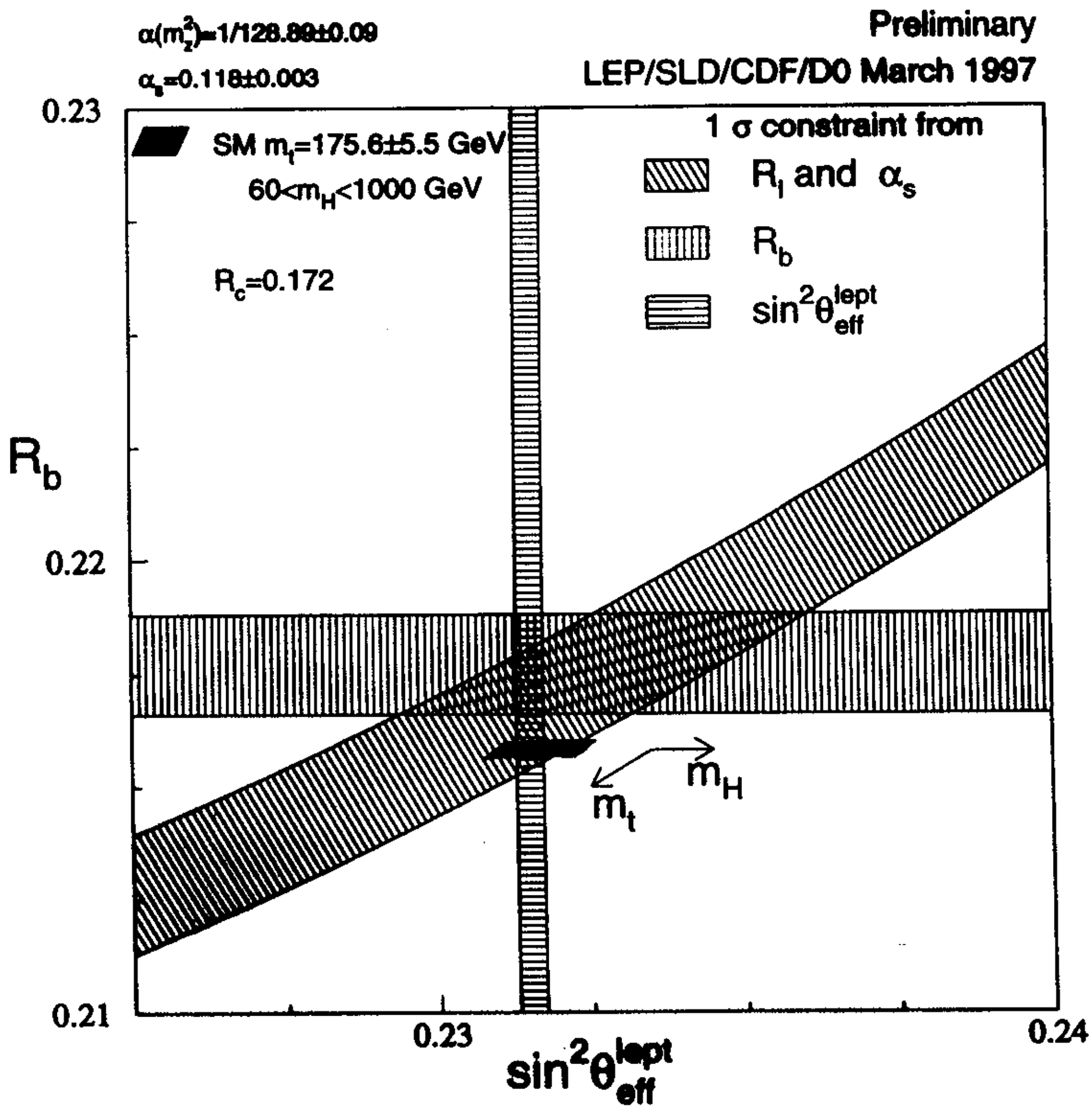


Figure 6: The LEP/SLD measurements of $\sin^2\theta_{\mathrm{eff}}^{\mathrm{lept}}$ (Table 20) and $R_b(R_c = 0.172)$ and the Standard Model prediction. Also shown is the constraint resulting from the measurement of $R_\ell$ on these variables, assuming $\alpha_s(m_Z^2) = 0.118 \pm 0.003$, as well as the Standard Model dependence of light-quark partial widths on $\sin^2\theta_{\mathrm{eff}}^{\mathrm{lept}}$. The Standard Model value for $R_c$ is assumed.

| | LEP (including LEP-II $m_W$) | all data except $m_t$ and $m_W$ | all data |
|---|---|---|---|
| $m_t$ [GeV] | $155^{+15}_{-11}$ | $155^{+10}_{-9}$ | $172.7 \pm 5.4$ |
| $m_H$ [GeV] | $70^{+147}_{-40}$ | $36^{+52}_{-18}$ | $127^{+127}_{-72}$ |
| $\log(m_H)$ | $1.85^{+0.49}_{-0.38}$ | $1.56^{+0.39}_{-0.30}$ | $2.10^{+0.30}_{-0.36}$ |
| $\alpha_s(m_Z^2)$ | $0.122 \pm 0.003$ | $0.121 \pm 0.003$ | $0.120 \pm 0.003$ |
| $\chi^2$/d.o.f. | 10/9 | 18/12 | 21/15 |
| $\sin^2\theta_{\rm eff}^{\rm lept}$ | $0.23190 \pm 0.00026$ | $0.23152 \pm 0.00022$ | $0.23157 \pm 0.00022$ |
| $1 - m_W^2/m_Z^2$ | $0.2248 \pm 0.0009$ | $0.2241 \pm 0.0008$ | $0.2232 \pm 0.0006$ |
| $m_W$ (GeV) | $80.285 \pm 0.045$ | $80.323 \pm 0.042$ | $80.366 \pm 0.031$ |

Table 22: Results of the fits to LEP data alone, to all data except the direct determinations of $m_t$ and $m_W$ (Tevatron and LEP-II) and to all data including the top quark mass determination. As the sensitivity to $m_H$ is logarithmic, both $m_H$ as well as $\log(m_H)$ are quoted. The bottom part of the table lists derived results for $\sin^2\theta_{\rm eff}^{\rm lept}$, $1 - m_W^2/m_Z^2$ and $m_W$. See text for a discussion of theoretical errors not included in the errors above.

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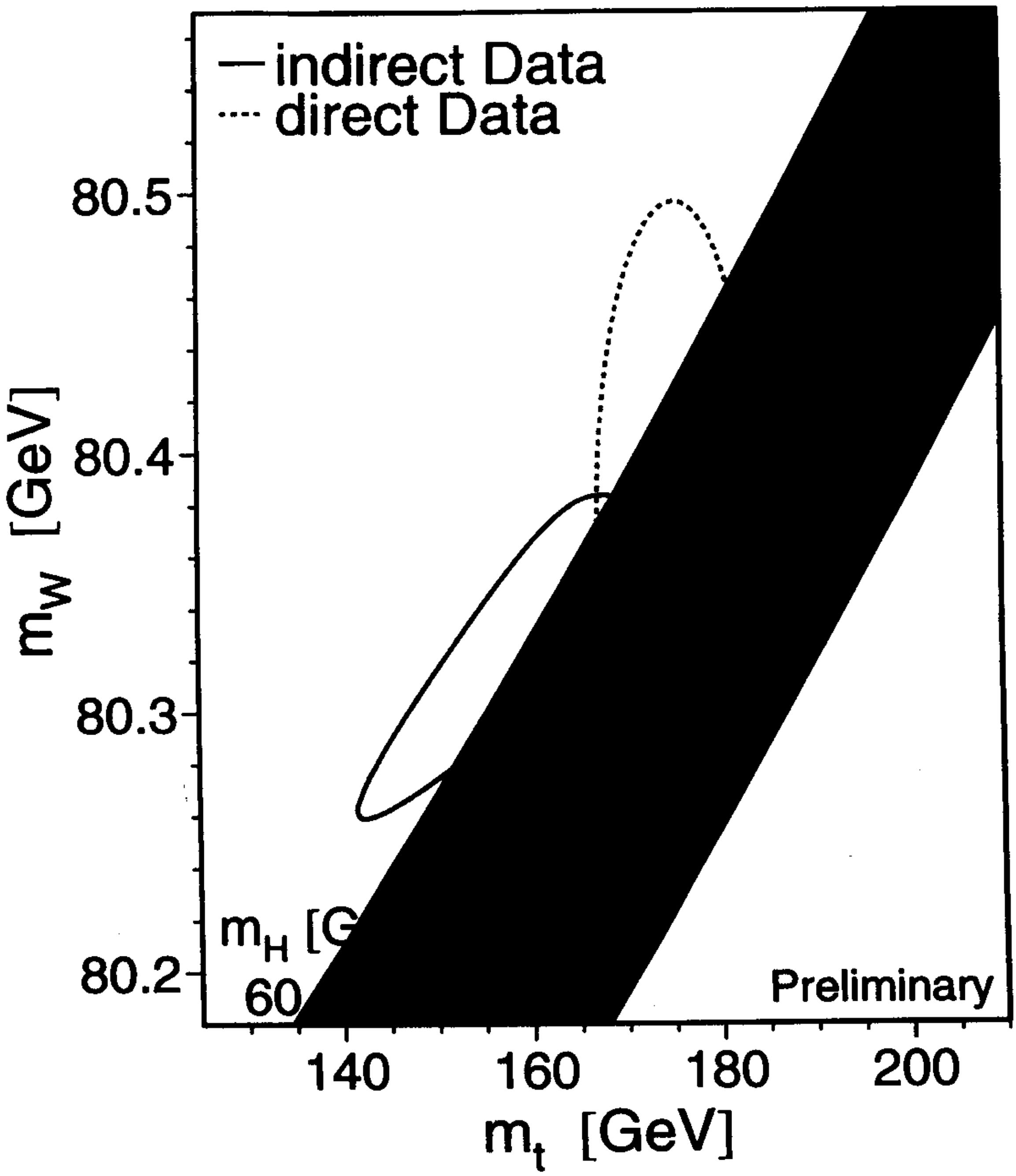



Figure 7: The comparison of the indirect measurements of $m_W$ and $m_t$ (LEP+SLD+$\nu$N data) (solid contour) and the direct measurements (Tevatron and LEPII data) (dashed contour). In both cases the 68% CL contours are plotted. Also shown is the Standard Model relationship for the masses as a function of the Higgs mass.

LEWWG 1997

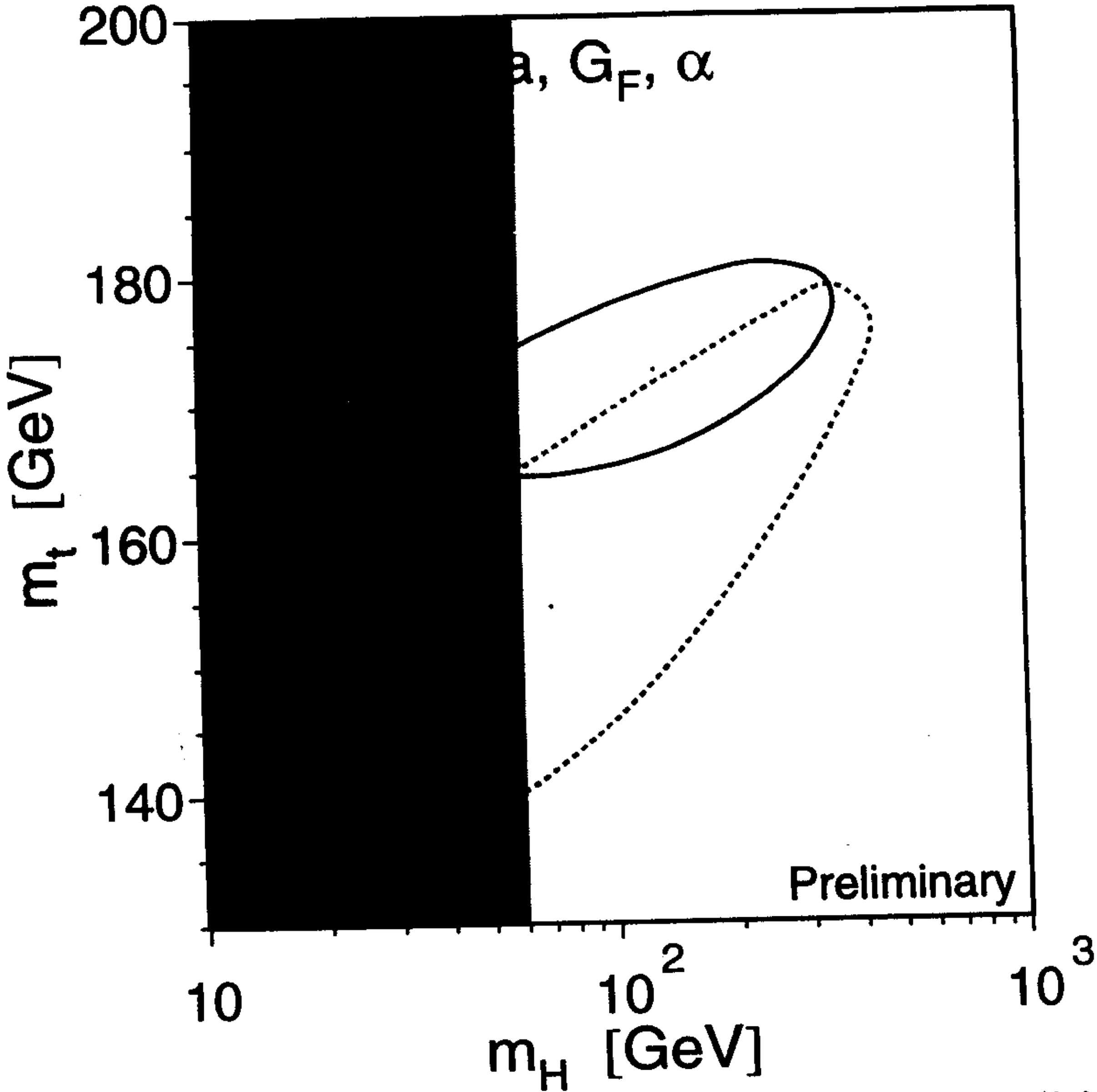



Figure 8: The 68% confidence level contours in $m_t$ and $m_H$ for the fits to LEP data only (dashed curve) and to all data including the CDF/DØ $m_t$ measurement (solid curve).

LEWWG 1997



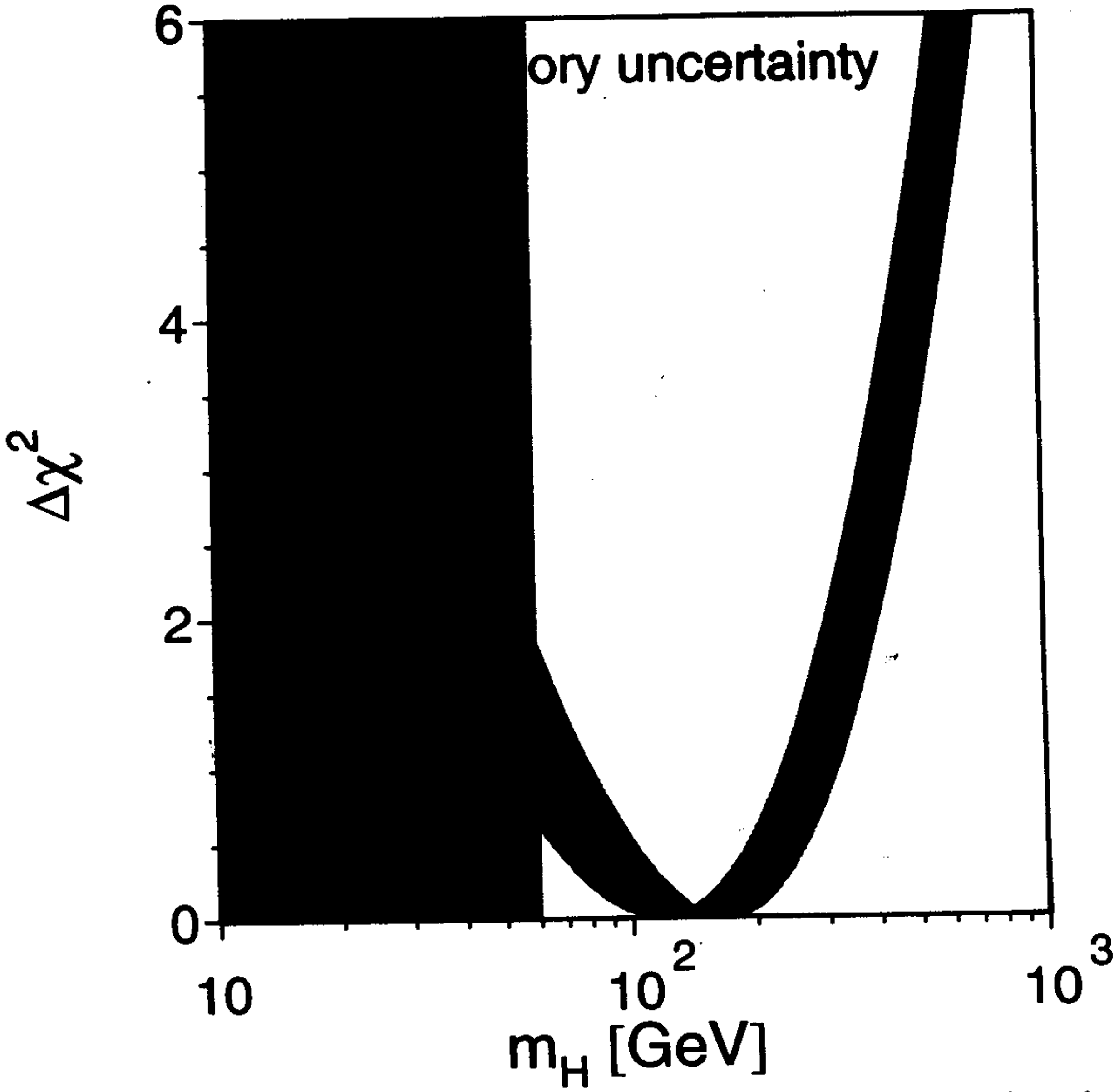



Figure 11: $\Delta\chi^2 = \chi^2 - \chi^2_{min}$ *vs.* $m_H$ curve. The line is the result of the fit using all data (last column of Table 22); the band represents an estimate of the theoretical error due to missing higher order corrections.

# W-mass prediction in electroweak theory

## Preliminaries:

$m_W^2, m_Z^2$ parameters arising from the bare (tree-level) Lagrangian

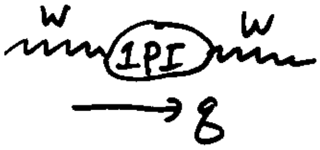
$m_{WP}^2, m_{ZP}^2$ physical masses

$m_{ZP}^2$ is an input parameter while $m_{WP}^2$ is what we shall predict.

Tree-level gauge boson propagator (in $R_\xi$-gauge):

$$D_{\mu\nu}(q) = \frac{i}{q^2 - m_W^2 + i\epsilon}\left[-g_{\mu\nu} + (1-\xi)\frac{q_\mu q_\nu}{q^2 - \xi m_W^2}\right]$$

Sum of all 1PI graphs:

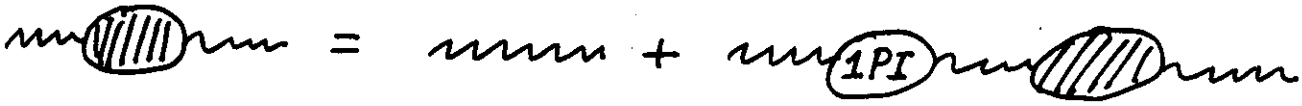



$$= i\Pi_{WW}^{\mu\nu} = -ig^{\mu\nu}A_{WW}(q^2) - iq^\mu q^\nu B_{WW}(q^2)$$

$$= -i\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)A_{WW}(q^2)$$

$$-iq^\mu q^\nu\left[\frac{1}{q^2}A_{WW}(q^2) + B_{WW}(q^2)\right]$$

Full propagator




$$\mathcal{D}_{\mu\nu}^{-1}(q) = D_{\mu\nu}^{-1}(q) - i\Pi_{\mu\nu}(q)$$

We decompose into transverse and longitudinal pieces

$$\mathcal{D}_{\mu\nu}(q) = \mathcal{D}(q^2)\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \mathcal{D}^{(0)}(q^2)\,\frac{q_\mu q_\nu}{q^2}$$

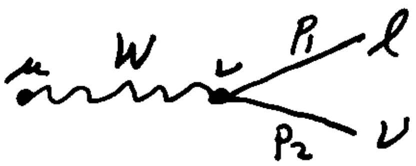
etc. Then,

$$\mathcal{D}^{-1}(q^2) = D^{-1}(q^2) + i A_{WW}(q^2)$$
$$= i\left(q^2 - m_W^2 + A_{WW}(q^2)\right)$$

Hence,

$$\mathcal{D}_{\mu\nu}(q) = \frac{-i g_{\mu\nu}}{q^2 - m_W^2 + A_{WW}(q^2)} + q_\mu q_\nu [\cdots]$$

In Feynman graphs where the gauge boson line is attached to an on-shell fermion




the effect of $q_\nu\, \bar{u}(p_1)\gamma^\nu(1-\gamma_5)v(p_2)$

$$= (p_1+p_2)_\nu\, \bar{u}(p_1)\gamma^\nu(1-\gamma_5)v(p_2)$$
$$= \bar{u}(p_1)(\not{p}_1+\not{p}_2)(1-\gamma_5)v(p_2)$$
$$= -m_\ell\, \bar{u}(p_1)(1-\gamma_5)v(p_2)$$

using the Dirac equation. For $m_\ell \ll m_W$, this is negligible, and we can simply drop the $q_\mu q_\nu$ pieces in $\mathcal{D}_{\mu\nu}(q)$ for all practical purposes.

The physical mass corresponds to the pole in the propagator:

$$D_{\mu\nu}(q) = \frac{-i g_{\mu\nu}}{q^2 - m_W^2 + A_{WW}(q^2)}$$

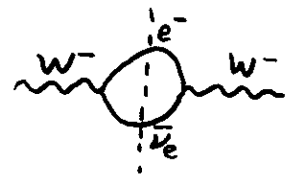
Actually, since the W is unstable, the pole will lie slightly off the real axis. In general,

$$A_{WW}(q^2) = \mathrm{Re}\, A_{WW}(q^2) + i\, \mathrm{Im}\, A_{WW}(q^2)$$

Thus, with one-loop accuracy, we identify:

$$\mathrm{Im}\, A_{WW}(m_W^2) = m_W \Gamma_W$$

$$m_W^2 - \mathrm{Re}\, A_{WW}(m_W^2) = m_{WP}^2$$




$\mathrm{Im}\, A_{WW}(q^2) \neq 0$ if dashed line cuts through a possible physical decay; i.e. if $q^2 \geq m_e^2$

<u>Definition:</u>

$$\delta m_W^2 \equiv m_W^2 - m_{WP}^2$$

Then,

$$\boxed{\delta m_W^2 = \mathrm{Re}\, A_{WW}(m_W^2)}$$

Similarly, repeating the above steps for the Z-propagator yields:

$$\boxed{\delta m_Z^2 = \mathrm{Re}\, A_{ZZ}(m_Z^2)}$$

To predict $m_{WP}^2$, start with the tree-level relation:

$$m_W^2 = \frac{\pi\alpha}{\sqrt{2}G_F \sin^2\theta_W}, \qquad \sin^2\theta_W = 1 - \frac{m_W^2}{m_Z^2}$$

Then, rewrite in terms of the physical parameters:

$$m_W^2 = m_{WP}^2 + \delta m_W^2$$

$$\alpha = \alpha_P + \delta\alpha$$

etc.

Then,

$$m_{WP}^2 + \delta m_W^2 = \frac{\pi(\alpha_P + \delta\alpha)}{\sqrt{2}(G_{FP} + \delta G_F)(s_P^2 + \delta s^2)}, \qquad (s \equiv \sin\theta_W)$$

That is,

$$m_{WP}^2 = \frac{\pi\alpha_P}{\sqrt{2}G_{FP}\left(1 - \frac{m_{WP}^2}{m_{ZP}^2}\right)}(1 + \Delta r)$$

where:

$$\Delta r = \frac{\delta\alpha}{\alpha} - \frac{\delta G_F}{G_F} - \frac{\delta s^2}{s^2} - \frac{\delta m_W^2}{m_W^2}$$

<u>Note</u>: since $\delta\alpha$ is a one loop quantity, if one is doing a one-loop computation, then there is no distinction between $\frac{\delta\alpha}{\alpha_P}$ and $\frac{\delta\alpha}{\alpha}$.

From $s^2 = 1 - \dfrac{m_W^2}{m_Z^2}$,

$$s_P^2 + \delta s^2 = 1 - \frac{m_{WP}^2 + \delta m_W^2}{m_{ZP}^2 + \delta m_Z^2}.$$

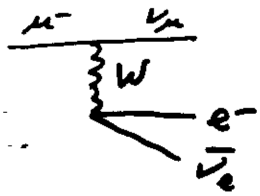
which can be reduced to:

$$\frac{\delta s^2}{s^2} = -\frac{c^2}{s^2}\left(\frac{\delta m_W^2}{m_W^2} - \frac{\delta m_Z^2}{m_Z^2}\right)$$

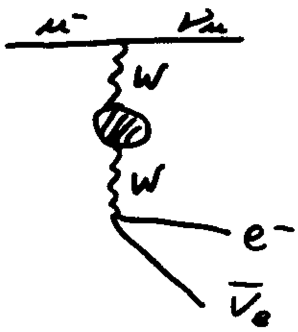
where $c^2 = 1 - s^2 = \dfrac{m_W^2}{m_Z^2}$.

Next, we must work out $\dfrac{\delta G_F}{G_F}$. This requires a full one-loop computation of $\mu$-decay:

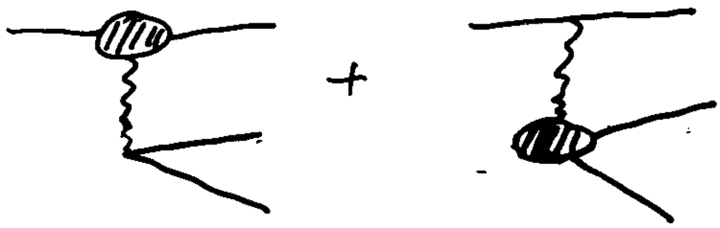
<u>tree level</u>:



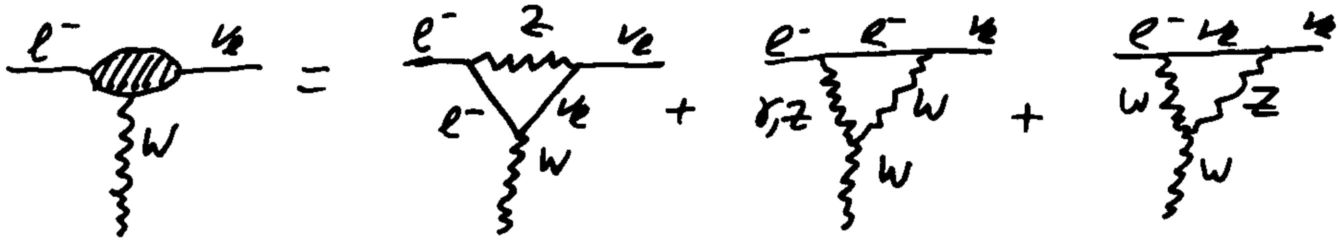

<u>propagator corrections</u>:



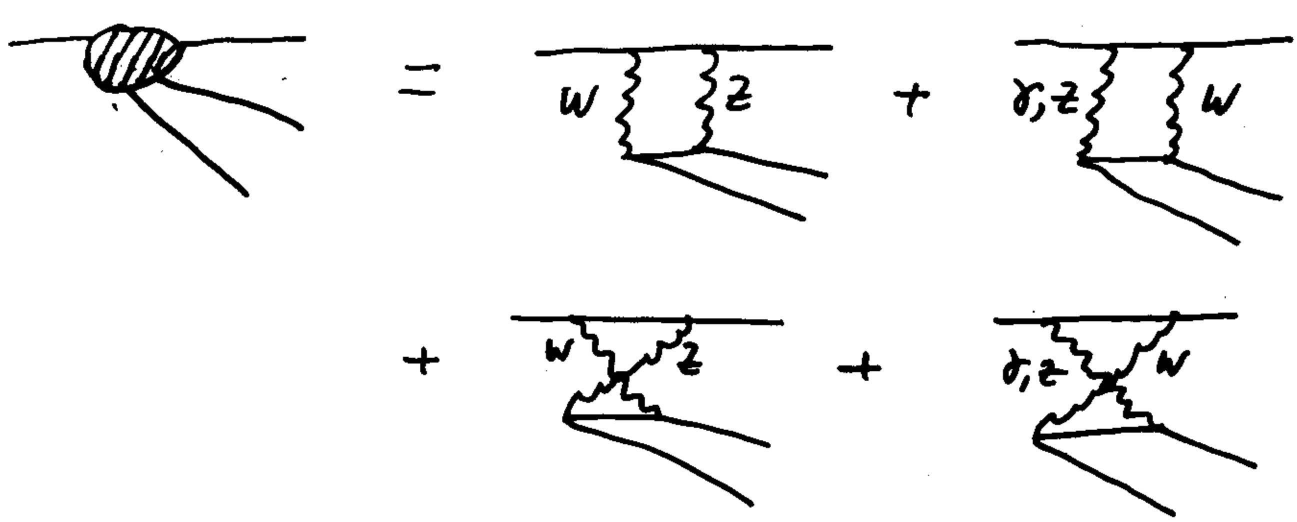
## vertex corrections:



where

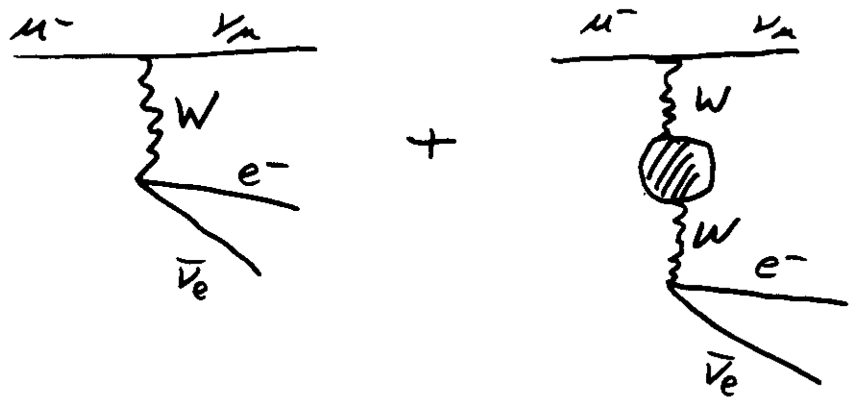



## box diagrams:




(cross lines above do not touch)

The effect of




in the limit of $m_W \gg m_\mu$ is to replace

$$\left.\frac{-ig_{\mu\nu}}{q^2-m_W^2}\right|_{q^2\simeq 0} \longrightarrow \left.\frac{-ig_{\mu\nu}}{q^2-m_W^2+A_{WW}(q^2)}\right|_{q^2\simeq 0}$$

That is:

$$\frac{1}{m_W^2} \longrightarrow \frac{1}{m_W^2-A_{WW}(0)} \simeq \frac{1}{m_W^2}\left(1+\frac{A_{WW}(0)}{m_W^2}\right)$$

or since $\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2}$,

$$G_F \longrightarrow G_F\left(1+\frac{A_{WW}(0)}{m_W^2}\right) \equiv G_{FP}$$

Writing $G_F = G_{FP} + \delta G_F$, we end up with

$$\boxed{\frac{\delta G_F}{G_F} = -\frac{A_{WW}(0)}{m_W^2} + \text{vertex corrections} + \text{box corrections}}$$

Collecting all the pieces:

$$\Delta r = \frac{\delta\alpha}{\alpha} - \frac{c^2}{s^2}\left(\frac{\delta m_Z^2}{m_Z^2} - \frac{\delta m_W^2}{m_W^2}\right) + \frac{A_{WW}(0) - \delta m_W^2}{m_W^2} + \text{vertex} + \text{box}$$

Finally, we need to work out $\delta\alpha$. The fine structure constant is defined by Thomson scattering, e.g. $e^-e^- \to e^-e^-$ as $q^2 \to 0$.

In QED, the relation is rather simple. This is because of the QED Ward identity which leads to the simple relation:

$$e = Z_3^{-1/2} e_p$$

where $Z_3$ (photon wave-function renormalization) is the residue of the pole at $q^2=0$ in the photon propagator:

$$\mathcal{D}_{\mu\nu}(q) = \frac{-i g_{\mu\nu}}{q^2\left[1 + \frac{A_{\gamma\gamma}(q^2)}{q^2}\right]}$$

Recall that for the massless photon,

$$i\Pi^{\mu\nu}(q) = -i(q^2 g^{\mu\nu} - q^\mu q^\nu)\,\Pi(q^2)$$

which means that $\boxed{A_{\gamma\gamma}(0) = 0}$ [this is EM gauge invariance], so

$$\lim_{q^2\to 0} \frac{A_{\gamma\gamma}(q^2)}{q^2} = \left.\frac{dA_{\gamma\gamma}(q^2)}{dq^2}\right|_{q^2=0} \equiv A'_{\gamma\gamma}(0)$$

Thus, $Z_3 = \dfrac{1}{1 + A'_{\gamma\gamma}(0)}$.

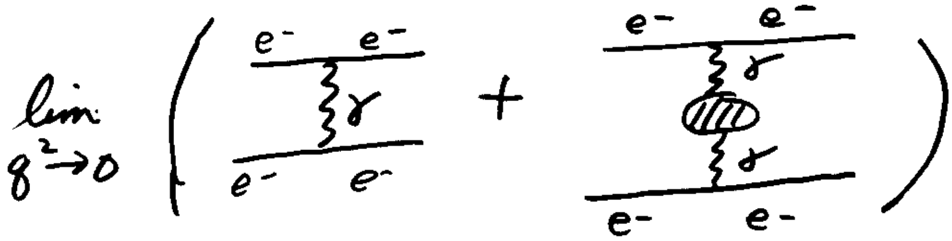
$$e = \left(1 + A'_{\gamma\gamma}(0)\right)^{1/2} e_p$$

$$\alpha = \left(1 + A'_{\gamma\gamma}(0)\right)\alpha_p = \alpha_p + \delta\alpha$$

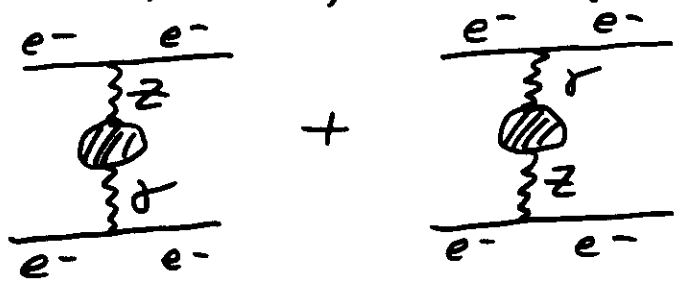
and so:

$$\frac{\delta\alpha}{\alpha} = A'_{\gamma\gamma}(0)$$

in QED. In effect, we have defined the Thomson limit via:




In the electroweak model, two extra graphs must be considered:




The end result is that in the electroweak model,

$$\boxed{\frac{\delta\alpha}{\alpha} = A'_{\gamma\gamma}(0) - \frac{2s}{c}\frac{A_{Z\gamma}(0)}{m_Z^2}}$$

Notation.

$$A_{ij}(q^2) \equiv A_{ij}(0) + q^2 F_{ij}(q^2)$$

e.g. $A'_{\gamma\gamma}(0) \equiv F_{\gamma\gamma}(0)$.

Collecting once more our results:

$$\boxed{\begin{aligned}\Delta r = {} & F_{\gamma\gamma}(0) - \frac{2s}{c}\frac{A_{Z\gamma}(0)}{m_Z^2} - \frac{c^2}{s^2}\left(\frac{A_{ZZ}(0)}{m_Z^2} - \frac{A_{WW}(0)}{m_W^2}\right) \\ & - \frac{c^2}{s^2}\Big(\mathrm{Re}\left(F_{ZZ}(m_Z^2) - F_{WW}(m_W^2)\right)\Big) - \mathrm{Re}\, F_{WW}(m_W^2) \\ & + \text{vertex} + \text{box}\end{aligned}}$$

This can be written as:

$$\Delta r \equiv \Delta\alpha - \frac{c^2}{s^2}\Delta\rho + \Delta r_{rem}$$

with

$$\Delta\alpha \equiv F_{\gamma\gamma}(0) - \mathrm{Re}\, F_{\gamma\gamma}(m_Z^2)$$

$$\Delta\rho \equiv \frac{A_{ZZ}(0)}{m_Z^2} - \frac{A_{WW}(0)}{m_W^2} + \frac{2s}{c}\frac{A_{Z\gamma}(0)}{m_Z^2}$$

and $\Delta r_{rem}$ turns out to be numerically very small.

So,

$$m_{Wp}^2 = \frac{\pi \alpha_p}{\sqrt{2}\, G_{Fp} \left(1 - \frac{m_{Wp}^2}{m_{Zp}^2}\right)} \left[1 + \Delta\alpha - \frac{c^2}{s^2}\Delta\rho + \Delta r_{rem}\right]$$

One can recognize

$$\alpha_p (1 + \Delta\alpha) \simeq \frac{\alpha_p}{1 - \Delta\alpha} = \alpha(m_Z) \simeq (128.9 \pm 0.09)$$

So, to a good approximation:

$$m_{Wp}^2 \simeq \frac{\pi \alpha(m_Z)}{\sqrt{2}\, G_{Fp} \left(1 - \frac{m_{Wp}^2}{m_{Zp}^2}\right)} \left(1 - \frac{c^2}{s^2}\Delta\rho\right)$$

I will show in a moment that the dominant contribution to $\Delta\rho$ comes from top-quark loops:

$$\Delta\rho \simeq \frac{3\alpha}{16\pi s^2 c^2} \frac{m_t^2}{m_Z^2} \simeq 0.009$$

Thus,

$$1 - \frac{c^2}{s^2}\Delta\rho \simeq 0.97$$

$$\frac{\alpha(m_Z)}{\alpha_p} \simeq 1.063$$

So,

$$\frac{\pi \alpha(m_Z)}{\sqrt{2}\, G_{Fp}} \left(1 - \frac{c^2}{s^2}\Delta\rho\right) \simeq (37.86\ \text{GeV})^2$$

$$\Rightarrow m_W = 80.47\ \text{GeV}$$

# More on $\Delta\rho$

$$\Delta\rho = \frac{A_{ZZ}(0)}{m_Z^2} - \frac{A_{WW}(0)}{m_W^2} + \frac{2s}{c}\frac{A_{Z\gamma}(0)}{m_Z^2}.$$

Let us define:


$$= i\Pi^{\mu\nu}_{33}$$

Using:

$$W^3_\mu = A_\mu s + Z_\mu c,$$

$$i\Pi^{\mu\nu}_{33} = i\Pi^{\mu\nu}_{\gamma\gamma} s^2 + i\Pi^{\mu\nu}_{ZZ} c^2 + 2isc\,\Pi^{\mu\nu}_{Z\gamma}$$

It follows that:

$$A_{33}(0) = c^2 A_{ZZ}(0) + 2sc\,A_{Z\gamma}(0)$$

using $A_{\gamma\gamma}(0) = 0$. Thus,

$$\boxed{\Delta\rho = \frac{1}{m_W^2}\left[A_{33}(0) - A_{WW}(0)\right]}.$$

Clearly, $\Delta\rho$ is a measure of weak isospin breaking.
If $SU(2)_L$ were unbroken then $\Delta\rho = 0$.

$\Delta\rho$ is also related to the $\rho$-parameter.

Consider

$$\rho_{NC} = \frac{G_{NC}}{G_F}$$

We found that

$$\frac{\delta G_F}{G_F} = -\frac{A_{WW}(0)}{m_W^2} + \text{vertex} + \text{box}$$

Likewise,

$$\frac{\delta G_{NC}}{G_{NC}} = -\frac{A_{ZZ}(0)}{m_Z^2} + \text{vertex} + \text{box}$$

So,

$$\frac{\delta\rho_{NC}}{\rho_{NC}} = \frac{\delta G_{NC}}{G_{NC}} - \frac{\delta G_F}{G_F}$$

$$= -\Delta\rho + \cdots$$

The minus sign is a result of my conversions. The tree-level $\rho$-parameter is 1, so I must write:

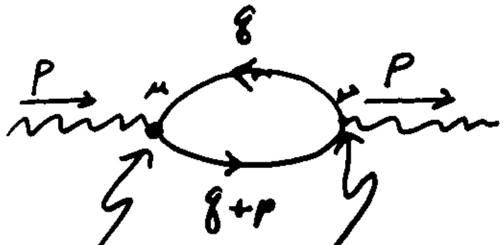
$$1 = \rho_{NC} + \delta\rho_{NC}$$

whereas

$$\rho_{NC} = 1 + \Delta\rho$$

# Large $m_t$ contribution to $\Delta\rho$

For fermion loops, $A_{Z\gamma}(0)=0$. So all we need to evaluate is:




$$\frac{-ig\gamma_\mu}{2}(G_L P_L + G_R P_R)$$

$$P_L = \tfrac{1}{2}(1-\gamma_5)$$
$$P_R = \tfrac{1}{2}(1+\gamma_5)$$

Compute using dimensional regularization. $n = 4-2\varepsilon$ dimensions

$$i\Pi_{\mu\nu}(p) = (-1)\int \frac{d^n q}{(2\pi)^n}\, \mathrm{Tr}\left\{ \frac{ig}{2}\gamma_\mu (G_L P_L + G_R P_R)\frac{i(\not{q}+m_{q_1})}{q^2-m_{q_1}^2} \right.$$

fermion loop factor

$$\left. \times \frac{ig}{2}\gamma_\nu (G_L P_L + G_R P_R)\frac{i(\not{q}+\not{p}+m_{q_2})}{(q+p)^2-m_{q_2}^2} \right\}$$

$$= -\frac{1}{4}g^2 \int \frac{d^n q}{(2\pi)^n}\, \frac{\mathrm{Tr}[\gamma_\mu (G_L P_L + G_R P_R)(\not{q}+m_{q_1})(G_L P_L + G_R P_R)(\not{q}+\not{p}+m_{q_2})}{(q^2-m_{q_1}^2)[(q+p)^2-m_{q_2}^2]}$$

$$\frac{1}{4}\mathrm{Tr}[\cdots] = m_{q_1} m_{q_2} G_L G_R\, g_{\mu\nu}$$
$$+ \frac{1}{2}(G_L^2+G_R^2)(2q_\mu q_\nu - g_{\mu\nu}q^2 + q_\mu p_\nu + q_\nu p_\mu - g_{\mu\nu}\, q\cdot p)$$
$$+ \varepsilon_{\mu\nu\alpha\beta}\, q^\alpha p^\beta (\cdots\cdot)$$

For the calculation of $\Delta\rho$, we need to extract the term proportional to $g_{\mu\nu}$ and evaluate it at $p^2=0$.

In particular, noting that

$$\int d^n q\, q_\mu q_\nu f(q^2) = \frac{1}{n} g_{\mu\nu} \int d^n q\, q^2 f(q^2)$$

a little algebra leads to:

$$i\Pi_{\mu\nu}(0) = -g^2 g_{\mu\nu} \Big\{ m_{\tilde{q}_1} m_{\tilde{q}_2} G_L G_R \int \frac{d^n q}{(2\pi)^n} \frac{1}{(q^2-m_{\tilde{q}_1}^2)(q^2-m_{\tilde{q}_2}^2)}$$

$$+ \frac{1}{2}(G_L^2+G_R^2)\left(\frac{2}{n}-1\right) \int \frac{d^n q}{(2\pi)^n} \frac{q^2}{(q^2-m_{\tilde{q}_1}^2)(q^2-m_{\tilde{q}_2}^2)} \Big\}$$

$$+ q_\mu q_\nu (\cdots)$$

The rest is algebra (with some help from Feynman's trick).
Some of the steps:

$$\int \frac{d^n q}{(2\pi)^n} \frac{1}{(q^2-m_{\tilde{q}_1}^2)(q^2-m_{\tilde{q}_2}^2)} = \int_0^1 dx \int \frac{d^n q}{(2\pi)^n} \frac{1}{\left[q^2 - x m_{\tilde{q}_1}^2 - (1-x) m_{\tilde{q}_2}^2\right]^2}$$

$\varepsilon \equiv 2 - \frac{n}{2}$

$$= \frac{i}{16\pi^2} (4\pi)^\varepsilon \Gamma(\varepsilon) \int_0^1 dx \left[x m_{\tilde{q}_1}^2 + (1-x) m_{\tilde{q}_2}^2\right]^{-\varepsilon}$$

$$= \frac{i}{16\pi^2} \left[\Delta - \int_0^1 dx \ln\left[x m_{\tilde{q}_1}^2 + (1-x) m_{\tilde{q}_2}^2\right] + O(\varepsilon)\right]$$

$$\Delta \equiv (4\pi)^\varepsilon \Gamma(\varepsilon) \simeq \frac{1}{\varepsilon} - \gamma + \ln 4\pi + O(\varepsilon)$$

At the end of the day, two integrals emerge:

$$f(m_1^2, m_2^2) \equiv \int_0^1 dx \ln\left[x m_1^2 + (1-x) m_2^2\right]$$

$$= \frac{1}{m_1^2 - m_2^2}\left[m_1^2 \ln m_1^2 - m_2^2 \ln m_2^2\right] - 1$$

$$g(m_1^2, m_2^2) \equiv 2\int_0^1 \left[x m_1^2 + (1-x) m_2^2\right] \ln\left[x m_1^2 + (1-x) m_2^2\right] dx$$

$$= \frac{m_1^4 \ln m_1^2 + m_2^4 \ln m_2^2}{m_1^2 - m_2^2} - \frac{1}{2}(m_1^2 + m_2^2)$$

Be careful to write:

$$\frac{2}{n} - 1 = -\frac{1}{2}\left(1 - \frac{\varepsilon}{2}\right)$$

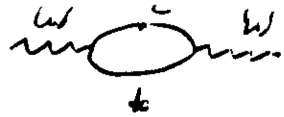
since $\varepsilon \Gamma(\varepsilon) = \Gamma(1+\varepsilon) = 1 + O(\varepsilon)$.

<u>End result:</u>

$$i\Pi_{\mu\nu}(0) = \frac{-i g^2 g_{\mu\nu}}{16\pi^2}\left\{ m_{\chi_1} m_{\chi_2} G_L G_R \left(\Delta - f(m_{\chi_1}^2, m_{\chi_2}^2)\right) - \frac{1}{4}(G_L^2 + G_R^2)\left[(m_1^2 + m_2^2)\Delta - g(m_1^2, m_2^2)\right]\right\} + \ldots$$

$$= -i g_{\mu\nu} A(0) + \ldots$$

## Wtb loop



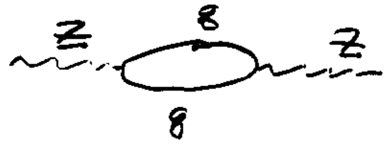

$G_L = \sqrt{2}$

$G_R = 0$

$N_c = 3$ colors

$$A_{WW}(0) = \frac{-g^2 N_c (m_t^2 + m_b^2)}{32\pi^2}\left[\Delta + \frac{1}{2} - \frac{m_t^4 \ln m_t^2 - m_b^4 \ln m_b^2}{m_t^4 - m_b^4}\right]$$

## $Zq\bar{q}$ loop $(q = t, b)$




Since $m_{q_1} = m_{q_2}$, we recompute:

$$f(m^2, m^2) = \ln m^2$$

$$g(m^2, m^2) = 2m^2 \ln m^2$$

$$A_{ZZ}(0) = \frac{-g^2 N_c}{16\pi^2}\left[G_L G_R m^2(\Delta - \ln m^2) - \frac{1}{2}(G_L^2 + G_R^2) m^2 (\Delta - \ln m^2)\right]$$

$$= \frac{-g^2 (G_L - G_R)^2 N_c m^2 (\Delta - \ln m^2)}{32\pi^2}$$

But $G_L - G_R = \dfrac{\pm 1}{\cos\theta_W}$ $\quad$ $+$ for $b$, $-$ for $t$

So,

$$A_{ZZ}^b(0) + A_{ZZ}^t(0) = \frac{-g^2 N_c}{32\pi^2 \cos^2\theta_W}\left[m_b^2(\Delta - \ln m_b^2) + m_t^2(\Delta - \ln m_t^2)\right]$$

$$\Delta \rho = \frac{A_{ZZ}(0)}{m_Z^2} - \frac{A_{WW}(0)}{m_W^2} + \cancel{\frac{2s}{c}\frac{A_{Z\gamma}(0)}{m_Z^2}}^{\,0}$$

Combining our results, we see that $\Delta$ cancels exactly. Thus, the end result is <u>finite</u>.

$$\boxed{\Delta \rho = \frac{g^2 N_c}{64\pi^2 m_W^2}\left[m_t^2 + m_b^2 - \frac{2m_t^2 m_b^2}{m_t^2 - m_b^2}\ln\left(\frac{m_t^2}{m_b^2}\right)\right]}$$

<u>Notes:</u>

1. For $m_t = m_b$, $\Delta \rho = 0$

   In this limit, there is no weak isospin breaking from a $\binom{t}{b}$ doublet.

2. For $m_t \neq m_b$, $\Delta \rho > 0$

3. For $m_b = 0$,

$$\Delta \rho = \frac{3g^2 m_t^2}{64\pi^2 m_W^2} = \frac{3\alpha m_t^2}{16 s^2 c^2 m_Z^2}$$

as previously quoted.

Start with:

$$\Delta r = F_{\gamma\gamma}(0) - \frac{2s}{c}\frac{A_{Z\gamma}(0)}{m_Z^2} - \frac{c^2}{s^2}\left(\frac{A_{ZZ}(0)}{m_Z^2} - \frac{A_{WW}(0)}{m_W^2}\right)$$

$$- \frac{c^2}{s^2}\Big(Re\,(F_{ZZ}(m_Z^2) - F_{WW}(m_W^2))\Big) - Re\,F_{WW}(m_W^2)$$

$$+ \text{vertex} + \text{box}$$

With only a few exceptions, new physics, which can contribute virtually through loops, does not enter in the vertex + box pieces. They can couple to $W, Z, \gamma$ so they will enter the remaining terms via the propagator (sometimes called the "oblique") corrections.

## Definitions

$$\varepsilon_1 \equiv F_{\gamma\gamma}(0) - F_{\gamma\gamma}(m_Z^2)$$

$$\varepsilon_2 \equiv \tilde{F}_{WW}(m_W^2) - \tilde{F}_{33}(m_Z^2)$$

$$= \tilde{F}_{WW}(m_W^2) - c^2\tilde{F}_{ZZ}(m_Z^2) - 2sc\,F_{Z\gamma}(m_Z^2) - s^2 F_{\gamma\gamma}(m$$

$$\varepsilon_3 = \frac{1}{s}F_{3\gamma}(m_Z^2) - \tilde{F}_{33}(m_Z^2)$$

$$= c^2 F_{\gamma\gamma}(m_Z^2) + \frac{c}{s}(1-2s^2)F_{Z\gamma}(m_Z^2) - c^2\tilde{F}_{ZZ}(m_Z^2)$$

[where $\tilde{F}_{WW}(m_W^2) \equiv Re\,F_{WW}(m_W^2)$, etc.]

Note: only $F_{WW}$ and $F_{ZZ}$ have imaginary parts, corresponding to "real" intermediate sta

A little algebra yields:

$$\varepsilon_1 + \frac{1-2s^2}{s^2}\varepsilon_2 + 2\varepsilon_3 = F_{\gamma\gamma}(0) - \frac{c^2}{s^2}\left(\tilde{F}_{ZZ}(m_Z^2) - \tilde{F}_{WW}(m_W^2)\right) - \tilde{F}_{WW}(m_W^2)$$

In addition, new physics contributions contribute nothing to $A_{Z\gamma}(0)$ due to gauge invariance, so we drop it.

Thus,

$$\boxed{\Delta r = \Delta\alpha - \frac{c^2}{s^2}\Delta\rho + \left(\frac{1-2s^2}{s^2}\right)\varepsilon_2 + 2\varepsilon_3}$$

Peskin and Takeuchi introduced $S, T, U$:

$$\alpha T \equiv \Delta\rho$$

$$\frac{g^2}{16\pi} S \equiv \varepsilon_3$$

$$\frac{-g^2}{16\pi} U \equiv \varepsilon_2$$

and so,

$$\Delta r = \frac{g^2}{8\pi}\left[S - 2c^2 T + \left(\frac{2s^2-1}{2s^2}\right)U\right] + \Delta\alpha$$

In Peskin and Takeuchi's approximation, they ignored the momentum dependence of the $F_{ij}(q)$, although one need not do so here.

# A few results regarding S, T, U

1. S, T, U are finite. All divergence cancel out exactly.

2. Non-decoupling

   We have already seen non-decoupling in action in our result for $\Delta\rho$. One finds a related behavior in S. Consider adding a fourth generation of fermions $\begin{pmatrix} U \\ D \end{pmatrix}$ with charges $\begin{Bmatrix} e_D+1 \\ e_D \end{Bmatrix}$.

Then:

$$S_{new} = \frac{N_c}{6\pi}\left[1 + (1+2e_D)\ln\frac{m_D^2}{m_U^2}\right]$$

$$U_{new} \simeq 0.$$

## Analysis of LEP electroweak data (Erler)

$$S_{new} = -0.10 \pm 0.14$$

$$T_{new} = -0.09 \pm 0.15$$

$$U_{new} = 0.13 \pm 0.24$$

LIMITS ON THE VIRTUAL CONTRIBUTIONS OF NEW PHYSICS

assuming $m_{H^0} = 100$ GeV (with $\ln m_H^2$ sensitivity to changes)

Erler and Pierce