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I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

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INTRODUCTION TO SUSY

A. BRIGNOLE
CERN
GENEVA
SWITZERLAND

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Globally Supersymmetric Theories in Four and Two Dimensions

Jean-Pierre Derendinger

Institut für Theoretische Physik
E. T. H. - Hönggerberg
8093 Zürich, Switzerland

These lecture notes provide a detailed and self-contained discussion of globally supersymmetric field theories in four and two space-time dimensions, starting at an elementary level. The aim is to give a rather complete presentation of the theoretical background used in the construction of unified supersymmetric models of particle interactions, with many technical and computational details. The emphasis is more on the algebraic aspects than on quantum field theoretical problems. For instance, the superfield formalism is developed and extensively used, but its applications to the calculation of perturbative quantum corrections (superfield Feynman rules and diagrams) are not presented. Supergravity theories (local supersymmetry), or superstrings are however not discussed.

Only a limited set of references directly related to the material discussed in each chapter is given. The intention is not to provide a survey of the published literature, or to give complete historical credits, but only to indicate some useful complements to the notes.

The organization of the notes is as follows:

Chapter 1

Introduction

Symmetries play a crucial role in the theory of strong, weak and electromagnetic interactions of elementary particles. The Standard (Glashow, Salam, Weinberg) Model is a local gauge theory, which is fully defined once the gauge group $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ and the transformations of the fundamental fields (quarks, leptons, Higgs bosons) are given. The symmetry structure of the theory is of the form $P \times G$, where P is the Poincaré group, containing global space-time translations P^μ and Lorentz rotations generated by operators $M^{\mu\nu} = -M^{\nu\mu}$, and G is the gauge group. Because of the direct product, G is a purely internal symmetry. It commutes with transformations of the Poincaré group. As long as G is a compact Lie group, one can construct a consistent quantum field theory preserving invariance under local transformations of G.

This scheme is fully satisfactory as long as one does not try to incorporate gravitational interactions. According to general relativity, the symmetry principle which governs gravity is invariance under general (local) coordinate transformations. Gravity can be viewed as a gauge theory of the Poincaré group. It is however impossible to construct a consistent quantum theory of gravitation along these lines. Such a theory is desperately non renormalizable. Notice also that the Poincaré group is not compact. The commutation relations of its algebra are

$$\begin{aligned}
[M^{\mu\nu}, M^{\rho\sigma}] &= -i\,(\eta^{\mu\rho}M^{\nu\sigma} + \eta^{\nu\sigma}M^{\mu\rho} - \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\rho}M^{\mu\sigma}), \\
[P^\mu, M^{\nu\rho}] &= i\,(\eta^{\mu\nu}P^\rho - \eta^{\mu\rho}P^\nu), \\
[P^\mu, P^\nu] &= 0, \qquad (\mu,\nu,\rho,\sigma = 0,1,2,3)
\end{aligned} \tag{1.1}$$

where the flat space-time metric is

$$\eta^{\mu\nu} = diag(+1,-1,-1,-1). \tag{1.2}$$

The commutation relations of Lorentz generators $M^{\mu\nu}$ correspond to the pseudo-orthogonal algebra $SO(1,3)$.

The presence of the internal symmetries contained in G does not help to cure the problem of quantum gravity. The generators T^a of G satisfy

$$\begin{aligned}
[T^a, P^\mu] &= [T^a, M^{\mu\nu}] = 0, \\
[T^a, T^b] &= i f^{abc} T^c,
\end{aligned} \tag{1.3}$$

where f^{abc} are the (antisymmetric) structure constants of the compact group G.

Since symmetries have proved to be so important in the description of strong, weak and electromagnetic interactions, one may be tempted to look for enlarged symmetry structures, incorporating the Poincaré group and an internal gauge group in a less trivial structure than a direct product. Specifically, one tries to find new symmetries Q^i such that for instance

$$[Q^i, M^{\mu\nu}] \neq 0, \tag{1.4}$$

which means that the generators of these new symmetries have non trivial Lorentz transformations, and that they have a spin different from zero. They will then relate *particles of different spins.*

The symmetries Q^i fall naturally in two classes: integer spin symmetries (bosonic symmetries) or half-integer (fermionic) symmetries. These two classes will lead to different algebraic structures. What we are interested in is to find symmetries of a possibly interacting quantum field theory. According to the Noether theorem, these symmetries will correspond to conserved currents and will be generated by charges which are space integrals of some combination of fields and their derivatives. It is then natural to expect that a bosonic charge is a space integral of a bosonic local field, while a fermionic charge is a space integral of fermionic local fields. But canonical quantization prescribes commutators of bosonic fields and anticommutators of fermionic fields. One then infers that commutators $[Q^i, Q^j]$ of integer spin symmetries will be determined, leading to a Lie algebra extending the $P \times G$ structure. On the contrary, fermionic symmetries should be characterized by anticommutation rules $\{Q^i, Q^j\}$, leading to a new algebraic structure called *Lie superalgebra* [1] (which is a particular class of graded Lie algebras).

There exists however a no-go theorem concerning bosonic symmetries. The Coleman-Mandula theorem [2] forbids the existence, in a relativistic, interacting quantum theory with a discrete spectrum of massive one-particle states, of any conserved charges that are not Lorentz scalars, other than those belonging to the Poincaré group. This theorem does not apply to fermionic symmetries since it holds only for Lie algebras. In the completely massless case however, the Poincaré group can be extended to the conformal group.

One is then led to consider new fermionic symmetries Q^i, for which

$$[M^{\mu\nu}, Q^i] = (b^{\mu\nu} Q)^i, \tag{1.5}$$

where $b^{\mu\nu}$ is a matrix specifying a spinorial (half-integer spin) representation of the Lorentz group and

$$\{Q^i, Q^j\} = T^{ij}, \tag{1.6}$$

where T^{ij} is some combination of Poincaré generators and possibly some internal (spin zero) symmetry generators. The only relevant case turns out to correspond to spin 1/2 charges Q^i, and it is the case of the supersymmetry algebra. Supersymmetry transformations are of the form

$$\begin{aligned} \delta(\text{boson, spin } s) &= (\text{fermion, spin } s \pm 1/2), \\ \delta(\text{fermion, spin } s') &= (\text{boson, spin } s' \pm 1/2). \end{aligned} \tag{1.7}$$

The supersymmetry algebra is then the only possible non trivial unification of internal and space-time symmetries compatible with quantum field theory.

Let us conclude this introduction by a brief historical note. The supersymmetry algebra, as a possible physical extension of the Poincaré group was discovered in 1971 by Gol'fand and Likhtman [3]. It was also found in 1971 by Ramond, Neveu and Schwarz [4] that fermionic strings possess a two-dimensional (world-sheet) supersymmetry. Supersymmetry, as an invariance of field theory (in four dimensions) was introduced in 1973 by Volkov and Akulov [5], in an attempt to understand the apparent absence of mass of neutrinos by assuming that they are Goldstone particles. In this article, supersymmetry was non linearly realized. The first field theories with linear supersymmetry were discovered by Wess and Zumino in a series of articles [6] which have built the foundations of all subsequent developments of supersymmetric field theories.

One of the main motivation for the study of supersymmetric theories is that they could bring new insights on the unification of strong, weak and electromagnetic interactions with gravity and on the difficulties of quantum gravity. This however requires that one finds theories invariant under *local*, and not only *global* supersymmetry transformations. Locally supersymmetric theories are called *supergravities*, and have been invented by Freedman, Ferrara and van Nieuwenhuizen [7], and also by Deser and Zumino [8]. It is now admitted that all supergravity theories are non renormalisable: quantizing gravity destroys their consistency. The expectations of solving the problems of quantum gravity with the help of supersymmetry have not been fulfilled, in the framework of quantum field theory, at least. At present, only superstring theories seem to describe quantum gravitation in a consistent way. The symmetry structure of these theories contains however always a supersymmetry algebra, and the interactions of the light string states corresponding hopefully to the states of the Standard Model are described by an effective supergravity Lagrangian, or, at low energy, by a softly broken globally supersymmetric theory.

Chapter 2

The Wess-Zumino model

The Wess-Zumino model [6] is the simplest example of a supersymmetric field theory. It describes the dynamics of n spin 1/2 fields and n complex scalar fields. A complex scalar field ϕ contains two real components A and B, defined by $\phi = (A+iB)/\sqrt{2}$. In general, A will behave under parity transformations $P:(x^0,x^i) \rightarrow (x^0,-x^i)$ like a scalar field, while B will be a pseudoscalar:

$$\begin{aligned} A &\xrightarrow{P} A, \\ B &\xrightarrow{P} -B. \end{aligned} \tag{2.1}$$

The free Lagrangian density for scalar fields is

$$\begin{aligned} \mathcal{L}_\phi &= (\partial_\mu\phi)^\dagger(\partial^\mu\phi) - m^2\phi^\dagger\phi \\ &= \frac{1}{2}\left[(\partial_\mu A)(\partial^\mu A) + (\partial_\mu B)(\partial^\mu B) - m^2(A^2+B^2)\right], \end{aligned} \tag{2.2}$$

and the corresponding equation of motion is the Klein-Gordon equation.

A spin 1/2 particle is described by a Dirac spinor ψ which, when subject to the Dirac equation, contains four real components describing the two helicity states of the particle and of its antiparticle. The spin 1/2 Lagrangian is

$$\mathcal{L}_\psi = \overline{\psi}\,(i\gamma^\mu\partial_\mu - m)\,\psi \tag{2.3}$$

when interactions are switched off. One can reduce the number of components to two by imposing either a Weyl or a Majorana condition. A Weyl spinor has a definite helicity: the two components of a left- (right-) handed spinor will be the left (right) helicity of the particle, and the right (left) helicity of the antiparticle. A Weyl spinor ψ_L or ψ_R satisfies

$$\psi_L = L\psi_L \quad , \quad \psi_R = R\psi_R \quad , \tag{2.4}$$

where

$$L = \frac{1}{2}(1+\gamma_5) \quad , \quad R = \frac{1}{2}(1-\gamma_5) \tag{2.5}$$

are the helicity projectors. Notice that a Weyl spinor is always massless. The Majorana condition corresponds to

$$\psi = \psi_c = C\overline{\psi}^T \tag{2.6}$$

where C is the charge conjugation matrix (see appendix A for conventions). This condition means that the particle is identical to the antiparticle.

A general property of all supersymmetric theories is that the number of bosonic states is always identical to the number of fermionic states (see ch. 3). Thus, to each complex scalar field ϕ, there will correspond a two-component spinor (Weyl or Majorana), forming a chiral multiplet which is the simplest supersymmetric multiplet.

In the next two sections, we will study in more details the two simplest cases, the free Wess-Zumino model for one chiral multiplet with a Weyl or a Majorana spinor. In each case, we will first write the most general transformation rules of the form

$$\begin{aligned} \delta(\text{scalar } \phi) &= O_\phi(\text{spinor } \psi), \\ \delta(\text{spinor } \psi) &= O_\psi(\text{scalar } \phi), \end{aligned} \tag{2.7}$$

where O_ϕ and O_ψ are the two spin 1/2 operators containing an infinitesimal spinorial transformation parameter ϵ. Recalling that the canonical dimension (in units of mass, $c=1$) of a Dirac spinor is 3/2 (i.e. ψ has dimension $(\text{mass})^{3/2}$) while a scalar has dimension 1 (see the Lagrangian densities 2.2 and 2.3: they have dimension four since the action $S=\int d^4x\mathcal{L}$ is a number), one finds:

$$\begin{aligned} dim[O_\phi] &= -1/2, \\ dim[O_\psi] &= +1/2. \end{aligned} \tag{2.8}$$

The parameter ϵ will turn out to have dimension $-1/2$. These three numbers are quite useful in writing the most general supersymmetry transformations for specific cases.

One could in principle take different masses for the scalar and the spin 1/2 states. This would however turn out to be incompatible with invariance under supersymmetry. Invariance of the Lagrangian will force the two masses to be identical (and in the Weyl case to vanish). This is also a general property of all supersymmetric theories (see ch. 3): all states belonging to a supersymmetric multiplet have the same mass. We will accept this result when establishing the supersymmetry transformations in the next section.

2.1. The Weyl case

The fields we consider are a complex scalar field ϕ and a left-handed spinor ψ_L satisfying

$$\psi_L = \frac{1}{2}(1+\gamma_5)\psi_L = L\psi_L. \tag{2.9}$$

Since $\overline{\psi}_L\psi_L = 0$, the spin 1/2 state is massless and supersymmetry will be possible only if the scalar state is also massless. We then consider the Lagrangian

$$\mathcal{L} = (\partial_\mu\phi)^\dagger(\partial^\mu\phi) + \overline{\psi_L}(i\gamma_\mu\partial^\mu)\psi_L. \tag{2.10}$$

Let us start by writing the most general transformation of the scalar ϕ. It is of the form $\bar{\epsilon}M\psi$, where M is a dimensionless matrix. Using Lorentz invariance and the fact that $\gamma_5\psi_L = \psi_L$, one gets that M is in fact proportional to the identity matrix. Absorbing the proportionality constant in the transformation parameter ϵ, one can write (with a factor $\sqrt{2}$ which will become clear later on)

$$\begin{aligned}\delta\phi &= \sqrt{2}\,\bar{\epsilon}\psi_L,\\ \delta\phi^\dagger &= \sqrt{2}\,\overline{\psi_L}\epsilon.\end{aligned} \tag{2.11}$$

The transformation of the spinor is more complicated. $\delta\psi_L$ should be a left-handed spinor. This is possible only if ϵ is also a Weyl spinor. Since we want to avoid $\delta\phi = 0$, Eq. (2.11) forces ϵ to be right-handed:

$$\epsilon = \epsilon_R. \tag{2.12}$$

The transformation $\delta\psi_L$ is then necessarily of the form (a and b are complex numbers):

$$\delta\psi_L = \sqrt{2}\left[a(\gamma^\mu\partial_\mu\phi) + b(\gamma^\nu\partial_\nu\phi^\dagger)\right]\epsilon_R. \tag{2.13}$$

This expression is the most general left-handed spinor of dimension 3/2 such that $L\delta\psi_L = \delta\psi_L$ and $-\gamma_5\epsilon_R = \epsilon_R$. Transforming the bosonic part of the Lagrangian (2.10) with Eq. (2.11), one obtains

$$\delta\mathcal{L}_{BOSE} = \sqrt{2}\left[(\partial_\mu\overline{\psi_L})\epsilon_R(\partial^\mu\phi) + (\partial_\mu\phi^\dagger)\overline{\epsilon_R}(\partial^\mu\psi_L)\right]. \tag{2.14}$$

From the transformation of the fermionic Lagrangian, it is apparent that since there is no term with $\phi^\dagger$ and $\overline{\psi_L}$ (or with ϕ and ψ_L) in Eq. (2.14), one should choose $b = 0$ in transformation (2.13). Requiring the invariance up to total derivatives of the Lagrangian will determine the value of the only free parameter, a. The transformation of the spin 1/2 Lagrangian can be written

$$\begin{aligned}\frac{1}{\sqrt{2}}\delta\left(\overline{\psi_L}i\gamma^\mu\partial_\mu\psi_L\right) =& ia^*\partial_\mu\left[\overline{\epsilon_R}\phi^\dagger\gamma^\mu(\gamma^\nu\partial_\nu\psi_L)\right]\\ &- ia^*\overline{\epsilon_R}\phi^\dagger\gamma^\mu\gamma^\nu(\partial_\mu\partial_\nu\psi_L)\\ &+ ia\overline{\psi_L}\gamma^\mu\gamma^\nu(\partial_\mu\partial_\nu\phi)\epsilon_R.\end{aligned} \tag{2.15}$$

This allows to eliminate the γ^μ matrices, using

$$\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu = \frac{1}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\mu\partial_\nu = \partial_\mu\partial^\mu. \tag{2.16}$$

Rearranging terms, one finally gets

$$\begin{aligned}\frac{1}{\sqrt{2}}\delta\left(\overline{\psi_L}i\gamma^\mu\partial_\mu\psi_L\right) =& ia^*\partial_\mu\left[\overline{\epsilon_R}\phi^\dagger\gamma^\mu(\gamma^\nu\partial_\nu\psi_L)\right]\\ &- ia^*\partial_\mu\left[\overline{\epsilon_R}\phi^\dagger(\partial^\mu\psi_L)\right]\\ &+ ia\partial_\mu\left[\overline{\psi_L}(\partial^\mu\phi)\epsilon_R\right]\\ &+ ia^*\overline{\epsilon_R}(\partial_\mu\phi^\dagger)(\partial^\mu\psi_L)\\ &- ia(\partial_\mu\overline{\psi_L})(\partial^\mu\phi)\epsilon_R.\end{aligned} \tag{2.17}$$

The last two terms cancel the transformation of the bosonic Lagrangian, Eq. (2.14), provided $a = -i$. The field theory is then invariant, up to total derivatives, under the supersymmetry transformations:

$$\begin{aligned}\delta\phi &= \sqrt{2}\,\overline{\epsilon_R}\psi_L, & \delta\phi^\dagger &= \sqrt{2}\,\overline{\psi_L}\epsilon_R,\\ \delta\psi_L &= -i\sqrt{2}\,\gamma^\mu\partial_\mu\phi\epsilon_R, & \delta\overline{\psi_L} &= i\sqrt{2}\,\overline{\epsilon_R}\gamma^\mu\partial_\mu\phi^\dagger.\end{aligned} \tag{2.18}$$

To close this section, let us compute the action on ϕ and ψ_L of the commutator of two supersymmetries (with parameters ϵ_R^1 and ϵ_R^2). One finds:

$$\begin{aligned}[\delta_1,\delta_2]\phi &= \delta_1(\delta_2\phi) - \delta_2(\delta_1\phi)\\ &= -2i\left(\overline{\epsilon_R}^2\gamma^\mu\epsilon_R^1 - \overline{\epsilon_R}^1\gamma^\mu\epsilon_R^2\right)\partial_\mu\phi\end{aligned} \tag{2.19}$$

and

$$[\delta_1,\delta_2]\psi_L = -2i\left(\overline{\epsilon_R}^2\gamma^\mu\epsilon_R^1 - \overline{\epsilon_R}^1\gamma^\mu\epsilon_R^2\right)\left[\partial_\mu\psi_L - \frac{1}{2}\gamma_\mu(\gamma^\nu\partial_\nu\psi_L)\right]. \tag{2.20}$$

To obtain this last formula, one must perform Fierz rearrangements, like

$$(\overline{\epsilon_R}^1\partial_\mu\psi_L)\gamma^\mu\epsilon_R^2 = -\frac{1}{2}(\overline{\epsilon_R}^1\gamma^\mu\epsilon_R^2)\gamma^\nu\gamma_\mu\partial_\nu\psi_L, \tag{2.21}$$

for anticommuting spinors (see appendix A). Thus, for on-shell fields ($\gamma^\mu\partial_\mu\psi_L = 0$), the commutator of two supersymmetries is a translation:

$$[\delta_1,\delta_2]\begin{pmatrix}\phi\\ \psi_L\end{pmatrix} = i\Delta_\mu P^\mu\begin{pmatrix}\phi\\ \psi_L\end{pmatrix} = \Delta_\mu\partial^\mu\begin{pmatrix}\phi\\ \psi_L\end{pmatrix} \tag{2.22}$$

with

$$\Delta_\mu = -2i(\overline{\epsilon_R}^2\gamma_\mu\epsilon_R^1 - \overline{\epsilon_R}^1\gamma_\mu\epsilon_R^2). \tag{2.23}$$

(Recall that translations P^μ are generated by $-i\partial^\mu$, since $\phi(x+\Delta) = (1 + i\Delta_\mu P^\mu)$ $\phi(x) = \phi(x) + \Delta_\mu \partial^\mu \phi(x)$ to first order). On-shell, the commutator of two supersymmetries is then a Poincaré transformation and the algebra closes. It does not close off-shell, because of the additional term in Eq. (2.20). Notice finally that Eqs. (2.19) and (2.20) justify the assertion that the spinorial parameter ϵ has dimension -1/2.

2.2. The Majorana case

We now turn to the other case where the spinor field ψ satisfies a Majorana condition (2.6). (A number of useful identities for Majorana spinors are given in appendix A). It will prove easier to split the complex scalar ϕ into its real components A (scalar) and B (pseudoscalar). The reason is that the free Majorana theory is parity-conserving, and supersymmetry transformations should respect this invariance: δA (δB) should then behave like a scalar (a pseudoscalar) under parity.

The Lagrangian we consider is

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}(\partial_\mu A)(\partial^\mu A) + \frac{1}{2}(\partial_\mu B)(\partial^\mu B) - \frac{1}{2}m^2(A^2 + B^2)\\ &+ \frac{1}{2}\overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi.\end{aligned} \tag{2.24}$$

The factor 1/2 in front of the Dirac Lagrangian avoids repetition of terms when the explicit form of a Majorana spinor is inserted (see appendix A, Eq. A17). We have chosen a common mass for spin 0 and spin 1/2 states, in order to have supersymmetry. Under parity, the fields behave according to

$$\begin{aligned} A &\xrightarrow{P} A,\\ B &\xrightarrow{P} -B,\\ \psi &\xrightarrow{P} \psi_P = \gamma^0\psi.\end{aligned}$$

The supersymmetry transformations of the scalars are then

$$\begin{aligned}\delta A &= \bar{\epsilon}\psi = \overline{\psi}\epsilon,\\ \delta B &= i\bar{\epsilon}\gamma_5\psi = i\overline{\psi}\gamma_5\epsilon,\end{aligned} \tag{2.25}$$

where ϵ is now a Majorana spinor. These transformations respect the parity behaviour:

$$\begin{aligned}\delta A &\xrightarrow{P} \delta A,\\ \delta B &\xrightarrow{P} -\delta B.\end{aligned}$$

They are deduced following the same line of reasoning as in the Weyl case. These transformations also preserve the reality of A and B (see appendix A). The bosonic part of the Lagrangian undergoes the supersymmetry transformation

$$\begin{aligned}\delta\mathcal{L}_{BOSE} =& \bar{\epsilon}(\partial^\mu\psi)(\partial_\mu A) + i\bar{\epsilon}\gamma_5(\partial^\mu\psi)(\partial_\mu B)\\ &- m^2(\bar{\epsilon}\psi)A - im^2(\bar{\epsilon}\gamma_5\psi)B,\end{aligned} \tag{2.26}$$

which must be compensated (up to total derivatives) by the transformation of the Dirac Lagrangian.

To find out how ψ transforms, we must first write the most general $\delta\psi$ consistent with linearity, Lorentz and parity invariances, the Majorana condition and the (canonical) dimension 3/2 of ψ. One finds

$$\delta\psi = [ia\gamma^\mu\partial_\mu A + b\gamma_5\gamma^\mu\partial_\mu B + m(cA + idB\gamma_5)]\epsilon, \tag{2.27}$$

where a, b, c, d are real constants to be determined with the help of the invariance of the Lagrangian. The behaviour under parity decides whether A or B appears in each term, by requiring

$$\delta\psi \xrightarrow{P} (\delta\psi)_P = \gamma^0\delta\psi.$$

The Majorana condition implies

$$(\delta\psi)_c = \delta\psi$$

or

$$i\gamma^2\delta\psi^* = \delta\psi$$

for our specific choice of γ-matrices (see appendix A, Eqs. A20 and A22). This condition determines the correct i factors.

The next step is to compute the transformation of the Dirac Lagrangian, using Eq. (2.27) and also

$$\delta\overline{\psi} = \bar{\epsilon}\,[-ia\gamma^\mu\partial_\mu A + b\gamma_5\gamma^\mu\partial_\mu B + m(cA + idB\gamma_5)] \tag{2.28}$$

and, in addition, properties of Majorana spinors (appendix A, Eq. A18). The terms containing m^2 cancel provided $c = d = -1$. Using the same rearrangement procedure as in the Weyl case, the terms without m cancel up to total derivatives if $a = b = -1$. Finally, the terms linear in m give a total derivative if $a = c$ and $b = d$, consistently with the other conditions. The supersymmetry transformations are then

$$\begin{aligned}\delta A &= \bar{\epsilon}\psi,\\ \delta B &= i\bar{\epsilon}\gamma_5\psi,\\ \delta\psi &= -[i\gamma^\mu\partial_\mu(A + iB\gamma_5) + m(A + iB\gamma_5)]\epsilon.\end{aligned} \tag{2.29}$$

They leave the Lagrangian invariant up to total derivatives:

$$\begin{aligned}\delta\mathcal{L} = &-\frac{1}{2}\bar{\epsilon}\partial_\nu\left[(\gamma^\mu\partial_\mu A + i\gamma_5\gamma^\mu\partial_\mu B + im(A+iB\gamma_5))\gamma^\nu\psi\right]\\ &+\bar{\epsilon}\partial_\nu\left[(\partial^\nu A + i\gamma_5\partial^\nu B)\psi\right].\end{aligned} \tag{2.30}$$

Let us now check whether the supersymmetry algebra closes under commutation. It is straightforward to verify that

$$\begin{aligned}[\delta_1,\delta_2]A &= \delta_1(\delta_2 A) - \delta_2(\delta_1 A)\\ &= -2i(\bar{\epsilon}_2\gamma^\mu\epsilon_1)\partial_\mu A,\end{aligned} \tag{2.31.a}$$

$$[\delta_1,\delta_2]B = -2i(\bar{\epsilon}_2\gamma^\mu\epsilon_1)\partial_\mu B. \tag{2.31.b}$$

The same computation for the spinor ψ is more complicated. It involves again Fierz rearrangement of terms. One obtains:

$$[\delta_1,\delta_2]\psi = -2i(\bar{\epsilon}_2\gamma^\mu\epsilon_1)\left[\partial_\mu\psi + \frac{i}{2}\gamma_\mu(i\gamma^\nu\partial_\nu - m)\psi\right]. \tag{2.31.c}$$

For the fields A and B, the commutator of two supersymmetries is a translation by an amount Δ_μ, with

$$\Delta_\mu = -2i(\bar{\epsilon}_2\gamma_\mu\epsilon_1) = -i(\bar{\epsilon}_2\gamma_\mu\epsilon_1 - \bar{\epsilon}_1\gamma_\mu\epsilon_2), \tag{2.32}$$

analogous to the result of the Weyl case, Eq. (2.23). The same result will hold for the spinor ψ only provided it satisfies the equation of motion $(i\gamma^\mu\partial_\mu - m)\psi = 0$.

2.3. The auxiliary fields

The results obtained in the two previous sections are disturbing for two reasons: firstly, the supersymmetry algebra closes only on-shell, using the equations of motion. It is only under this condition that commuting two supersymmetries gives a translation P^μ, belonging to the Poincaré algebra. The second disturbing fact is that the transformation rule of a Majorana spinor (Eq. 2.29) contains a parameter of the Lagrangian (the mass m), while the algebra is independent of m. One would then expect that a representation of this algebra does not contain explicitely m in the transformation rules.

Another observation is related to the number of degrees of freedom of spinors. The chiral multiplet always contains two bosonic degrees of freedom (A and B), on-shell and off-shell. A spinor, constrained to be Majorana or Weyl, has two (real) components only on-shell, when it satisfies the Dirac equation. Off-shell, it contains

four degrees of freedom. The equality of the number of bosons and fermions, which we already mentioned as being a necessary condition for all supersymmetric multiplets is then broken off-shell. To construct a supersymmetric quantum field theory, it is of crucial importance to also possess off-shell representations of the algebra. This is achieved with the introduction of new fields, which restore the equality between bosons and fermions and have algebraic equations of motion. They do not propagate and do not bring new on-shell degrees of freedom. They are called auxiliary fields.

For the chiral multiplet, one clearly needs to add two new (real) scalar fields F and G. The number of degrees of freedom for the chiral multiplet is then

4 bosons (A,B,F,G) + 4 fermions (ψ) OFF-SHELL,
2 bosons (A,B) + 2 fermions (ψ) ON-SHELL.

The presence of the auxiliary fields brings the solution of the two problems we have mentioned previously. The transformation rules of ψ will now contain the auxiliary fields and the term linear in m, in Eq. (2.29), is obtained only after one has solved the equations of motion of the auxiliary fields.

To be more precise, let us impose that the equations of motion of F and G are

$$\begin{aligned}F &= mA,\\ G &= mB\end{aligned} \tag{2.33}$$

(F is a scalar and G a pseudoscalar). This is obtained using the Lagrangian

$$\mathcal{L}_{AUX} = \frac{1}{2}(F^2+G^2) - m(AF+BG). \tag{2.34}$$

Notice that F and G have dimension two. This Lagrangian, when F and G are replaced by mA and mB respectively, generates the mass terms for A and B:

$$\mathcal{L}_{AUX}\Big|_{\substack{F=mA\\ G=mB}} = -\frac{1}{2}m^2(A^2+B^2), \tag{2.35}$$

so that the Lagrangian we now consider is

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}\bar{\psi}i\gamma^\mu\partial_\mu\psi + \frac{1}{2}(\partial_\mu A)^2 + \frac{1}{2}(\partial_\mu B)^2 + \frac{1}{2}(F^2+G^2)\\ &- m(\frac{1}{2}\bar{\psi}\psi + AF + BG),\end{aligned} \tag{2.36}$$

instead of the original Lagrangian (2.24) for the Majorana case (which is recovered by inserting $F = mA$ and $G = mB$). The supersymmetry transformations will now be independent of m. Both the 'kinetic' part

$$\mathcal{L}_{KIN} = \frac{1}{2}\bar{\psi}i\gamma^\mu\partial_\mu\psi + \frac{1}{2}(\partial_\mu A)^2 + \frac{1}{2}(\partial_\mu B)^2 + \frac{1}{2}(F^2+G^2), \tag{2.37}$$

and the mass Lagrangian

$$\mathcal{L}_{MASS} = -m(\frac{1}{2}\bar{\psi}\psi + FA + BG) \qquad (2.38)$$

will be supersymmetric invariant (up to total derivatives).

To proceed, let us first modify the transformation of ψ in an obvious way:

$$\begin{aligned} \delta\psi &= -\left[i\gamma^\mu\partial_\mu(A + iB\gamma_5) + F + iG\gamma_5\right]\epsilon, \\ \delta\overline{\psi} &= -\bar{\epsilon}\left[i\gamma^\mu\partial_\mu(-A + iB\gamma_5) + F + iG\gamma_5\right], \end{aligned} \qquad (2.39)$$

instead of Eqs. (2.29). The transformations of A and B cannot get contributions from F and G which have dimension two. They are again as in Eq. (2.29). As before, the transformation of the kinetic terms for A and B is compensated by the transformation of the Dirac Lagrangian using only the F and G independent parts of $\delta\psi$ and $\delta\overline{\psi}$. The contribution of the F and G depending parts is

$$-F\bar{\epsilon}(i\gamma^\mu\partial_\mu\psi) + G\bar{\epsilon}\gamma_5(\gamma^\mu\partial_\mu\psi) + \frac{1}{2}i\partial_\mu\left[\bar{\epsilon}(F + iG\gamma_5)\gamma^\mu\psi\right]. \qquad (2.40)$$

The last term is a total derivative, but the two first terms need to be compensated by

$$\delta\left(\frac{1}{2}F^2 + \frac{1}{2}G^2\right) = F\delta F + G\delta G. \qquad (2.41)$$

Then, clearly:

$$\begin{aligned} \delta F &= i\bar{\epsilon}\gamma^\mu\partial_\mu\psi, \\ \delta G &= -\bar{\epsilon}\gamma_5\gamma^\mu\partial_\mu\psi. \end{aligned} \qquad (2.42)$$

One can easily check that δF and δG are real (see appendix A, Eq. A19) and that δF is scalar and δG pseudoscalar under parity. It is then straightforward to verify that $\mathcal{L}_{MASS}$ is also invariant. To summarize, the supersymmetry transformations of the chiral multiplet (with Majorana spinor ψ) are:

$$\begin{aligned} \delta A &= \bar{\epsilon}\psi, \\ \delta B &= i\bar{\epsilon}\gamma_5\psi, \\ \delta\psi &= -\left[i\gamma^\mu\partial_\mu(A + iB\gamma_5) + F + iG\gamma_5\right]\epsilon, \\ \delta F &= i\bar{\epsilon}\gamma^\mu\partial_\mu\psi, \\ \delta G &= -\bar{\epsilon}\gamma_5\gamma^\mu\partial_\mu\psi. \end{aligned} \qquad (2.43)$$

The supersymmetry algebra closes correctly when acting on F and G:

$$\begin{aligned} [\delta_1, \delta_2]F &= -2i(\bar{\epsilon}_2\gamma^\mu\epsilon_1)\partial_\mu F, \\ [\delta_1, \delta_2]G &= -2i(\bar{\epsilon}_2\gamma^\mu\epsilon_1)\partial_\mu G, \end{aligned} \qquad (2.44)$$

in agreement with Eqs (2.31.a-b). However, instead of (2.31.c), one now gets

$$[\delta_1, \delta_2]\psi = -2i(\bar{\epsilon}_2\gamma^\mu\epsilon_1)\partial_\mu\psi, \qquad (2.45)$$

so that the algebra closes completely off-shell.

An interesting remark is that the auxiliary fields always transform with a total derivative. This is related to the fact that F and G have the highest dimension in the chiral multiplet. This also means that the Lagrangian

$$\mathcal{L}_{LINEAR} = \mu F \qquad (\text{or } \mu G) \qquad (2.46)$$

is by itself supersymmetric. When added to the Lagrangian (2.36), this term modifies the equation of motion of F, which becomes:

$$F = mA - \mu. \qquad (2.47)$$

Such a linear term can be removed by a redefinition of A:

$$A' = A - \mu/m. \qquad (2.48)$$

This corresponds to add a constant chiral multiplet

$$(A = -\mu/m, B = 0, \psi = 0, F = 0, G = 0)$$

to the chiral multiplet (A, B, ψ, F, G). Such a constant chiral multiplet is invariant under supersymmetry transformations (2.43).

Let us stress again that with the introduction of auxiliary fields, the Lagrangian (2.36) is the sum of two supersymmetric parts: the 'kinetic' Lagrangian (2.37) and the 'mass' Lagrangian (2.38), without derivatives and quadratic in the fields. One can in fact obtain supersymmetric Lagrangians analogous to (2.38) but containing product of an arbitrary high number of fields. This will allow to introduce interactions in the Wess-Zumino model.

To close this section, let us consider the introduction of auxiliary fields in the case of Weyl spinors. Since there is no mass, the transformation of the spinor ψ_L does not require at first sight a modification. It does not depend on a parameter of the Lagrangian. However, the supersymmetry algebra does not close off-shell (see Eq. 2.20) and, of course, the number of bosons (two) does not match fermions (four) off-shell. We then introduce a complex auxiliary field f (2 bosons), and we would like its equation of motion to be simply

$$f = 0. \qquad (2.49)$$

This is achieved by adding to the original Lagrangian (2.10) a new term

$$\mathcal{L}_{AUX} = ff^\dagger. \qquad (2.50)$$

We also need to modify $\delta\psi_L$ (Eq. 2.18) to obtain a closed algebra off-shell. It is then necessary to promote ϵ to a Majorana spinor, so that one can write

$$\delta\psi_L = -\sqrt{2}\,L\,(i\gamma^\mu\partial_\mu\phi + f)\,\epsilon. \qquad (2.51)$$

In Section 2.1, we only needed a right-handed transformation parameter ϵ_R. But there is then no way to add a term containing f and transforming the left-handed ψ_L. Because of the presence of the projector L, the left-handed part of ϵ (which is related to the right-handed part by the Majorana condition) only appears in the new term of $\delta\psi_L$. This term generates a new contribution to $\delta\left(\overline{\psi}_L i\gamma^\mu\partial_\mu\psi_L\right)$, which can be written

$$\begin{aligned}&-\sqrt{2}\,f\left[-i(\partial_\mu\overline{\psi}_L)\gamma^\mu\epsilon\right] - \sqrt{2}\,f^\dagger\left[i\overline{\epsilon}\gamma^\mu\partial_\mu\psi_L\right]\\ &-i\sqrt{2}\,\partial_\mu\left[\overline{\psi}_L\gamma^\mu f^\dagger\epsilon_L\right].\end{aligned} \qquad (2.52)$$

These terms must be compensated (apart from the total derivative) by

$$\delta\mathcal{L}_{AUX} = f(\delta f^\dagger) + f^\dagger(\delta f), \qquad (2.53)$$

so that

$$\begin{aligned}\delta f &= i\sqrt{2}\,\overline{\epsilon}\gamma^\mu\partial_\mu\psi_L,\\ \delta f^\dagger &= -i\sqrt{2}\,(\partial_\mu\overline{\psi}_L)\gamma^\mu\epsilon.\end{aligned} \qquad (2.54)$$

Only the left-handed part of ϵ enters in δf. One can easily check that

$$[\delta_1,\delta_2]f = -2i\,(\overline{\epsilon}_2\gamma^\mu L\epsilon_1 - \overline{\epsilon}_1\gamma^\mu L\epsilon_2)\,\partial_\mu f, \qquad (2.55)$$

analogous to Eq. (2.19). For a Majorana spinor (see Eqs. A18),

$$\overline{\epsilon}_1\gamma^\mu L\epsilon_2 = -\overline{\epsilon}_2\gamma^\mu R\epsilon_1 \qquad (2.56)$$

so that

$$\overline{\epsilon}_2\gamma^\mu L\epsilon_1 - \overline{\epsilon}_1\gamma^\mu L\epsilon_2 = \overline{\epsilon}_2\gamma^\mu\epsilon_1. \qquad (2.57)$$

Checking the closure of the algebra off-shell for the spinor ψ_L requires a Fierz rearrangement. One finally gets

$$[\delta_1,\delta_2]\begin{pmatrix}\phi\\ \psi_L\\ f\end{pmatrix} = -2i\,(\overline{\epsilon}_2\gamma^\mu\epsilon_1)\,\partial_\mu\begin{pmatrix}\phi\\ \psi_L\\ f\end{pmatrix}, \qquad (2.58)$$

in complete agreement with the Majorana case, Eqs. (2.44,45). This is the reason of the introduction of the factors $\sqrt{2}$ in the transformations of the Weyl multiplet

$$\begin{aligned}\delta\phi &= \sqrt{2}\,\overline{\epsilon}\psi_L,\\ \delta\psi_L &= -\sqrt{2}\,L\,(i\gamma^\mu\partial_\mu\phi + f)\,\epsilon,\\ \delta f &= i\sqrt{2}\,\overline{\epsilon}\gamma^\mu\partial_\mu\psi_L.\end{aligned} \qquad (2.59)$$

The invariant Lagrangian in the Weyl case is

$$\mathcal{L} = (\partial_\mu\phi^\dagger)(\partial^\mu\phi) + \overline{\psi}_L i\gamma^\mu\partial_\mu\psi_L + ff^\dagger. \qquad (2.60)$$

Again the auxiliary field f has dimension (mass)^2, and transforms like a total derivative, so that the Lagrangian

$$\mathcal{L}_{LIN} = \mu f + \mu^\dagger f^\dagger \qquad (2.61)$$

is supersymmetric. Adding $\mathcal{L}_{LIN}$ to $\mathcal{L}$ modifies the equation of motion of f which becomes

$$f = -\mu^\dagger. \qquad (2.62)$$

Solving for f in the Lagrangian introduces a constant term (irrelevant in the free case) $-|\mu|^2$.

2.4. Interactions and the tensor calculus

The introduction of the auxiliary fields has led us to two important observations. Firstly, auxiliary fields always transform with a total derivative. This follows trivially from the fact that they have the highest dimension in the multiplet (two for chiral multiplets) and, since ϵ has dimension -1/2, the transformation contains an expression whose dimension can then be correct only with a derivative. Secondly, the action of the free massive Wess-Zumino model with auxiliary fields is the sum of two parts which are separately supersymmetric invariant, the kinetic action and the mass terms. Plausibly, one should be able to construct interactions by obtaining supersymmetric actions which are trilinear, quadrilinear ... in the fields of the chiral multiplet. It would certainly be desirable to have a systematic method to construct such higher order actions. This method, the 'tensor calculus', relies upon our first observation. Assume we could construct a new chiral multiplet whose component fields would be quadratic combinations of the component fields of two chiral multiplets. Then, since its auxiliary fields would transform like total derivatives, they would provide us with supersymmetric invariant quadratic actions. In particular, we should recover our mass terms with this mechanism. Higher order invariant actions could then be obtained by iterating the process.

Historically, the method of tensor calculus was used to construct the first supersymmetric field theories. It was later replaced by the formalism of superfields which is more convenient and straightforward. We will only construct here the simplest part of the tensor calculus, involving only the chiral multiplets of the Wess-Zumino model. It can be generalized to more complicated multiplets, for which the superiority of superfields is however much more manifest.

Starting with two chiral multiplets $(A_1, B_1, \psi_1, F_1, G_1)$ and $(A_2, B_2, \psi_2, F_2, G_2)$, we must construct the components $(A_{12}, B_{12}, \psi_{12}, F_{12}, G_{12})$ of the product chiral multiplet. The scalar and pseudoscalar components will be quadratic combinations of A_1, A_2, B_1 and B_2, with the right parity properties. There is then a unique symmetric possibility for B_{12}:

$$B_{12} = A_1 B_2 + A_2 B_1. \tag{2.63}$$

Using the transformations (2.43), one immediately finds that its transformation is

$$\delta B_{12} = i\bar{\epsilon}\gamma_5 \left[(A_1 - iB_1\gamma_5)\psi_2 + (A_2 - iB_2\gamma_5)\psi_1\right]. \tag{2.64}$$

Comparing with (2.43), we must then define

$$\psi_{12} = (A_1 - iB_1\gamma_5)\psi_2 + (A_2 - iB_2\gamma_5)\psi_1. \tag{2.65}$$

The next step is to find A_{12} such that its transformation matches with our expression for ψ_{12}. The correct definition is

$$A_{12} = A_1 A_2 - B_1 B_2. \tag{2.66}$$

The expressions of the auxiliary fields F_{12} and G_{12} are then obtained from the transformation of ψ_{12}, which should correspond to the general transformation given in (2.43). One finds:

$$\begin{aligned}\delta\psi_{12} = &\left[(\bar{\epsilon}\psi_1) + (\bar{\epsilon}\gamma_5\psi_1)\gamma_5\right]\psi_2 + (1 \leftrightarrow 2)\\ &- (A_1 - iB_1\gamma_5)(F_2 + iG_2\gamma_5)\epsilon - (1 \leftrightarrow 2)\\ &- i\gamma^\mu\partial_\mu \left[(A_1 + iB_1\gamma_5)(A_2 + iB_2\gamma_5)\right]\epsilon.\end{aligned}$$

Using a Fierz rearrangement, the first line of this expression becomes simply

$$-(\overline{\psi}_1\psi_2)\epsilon - (\overline{\psi}_1\gamma_5\psi_2)\gamma_5\epsilon,$$

so that

$$\begin{aligned}\delta\psi_{12} = &- i\gamma^\mu\partial_\mu \left[(A_1A_2 - B_1B_2) + i(A_1B_2 + A_2B_1)\gamma_5\right]\epsilon\\ &- \left[(\overline{\psi}_1\psi_2) + A_1F_2 + A_2F_1 + B_1G_2 + B_2G_1\right]\epsilon\\ &- i\gamma_5\left[-i(\overline{\psi}_1\gamma_5\psi_2) + A_1G_2 + A_2G_1 - B_1F_2 - B_2F_1\right]\epsilon\\ = &- i\gamma^\mu\partial_\mu(A_{12} + iB_{12}\gamma_5)\epsilon - F_{12}\epsilon - iG_{12}\gamma_5\epsilon,\end{aligned} \tag{2.67}$$

with the definitions

$$\begin{aligned}F_{12} &= (\overline{\psi}_1\psi_2) + A_1F_2 + A_2F_1 + B_1G_2 + B_2G_1,\\ G_{12} &= -i(\overline{\psi}_1\gamma_5\psi_2) + A_1G_2 + A_2G_1 - B_1F_2 - B_2F_1.\end{aligned} \tag{2.68}$$

The last task is to verify that

$$\begin{aligned}\delta F_{12} &= i\bar{\epsilon}\gamma^\mu\partial_\mu\psi_{12},\\ \delta G_{12} &= -\bar{\epsilon}\gamma_5\gamma^\mu\partial_\mu\psi_{12}.\end{aligned}$$

It is now clear that the mass Lagrangian (2.38) precisely corresponds to the F component of the chiral multiplet obtained by taking the square (in the sense of tensor calculus) of the chiral multiplet (A, B, ψ, F, G):

$$\mathcal{L}_{MASS} = -\frac{1}{2}m\,F_{11} = -m(\frac{1}{2}\overline{\psi}\psi + FA + BG).$$

The rules of the tensor calculus for chiral multiplets are then the following:

$$\begin{aligned}A_{12} &= A_1A_2 - B_1B_2,\\ B_{12} &= A_1B_2 + A_2B_1,\\ \psi_{12} &= (A_1 - iB_1\gamma_5)\psi_2 + (A_2 - iB_2\gamma_5)\psi_1,\\ F_{12} &= (\overline{\psi}_1\psi_2) + A_1F_2 + A_2F_1 + B_1G_2 + B_2G_1,\\ G_{12} &= -i(\overline{\psi}_1\gamma_5\psi_2) + A_1G_2 + A_2G_1 - B_1F_2 - B_2F_1.\end{aligned} \tag{2.69}$$

One can then easily iterate the tensor product to get higher order invariant actions. For instance, a cubic, parity conserving supersymmetric action will then be

$$\begin{aligned}F_{123} =&(\overline{\psi}_1\psi_{23}) + A_1F_{23} + A_{23}F_1 + B_1G_{23} + B_{23}G_1\\ =&\overline{\psi}_1(A_2 - iB_2\gamma_5)\psi_3 + \overline{\psi}_2(A_3 - iB_3\gamma_5)\psi_1 + \overline{\psi}_3(A_1 - iB_1\gamma_5)\psi_2 +\\ &+ A_1A_2F_3 + A_1F_2A_3 + F_1A_2A_3 - B_1B_2F_3 - B_1F_2B_3 - F_1B_2B_3 +\\ &+ A_1B_2G_3 + B_1G_2A_3 + G_1A_2B_3 + A_1G_2B_3 + G_1B_2A_3 + B_1A_2G_3,\end{aligned} \tag{2.70}$$

or, for a single chiral multiplet,

$$\frac{1}{3}F_{111} = \overline{\psi}(A - iB\gamma_5)\psi + A^2F - B^2F + 2ABG. \tag{2.71}$$

Parity violating interactions are simply obtained by computing the G component of the corresponding tensor products.

Chapter 3

The supersymmetry algebra and its representations

We have constructed in chapter 2 the transformation rules of the chiral multiplet (A, B, ψ, F, G). These transformations contain a spinorial parameter ϵ and satisfy the following algebra:

$$[\delta_1, \delta_2] \begin{pmatrix} A, B \\ \psi \\ F, G \end{pmatrix} = -2i\,(\bar{\epsilon}_2 \gamma^\mu \epsilon_1)\,\partial_\mu \begin{pmatrix} A, B \\ \psi \\ F, G \end{pmatrix}, \tag{3.1}$$

where the two spinors ϵ_1 and ϵ_2 were assumed to anticommute.

The chiral multiplet is a first example of a linear representation of the supersymmetry algebra, given in Eq. (3.1). In this chapter, we will study the general construction of representations of supersymmetry and their particle content. In order to achieve this goal, we first need to obtain the complete structure of supersymmetry algebra.

3.1. The supersymmetry algebra

We have already written in chapter 1 the commutation relations of the Poincaré algebra (Eq. 1.1), generated by $M^{\mu\nu} = -M^{\nu\mu}$ and P^μ. The corresponding infinitesimal transformations are ($\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$ and Δ_μ are real parameters):

$$\begin{aligned} \text{Lorentz}: &\quad \delta\phi = i\frac{1}{2}\epsilon_{\mu\nu}M^{\mu\nu}\phi, \\ \text{Translations}: &\quad \delta\phi = i\Delta_\mu P^\mu \phi. \end{aligned} \tag{3.2}$$

The explicit form of the generators $M^{\mu\nu}$ and P^μ depends on the representation spanned by ϕ: ϕ can be a scalar field $\phi(x_\mu)$, a Lorentz vector, a spinor, a tensor ... Finite transformations are generated by group elements of the form

$$\begin{aligned} \text{Lorentz}: &\quad exp\left(i\frac{1}{2}\epsilon_{\mu\nu}M^{\mu\nu}\right)\phi = \phi', \\ \text{Translations}: &\quad exp(i\Delta_\mu P^\mu)\,\phi = \phi' \end{aligned} \tag{3.3}$$

(the factor 1/2 in Lorentz transformations avoids duplication of terms due to $M^{\mu\nu} = -M^{\nu\mu}$). If for instance ϕ is a scalar field $\phi(x)$, a translation by an amount Δ_μ acts

on ϕ according to

$$\phi'(x_\mu) = \phi(x_\mu + \Delta_\mu). \tag{3.4}$$

One can write a Taylor expansion

$$\phi(x_\mu + \Delta_\mu) = exp(\Delta_\mu \partial^\mu)\phi(x), \tag{3.5}$$

so that the appropriate form of P^μ is

$$P^\mu = -i\partial^\mu. \tag{3.6}$$

The corresponding form of Lorentz generators will be:

$$M^{\mu\nu} = i\,(x^\mu\partial^\nu - x^\nu\partial^\mu). \tag{3.7}$$

One easily checks that the Poincaré generators given in Eqs. (3.6) and (3.7) have the correct commutation relations, Eq. (1.1). Consider now a constant spinor λ. The appropriate representation of the Lorentz generators is:

$$M^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu], \tag{3.8}$$

as can be checked from the commutation relations. The spinor λ then transforms (infinitesimally) according to

$$\delta\lambda = -\frac{1}{8}\epsilon_{\mu\nu}[\gamma^\mu, \gamma^\nu]\lambda. \tag{3.9}$$

This transformation respects both the Weyl ($L\delta\lambda = \delta L\lambda$) and the Majorana ($\delta\lambda = \delta\lambda_c$ if $\lambda = \lambda_c$) conditions.

Turning to supersymmetry transformations, we need to express them in a form similar to (3.2):

$$\delta(\Phi) = i\bar{\epsilon}Q(\Phi), \tag{3.10}$$

where (Φ) denotes all fields belonging to a supermultiplet, a linear representation of supersymmetry. Since the parameter ϵ is a Majorana spinor, this is also the case of the supersymmetry generators denoted by Q. In chapter 2, we have chosen all spinors as being anticommuting. This is consistent with the canonical quantization procedure which specifies the anticommutators of fermion fields. Considering now two supersymmetry transformations with parameters ϵ_1 and ϵ_2, one gets

$$[\bar{\epsilon}_1 Q, \bar{\epsilon}_2 Q] = -\bar{\epsilon}_1^\alpha \bar{\epsilon}_2^\beta \{Q_\alpha, Q_\beta\}, \tag{3.11}$$

with the help of

$$\{\epsilon_1, \epsilon_2\} = 0, \qquad (3.12.a)$$
$$\{\epsilon, Q\} = 0. \qquad (3.12.b)$$

The indices α and β indicate explicitely the four components of the spinors. Eq. (3.12.a) tells us that the parameters of supersymmetry transformation are Grassmann (anticommuting) variables. As a result of Eq. (3.11), it is then natural to expect that the supersymmetry algebra will contain anticommutators of the Q^α's (and not commutators as for a Lie algebra). This is a reflexion of the fact that we are considering a superalgebra, whose generators are split into a bosonic sector B and a fermionic sector F. The superalgebra can be schematically written as

$$\begin{aligned} &[B, B] \subset B, \\ &[B, F] \subset F, \\ &\{F, F\} \subset B. \end{aligned} \qquad (3.13)$$

The bosonic sector B forms a closed Lie algebra. In our case, B is the direct sum of Poincaré algebra and a compact, internal Lie algebra, according to the Coleman-Mandula [2] theorem mentioned in chapter 1. The fermionic sector contains the supersymmetry generators Q. The Jacobi identities, generalized for superalgebras, become

$$\begin{aligned} &[[B_1, B_2], B_3] + [[B_2, B_3], B_1] + [[B_3, B_1], B_2] = 0, \\ &[[B_1, B_2], F_3] + [[B_2, F_3], B_1] + [[F_3, B_1], B_2] = 0, \\ &\{[B_1, F_2], F_3\} + [\{F_2, F_3\}, B_1] - \{[F_3, B_1], F_2\} = 0, \\ &[\{F_1, F_2\}, F_3] + [\{F_2, F_3\}, F_1] + [\{F_3, F_1\}, F_2] = 0, \end{aligned} \qquad (3.14)$$

where F_i and B_i ($i = 1, 2, 3$) are generators of the fermionic and bosonic sectors respectively. The second relation can be used to determine the commutator $[M^{\mu\nu}, Q_\alpha]$. It should be of the form

$$[M^{\mu\nu}, Q_\alpha] = (b^{\mu\nu})_\alpha{}^\beta Q_\beta. \qquad (3.15)$$

Using the second equation (3.14) and the Lorentz algebra leads to

$$[b^{\mu\nu}, b^{\rho\sigma}]_\alpha{}^\beta = -i\,(\eta^{\mu\rho} b^{\nu\sigma} + \eta^{\nu\sigma} b^{\mu\rho} - \eta^{\mu\sigma} b^{\nu\rho} - \eta^{\nu\rho} b^{\mu\sigma})_\alpha{}^\beta, \qquad (3.16)$$

which means that the matrices $b^{\mu\nu}$ form a representation of Lorentz algebra for spinors Q. We have already found this representation (Eq. 3.8), so that

$$\begin{aligned} [M^{\mu\nu}, Q] &= \frac{i}{4}[\gamma^\mu, \gamma^\nu] Q \\ &= \frac{i}{2}\gamma^{\mu\nu} Q, \end{aligned} \qquad (3.17)$$

defining $\gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$. We then conclude that the charges Q have the Lorentz transformations

$$\delta Q = -\frac{1}{4}\epsilon_{\mu\nu}\gamma^{\mu\nu} Q \qquad (3.18)$$

($\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$ are real parameters). As already indicated, this transformation preserves the Majorana character of Q, since $(\delta Q)_c = \delta Q$.

The result (3.16) reflects a general property of superalgebras, which says that generators of the fermionic sector span a representation of the Lie algebra forming the bosonic sector. This statement is also a consequence of the second Jacobi identity (3.14).

The commutators $[P^\mu, Q_\alpha]$ can be obtained in the same way, reproducing steps (3.15-16). But since $[P^\mu, P^\nu] = 0$, one concludes that

$$[P^\mu, Q_\alpha] = 0. \qquad (3.19)$$

The commutation relations (3.17) and (3.19) obviously remain the same if one has several supersymmetry charges

$$Q^i_\alpha \quad , \qquad i = 1, \ldots, N,$$

which is the case we will consider from now on.

We still have to find the anticommutators $\{Q^i_\alpha, Q^j_\beta\}$. We have seen in the explicit example of the chiral multiplet that commuting two supersymmetry transformations gives a translation. We then expect that $\{Q^i_\alpha, Q^j_\beta\}$ contains a P^μ piece, and no $M^{\mu\nu}$ term. Such a term is anyway forbidden by the Jacobi identity

$$\begin{aligned} 0 &= \left[\{Q^i_\alpha, Q^j_\beta\}, P^\mu\right] - \left\{[Q^j_\beta, P^\mu], Q^i_\alpha\right\} + \left\{[P^\mu, Q^i_\alpha], Q^j_\beta\right\} \\ &= \left[\{Q^i_\alpha, Q^j_\beta\}, P^\mu\right], \end{aligned} \qquad (3.20)$$

which implies that only generators commuting with P^μ can enter the anticommutators.

We now proceed to establish the most general admissible form for the anticommutators $\{Q^i_\alpha, Q^j_\beta\}$. Since they are symmetric under the interchange $\alpha, i \leftrightarrow \beta, j$, we must construct products of γ-matrices with definite symmetries. This involves the charge conjugation matrix C, since $(\gamma^\mu)^\tau = C\gamma^\mu C$. One shows easily that

$$C_{\alpha\beta}, \quad (\gamma_5 C)_{\alpha\beta}, \quad (\gamma^\mu\gamma_5 C)_{\alpha\beta}$$

are antisymmetric, while

$$(\gamma^\mu C)_{\alpha\beta}, \quad (\gamma^\mu\gamma^\nu C)_{\alpha\beta}, \quad \mu < \nu$$

are symmetric. Clearly the term of the anticommutator containing translations P^μ is associated with the symmetric matrix $\gamma^\mu C$. Since no bosonic generators transforming under Lorentz group other than P_μ and $M_{\mu\nu}$ are allowed by the Coleman-Mandula theorem, one concludes there is no term with $\gamma^\mu\gamma^\nu C$ or $\gamma^\mu\gamma_5 C$. (We have already argued that there is no $M_{\mu\nu}$ term). Then:

$$\{Q^i_\alpha, Q^j_\beta\} = (\gamma^\mu C)_{\alpha\beta} m^{ij} P_\mu + C_{\alpha\beta} V^{ij} + i(\gamma_5 C)_{\alpha\beta} Z^{ij}, \tag{3.21}$$

with

$$\begin{aligned} m^{ij} &= m^{ji}, \\ V^{ij} &= -V^{ji}, \\ Z^{ij} &= -Z^{ji}. \end{aligned} \tag{3.22}$$

m^{ij}, V^{ij} and Z^{ij} are real matrices. Their reality is fixed by the Majorana condition satisfied by the spinorial charges, $Q^i_\alpha = (C\gamma^0 Q^{i\dagger^T})_\alpha$. It implies for the anticommutators

$$\{Q^i_\alpha, Q^j_\beta\} = (C\gamma^0)_{\alpha\alpha'} \{Q^i_{\alpha'}, Q^j_{\beta'}\}^\dagger (C\gamma^0)_{\beta'\beta}. \tag{3.23}$$

Notice that one can always perform an orthogonal transformation and a rescaling of the charges Q^i_α, so that $m^{ij} = a\delta^{ij}$. Finally, a is fixed by requiring that two supersymmetry transformations generate the correct translation, as in Eq. (3.1). Considering the case $N = 1$, with only one spinorial charge Q, one gets

$$\begin{aligned} [\delta_1, \delta_2] &= [i\bar\epsilon_1 Q, i\bar\epsilon_2 Q] \\ &= \bar\epsilon_1^\alpha \bar\epsilon_2^\beta \{Q_\alpha, Q_\beta\} \\ &= \bar\epsilon_1^\alpha \bar\epsilon_2^\beta (\gamma^\mu C)_{\alpha\beta}\, a P_\mu. \end{aligned}$$

Recalling that $P_\mu = -i\partial_\mu$, one obtains

$$\begin{aligned} [\delta_1, \delta_2] &= -ia\left(\bar\epsilon_1 \gamma^\mu C \gamma^0 \epsilon_2^{\dagger^T}\right)\partial_\mu \\ &= -ia\,(\bar\epsilon_1 \gamma^\mu \epsilon_2)\,\partial_\mu \\ &= ia\,(\bar\epsilon_2 \gamma^\mu \epsilon_1)\,\partial_\mu, \end{aligned} \tag{3.24}$$

with the help of the Majorana condition $\epsilon_2 = C\gamma^0 \epsilon_2^{\dagger^T}$. Comparing with Eq. (3.1), one must choose $a = -2$ so that

$$\{Q_\alpha, Q_\beta\} = -2(\gamma^\mu C)_{\alpha\beta} P_\mu, \tag{3.25}$$

in the case of one supersymmetric charge ('$N = 1$ *supersymmetry*'), and

$$\{Q^i_\alpha, Q^j_\beta\} = -2(\gamma^\mu C)_{\alpha\beta}\delta^{ij} P_\mu + C_{\alpha\beta} V^{ij} + i(\gamma_5 C)_{\alpha\beta} Z^{ij}, \tag{3.26}$$

for '*extended supersymmetry*' ($N > 1$). The new generators V^{ij} and Z^{ij} are *central charges*: they commute with each other and with all generators of the superalgebra. This can be shown by using the Jacobi identities (3.14). To summarize, the full supersymmetry algebra is:

$$\begin{aligned} [M^{\mu\nu}, M^{\rho\sigma}] &= -i(\eta^{\mu\rho}M^{\nu\sigma} + \eta^{\nu\sigma}M^{\mu\rho} - \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\rho}M^{\mu\sigma}), \\ [M^{\mu\nu}, P^\sigma] &= -i(\eta^{\mu\rho}P^\nu - \eta^{\nu\rho}P^\mu), \\ [P^\mu, P^\nu] &= 0, \\ [M^{\mu\nu}, Q^i_\alpha] &= \frac{i}{2}(\gamma^{\mu\nu}Q^i)_\alpha, \qquad \gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu], \\ [P^\mu, Q^i_\alpha] &= 0, \\ \{Q^i_\alpha, Q^j_\beta\} &= -2(\gamma^\mu C)_{\alpha\beta}P_\mu\delta^{ij} + C_{\alpha\beta}V^{ij} + i(\gamma_5 C)_{\alpha\beta}Z^{ij}, \\ [V^{ij}, V^{kl}] &= [V^{ij}, Z^{kl}] = [Z^{ij}, Z^{kl}] = 0, \\ [M^{\mu\nu}, V^{ij}] &= [P^\mu, V^{ij}] = [Q^i_\alpha, V^{ij}] = 0, \\ [M^{\mu\nu}, Z^{ij}] &= [P^\mu, Z^{ij}] = [Q^i_\alpha, Z^{ij}] = 0. \end{aligned} \tag{3.27}$$

Since P_μ has dimension (mass)^1, the supersymmetry charges Q^i_α have dimension 1/2. This number is consistent with the choice we made in chapter 2, that the spinorial parameter ϵ has dimension $-1/2$ (the transformation $i\bar\epsilon Q$ is then dimensionless, as it should).

Notice also that the central charges have dimension (mass)^1. They will in general introduce mass parameters in supersymmetric multiplets. The form of the central charges can be freely chosen, since they belong to an abelian subalgebra. Any truncation of the algebra with general central charges is then also admissible.

The relations (3.27) correspond in fact to the simple *orthosymplectic superalgebra* $OSp(N|4)$. This algebra has a bosonic sector

$$Sp(4, R) \times SO(N).$$

$Sp(4, R)$ is isomorphic (as a Lie algebra) to the anti-de Sitter algebra $SO(3, 2)$ (pseudo-rotations with signature $(+1, -1, -1, -1, +1)$). The fermionic sector contains $4N$ generators transforming as a spinor of $SO(3, 2)$, and a vector (N) of $SO(N)$. The internal symmetry $SO(N)$ rotates the supersymmetry charges. We have omitted its generators in the relations (3.27).

The anti-de Sitter group $SO(3, 2)$ is not the same as the Poincaré group. $SO(3, 2)$ has a subgroup $SO(3, 1)$ (generated by, say, $M^{\mu\nu} = -M^{\nu\mu}, \mu, \nu = 0, 1, 2, 3$), identified with the Lorentz group. But the remaining generators $M^{\mu 5}$ are not translations: $[M^{\mu 5}, M^{\nu 5}] = -iM^{\mu\nu}$ and does not vanish when μ differs from ν. $SO(3, 2)$ would be appropriate to describe the isometries of anti-de Sitter space, with a negative cosmological constant ($SO(4,1)$, the de Sitter group, is relevant for a positive cosmological constant). Minkowski space corresponds however to a zero cosmological constant. The Poincaré algebra is then obtained in the zero curvature limit of de Sitter or anti-de Sitter space. In the language of group theory, the Poincaré group is the Inönü-Wigner contraction with respect to the Lorentz subgroup $SO(1,3)$ of $SO(2,3)$ (or $SO(1,4)$), denoted by $ISO(1,3)$. Schematically, one splits the anti-de Sitter algebra into two parts, Lorentz generators M and the remaining part P (the generators $M^{\mu 5}$). The $SO(2,3)$ algebra is

$$\begin{aligned} [M,M] &= M, \\ [M,P] &= P, \\ [P,P] &= M. \end{aligned} \tag{3.28}$$

Define now generators $\overline{P}$ by

$$\overline{P} = \lambda P. \tag{3.29}$$

In the contracted limit, $\lambda \to 0$, one gets

$$\begin{aligned} [M,M] &= M, \\ [M,\overline{P}] &= \overline{P}, \\ [\overline{P},\overline{P}] &= 0, \end{aligned} \tag{3.30}$$

corresponding to the Poincaré algebra, $\overline{P}$ being the translation generators.

In the case of the superalgebra, one must perform an analogous contraction. The $OSp(4|N)$ algebra is, schematically,

$$\begin{aligned} &[M,M] = M, \qquad && [M,Q] = Q, \\ &[M,P] = P, && [P,Q] = Q, \\ &[P,P] = M, && [T,Q] = Q, \\ &[T,T] = T, && \\ &[T,M] = [T,P] = 0, && \\ &\{Q,Q\} = `P + M + T', && \end{aligned} \tag{3.31}$$

where T denotes the $SO(N)$ generators and '$P + M + T$' a linear combination of bosonic generators. The contraction (3.29) cannot be used since it would give singular terms in $\{Q,Q\}$ when $\lambda \to 0$. One clearly must also contract the fermionic sector Q. There are two relevant possibilities, differing by the contraction of internal symmetries T. Define:

$$1) \quad \overline{P} = \lambda P, \quad \overline{Q} = \sqrt{\lambda} Q, \quad \overline{T} = T. \tag{3.32}$$

One gets, in the limit $\lambda \to 0$,

$$\begin{aligned} &[M,M] = M, \quad [M,\overline{P}] = \overline{P}, \quad [\overline{P},\overline{P}] = 0, \\ &[\overline{T},\overline{T}] = \overline{T}, \\ &[\overline{T},M] = [\overline{T},\overline{P}] = 0, \\ &[M,\overline{Q}] = \overline{Q}, \quad [\overline{P},\overline{Q}] = 0, \quad [\overline{T},\overline{Q}] = \overline{Q}, \\ &\{\overline{Q},\overline{Q}\} = \overline{P}. \end{aligned} \tag{3.33}$$

These relations correspond to the direct product of the super-Poincaré algebra without central charges and the internal $SO(N)$ group. It is clear that one can freely truncate (or enlarge) $SO(N)$ without doing any harm, as long as the internal symmetry can be represented on the fermionic sector: $[\overline{T},\overline{Q}] = \overline{Q}$.

The second possibility is:

$$2) \quad \overline{P} = \lambda P, \quad \overline{Q} = \sqrt{\lambda} Q, \quad \overline{T} = \lambda T. \tag{3.34}$$

One finds:

$$\begin{aligned} &[M,M] = M, \quad [M,\overline{P}] = \overline{P}, \quad [\overline{P},\overline{P}] = 0, \\ &[\overline{T},\overline{T}] = 0, \\ &[\overline{T},M] = [\overline{T},\overline{P}] = 0, \\ &[M,\overline{Q}] = \overline{Q}, \quad [\overline{P},\overline{Q}] = [\overline{T},\overline{Q}] = 0, \\ &\{\overline{Q},\overline{Q}\} = `\overline{P} + \overline{T}'. \end{aligned} \tag{3.35}$$

The generators $\overline{T}$ are now central charges. It is clear that one can obtain simultaneously (3.33) and (3.35) by contracting only some of the T's.

We now turn to general results on the representations of supersymmetry. Three important properties of supermultiplets can be easily proven:

1. All particles belonging to an irreducible representation of supersymmetry have *the same mass*.

This is because $P^2 = P_\mu P^\mu$ is a Casimir operator of the super-Poincaré algebra. It is straightforward, using Eqs. (3.27), to verify that

$$[P^2, P^\mu] = [P^2, M^{\mu\nu}] = [P^2, Q^i_\alpha] = 0. \tag{3.36}$$

2. The energy P_0 in supersymmetric theories is always *positive*.

Using the algebra (3.27) and the Majorana condition $\overline{Q}^i = Q^{i\tau} C$, one gets:

$$\{Q^i_\alpha, \overline{Q}^{j\beta}\} = 2(\gamma^\mu)_\alpha{}^\beta P_\mu \delta^{ij} - V^{ij}\delta^\beta_\alpha - i(\gamma_5)^\beta_\alpha Z^{ij}. \tag{3.37}$$

Then:

$$\sum_i Tr\left(\{Q^i_\alpha, \overline{Q}^{i\alpha}\}\gamma^0\right) = \sum_i \left(Q^i_\alpha (Q^i_\alpha)^\dagger + (Q^i_\alpha)^\dagger Q^i_\alpha\right) = \\ = 2N\, Tr(\gamma^\mu \gamma^0 P_\mu) = 8N P_0. \tag{3.38}$$

Since $Q^i_\alpha (Q^i_\alpha)^\dagger + (Q^i_\alpha)^\dagger Q^i_\alpha$ is positive or zero, then

$$E = P^0 \geq 0. \tag{3.39}$$

3. A supermultiplet always contains an equal number of fermion and boson degrees of freedom.

Let us introduce a fermion number operator N_F such that $(-1)^{N_F}$ is either +1 on bosonic states or -1 on fermionic states. It has the property that

$$(-1)^{N_F} Q^i = -Q^i (-1)^{N_F}. \tag{3.40}$$

Then, on any finite dimensional representation,

$$Tr\left((-1)^{N_F}\{Q^i_\alpha, \overline{Q}^{i\beta}\}\right) = Tr\left(-Q^i_\alpha (-1)^{N_F}\overline{Q}^{i\beta} + Q^i_\alpha (-1)^{N_F}\overline{Q}^{i\beta}\right) = 0. \tag{3.41}$$

Using now (3.37) for $i = j$, one finds

$$0 = Tr\left((-1)^{N_F}\{Q^i_\alpha, \overline{Q}^{i\beta}\}\right) = 2(\gamma^\mu)_\alpha{}^\beta\, Tr\left((-1)^{N_F} P_\mu\right). \tag{3.42}$$

Choosing now a fixed non zero momentum P_μ leads to

$$Tr(-1)^{N_F} = 0, \tag{3.43}$$

which is the desired result. The case of central charges (or $i \neq j$) is exactly the same.

Notice that we have used two different traces in proving properties 2 and 3. In Eq. (3.38), the trace is taken over spinor indices α, β while in Eqs. (3.41,42), one sums over all states belonging to the representation.

Another important result has been proven by Haag, Lopuszanski and Sohnius [9]. They have shown that the supersymmetry algebra (3.27) is in fact the most general superalgebra admissible as a symmetry of an interacting quantum field theory. This result is in fact the extension to superalgebra of the Coleman-Mandula theorem mentioned in chapter 1.

We will now proceed to determine the particle content of representations of supersymmetry. They will be characterized by the values of the Casimir operators of the algebra. In the Poincaré case, the two Casimir are the mass and the spin of the

particle. We have already shown that the mass is also a Casimir of the super-Poincaré algebra. We will see that one can define a 'superspin' which will be essentially the highest spin of all particles belonging to the supermultiplet. More rigourously, one should speak of helicity or super-helicity in the massless case. In addition, the supersymmetry algebra has an internal symmetry part. A supermultiplet will then also be characterized by the Casimir operators of the internal symmetry.

In the next two sections, we will discuss massless and massive supersymmetry representations for on-shell states (following a line of reasoning given in [11,12]). In general, there is no method to find the necessary auxiliary fields. It is often impossible to extend an on-shell multiplet to a (linear) off-shell representation. We will neglect central charges (see [12]). They generate only massive representations.

3.2. Massless supermultiplets

We shall study first irreducible massless representations of supersymmetry. To do this, we first choose a special frame for a massless state $P^\mu P_\mu = 0$, and generate the states of the supermultiplet in that frame. We choose

$$P_\mu = (E, 0, 0, E). \tag{3.44}$$

In this frame, the supersymmetry algebra (3.27), without central charges, reads

$$\{Q^i_\alpha, Q^j_\beta\} = -4E \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \delta^{ij}, \tag{3.45}$$

so that one gets

$$\{Q^i_1, Q^j_4\} = \{Q^i_4, Q^j_1\} = -4E\delta^{ij}, \tag{3.46}$$

all others anticommutators being zero. Since Q^i is a Majorana spinor, we also have

$$Q^i_3 = Q^{i\,\dagger}_2 \quad , \quad Q^i_4 = -Q^{i\,\dagger}_1. \tag{3.47}$$

The supersymmetry algebra (3.27) is then equivalent to

$$\begin{aligned} &\{Q^i_1, Q^{j\,\dagger}_1\} = 4E\delta^{ij}, \\ &\{Q^i_2, Q^{j\,\dagger}_2\} = \{Q^i_1, Q^{j\,\dagger}_2\} = \{Q^i_2, Q^{j\,\dagger}_1\} = 0, \\ &\{Q^i_1, Q^j_1\} = \{Q^i_2, Q^j_2\} = 0, \\ &\{Q^i_1, Q^j_2\} = 0. \end{aligned} \tag{3.48}$$

Consider now only Q_1^i, $Q_1^{j\dagger}$. One can easily absorb the factor $4E$ into the charges (or choose $E = 1/4$) so that $\{Q_1^i, Q_1^{j\dagger}\} = \delta^{ij}$. Define now the $2N$ quantities Γ_A by

$$\begin{aligned} \Gamma_{2i} &= Q_1^i + Q_1^{i\dagger} \\ \Gamma_{2i+1} &= i(Q_1^i - Q_1^{i\dagger}) \end{aligned} \qquad i = 1, \ldots, N. \tag{3.49}$$

They form a Clifford algebra, with anticommutation relations

$$\{\Gamma_A, \Gamma_B\} = 2\delta_{AB}; \qquad A, B = 1, \ldots, 2N. \tag{3.50}$$

The automorphism group of this Clifford algebra is $SO(2N)$. It is well known that this Clifford algebra has a unique representation of dimension 2^N. The states of the representation are easily constructed using the $Q_1^{i\dagger}$. The relations $\{Q_1^i, Q_1^{j\dagger}\} = \delta^{ij}$ correspond to a set of N Fermi creation (the $Q_1^{i\dagger}$'s) and N annihilation (the Q_1^i's) operators. One then starts from a Clifford vacuum $|\Omega>$ such that

$$Q_1^i|\Omega> = 0. \tag{3.51}$$

The states of the representation of the Clifford algebra are then obtained by acting on $|\Omega>$ with $Q_1^{i\dagger}$:

$$\begin{aligned} &|\Omega> && : 1 \text{ state} \\ &Q_1^{i\dagger}|\Omega> && : N \text{ states} \\ &Q_1^{i\dagger}Q_1^{j\dagger}|\Omega> && : \frac{1}{2}N(N-1) \text{ states} \\ &\vdots \\ &Q_1^{i_1\dagger}\ldots Q_1^{i_k\dagger}|\Omega> && : \binom{N}{k} = \frac{N!}{k!(N-k)!} \text{ states} \\ &\vdots \\ &Q_1^{1\dagger}Q_1^{2\dagger}\ldots Q_1^{N\dagger}|\Omega> && : \binom{N}{N} = 1 \text{ state} \end{aligned}$$

Since

$$\sum_{k=0}^{[N/2]} \binom{N}{2k} = \sum_{\substack{k=0 \\ (2k+1 \le N)}}^{[N/2]} \binom{N}{2k+1} = 2^{N-1}$$

($[\ldots]$ means integer part) we have precisely constructed 2^N states, falling naturally into two classes:

2^{N-1} states obtained from $|\Omega>$ by acting with an odd number of $Q_1^{i\dagger}$,

2^{N-1} states obtained from $|\Omega>$ by acting with an even number of $Q_1^{i\dagger}$.

All this states are characterized by a vanishing mass. They also possess a helicity, generated by $M^{12} = -M^{21}$. This generator leaves our choice of P_μ, Eq. (3.44), invariant. It is then an admissible quantum number. So the vacuum state $|\Omega>$ possesses a helicity λ_{MAX}. Then, according to Eq. (3.27),

$$\begin{aligned} [M^{12}, Q_1{}^\dagger] &= -\frac{1}{2}Q_1{}^\dagger, \\ [M^{12}, Q_1] &= +\frac{1}{2}Q_1. \end{aligned} \tag{3.52}$$

A state $Q_1^{i_1\dagger}\ldots Q_1^{i_k\dagger}|\Omega>$ has then a helicity

$$\lambda = \lambda_{MAX} - k/2, \tag{3.53}$$

which can be written in an operator form

$$\Lambda = \lambda_{MAX} - \frac{1}{2}\sum_{i=1}^{N} Q_1^{i\dagger}Q_1^i. \tag{3.54}$$

The 2^N states representing the Clifford algebra (3.50) have then helicities ranging from λ_{MAX} to $\lambda_{MAX} - N/2$. Returning to Eq. (3.48), one sees that

$$Q_2^i = Q_2^{i\dagger} = 0. \tag{3.55}$$

This result follows from the following argument: a physical state should have positive norm and all states of the form $Q_2^i|\phi>$ or $Q_2^{i\dagger}|\phi>$, where $|\phi>$ is a physical state, have necessarily zero norm due to the vanishing anticommutators in Eqs. (3.48). The 2^N states constructed with $Q_1^{i\dagger}$ are then the basic blocks of each supermultiplet. There is a further constraint due to the CPT theorem of field theory. A physical massless state always contains two helicities $+\lambda$ and $-\lambda$ (except $\lambda = 0$). In terms of our 2^N states, one checks easily that this constraint will be satisfied only if

$$\lambda_{MAX} = \frac{N}{4}, \tag{3.56}$$

for N even. These massless supermultiplets, containing 2^N helicity states, will be called CPT-self-conjugate multiplets. If Eq. (3.56) is not verified, one gets all necessary helicity states by doubling the multiplet, choosing a second Clifford vacuum $|\Omega'>$ with

$$\lambda_{MAX}{}' = \frac{N}{2} - \lambda_{MAX}. \tag{3.57}$$

A massless, non CPT-self-conjugate, supermultiplet contains then 2^{N+1} helicity states.

The last points to investigate are the internal quantum numbers. The algebra (3.48) is clearly invariant under

$$Q_1^i \rightarrow U^i{}_j Q_1^j, \tag{3.58}$$

where U is a unitary matrix. This gives a $U(N)$ invariance group generated by

$$U^i{}_j = Q_1^i Q_1^{j\dagger}, \tag{3.59}$$

with the algebra

$$[U^i{}_j, U^k{}_l] = \delta_j^k U^i{}_l - \delta_l^i U^k{}_j. \tag{3.60}$$

The abelian part of $U(N) \sim SU(N) \times U(1)$ is generated by the number operator $\sum_{i=1}^N Q_1^{i\dagger} Q_1^i$ which is part of the helicity operator, Eq. (3.54). $SU(N)$ is then the natural symmetry to classify states of given helicity. If one assumes that the Clifford vacuum is an $SU(N)$ singlet, the states $Q_1^{i_1\dagger} Q_1^{i_2\dagger} \ldots Q_1^{i_k\dagger}|\Omega>$, with helicity $\lambda_{MAX} - k/2$ belong to the fully antisymmetric tensor representation of rank k of $SU(N)$, denoted by $[k]_N$. Its dimension is:

$$dim\,[k]_N = \binom{N}{k} = \frac{N!}{k!(N-k)!}. \tag{3.61}$$

One could however assume that $|\Omega>$ belongs to a representation R of $SU(N)$. The states $Q_1^{i_1\dagger} Q_1^{i_2\dagger} \ldots Q_1^{i_k\dagger}|\Omega>$ are then in the representation $R \times [k]_N$ of $SU(N)$.

A massless supermultiplet is then fully determined by the helicity λ_{MAX} of the Clifford vacuum $|\Omega>$ and by its $SU(N)$ representation R. CPT corresponds to the condition that to each state with helicity λ and $SU(N)$ representation r, there corresponds a state ('antiparticle') with helicity $-\lambda$ and representation $\overline{r}$. The condition is satisfied by associating to the vacuum a second vacuum with helicity $N/4 - \lambda_{MAX}$ and representation $\overline{R}$. Clearly, CPT doubling is not necessary for the self-conjugate multiplets with

$$\begin{aligned} \lambda_{MAX} &= N/4, \\ R &= \overline{R}. \end{aligned} \tag{3.62}$$

When constructing supersymmetric gauge theories, we will consider only states with helicities 0 (scalar fields), $\pm 1/2$ (spin 1/2 fields), ± 1 (vector fields, spin 1). This implies that the helicity extension of the multiplet is then at most two, which means $N \leq 4$. The complete list of acceptable supermultiplets is then given by the following table:

Helicity states of supersymmetric gauge theory multiplets:

λ	[1] $N=1$ $\lambda_{MAX}=\frac{1}{2}$	[2] $N=1$ $\lambda_{MAX}=1$	[3] $N=2$ $\lambda_{MAX}=\frac{1}{2}$	[4] $N=2$ $\lambda_{MAX}=1$	[5] $N=3$ $\lambda_{MAX}=1$	[6] $N=4$ $\lambda_{MAX}=1$
1		1		1	1	1
1/2	1	1	$1+(1)$	2	$3+1$	4
0	$1+1$		$2+(2)$	$1+1$	$\overline{3}+3$	6
−1/2	1	1	$1+(1)$	2	$1+\overline{3}$	$\overline{4}$
1		1		1	1	1

(The numbers of states correspond to $SU(N)$ representations. They are given for a $SU(N)$-invariant Clifford vacuum).

[1] $N=1$ chiral multiplet: $2^{N+1}=4$ states.

[2] $N=1$ vector multiplet: $2^{N+1}=4$ states.

[3] $N=2$ 'hypermultiplet': $2^{N+1}=8$ states.

[4] $N=2$ vector multiplet: $2^{N+1}=8$ states.

[5] $N=3$ vector multiplet: $2^{N+1}=16$ states.

[6] $N=4$ vector multiplet: this multiplet is CPT-self-conjugate, without doubling ($2^N=16$ states).

Two remarks are in order:

The $N=2$ hypermultiplet has a peculiarity. Since Eq. (3.56) is satisfied, there is at first sight no doubling. However scalars transform as an $SU(2)$-doublet which is a *pseudoreal* representation. CPT however requires scalar fields to belong to real representations. One must then double the states.

When constructing a $N=3$ supersymmetric gauge theory, one gets automatically a fourth supersymmetry. The states in multiplets [5] and [6] are in one to one correspondance and it turns out that the most general renormalizable field theory contructed with the $N=3$ multiplet is also the same as the $N=4$ theory.

The other massless supermultiplets of interest are those containing one (and only one) spin two state (with helicities ± 2) to be identified with the graviton. These multiplets will be used to construct the supergravity theories:

λ	$N=1,$	$2,$	$3,$	$4,$	$5,$	$6,$	$7,$	8
2	1	1	1	1	1	1	1	1
$\frac{3}{2}$	1	2	3	4	5	6	$7+1$	8
1		1	$\overline{3}$	6	10	$15+1$	$21+7$	28
$\frac{1}{2}$			1	$\overline{4}$	$\overline{10}+1$	$20+6$	$35+21$	56
0				$1+1$	$\overline{5}+5$	$\overline{15}+15$	$\overline{35}+35$	70
$-\frac{1}{2}$			1	4	$1+10$	$\overline{6}+20$	$\overline{21}+\overline{35}$	$\overline{56}$
-1		1	3	6	$\overline{10}$	$1+\overline{15}$	$\overline{7}+\overline{21}$	$\overline{28}$
$-\frac{3}{2}$	1	2	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$	$\overline{7}+1$	$\overline{8}$
-2	1	1	1	1	1	1	1	1

Again the $N = 7$ supergravity theory is in fact the $N = 8$ theory (the numbers of states also match). But the $N = 3$ supergravity theory is now different from $N = 4$ theory.

3.3. Massive supermultiplets

Since all particles in the representation will have a mass M, one can simply go to the rest frame of the multiplet and choose

$$P_\mu = (M, 0, 0, 0). \tag{3.63}$$

The supersymmetry algebra (3.27) without central charges reads then:

$$\{Q^i_\alpha, Q^j_\beta\} = -2M(\gamma^0 C)_{\alpha\beta}\delta^{ij} = -2M\delta^{ij}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{3.64}$$

Using the Majorana condition, one finds

$$\begin{aligned} \{Q^i_1, Q^{j\dagger}_1\} &= \{Q^i_2, Q^{j\dagger}_2\} = 2M\delta^{ij}, \\ \{Q^i_1, Q^{j\dagger}_2\} &= \{Q^i_2, Q^{j\dagger}_1\} = 0, \\ \{Q^i_1, Q^j_1\} &= \{Q^i_2, Q^j_2\} = \{Q^i_1, Q^j_2\} = 0. \end{aligned} \tag{3.65}$$

Absorbing the factor $2M$ in the normalization of charges (or choosing $M = 1/2$), one realizes, comparing with the massless case, that relations (3.65) are equivalent to a Clifford algebra of $4N$ operators defined by

$$\begin{aligned} \Gamma_{2k} &= Q^i_1 + Q^{i\dagger}_1; & \Gamma_{2N+2k} &= Q^i_2 + Q^{i\dagger}_2; \\ \Gamma_{2k-1} &= i(Q^i_1 - Q^{i\dagger}_1); & \Gamma_{2N+2k-1} &= i(Q^i_2 - Q^{i\dagger}_2); \end{aligned} \qquad k = 1, \ldots, N. \tag{3.66}$$

The representation of this algebra will have dimension 2^{2N}. Its automorphism group is $SO(4N)$. States are again easily constructed by defining a Clifford vacuum $|\Omega>$ such that

$$Q^i_1|\Omega> = Q^i_2|\Omega> = 0, \tag{3.67}$$

and acting on $|\Omega>$ with the $2N$ creation operators $Q^{i\dagger}_1$ and $Q^{i\dagger}_2$.

In the massive case, states are no more classified with an helicity. The Lorentz generators leaving our choice of P_μ invariant are now

$$M^{12}, M^{13}, M^{23}.$$

They form an $SO(3) \sim SU(2)$ algebra. Each particle state will then be characterized by its spin, the $SU(2)$ Casimir, and by the quantum numbers of the subgroup of $SO(4N)$ which commutes with the $SU(2)$ spin group.

Under the spin group, the creation operators behave like a doublet (spin 1/2):

$$\begin{aligned} Q^{i\dagger}_1 &: \quad J = 1/2, \quad J_3 = -1/2, \\ Q^{i\dagger}_2 &: \quad J = 1/2, \quad J_3 = +1/2. \end{aligned} \tag{3.68}$$

$SU(2)$ is a subgroup of $SO(4N)$. The embedding is defined by the observation that all states of the 4N vector representation (which contains the $4N$ operators $(Q^i_1, Q^i_2, Q^{i\dagger}_1, Q^{i\dagger}_2)$) belong to an $SU(2)$ doublet:

$$\mathbf{4N} = (\mathbf{2}, \mathbf{2N}). \tag{3.69}$$

The group acting on the representation **2N** is the symplectic group $USp(2N)$. It is the largest subgroup of $SO(4N)$ which commutes with $SU(2)$, since $SO(4N)$ has a maximal subalgebra $SO(4N) \supset SU(2) \times USp(2N)$.

States with a given spin will then be classified in representations of the symplectic algebra $USp(2N)$ (to be compared with the massless case where states of given helicity are in $SU(N)$ multiplets).

The explicit construction of the states of the fundamental supermultiplets goes along the same line as for the massless case. One starts with a Clifford vacuum $|\Omega>$ which, in the simplest case, has spin zero ($SU(2)$-invariant) and no internal ($USp(2N)$) quantum numbers. A more general situation, which we will only shortly consider, would be to start from a vacuum with non zero spin and/or with non trivial $USp(2N)$ quantum numbers. One then acts on $|\Omega>$ with the creation operators $Q^{i\dagger}_1$

(with $J_3 = -1/2$) and with $Q_2^{i\dagger}$ (with $J_3 = +1/2$). The states

$$
\begin{gathered}
|\Omega> \\
Q_\alpha^{i\dagger}|\Omega> \\
Q_\alpha^{i\dagger}Q_\beta^{j\dagger}|\Omega> \\
\vdots \\
Q_1^{1\dagger}Q_2^{1\dagger}\dots Q_1^{N\dagger}Q_2^{N\dagger}|\Omega>
\end{gathered}
$$

form the 2^{2N}-dimensional representation of the Clifford algebra, which contains the two fundamental spinorial representations of $SO(4N)$, both with dimension 2^{2N-1}. The states belonging to each spinorial representation are constructed with an even or odd number of creation operators. Each representation contains then only fermions or bosons.

To determine the particle content of the supermultiplet, we need the $SU(2) \times USp(2N)$ quantum numbers of the states. One finds that

$$
\begin{aligned}
2^{2N} = & \left(\frac{N}{2},[0]\right) + \left(\frac{N-1}{2},[1]\right) + \left(\frac{N-2}{2},[2]\right) + \\
& + \cdots + (0,[N]),
\end{aligned} \tag{3.70}
$$

where the notation (spin, $USp(2N)$ representation) is used. $[k]$ denotes the k-fold antisymmetric traceless irreducible representation of $USp(2N)$, with dimension

$$
dim[k] = \binom{2N}{k} - \binom{2N}{k-2}, \tag{3.71}
$$

The simplest massive supermultiplets, with a spin zero vacuum without internal quantum numbers, contain states with spins ranging from zero up to $N/2$.

If one uses a Clifford vacuum with a spin J_Ω, one constructs in the usual way $(2J_\Omega + 1)2^{2N}$ states. All massive supermultiplets with spin up to two are given in the following table:

Massive supermultiplets with maximal spin two:
(n is the number of states and s the spin of $|\Omega>$)

J	$N=1$	1	1	1	2	2	2	3	3	4
2				1			1		1	1
$\frac{3}{2}$			1	2		1	4	1	6	8
1		1	2	1	1	4	$5+1$	6	$14+1$	27
$\frac{1}{2}$	1	2	1		4	$5+1$	4	14	$14'+6$	48
0	2	1			5	4	1	$14'$	14	42
n	4	8	12	16	16	32	48	64	128	256
s	0	$\frac{1}{2}$	1	$\frac{3}{2}$	0	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0

The internal quantum numbers are the $USp(2N)$ representations. Notice however that $USp(2N)$ cannot be used as an internal symmetry since it does not exist in a general frame. Lorentz transformations only preserve the $U(N)$ subgroup of $USp(2N)$. $USp(2N)$ can only be used for classification purposes. Also, a general Clifford vacuum can only carry a non trivial representation R of $U(N)$. CPT invariance requires R to be real (if not one must also take the conjugate representation $\overline{R}$). The quantum numbers of states of given spin are then obtained by multiplying those given in the table by R (or $R + \overline{R}$).

Chapter 4

Superfields

As we have seen in chapter 2, constructing supersymmetric field theories requires in general painful calculations. This is particularly true for interacting theories. This observation is to be compared with the case of translations. Invariance under translations is always manifest, since P_μ acts with a derivative, $P_\mu = -i\partial_\mu$:

$$\begin{aligned} \delta\phi(x) &= \Delta_\mu \partial^\mu \phi, \\ \delta\mathcal{L} &= \Delta_\mu \partial^\mu \mathcal{L} = \text{ total derivative.} \end{aligned} \tag{4.1}$$

However, supersymmetries are 'square root of translations', as is clear from the super-Poincaré algebra. It is then attractive to try to represent supersymmetry transformations as being some generalized translations. If possible, such an idea can be achieved only in an enlarged space with some new coordinates which are translated by supersymmetry. This enlarged space is called *superspace*.

Usual translations in space-time act on a vector of coordinates x^μ. Their parameters then also form a vector Δ^μ and the translation is

$$x^\mu \rightarrow x^\mu + \Delta^\mu.$$

We know already that the parameter of a supersymmetry transformation (we consider only the $N = 1$ case) is an anticommuting (Grassmann variable) spinor ϵ, subject to the Majorana condition. It is then natural to expect that the coordinates of a point in superspace will be given by

$$(x^\mu, \theta),$$

where θ is also a Majorana anticommuting spinor. The spinors ϵ and θ have only two independent components. We will then adopt a two-component notation which will prove much useful to handle the algebra of the Grassmann variables contained in θ.

4.1. The two component notation

We have chosen γ-matrices such that a Dirac spinor ψ can be written

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}. \tag{4.2}$$

ψ_L is a left-handed two component Weyl spinor, transforming under the Lorentz group $SO(1,3) \sim Sl(2,C)$ according to the (1/2,0) representation (i.e. like a spin 1/2 for 'left' transformations and a scalar for 'right' transformations). Accordingly, ψ_R transforms like the (0,1/2) representation. We will then use the notation

$$\begin{aligned} \psi_L &= \psi_\alpha, \qquad \alpha = 1,2, \\ \psi_R &= \overline{\psi}^{\dot\alpha}, \qquad \dot\alpha = 1,2, \end{aligned} \tag{4.3}$$

to indicate these transformation properties. The Lorentz group $SO(1,3) \sim Sl(2,C)$ acts on these two-component spinors via 2×2 matrices M of unit determinant (i.e. $Sl(2,C)$ matrices) which represent the Lorentz group. Since M^*, $(M^\tau)^{-1}$ and $(M^\dagger)^{-1}$ are also $Sl(2,C)$ matrices, they also represent the action of the Lorentz group on two-component spinors. The indices α and $\dot\alpha$ correspond to the following transformations:

$$\begin{aligned} \psi'_\alpha &= M_\alpha{}^\beta \psi_\beta, \qquad & \psi'^\alpha &= (M^{-1})_\beta{}^\alpha \psi^\beta, \\ \overline{\psi}'_{\dot\alpha} &= (M^*)_{\dot\alpha}{}^{\dot\beta} \overline{\psi}_{\dot\beta}, \qquad & \overline{\psi}'^{\dot\alpha} &= (M^{*-1})_{\dot\beta}{}^{\dot\alpha} \overline{\psi}^{\dot\beta}. \end{aligned} \tag{4.4}$$

This definition means that $\psi_1^\alpha \psi_{2\alpha}$ and $\overline{\psi}_1^{\dot\alpha} \overline{\psi}_{2\dot\alpha}$ are Lorentz invariant:

$$\begin{aligned} \psi_1'^\alpha \psi'_{2\alpha} &= \psi_1^\beta (M^{-1})_\beta{}^\alpha M_\alpha{}^\gamma \psi_{2\gamma} &= \psi_1^\alpha \psi_{2\alpha}, \\ \overline{\psi}_1'^{\dot\alpha} \overline{\psi}'_{2\dot\alpha} &= \overline{\psi}_1^{\dot\beta} (M^{*-1})_{\dot\beta}{}^{\dot\alpha} (M^*)_{\dot\alpha}{}^{\dot\gamma} \overline{\psi}_{2\dot\gamma} &= \overline{\psi}_1^{\dot\alpha} \overline{\psi}_{2\dot\alpha}. \end{aligned} \tag{4.5}$$

We now have the identity

$$\epsilon_{\alpha\gamma} M_\rho{}^\alpha M_\delta{}^\gamma = det(M)\, \epsilon_{\rho\delta}, \tag{4.6}$$

where $\epsilon_{\alpha\beta}$ is the antisymmetric tensor. This identity means that $\epsilon_{\alpha\beta}$ is Lorentz invariant, since $det\,(M) = 1$. Defining now $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$ such that

$$\epsilon_{\alpha\beta} \epsilon^{\beta\gamma} = \delta_\alpha^\gamma, \tag{4.7}$$

one gets

$$\epsilon_{\alpha\gamma} M_\rho{}^\alpha M_\delta{}^\gamma \epsilon^{\delta\beta} = \delta_\rho^\beta, \tag{4.8}$$

or

$$(M^{-1})_\alpha{}^\beta = \epsilon_{\alpha\gamma} M_\delta{}^\gamma \epsilon^{\delta\beta}. \tag{4.9}$$

One can then use $\epsilon^{\alpha\beta}$ ($\epsilon_{\alpha\beta}$) to raise (lower) indices:

$$\begin{aligned} \psi^\alpha &= \epsilon^{\alpha\beta} \psi_\beta, \\ \psi_\alpha &= \epsilon_{\alpha\beta} \psi^\beta. \end{aligned} \tag{4.10}$$

Notice that for anticommuting spinors

$$\psi_1^\alpha\psi_{2\alpha} = \epsilon^{\alpha\beta}\epsilon_{\alpha\gamma}\psi_{1\beta}\psi_2^\gamma = -\psi_{1\beta}\psi_2^\beta = \psi_2^\beta\psi_{1\beta}. \tag{4.11}$$

We will use the convention

$$\psi_1\psi_2 \equiv \psi_1^\alpha\psi_{2\alpha} = \psi_2\psi_1. \tag{4.12}$$

Four component spinors are then written

$$\begin{aligned}\psi &= \begin{pmatrix}\psi_\alpha\\ \overline{\chi}^{\dot\alpha}\end{pmatrix},\\ \overline{\psi} &= \begin{pmatrix}\chi^\alpha & \overline{\psi}_{\dot\alpha}\end{pmatrix},\end{aligned} \tag{4.13}$$

which implies that

$$\overline{\psi}\psi = \chi^\alpha\psi_\alpha + \overline{\psi}_{\dot\alpha}\overline{\chi}^{\dot\alpha}. \tag{4.14}$$

It is then natural to choose the convention

$$\overline{\psi}_1\overline{\psi}_2 \equiv \overline{\psi}_{1\dot\alpha}\overline{\psi}_2^{\dot\alpha} = \overline{\psi}_2\overline{\psi}_1, \tag{4.15}$$

with

$$\begin{aligned}\overline{\psi}_{\dot\alpha} &= \epsilon_{\dot\alpha\dot\beta}\overline{\psi}^{\dot\beta},\\ \overline{\psi}^{\dot\alpha} &= \epsilon^{\dot\alpha\dot\beta}\overline{\psi}_{\dot\beta},\\ \epsilon_{\dot\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma} &= \delta_{\dot\alpha}^{\dot\gamma}.\end{aligned} \tag{4.16}$$

A Majorana spinor satisfies the condition

$$\begin{aligned}\overline{\chi}^{\dot\alpha} &= (i\sigma^2\overline{\psi}^\tau)^{\dot\alpha}\\ &= \epsilon^{\dot\alpha\dot\beta}\overline{\chi}_{\dot\beta} = \epsilon^{\dot\alpha\dot\beta}(\chi_\beta)^*.\end{aligned} \tag{4.17}$$

The last equality is due to Eq. (4.4). This forces us to choose $\epsilon^{\dot\alpha\dot\beta} = (i\sigma^2)^{\dot\alpha\dot\beta}$, or

$$\begin{aligned}\epsilon^{\dot1\dot2} &= -\epsilon^{\dot2\dot1} = 1,\\ \epsilon_{\dot1\dot2} &= -\epsilon_{\dot2\dot1} = -1,\\ \epsilon^{\dot1\dot1} &= \epsilon^{\dot2\dot2} = \epsilon_{\dot1\dot1} = \epsilon_{\dot2\dot2} = 0.\end{aligned} \tag{4.18}$$

The set of γ-matrices appropriate to the two-component notation reads

$$\gamma^\mu = \begin{pmatrix}0 & (\sigma^\mu)_{\alpha\dot\alpha}\\ (\overline{\sigma}^\mu)^{\dot\alpha\alpha} & 0\end{pmatrix}, \tag{4.19}$$

with

$$(\overline{\sigma}^\mu)^{\dot\alpha\alpha} = \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}(\sigma^\mu)_{\beta\dot\beta}. \tag{4.20}$$

In agreement with the γ-matrices of appendix A, we choose

$$\sigma^\mu = (1, -\sigma^i), \tag{4.21}$$

and we would like to find

$$\overline{\sigma}^\mu = (1, \sigma^i). \tag{4.22}$$

This is the case if $\epsilon^{\alpha\beta} = (i\sigma^2)^{\alpha\beta}$ or

$$\begin{aligned}\epsilon^{12} &= -\epsilon^{21} = 1,\\ \epsilon_{12} &= -\epsilon_{21} = -1,\\ \epsilon^{11} &= \epsilon^{22} = \epsilon_{11} = \epsilon_{22} = 0.\end{aligned} \tag{4.23}$$

Then, in matrix notation, Eq. (4.20) reads

$$(\overline{\sigma}^\mu)^\tau = \sigma^2\sigma^\mu\sigma^2, \tag{4.24}$$

which leads to the result (4.22).

One can prove several useful identities in this formalism. Some of them are collected in appendix B.

We can now rewrite the supersymmetry algebra in two component notation. Since the superfield formalism is available essentially only for $N = 1$ supersymmetry, we consider only one supersymmetry charge Q. It is a Majorana spinor:

$$Q = \begin{pmatrix}Q_\alpha\\ \overline{Q}^{\dot\alpha}\end{pmatrix}. \tag{4.25}$$

The anticommutator in Eqs. (3.27) reads

$$\begin{pmatrix}\{Q_\alpha, Q_\beta\} & \{Q_\alpha, \overline{Q}^{\dot\beta}\}\\ \{\overline{Q}^{\dot\alpha}, Q_\beta\} & \{\overline{Q}^{\dot\alpha}, \overline{Q}^{\dot\beta}\}\end{pmatrix} = \begin{pmatrix}0 & -2P_\mu(i\sigma^\mu\sigma^2)\\ 2P_\mu(i\overline{\sigma}^\mu\sigma^2) & 0\end{pmatrix}, \tag{4.26}$$

where $(i\sigma^\mu\sigma^2) = \sigma^\mu{}_{\alpha\dot\alpha}\epsilon^{\dot\alpha\dot\beta}$. One then deduces that

$$\begin{aligned}\{Q_\alpha, \overline{Q}_{\dot\beta}\} &= \epsilon_{\dot\beta\dot\gamma}\{Q_\alpha, \overline{Q}^{\dot\gamma}\} = -2P_\mu(\sigma^\mu)_{\alpha\dot\alpha}\epsilon^{\dot\alpha\dot\gamma}\epsilon_{\dot\beta\dot\gamma}\\ &= 2P_\mu(\sigma^\mu)_{\alpha\dot\beta},\\ \{Q_\alpha, \overline{Q}_{\dot\beta}\} &= \epsilon_{\dot\beta\dot\alpha}\{\overline{Q}^{\dot\alpha}, Q_\alpha\} = -2P_\mu(\overline{\sigma}^\mu)^{\dot\alpha\gamma}\epsilon_{\gamma\alpha}\epsilon_{\dot\beta\dot\alpha}\\ &= -2P_\mu\epsilon^{\dot\alpha\dot\delta}\epsilon^{\gamma\delta}(\sigma^\mu)_{\delta\dot\delta}\epsilon_{\gamma\alpha}\epsilon_{\dot\beta\dot\alpha}\\ &= 2P_\mu(\sigma^\mu)_{\alpha\dot\beta}.\end{aligned}$$

The supersymmetry algebra is then

$$\{Q_\alpha, Q_\beta\} = \{\overline{Q}_{\dot\alpha}, \overline{Q}_{\dot\beta}\} = 0, \qquad \{Q_\alpha, \overline{Q}_{\dot\beta}\} = 2P_\mu(\sigma^\mu)_{\alpha\dot\beta}. \tag{4.27}$$

(In trying to compare this algebra with, for instance, (3.45) or (3.64), one must remember that $\overline{Q}_{\dot 1} = -\overline{Q}^{\dot 2} = -Q_4$ and $\overline{Q}_{\dot 2} = \overline{Q}^{\dot 1} = Q_3$). Introducing a spinorial parameter

$$\epsilon = \begin{pmatrix} \epsilon_\alpha \\ \overline{\epsilon}^{\dot\alpha} \end{pmatrix}, \tag{4.28}$$

a generic supersymmetry transformation (see Eq. 3.10) will read

$$\delta(\phi) = i\left(\epsilon Q + \overline{\epsilon}\overline{Q}\right)(\phi) = i\left(\epsilon^\alpha Q_\alpha + \overline{\epsilon}_{\dot\alpha}\overline{Q}^{\dot\alpha}\right)(\phi), \tag{4.29}$$

and the algebra is

$$[\epsilon_1 Q + \overline{\epsilon}_1\overline{Q}, \epsilon_2 Q + \overline{\epsilon}_2\overline{Q}] = \left(\epsilon_1^\alpha\overline{\epsilon}_2^{\dot\alpha} + \overline{\epsilon}_1^{\dot\alpha}\epsilon_2^\alpha\right)\{Q_\alpha, \overline{Q}_{\dot\alpha}\} = 2\left(\epsilon_1\sigma^\mu\overline{\epsilon}_2 - \epsilon_2\sigma^\mu\overline{\epsilon}_1\right)P_\mu, \tag{4.30}$$

so that

$$[\delta_1, \delta_2](\phi) = -2(\epsilon_1\sigma^\mu\overline{\epsilon}_2 - \epsilon_2\sigma^\mu\overline{\epsilon}_1)P_\mu, \tag{4.31}$$

where

$$\epsilon\sigma^\mu\overline{\epsilon} = \epsilon^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\overline{\epsilon}^{\dot\alpha}.$$

In order to obtain the algebra (4.30), one assumes that the parameters ϵ^α and $\overline{\epsilon}^{\dot\alpha}$ are Grassmann variables:

$$\{\epsilon^\alpha, \epsilon^\beta\} = \{\epsilon^\alpha, \overline{\epsilon}^{\dot\alpha}\} = \{\overline{\epsilon}^{\dot\alpha}, \overline{\epsilon}^{\dot\beta}\} = 0, \qquad \{\epsilon^\alpha, Q^\alpha\} = \{\epsilon^\alpha, \overline{Q}^{\dot\alpha}\} = \{\overline{\epsilon}^{\dot\alpha}, Q^\alpha\} = \{\overline{\epsilon}^{\dot\alpha}, \overline{Q}^{\dot\alpha}\} = 0. \tag{4.32}$$

4.2. Superfields

The parameters of supersymmetry transformations are the two component spinors ϵ^α and $\overline{\epsilon}^{\dot\alpha}$. We want to introduce new fields (*superfields*) ϕ such that the charges Q_α and $\overline{Q}_{\dot\alpha}$ act on ϕ through derivatives only, in analogy with translations. We then enlarge space-time to *superspace*, with new coordinates θ^α and $\overline{\theta}^{\dot\alpha}$ which are Grassmann variables:

$$0 = \{\theta^\alpha, \theta^\beta\} = \{\overline{\theta}^{\dot\alpha}, \overline{\theta}^{\dot\beta}\} = \{\theta^\alpha, \overline{\theta}^{\dot\beta}\}. \tag{4.33}$$

A point in superspace has coordinates

$$X = (x^\mu, \theta^\alpha, \overline{\theta}^{\dot\alpha}). \tag{4.34}$$

We must then find the representation of the charges Q^α, $\overline{Q}^{\dot\alpha}$ acting on the superfield $\phi(x, \theta, \overline{\theta})$, and satisfying the correct algebra (4.27). Notice that since ϵ^α and $\overline{\epsilon}^{\dot\alpha}$ have dimension $(\mathrm{mass})^{-1/2}$, θ_α and $\overline{\theta}_{\dot\alpha}$ should have the same dimension.

We want to find charges Q_α and $\overline{Q}_{\dot\alpha}$ (dimension 1/2) containing derivatives ∂_μ (dimension 1), $\frac{d}{d\theta^\alpha}$ and $\frac{d}{d\overline{\theta}^{\dot\alpha}}$ (dimension 1/2). Then:

$$Q_\alpha = a^\mu_\alpha\partial_\mu + b\frac{\partial}{\partial\theta^\alpha} + c_{\alpha\dot\alpha}\frac{\partial}{\partial\overline{\theta}_{\dot\alpha}}, \qquad \overline{Q}_{\dot\alpha} = \overline{a}^\mu_{\dot\alpha}\partial_\mu + \overline{b}\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} + \overline{c}^\alpha_{\dot\alpha}\frac{\partial}{\partial\theta^\alpha}. \tag{4.35}$$

In these expressions, b, $\overline{b}$, $c_{\alpha\dot\alpha}$ and $\overline{c}^\alpha_{\dot\alpha}$ are dimensionless, while a^μ_α and $\overline{a}^\mu_{\dot\alpha}$ have dimension -1/2. Their natural form is:

$$a^\mu_\alpha = a(\sigma^\mu)_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}, \qquad \overline{a}^\mu_{\dot\alpha} = \overline{a}\theta^\alpha(\sigma^\mu)_{\alpha\dot\alpha}, \tag{4.36}$$

with a and $\overline{a}$ dimensionless. Then, observe that

$$\left\{\frac{\partial}{\partial\theta^\alpha}, (\sigma^\mu)_{\beta\dot\alpha}\overline{\theta}^{\dot\alpha}\partial_\mu\right\} = -(\sigma^\mu)_{\beta\dot\alpha}\overline{\theta}^{\dot\alpha}\left[\frac{\partial}{\partial\theta^\alpha}, \partial_\mu\right] = 0,$$
$$\left\{\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}}, \theta^\alpha(\sigma^\mu)_{\alpha\dot\beta}\partial_\mu\right\} = 0, \tag{4.37}$$
$$\left\{\frac{\partial}{\partial\theta^\alpha}, \theta^\beta(\sigma^\mu)_{\beta\dot\beta}\partial_\mu\right\} = (\sigma^\mu)_{\alpha\dot\beta}\partial_\mu.$$

Also

$$\left[\epsilon^\alpha\frac{\partial}{\partial\theta^\alpha}, (\theta^\beta\sigma^\mu_{\beta\dot\beta}\overline{\epsilon}^{\dot\beta})\partial_\mu\right] = (\epsilon^\alpha\sigma^\mu_{\alpha\dot\alpha}\overline{\epsilon}^{\dot\alpha})\partial_\mu, \tag{4.38}$$

which is precisely of the form required by the supersymmetry algebra (4.30). To avoid other unwanted terms, we then assume that a, $\overline{a}$, b and $\overline{b}$ are complex numbers, and, in order to obtain $\{Q_\alpha, Q_\beta\} = \{\overline{Q}_{\dot\alpha}, \overline{Q}_{\dot\beta}\} = 0$, one must choose $c_{\alpha\dot\alpha} = \overline{c}^\alpha_{\dot\alpha} = 0$. The only non trivial anticommutator is then

$$\{Q_\alpha, \overline{Q}_{\dot\beta}\} = (\overline{a}b + a\overline{b})(\sigma^\mu)_{\alpha\dot\beta}\partial_\mu, \qquad [\epsilon_1^\alpha Q_\alpha, \overline{\epsilon}_{2\dot\alpha}\overline{Q}^{\dot\alpha}] = (\overline{a}b + a\overline{b})(\epsilon_1\sigma^\mu\overline{\epsilon}_2)\partial_\mu. \tag{4.39}$$

Then:

$$[\epsilon_1^\alpha Q_\alpha + \bar\epsilon_{1\dot\alpha}\overline{Q}^{\dot\alpha}, \epsilon_2^\beta Q_\beta + \bar\epsilon_{2\dot\beta}\overline{Q}^{\dot\beta}] = (\bar a b + a\bar b)(\epsilon_1\sigma^\mu\bar\epsilon_2 - \epsilon_2\sigma^\mu\bar\epsilon_1)\partial_\mu. \tag{4.40}$$

To reproduce (4.27) (with $P_\mu = -i\partial_\mu$) and (4.30), we must choose

$$\bar a b + a\bar b = -2i \tag{4.41}$$

(Recall that a and $\bar a$ or b and $\bar b$ are independent complex numbers and not conjugate each other). The form of the anticommutators is not strong enough to completely determine the charges. One can choose:

$$\begin{aligned} Q_{1\alpha} &= -i\left(\frac{\partial}{\partial\theta^\alpha} - i(\sigma^\mu)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu\right), \\ \overline{Q}_{1\dot\alpha} &= -i\left(-\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i\theta^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu\right). \end{aligned} \tag{4.42}$$

(The index 1 indicates that other choices exist).

Another approach is to write (formal) group elements generalizing again the case of translations for which

$$\phi(x_\mu + \Delta_\mu) = e^{i\Delta_\mu(-i\partial^\mu)}\phi(x_\mu).$$

The group elements we are looking for will be of the form

$$G(\Delta,\epsilon,\bar\epsilon) = e^{i(\Delta_\mu P^\mu + \epsilon Q + \bar\epsilon\overline{Q})}. \tag{4.43}$$

As in the case of translations, the action of a group element G on a superfield ϕ will induce a motion in parameter space. Since $[P^\mu, Q_\alpha] = [P^\mu, \overline{Q}_{\dot\alpha}] = 0$, one can easily calculate this motion. One has

$$e^A e^B = a^{A+B+\frac{1}{2}[A,B]} \tag{4.44}$$

since all higher commutators vanish. The action of the group element $G(0,\epsilon,\bar\epsilon)$ on the superfield and group element $G(x,\theta,\bar\theta)$ is

$$G(0,\epsilon,\bar\epsilon)G(x,\theta,\bar\theta) = G(x^\mu + i\epsilon\sigma^\mu\bar\theta - i\theta\sigma^\mu\bar\epsilon, \epsilon+\theta, \bar\epsilon+\bar\theta) \tag{4.45}$$

since in this case $A + B + [A,B]/2$ is

$$i\left(x_\mu P^\mu + (\epsilon+\theta)Q + (\bar\epsilon+\bar\theta)\overline{Q} + i\epsilon\sigma^\mu\bar\theta P_\mu - i\theta\sigma^\mu\bar\epsilon P_\mu\right).$$

Then, Q generates a translation $i\epsilon\sigma^\mu\bar\theta$ in x^μ-space and a translation ϵ of the coordinate θ. Since x^μ translations are generated by $-i\partial_\mu$, Q_α should contain a term $(\sigma^\mu)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu$. Analogously, $\overline{Q}_{\dot\alpha}$ should contain a term $-\theta^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu$. To obtain the translation operators for the spinorial coordinates, we study the case of functions $F(\theta)$ and $\overline{F}(\bar\theta)$ which, since $\theta^3 = \bar\theta^3 = 0$ have the power expansions

$$\begin{aligned} F(\theta) &= a + b^\alpha\theta_\alpha + c\theta^\alpha\theta_\alpha, \\ \overline{F}(\bar\theta) &= \bar a + \bar b_{\dot\alpha}\bar\theta^{\dot\alpha} + \bar c\bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}. \end{aligned} \tag{4.46}$$

Then, using the identities listed in appendix B, one gets the Taylor expansions

$$\begin{aligned} F(\theta+\epsilon) &= F(\theta) + \epsilon^\alpha\frac{\partial}{\partial\theta^\alpha}F + \frac{1}{2}\left(\epsilon^\alpha\frac{\partial}{\partial\theta^\alpha}\right)\left(\epsilon^\beta\frac{\partial}{\partial\theta^\beta}\right)F \\ &= F(\theta) + \epsilon_\alpha\frac{\partial}{\partial\theta_\alpha}F + \frac{1}{2}\left(\epsilon_\alpha\frac{\partial}{\partial\theta_\alpha}\right)\left(\epsilon_\beta\frac{\partial}{\partial\theta_\beta}\right)F, \\ \overline{F}(\bar\theta+\bar\epsilon) &= \overline{F}(\bar\theta) + \bar\epsilon^{\dot\alpha}\frac{\partial}{\partial\bar\theta^{\dot\alpha}}\overline{F} + \frac{1}{2}\left(\bar\epsilon^{\dot\alpha}\frac{\partial}{\partial\bar\theta^{\dot\alpha}}\right)\left(\bar\epsilon^{\dot\beta}\frac{\partial}{\partial\bar\theta^{\dot\beta}}\right)\overline{F} \\ &= \overline{F}(\bar\theta) + \bar\epsilon_{\dot\alpha}\frac{\partial}{\partial\bar\theta_{\dot\alpha}}\overline{F} + \frac{1}{2}\left(\bar\epsilon_{\dot\alpha}\frac{\partial}{\partial\bar\theta_{\dot\alpha}}\right)\left(\bar\epsilon_{\dot\beta}\frac{\partial}{\partial\bar\theta_{\dot\beta}}\right)\overline{F}. \end{aligned} \tag{4.47}$$

To first order, it is apparent that θ-translations are generated by

$$-i\frac{\partial}{\partial\theta^\alpha} \equiv (T_\theta)_\alpha \tag{4.48}$$

and $\bar\theta$-translations by

$$+i\frac{\partial}{\partial\bar\theta^{\dot\alpha}} \equiv (\overline{T}_{\bar\theta})_{\dot\alpha}, \tag{4.49}$$

so that a general translation group element to first order is

$$\begin{aligned} 1 + i(\epsilon T_\theta + \bar\epsilon\overline{T}_{\bar\theta}) &= 1 + i(\epsilon^\alpha T_{\theta\alpha} + \bar\epsilon_{\dot\alpha}\overline{T}^{\dot\alpha}_{\bar\theta}) \\ &= 1 + i(\epsilon^\alpha T_{\theta\alpha} - \bar\epsilon^{\dot\alpha}\overline{T}_{\bar\theta\dot\alpha}) = 1 + \epsilon^\alpha\frac{\partial}{\partial\theta^\alpha} + \bar\epsilon^{\dot\alpha}\frac{\partial}{\partial\bar\theta^{\dot\alpha}} \end{aligned} \tag{4.50}$$

as required by expansions (4.47). One then obtains

$$\begin{aligned} Q_\alpha &= -i\left(\frac{\partial}{\partial\theta^\alpha} + i(\sigma^\mu)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu\right), \\ \overline{Q}_{\dot\alpha} &= -i\left(-\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu\right). \end{aligned} \tag{4.51}$$

Notice that (see appendix B):

$$\overline{Q}^{\dot\alpha} = -i\left(\frac{\partial}{\partial\bar\theta_{\dot\alpha}} + i\theta^\beta(\sigma^\mu)_{\beta\dot\beta}\epsilon^{\dot\beta\dot\alpha}\partial_\mu\right), \tag{4.52}$$

so that

$$\epsilon Q + \bar{\epsilon}\overline{Q} = -i\left(\epsilon^\alpha \frac{\partial}{\partial\theta^\alpha} + \bar{\epsilon}_{\dot\alpha}\frac{\partial}{\partial\overline{\theta}_{\dot\alpha}} + i(\epsilon\sigma^\mu\overline{\theta} - \theta\sigma^\mu\bar{\epsilon})\partial_\mu\right). \tag{4.53}$$

The supersymmetry charges (4.51) satisfy an algebra with the wrong sign of P_μ (compare with the condition 4.41):

$$\{Q_\alpha, \overline{Q}_{\dot\alpha}\} = 2(\sigma^\mu)_{\alpha\dot\alpha}(i\partial_\mu). \tag{4.54}$$

This sign arises only because we have chosen to act on the left with the group elements. We could also choose to act on the right. Since

$$G(x,\theta,\overline{\theta})G(0,\epsilon,\bar{\epsilon}) = G(x^\mu - i\epsilon\sigma^\mu\overline{\theta} + i\theta\sigma^\mu\bar{\epsilon}, \epsilon+\theta, \bar{\epsilon}+\overline{\theta}),$$

we find new charges with reversed ∂_μ parts:

$$\begin{aligned} Q'_\alpha &= -i\left(\frac{\partial}{\partial\theta^\alpha} - i(\sigma^\mu)_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}\partial_\mu\right), \\ \overline{Q}'_{\dot\alpha} &= -i\left(-\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} + i\theta^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu\right). \end{aligned} \tag{4.55}$$

These charges are identical to those given in Eq. (4.42). They then satisfy the algebra

$$\{Q'_\alpha, \overline{Q}'_{\dot\alpha}\} = -2(\sigma^\mu)_{\alpha\dot\alpha}(i\partial_\mu),$$

with the usual sign. Conventionnally, the charges (4.51) are used in the literature. In the following we will accept and follow this convention.

A general superfield $F(x,\theta,\overline{\theta})$ is defined by its expansion in powers of θ and $\overline{\theta}$:

$$\begin{aligned} F(x,\theta,\overline{\theta}) =& f(x) + \theta\phi(x) + \overline{\theta}\overline{\chi}(x) + \theta\theta m(x) + \overline{\theta}\overline{\theta}n(x) + \\ &+ \theta\sigma^\mu\overline{\theta}v_\mu(x) + \theta\theta\overline{\theta}\overline{\lambda}(x) + \overline{\theta}\overline{\theta}\theta\psi(x) + \theta\theta\overline{\theta}\overline{\theta}d(x). \end{aligned} \tag{4.56}$$

In this expression:

f, m, n, d are complex scalar fields,

$\phi_\alpha, \overline{\chi}^{\dot\alpha}, \overline{\lambda}^{\dot\alpha}, \psi_\alpha$ are two-component spinors,

v_μ is a complex vector field.

We then have in general 16 real bosonic components (two for each scalar, eight for v_μ) and 16 real fermions (four for each spinor). We have an equal number of fermions and bosons, but far too many fields to describe small multiplets like for instance the chiral multiplet. In general, F can also have Lorentz quantum numbers (F_μ, $F_{\mu\nu}$, F_α,

F^α_μ, ...). Each field ($f, m, \ldots$) of the expansion will then carry the same indices as F, in addition to its intrinsic quantum numbers as given in (4.56). Such a multiplet has then $k(16_{bos.} + 16_{fer.})$ real components, k being the number of Lorentz components of F.

The supersymmetry transformations of the components of F are simply given by

$$\delta F = i(\epsilon Q + \bar{\epsilon}\overline{Q})F(x,\theta,\overline{\theta}). \tag{4.57}$$

The transformation of each component is obtained by identifying the term of δF with the same θ and $\overline{\theta}$ structure as the component. One can construct the superfield corresponding to any component supermultiplet by starting from one of the components and acting on this component with the transformation (4.57) iteratively until the multiplet is closed.

The superfield F contains too many degrees of freedom to describe small multiplets like the chiral multiplet. It is however reducible in the sense that one can impose constraints which are preserved by supersymmetry transformations (4.57). To do this, we first need to introduce *covariant derivatives*. These covariant derivatives D_α and $\overline{D}_{\dot\alpha}$ will be used to impose constraints of the form $D_\alpha F = 0$ or $\overline{D}_{\dot\alpha}F = 0$. These derivative constraints will prove to have no dynamical content. They only reduce the number of components. Covariant derivatives will also play a central role in the construction of supersymmetric actions in the superfield formalism. To construct them, one must require that they transform covariantly under supersymmetry:

$$D_\alpha(\delta F) = \delta(D_\alpha F), \qquad \overline{D}_{\dot\alpha}(\delta F) = \delta(\overline{D}_{\dot\alpha}F). \tag{4.58}$$

These conditions correspond to

$$\{Q_\alpha, D_\beta\} = \{\overline{Q}_{\dot\alpha}, D_\beta\} = \{Q_\alpha, \overline{D}_{\dot\beta}\} = \{\overline{Q}_{\dot\alpha}, \overline{D}_{\dot\beta}\} = 0. \tag{4.59}$$

We have

$$\begin{aligned} \{Q_\alpha, \frac{\partial}{\partial\theta^\beta}\} &= \{\overline{Q}_{\dot\alpha}, \frac{\partial}{\partial\overline{\theta}^{\dot\beta}}\} = 0, \\ \{Q_\alpha, \frac{\partial}{\partial\overline{\theta}^{\dot\beta}}\} &= \frac{\partial}{\partial\overline{\theta}^{\dot\beta}}(\sigma^\mu)_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}\partial_\mu = (\sigma^\mu)_{\alpha\dot\beta}\partial_\mu, \\ \{\overline{Q}_{\dot\alpha}, \frac{\partial}{\partial\theta^\beta}\} &= \frac{\partial}{\partial\theta^\beta}\theta^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu = -(\sigma^\mu)_{\beta\dot\alpha}\partial_\mu. \end{aligned} \tag{4.60}$$

Also:

$$\begin{aligned} \{Q_\alpha, \theta^\beta(\sigma^\mu)_{\beta\dot\beta}\partial_\mu\} &= -i\frac{\partial}{\partial\theta^\alpha}\theta^\beta(\sigma^\mu)_{\beta\dot\beta}\partial_\mu = -i(\sigma^\mu)_{\alpha\dot\beta}\partial_\mu, \\ \{\overline{Q}_{\dot\alpha}, (\sigma^\mu)_{\beta\dot\beta}\overline{\theta}^{\dot\beta}\partial_\mu\} &= i\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}}(\sigma^\mu)_{\beta\dot\beta}\overline{\theta}^{\dot\beta}\partial_\mu = i(\sigma^\mu)_{\beta\dot\alpha}\partial_\mu, \\ \{Q_\alpha, (\sigma^\mu)_{\beta\dot\beta}\overline{\theta}^{\dot\beta}\partial_\mu\} &= \{\overline{Q}_{\dot\alpha}, \theta^\beta(\sigma^\mu)_{\beta\dot\beta}\partial_\mu\} = 0. \end{aligned} \tag{4.61}$$

It then follows that the expressions

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial\theta^\alpha} - i(\sigma^\mu)_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}\partial_\mu, \\ \overline{D}_{\dot\alpha} &= \frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} - i\theta^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu, \end{aligned} \tag{4.62}$$

satisfy Eqs. (4.59). Comparing with (4.55), we see that $Q'_\alpha = -iD_\alpha$ and $\overline{Q}'_{\dot\alpha} = i\overline{D}_{\dot\alpha}$, so that D_α and $\overline{D}_{\dot\alpha}$ also verify

$$\begin{aligned} \{D_\alpha, \overline{D}_{\dot\alpha}\} &= 2(\sigma^\mu)_{\alpha\dot\alpha}(-i\partial_\mu), \\ \{D_\alpha, D_\beta\} &= \{\overline{D}_{\dot\alpha}, \overline{D}_{\dot\beta}\} = 0. \end{aligned} \tag{4.63}$$

Constraints like $D_\alpha F = 0$ or $\overline{D}_{\dot\alpha} F = 0$ are then preserved by supersymmetry transformations. They give rise to chiral multiplets (see the next section).

One can also impose a reality constraint, $F = F^*$, which corresponds to vector multiplets (section 4.4).

The superfield formalism is particularly convenient to combine supersymmetry representations. This is related to the fact that supersymmetry generators Q_α and $\overline{Q}_{\dot\alpha}$ are derivatives. Then, clearly, any linear combination of superfields is a superfield. Also, Q_α and $\overline{Q}_{\dot\alpha}$ act with a chain rule:

$$\begin{aligned} Q_\alpha(F_1F_2) &= (Q_\alpha F_1)F_2 \pm F_1(Q_\alpha F_2), \\ \overline{Q}_{\dot\alpha}(F_1F_2) &= (\overline{Q}_{\dot\alpha}F_1)F_2 \pm F_1(\overline{Q}_{\dot\alpha}F_2), \end{aligned} \tag{4.64}$$

where the plus and minus signs apply if F_1 is a boson or a fermion respectively. It follows that a product of superfields is also a superfield.

This calculus prescription suggests a systematic method to obtain supersymmetric Lagrangians. The $\theta\theta\overline{\theta\theta}$ component of a superfield always transforms with a total derivative: in transformation (4.57), the $\theta\theta\overline{\theta\theta}$ component of δF arises either through the $\epsilon\sigma^\mu\overline{\theta}\partial_\mu$ part of ϵQ acting on the $\theta\overline{\theta\theta}$ component of F, or through the $\theta\sigma^\mu\overline{\epsilon}\partial_\mu$ part of $\overline{\epsilon Q}$ acting on the $\overline{\theta}\theta\theta$ component. This observation leads to the superfield formulation of the tensor calculus method outlined in ch. 2. To construct supersymmetric Lagrangians for a set of general superfields F_i, one first writes a function of these superfields (in the simplest cases, this function is a linear combination of products of superfields). Since supersymmetry transformations act on this function of superfields with derivatives only, the function itself is a superfield (this generalizes eqs. 4.64). One then extracts the $\theta\theta\overline{\theta\theta}$ component of the function, which according to the previous remark always transform with a total derivative: it is then a supersymmetric Lagrangian.

Notice also that a constant c is a superfield. Since Q_α and $\overline{Q}_{\dot\alpha}$ are derivatives, c is invariant under supersymmetry transformations.

4.3. Chiral superfields

Chiral (or scalar) superfields are defined by the condition:

$$\overline{D}_{\dot\alpha}\phi = 0 \tag{4.65}$$

for left-handed chiral superfields and

$$D_\alpha\overline{\phi} = 0 \tag{4.66}$$

for right-handed chiral superfields. These conditions are easily solved by observing that

$$\begin{aligned} D_\alpha\overline{y}^\mu = 0, &\qquad \overline{y}^\mu = x^\mu + i\theta\sigma^\mu\overline{\theta}, \\ \overline{D}_{\dot\alpha}y^\mu = 0, &\qquad y^\mu = x^\mu - i\theta\sigma^\mu\overline{\theta}, \\ D_\alpha\overline{\theta} &= \overline{D}_{\dot\alpha}\theta = 0. \end{aligned} \tag{4.67}$$

A left-handed chiral superfield is then a fonction of y and θ only:

$$\phi(y,\theta) = z(y) + \sqrt{2}\theta\psi(y) - \theta\theta f(y). \tag{4.68}$$

The factor $\sqrt{2}$ and the minus sign are introduced for convenience: kinetic terms will be correctly normalized when constructing Lagrangians. We can Taylor expand ϕ around x^μ with the help of

$$\begin{aligned} z(y) &= z(x) - i\theta\sigma^\mu\overline{\theta}\partial_\mu z(x) - \frac{1}{4}\theta\theta\overline{\theta\theta}\partial^\mu\partial_\mu z(x), \\ \theta\psi(y) &= \theta\psi(x) - i(\theta\sigma^\mu\overline{\theta})(\theta\partial_\mu\psi(x)) = \theta\psi(x) + \frac{1}{2}i\theta\theta(\partial_\mu\psi(x)\sigma^\mu\overline{\theta}), \\ \theta\theta f(y) &= \theta\theta f(x). \end{aligned} \tag{4.69}$$

(See appendix B for the necessary identities). Then:

$$\begin{aligned} \phi(x,\theta,\overline{\theta}) =& z(x) + \sqrt{2}\theta\psi(x) - \theta\theta f(x) - i(\theta\sigma^\mu\overline{\theta})\partial_\mu z(x) + \\ &+ \frac{1}{\sqrt{2}}i\theta\theta(\partial_\mu\psi(x)\sigma^\mu\overline{\theta}) - \frac{1}{4}\theta\theta\overline{\theta\theta}\partial^\mu\partial_\mu z(x). \end{aligned} \tag{4.70}$$

One can derive the same results for right-handed chiral superfields $\overline{\phi}$:

$$\overline{\phi}(\overline{y},\overline{\theta}) = \overline{z}(\overline{y}) + \sqrt{2}\overline{\theta\psi}(\overline{y}) - \overline{\theta\theta f}(\overline{y}). \tag{4.71}$$

The expansion is

$$\begin{aligned}\overline{\phi}(x,\theta,\overline{\theta}) =& \overline{z}(x) + \sqrt{2}\overline{\theta}\overline{\psi}(x) - \overline{\theta}\overline{\theta}\overline{f}(x) + i(\theta\sigma^\mu\overline{\theta})\partial_\mu\overline{z}(x) - \\ &- \frac{1}{\sqrt{2}}i\overline{\theta}\overline{\theta}(\theta\sigma^\mu\partial_\mu\overline{\psi}(x)) - \frac{1}{4}\theta\theta\overline{\theta}\overline{\theta}\partial^\mu\partial_\mu\overline{z}(x),\end{aligned} \tag{4.72}$$

using the power expansions

$$\begin{aligned}\overline{z}(\overline{y}) &= \overline{z}(x) + i(\theta\sigma^\mu\overline{\theta})\partial_\mu\overline{z}(x) - \frac{1}{4}\theta\theta\overline{\theta}\overline{\theta}\partial^\mu\partial_\mu\overline{z}(x), \\ \overline{\theta}\overline{\psi}(\overline{y}) &= \overline{\theta}\overline{\psi}(x) + i(\theta\sigma^\mu\overline{\theta})(\overline{\theta}\partial_\mu\overline{\psi}(x) \\ &= \overline{\theta}\overline{\psi}(x) - \frac{1}{2}i\overline{\theta}\overline{\theta}(\theta\sigma^\mu\partial_\mu\overline{\psi}(x)), \\ \overline{\theta}\overline{\theta}\overline{f}(\overline{y}) &= \overline{\theta}\overline{\theta}\overline{f}(x).\end{aligned} \tag{4.73}$$

One immediately notices that

$$\overline{\phi}(x,\theta,\overline{\theta}) = \phi(x,\theta,\overline{\theta})^\dagger. \tag{4.74}$$

The left-handed chiral superfield ϕ contains a complex scalar z, a left-handed Weyl spinor ψ_α and a complex scalar field f. Choosing ϕ to have dimension $(\text{mass})^1$ gives the standard chiral multiplet (in the Weyl formalism) described in ch. 2. The transformations are easily obtained by using the charges (4.51). Firstly, observe that

$$\begin{aligned}Q_\alpha y^\mu &= \overline{Q}_{\dot\alpha}\overline{y}^\mu = 0, \\ Q_\alpha \overline{y}^\mu &= 2(\sigma^\mu\overline{\theta})_\alpha, \\ \overline{Q}_{\dot\alpha} y^\mu &= -2(\theta\sigma^\mu)_{\dot\alpha}.\end{aligned} \tag{4.75}$$

Then, acting on a superfield F depending on the variables y, θ and $\overline{\theta}$, one gets

$$\begin{aligned}Q_\alpha F(y,\theta,\overline{\theta}) &= (Q_\alpha y^\mu)\frac{\partial F}{\partial y^\mu} - i\frac{\partial}{\partial\theta^\alpha}F(y,\theta,\overline{\theta}) \\ &= -i\frac{\partial}{\partial\theta^\alpha}F(y,\theta,\overline{\theta}), \\ \overline{Q}_{\dot\alpha}F(y,\theta,\overline{\theta}) &= (\overline{Q}_{\dot\alpha}y^\mu)\frac{\partial F}{\partial y^\mu} + i\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}}F(y,\theta,\overline{\theta}) \\ &= -i\left(-\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} - 2i(\theta\sigma^\mu)_{\dot\alpha}\frac{\partial}{\partial y^\mu}\right)F(x,\theta,\overline{\theta}).\end{aligned} \tag{4.76}$$

Also:

$$\begin{aligned}Q_\alpha\overline{F}(\overline{y},\theta,\overline{\theta}) &= (Q_\alpha\overline{y}^\mu)\frac{\partial\overline{F}}{\partial\overline{y}^\mu} - i\frac{\partial}{\partial\theta^\alpha}\overline{F}(\overline{y},\theta,\overline{\theta}) \\ &= -i\left(\frac{\partial}{\partial\theta^\alpha} + 2i(\sigma^\mu\overline{\theta})_\alpha\frac{\partial}{\partial\overline{y}^\mu}\right)\overline{F}(\overline{y},\theta,\overline{\theta}), \\ \overline{Q}_{\dot\alpha}\overline{F}(\overline{y},\theta,\overline{\theta}) &= (\overline{Q}_{\dot\alpha}\overline{y}^\mu)\frac{\partial\overline{F}}{\partial\overline{y}^\mu} + i\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}}\overline{F}(\overline{y},\theta,\overline{\theta}) \\ &= i\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}}\overline{F}(\overline{y},\theta,\overline{\theta}).\end{aligned} \tag{4.77}$$

These results mean that the supersymmetric charges can be written

$$\begin{aligned}Q_\alpha &= -i\frac{\partial}{\partial\theta^\alpha}, \\ \overline{Q}_{\dot\alpha} &= -i\left(-\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} - 2i(\theta\sigma^\mu)_{\dot\alpha}\frac{\partial}{\partial y^\mu}\right),\end{aligned} \tag{4.78}$$

when acting on a superfield expressed in terms of variables $y, \theta, \overline{\theta}$. For left-handed chiral superfields,

$$\frac{\partial\phi}{\partial\overline{\theta}^{\dot\alpha}} = 0.$$

If the superfield is expressed in terms of $\overline{y}, \theta, \overline{\theta}$, the appropriate supercharges are

$$\begin{aligned}Q_\alpha &= -i\left(\frac{\partial}{\partial\theta^\alpha} + 2i(\sigma^\mu\overline{\theta})_\alpha\frac{\partial}{\partial\overline{y}^\mu}\right), \\ \overline{Q}_{\dot\alpha} &= i\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}}.\end{aligned} \tag{4.79}$$

These expressions allow an easier computation of the transformation rules of the chiral multiplet:

$$\begin{aligned}\delta\phi(y,\theta) &= i(\epsilon Q + \overline{\epsilon}\overline{Q})\phi(y,\theta) \\ &= \left(\epsilon^\alpha\frac{\partial}{\partial\theta^\alpha} - 2i(\theta\sigma^\mu\overline{\epsilon})\frac{\partial}{\partial y^\mu}\right)\phi(y,\theta) \\ &= \sqrt{2}\epsilon\psi + \sqrt{2}\theta(-\sqrt{2}\epsilon f - \sqrt{2}i(\sigma^\mu\overline{\epsilon})\partial_\mu z) + \sqrt{2}i\theta\theta(\partial_\mu\psi\sigma^\mu\overline{\epsilon}).\end{aligned} \tag{4.80}$$

The transformation rules are then

$$\begin{aligned}\delta z &= \sqrt{2}\epsilon\psi, \\ \delta\psi_\alpha &= -\sqrt{2}f\epsilon_\alpha - \sqrt{2}i(\sigma^\mu\overline{\epsilon})_\alpha\partial_\mu z, \\ \delta f &= -\sqrt{2}i(\partial_\mu\psi\sigma^\mu\overline{\epsilon}).\end{aligned} \tag{4.81}$$

These transformation rules can easily be compared with those of ch. 2 by rewriting Eqs. (2.59) in the two component notation. One finds that the multiplet (ϕ, ψ_L, f) corresponds to (z, ψ_α, f).

If ϕ is a left-handed chiral superfield, then ϕ^n is also a left-handed chiral superfield:

$$\overline{D}_{\dot\alpha}(\phi^n) = n\phi^{n-1}(\overline{D}_{\dot\alpha}\phi) = 0. \tag{4.82}$$

This is however not the case of $\phi^n\overline{\phi}^m$ $(m \geq 1)$. Since ϕ^n is a left-handed chiral superfield, its $\theta\theta$ component transforms like a total derivative according to Eq. (4.81). This suggests a systematic method to construct supersymmetric Lagrangians by taking

$$\mathcal{L} = \sum_{n\geq 1}[a_n\phi^n]_{\theta\theta} + \text{c.c.}, \tag{4.83}$$

where $[\dots]_{\theta\theta}$ indicates that the $\theta\theta$ component only is retained. This expression is clearly not sufficient: ϕ^n does not contain derivatives of fields and there is no kinetic Lagrangian. Kinetic terms are obtained from the $\theta\theta\overline{\theta\theta}$ component of $\phi^\dagger\phi$, which is not a chiral superfield. Computing $\phi^\dagger\phi$ (this should be done in variables $x, \theta, \overline{\theta}$) leads to

$$\begin{aligned}\phi^\dagger\phi = {} & z^\dagger z + \sqrt{2}z^\dagger(\theta\psi) + \sqrt{2}z(\overline{\theta\psi}) - fz^\dagger(\theta\theta) - f^\dagger z(\overline{\theta\theta}) + 2(\theta\psi)(\overline{\theta\psi}) \\ & + (\theta\sigma^\mu\overline{\theta})(iz\partial_\mu z^\dagger - iz^\dagger\partial_\mu z) + \frac{i}{\sqrt{2}}(\theta\theta)\left[z^\dagger(\partial_\mu\psi\sigma^\mu\overline{\theta}) - (\partial_\mu z^\dagger)(\psi\sigma^\mu\overline{\theta})\right] \\ & - \frac{i}{\sqrt{2}}(\overline{\theta\theta})\left[z(\theta\sigma^\mu\partial_\mu\overline{\psi}) - (\partial_\mu z)(\theta\sigma^\mu\overline{\psi})\right] - \sqrt{2}f(\theta\theta)(\overline{\theta\psi}) - \sqrt{2}f^\dagger(\overline{\theta\theta})(\theta\psi) \\ & + \mathcal{L}_K\,\theta\theta\overline{\theta\theta},\end{aligned} \tag{4.84}$$

with

$$\begin{aligned}\mathcal{L}_K = {} & ff^\dagger - \frac{1}{4}(z\partial_\mu\partial^\mu z^\dagger + z^\dagger\partial_\mu\partial^\mu z) + \frac{1}{2}(\partial_\mu z)(\partial^\mu z^\dagger) \\ & + \frac{i}{2}(\psi\sigma^\mu\partial_\mu\overline{\psi}) - \frac{i}{2}(\partial_\mu\psi\sigma^\mu\overline{\psi}) \\ = {} & ff^\dagger + (\partial_\mu z)(\partial^\mu z^\dagger) + \frac{i}{2}(\psi\sigma^\mu\partial_\mu\overline{\psi} - \partial_\mu\psi\sigma^\mu\overline{\psi}) \\ & + \text{ total derivative.}\end{aligned} \tag{4.85}$$

Recalling that in four component notation $\overline{\psi}\gamma^\mu\partial_\mu\psi = \psi\sigma^\mu\partial_\mu\overline{\psi} - \partial_\mu\psi\sigma^\mu\overline{\psi}$, $\mathcal{L}_K$ corresponds precisely to the kinetic Wess-Zumino Lagrangian (2.37) with $z = (A+iB)/\sqrt{2}$, $f = (F-iG)/\sqrt{2}$. Interactions and mass terms are introduced via $\mathcal{L}_c$. In particular:

$$\phi_i\phi_j = z_iz_j + \sqrt{2}(z_i\theta\psi_j + z_j\theta\psi_i) - \theta\theta(z_if_j + z_jf_i + \psi_i\psi_j), \tag{4.86}$$

$$\begin{aligned}\phi_i\phi_j\phi_k = {} & z_iz_jz_k + \sqrt{2}(z_iz_j\theta\psi_k + z_jz_k\theta\psi_i + z_kz_i\theta\psi_j) \\ & - \theta\theta(z_iz_jf_k + z_jz_kf_i + z_kz_if_j + z_i\psi_j\psi_k + z_j\psi_k\psi_i + z_k\psi_i\psi_j).\end{aligned} \tag{4.87}$$

The most general supersymmetric Lagrangian for chiral multiplets is then

$$\begin{aligned}\mathcal{L} = \sum_i \Big(& [\phi_i^\dagger\phi_i]_{\theta\theta\overline{\theta\theta}} + \\ & + [a_i\phi_i + \frac{1}{2}m_{ij}\phi_i\phi_j + \frac{1}{3}\lambda_{ijk}\phi_i\phi_j\phi_k]_{\theta\theta} + \\ & + [a_i^\dagger\phi_i^\dagger + \frac{1}{2}m_{ij}^\dagger\phi_i^\dagger\phi_j^\dagger + \frac{1}{3}\lambda_{ijk}^\dagger\phi_i^\dagger\phi_j^\dagger\phi_k^\dagger]_{\overline{\theta\theta}}\Big).\end{aligned} \tag{4.88}$$

Terms of higher order in ϕ_i are allowed by supersymmetry but lead to non renormalizable field theories. Explicitly,

$$\begin{aligned}\mathcal{L} = {} & \frac{i}{2}(\psi_i\sigma^\mu\partial_\mu\overline{\psi}_i - \partial_\mu\psi_i\sigma^\mu\overline{\psi}_i) + (\partial_\mu z_i)(\partial^\mu z_i^\dagger) + f_if_i^\dagger \\ & - m_{ij}(z_if_j + \frac{1}{2}\psi_i\psi_j) \quad + \quad \text{c.c.} \\ & - \lambda_{ijk}(z_iz_jf_k + z_i\psi_j\psi_k) \quad + \quad \text{c.c.} \\ & - a_if_i \quad + \quad \text{c.c.}\end{aligned} \tag{4.89}$$

The equations of motion of the auxiliary fields are

$$f_i^\dagger = m_{ij}z_j + \lambda_{ijk}z_jz_k + a_i. \tag{4.90}$$

They can be substituted in the Lagrangian to obtain

$$\begin{aligned}\mathcal{L} = {} & \frac{i}{2}(\psi_i\sigma^\mu\partial_\mu\overline{\psi}_i - \partial_\mu\psi_i\sigma^\mu\overline{\psi}_i) - \frac{1}{2}m_{ij}(\psi_i\psi_j + \overline{\psi}_i\overline{\psi}_j) + \\ & + (\partial_\mu z_i)(\partial^\mu z_i^\dagger) - \sum_i |a_i + m_{ij}z_j + \lambda_{ijk}z_jz_k|^2 - \\ & - \lambda_{ijk}z_i\psi_j\psi_k - \lambda_{ijk}^\dagger z_i^\dagger\overline{\psi}_j\overline{\psi}_k,\end{aligned} \tag{4.91}$$

(choosing $m_{ij} = m_{ij}^\dagger$). The scalar potential is a sum of squares. It is of the form

$$V_c = \sum_i \left|\frac{\partial W}{\partial z_i}\right|^2, \tag{4.92}$$

where

$$W = a_iz_i + \frac{1}{2}m_{ij}z_iz_j + \frac{1}{3}\lambda_{ijk}z_iz_jz_k, \tag{4.93}$$

i.e. W is the same function (of z_i) as the chiral Lagrangian $\mathcal{L}_c$ used in Eq. (4.88). Also, the Yukawa couplings and fermion mass terms are

$$-(\lambda_{ijk}z_i\psi_j\psi_k + \frac{1}{2}m_{ij}\psi_i\psi_j + \text{c.c.}) = -\frac{1}{2}\frac{\partial^2 W(z)}{\partial z_i\partial z_j}\psi_i\psi_j - \frac{1}{2}\frac{\partial^2\overline{W}}{\partial z_i^\dagger\partial z_j^\dagger}, \tag{4.94}$$

so that the Lagrangian is

$$\begin{aligned}\mathcal{L} = {} & \frac{i}{2}(\psi_i\sigma^\mu\partial_\mu\overline{\psi}_i - \partial_\mu\psi_i\sigma^\mu\overline{\psi}_i) + (\partial_\mu z_i)(\partial^\mu z_i^\dagger) \\ & - \sum_i\left|\frac{\partial W}{\partial z_i}\right|^2 - \frac{1}{2}\frac{\partial^2 W}{\partial z_i\partial z_j}\psi_i\psi_j - \frac{1}{2}\frac{\partial^2\overline{W}}{\partial z_i^\dagger\partial z_j^\dagger}\overline{\psi}_i\overline{\psi}_j.\end{aligned} \tag{4.95}$$

The function $W(z_i)$ (W does not depend on $z_i^\dagger$) is called the superpotential.

For one superfield ϕ only, and with $a_i = \lambda_{ijk} = 0$, one gets the Lagrangian

$$\begin{aligned}\mathcal{L} =& \frac{i}{2}(\psi\sigma^\mu\partial_\mu\overline{\psi} - \partial_\mu\psi\sigma^\mu\overline{\psi}) + (\partial_\mu z)(\partial^\mu z^\dagger) + ff^\dagger \\ &- m(zf + z^\dagger f^\dagger) - \frac{1}{2}m(\psi\psi + \overline{\psi}\overline{\psi}).\end{aligned} \tag{4.96}$$

To compare with the results obtained in ch. 2, one must use the following definitions:

$$\begin{aligned} z &= \frac{1}{\sqrt{2}}(A - iB), \\ f &= \frac{1}{\sqrt{2}}(F + iG), \end{aligned} \qquad \psi = \begin{pmatrix} \psi_\alpha \\ \overline{\psi}^{\dot\alpha} \end{pmatrix}. \tag{4.97}$$

With these expressions, the Lagrangian (4.96) corresponds precisely to (2.36), and the transformation rules (4.81) are equivalent with (2.43). In fact, the expressions (4.97) are fully determined by matching the transformation rules, but not completely when comparing quadratic Lagrangians.

4.4. Vector superfields

Vector superfields are real superfields: $V(x,\theta,\overline{\theta}) = V(x,\theta,\overline{\theta})^\dagger$. We will write their expansion in the form:

$$\begin{aligned} V(x,\theta,\overline{\theta}) =& C(x) + i\theta\chi(x) - i\overline{\theta}\overline{\chi}(x) + \theta\sigma^\mu\overline{\theta}v_\mu(x) + \\ &+ \frac{i}{2}\theta\theta[M(x) + iN(x)] - \frac{i}{2}\overline{\theta}\overline{\theta}[M(x) - iN(x)] + \\ &+ i\theta\theta\overline{\theta}[\overline{\lambda}(x) + \frac{i}{2}\partial_\mu\chi(x)\sigma^\mu] - i\overline{\theta}\overline{\theta}\theta[\lambda(x) - \frac{i}{2}\sigma^\mu\partial_\mu\overline{\chi}(x)] + \\ &+ \frac{1}{2}\theta\theta\overline{\theta}\overline{\theta}[D(x) - \frac{1}{2}\partial^\mu\partial_\mu C(x)]. \end{aligned} \tag{4.98}$$

The vector multiplet contains 8 bosons (C, D, M, N and the four components of v_μ) and 8 fermions (the 2 two-component spinors χ_α and λ_α). An expansion using peculiar combinations of fields has been chosen to make apparent that the components C, M, N and χ can be transformed to zero by a gauge transformation of the form (see ch. 5)

$$V \to V + \phi + \phi^\dagger \tag{4.99}$$

where ϕ is a chiral superfield. Using (4.70) and (4.72), one finds

$$\begin{aligned} \phi + \phi^\dagger =& 2Re\, z + \sqrt{2}\theta\psi + \sqrt{2}\overline{\theta}\overline{\psi} - \theta\theta f - \overline{\theta}\overline{\theta}f^\dagger - i\theta\sigma^\mu\overline{\theta}\partial_\mu(z - z^\dagger) + \\ &+ \frac{i}{\sqrt{2}}\theta\theta(\partial_\mu\psi\sigma^\mu\overline{\theta}) - \frac{i}{\sqrt{2}}\overline{\theta}\overline{\theta}(\theta\sigma^\mu\partial_\mu\overline{\psi}) - \frac{1}{2}\theta\theta\overline{\theta}\overline{\theta}\partial^\mu\partial_\mu Re\, z, \end{aligned} \tag{4.100}$$

so that

$$\begin{aligned} V + \phi + \phi^\dagger =& (C + 2Re\, z) + i\theta(\chi - i\sqrt{2}\psi) - i\overline{\theta}(\overline{\chi} + i\sqrt{2}\overline{\psi}) + \\ &+ \theta\sigma^\mu\overline{\theta}(v_\mu - i\partial_\mu(z - z^\dagger)) + \\ &+ \frac{i}{2}\theta\theta(M + iN + 2if) - \frac{i}{2}\overline{\theta}\overline{\theta}(M - iN - 2if^\dagger) + \\ &+ i\theta\theta\overline{\theta}[\overline{\lambda} + \frac{i}{2}\partial_\mu\chi\sigma^\mu + \frac{1}{\sqrt{2}}\partial_\mu\psi\sigma^\mu] - i\overline{\theta}\overline{\theta}\theta[\lambda - \frac{i}{2}\sigma^\mu\partial_\mu\overline{\chi} + \frac{1}{\sqrt{2}}\sigma^\mu\partial_\mu\overline{\psi}] + \\ &+ \frac{1}{2}\theta\theta\overline{\theta}\overline{\theta}[D - \frac{1}{2}\partial^\mu\partial_\mu(C + 2Re\, z)]. \end{aligned} \tag{4.101}$$

The transformation (4.99) is then equivalent to

$$\begin{aligned} C &\to C + 2Re\, z, \\ \chi &\to \chi - i\sqrt{2}\psi, \\ \overline{\chi} &\to \overline{\chi} + i\sqrt{2}\overline{\psi}, \\ M + iN &\to M + iN + 2if, \\ v_\mu &\to v_\mu - i\partial_\mu(z - z^\dagger), \\ \lambda &\to \lambda, \\ D &\to D. \end{aligned} \tag{4.102}$$

In particular one can choose:

$$\begin{aligned} Re\, z &= -\frac{1}{2}C, \\ \psi &= -\frac{i}{\sqrt{2}}\chi, \\ f &= \frac{i}{2}(M + iN), \end{aligned} \tag{4.103}$$

to eliminate C, M, N, χ. The imaginary part of z can also be used to eliminate one component of v_μ. In this gauge, which is called the Wess-Zumino gauge, the vector multiplet reduces to four bosons (D and the three remaining components of v_μ) and four fermions (the Majorana spinor λ). Notice however that this choice of gauge is not preserved by supersymmetry transformations which break relations (4.103).

The transformation (4.99) is the starting point of the construction of supersymmetric gauge theories, which will be discussed in the next chapter.

Since V is only constrained to be real, a linear combination (with real coefficients) of vector superfields is again a vector superfield. Also, if V is a vector superfield, then V^n is a vector superfield. In the Wess-Zumino gauge,

$$V_{WZ} = \theta\sigma^\mu\overline{\theta}v_\mu + i\theta\theta\overline{\theta}\overline{\lambda} - i\overline{\theta}\overline{\theta}\theta\lambda + \frac{1}{2}\theta\theta\overline{\theta}\overline{\theta}D. \tag{4.104}$$

Then $V^n_{WZ} = 0$ for $n \geq 3$. The only non zero product reads

$$V^2_{WZ} = (\theta\sigma^\mu\bar{\theta})(\theta\sigma^\nu\bar{\theta})v_\mu v_\nu = \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\, v^\mu v_\mu. \tag{4.105}$$

The vector multiplet contains the necessary fields to construct gauge theories. It is however clear that V and powers of V will never generate the kinetic terms for vector fields v_μ and spinors λ. These terms will be obtained by acting on V with covariant derivatives D_α and $\bar{D}_{\dot\alpha}$. Covariant derivatives introduce naturally space-time derivatives of the fields which are needed in the kinetic terms. By inspection, one finds that the chiral superfields

$$\begin{aligned} W_\alpha &= -\frac{1}{4}(\bar{D}\bar{D})D_\alpha V, \\ \bar{W}_{\dot\alpha} &= -\frac{1}{4}(DD)\bar{D}_{\dot\alpha}V, \end{aligned} \tag{4.106}$$

have the form required for kinetic terms. They are chiral, since $D^3 = \bar{D}^3 = 0$:

$$\bar{D}_{\dot\alpha}W_\alpha = D_\alpha \bar{W}_{\dot\alpha} = 0. \tag{4.107}$$

W_α and $\bar{W}_{\dot\alpha}$ are anticommuting Lorentz spinors. Under the transformation (4.99),

$$\begin{aligned} W_\alpha \rightarrow & W_\alpha - \frac{1}{4}(\bar{D}\bar{D})D_\alpha(\phi + \phi^\dagger) \\ &= W_\alpha + \frac{1}{4}\bar{D}^{\dot\alpha}\bar{D}_{\dot\alpha}D_\alpha\phi \\ &= W_\alpha + \frac{1}{4}\bar{D}^{\dot\alpha}\{D_\alpha, \bar{D}_{\dot\alpha}\}\phi \\ &= W_\alpha - \frac{i}{2}(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu\bar{D}^{\dot\alpha}\phi \\ &= W_\alpha, \end{aligned} \tag{4.108}$$

with the help of $D_\alpha\phi^\dagger = \bar{D}_{\dot\alpha}\phi = 0$. W_α and $\bar{W}_{\dot\alpha}$ are then invariant under (4.99).

We will only compute W_α and $\bar{W}_{\dot\alpha}$ in the Wess-Zumino gauge where Eq. (4.104) holds. Since W_α is a left-handed chiral superfield, it can be expressed in terms of the variables y and θ. The computation is straightforward and gives

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D(y) + \frac{i}{2}(\theta\sigma^\mu\bar{\sigma}^\nu)_\alpha F_{\mu\nu}(y) - \theta\theta(\sigma^\mu\partial_\mu\bar{\lambda}(y))_\alpha, \tag{4.109}$$

where

$$F_{\mu\nu}(y) = \partial_\mu v_\nu(y) - \partial_\nu v_\mu(y). \tag{4.110}$$

One also finds that

$$\bar{W}_{\dot\alpha} = i\bar{\lambda}_{\dot\alpha}(\bar{y}) + \bar{\theta}_{\dot\alpha}D(\bar{y}) - \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu\bar{\theta})_{\dot\alpha}F_{\mu\nu}(\bar{y}) - \bar{\theta}\bar{\theta}(\partial_\mu\lambda(\bar{y})\sigma^\mu)_{\dot\alpha}. \tag{4.111}$$

The lowest component of W_α is λ_α and it contains the familiar (abelian) field strength $F_{\mu\nu}$. Taking the 'square' of W_α gives, keeping only the $\theta\theta$ component which transforms like a total derivative

$$[W^\alpha W_\alpha]_{\theta\theta} = 2i\lambda\sigma^\mu\partial_\mu\bar{\lambda} + D^2 - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}. \tag{4.112}$$

To obtain this result, one uses

$$Tr(\sigma^\mu\bar{\sigma}^\nu\sigma^\rho\bar{\sigma}^\chi) = 2(\eta^{\mu\nu}\eta^{\rho\chi} - \eta^{\mu\rho}\eta^{\nu\chi} + \eta^{\mu\chi}\eta^{\nu\rho} - i\epsilon^{\mu\nu\rho\chi}). \tag{4.113}$$

Eq. (4.112) contains all terms needed in the kinetic Lagrangian of the vector multiplet, which reads

$$\begin{aligned} \mathcal{L}_V &= \frac{1}{4}[W^\alpha W_\alpha]_{\theta\theta} + \frac{1}{4}[\bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}]_{\bar{\theta}\bar{\theta}} \\ &= \frac{i}{2}(\lambda\sigma^\mu\partial_\mu\bar{\lambda} - \partial_\mu\lambda\sigma^\mu\bar{\lambda}) + \frac{1}{2}D^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \end{aligned} \tag{4.114}$$

To obtain the correct physical dimensions for v_μ and λ, one must assume that V is dimensionless. The real scalar field D (dimension $(\text{mass})^2$) is an auxiliary field. This is also the case of the states C, M, N, χ, which are absent in the Wess-Zumino gauge.

We close this section by giving the transformation rules of the vector multiplet. They are obtained by computing

$$\delta V = i(\epsilon Q + \bar{\epsilon}\bar{Q})V. \tag{4.115}$$

One finds:

$$\begin{aligned} \delta C &= i\epsilon\chi - i\bar{\epsilon}\bar{\chi}, \\ \delta\chi &= (M + iN)\epsilon - i(\sigma^\mu\bar{\epsilon})(v_\mu - i\partial_\mu C), \\ \delta(M + iN) &= 2\bar{\epsilon}\bar{\lambda} + 2i(\partial_\mu\chi\sigma^\mu\bar{\epsilon}), \\ \delta v_\mu &= i\epsilon\sigma^\mu\bar{\lambda} - i\lambda\sigma^\mu\bar{\epsilon} + \partial_\mu\chi\epsilon + \bar{\epsilon}\partial_\mu\bar{\chi}, \\ \delta\lambda &= iD\epsilon - \frac{1}{2}(\sigma^\mu\bar{\sigma}^\nu\epsilon)(\partial_\mu v_\nu - \partial_\nu v_\mu), \\ \delta D &= \epsilon\sigma^\nu\partial_\nu\bar{\lambda} + \partial_\nu\lambda\sigma^\nu\bar{\epsilon}. \end{aligned} \tag{4.116}$$

Chapter 5

Supersymmetric gauge theories

We now need to generalize the formalism of gauge theories to the supersymmetric case. This will lead us to the most general renormalizable supersymmetric gauge theory, describing the interactions of a set of left-handed chiral superfields ϕ^i, transforming according to an arbitrary representation **R** of the gauge group G, and a set of vector multiplets V^a, belonging to the adjoint representation of G ($a = 1, \dots, dim\, G$).

Let us first discuss shortly the case of a theory invariant under global transformations of the symmetry group G. The transformations of the chiral multiplets are

$$\phi'^i = [exp\; i\Lambda^a(T^a)]^i{}_j \phi^j, \tag{5.1}$$

or, infinitesimally,

$$\delta\phi^i = i\Lambda^a(T^a)^i{}_j\phi^j. \tag{5.2}$$

The matrices T^a are the hermitian generators of G for the representation **R** of the chiral multiplets. The transformation parameters Λ^a are real constants. Since a constant is also a superfield, they can be considered as being left-handed chiral superfields so that Eq. (5.1) is a superfield equation. The most general renormalizable Wess-Zumino Lagrangian for the chiral multiplets is

$$\mathcal{L}_c = [\phi_i^\dagger\phi^i]_{\theta\theta\overline{\theta}\overline{\theta}} + [W(\phi^i)]_{\theta\theta} + [\overline{W}(\phi_i^\dagger)]_{\overline{\theta}\overline{\theta}}, \tag{5.3}$$

where the superpotential $W(\phi^i)$ is

$$W(\phi^i) = a_i\phi^i + \frac{1}{2}m_{ij}\phi^i\phi^j + \frac{1}{3}\lambda_{ijk}\phi^i\phi^j\phi^k. \tag{5.4}$$

The kinetic terms $[\phi_i^\dagger\phi^i]_{\theta\theta\overline{\theta}\overline{\theta}}$ are naturally invariant under the transformations (5.1). The requirement of invariance imposes constraints on the superpotential. Each term of $W(\phi^i)$ must be a group invariant. Then, for instance, a_i can be non zero only for fields ϕ^i which are themselves invariant under G.

When going from global to local invariance, one encounters two difficulties. Firstly, the local transformation parameters $\Lambda^a(x)$ are no longer superfields. The transformations (5.1) and (5.2) are consistent with supersymmetry only if one allows

the parameters Λ^a to be complete left-handed chiral superfields. The second difficulty is that when Λ^a is promoted to a chiral superfield, $\Lambda^{a\dagger} \neq \Lambda^a$ and the kinetic terms $[\phi_i^\dagger\phi^i]_{\theta\theta\overline{\theta}\overline{\theta}}$ are not invariant:

$$(\phi^\dagger\phi)' = \phi^\dagger e^{-i\Lambda^\dagger}e^{i\Lambda}\phi, \tag{5.5}$$

omitting indices and using a matrix notation $\Lambda = \Lambda^a T^a$. To restore the invariance, one needs to introduce a vector multiplet

$$V = V^aT^a, \tag{5.6}$$

with the transformation

$$e^V \to e^{i\Lambda^\dagger}e^Ve^{-i\Lambda}. \tag{5.7}$$

This transformation is chosen to make the new kinetic terms

$$\mathcal{L}_{KIN} = [\phi^\dagger e^V\phi]_{\theta\theta\overline{\theta}\overline{\theta}} \tag{5.8}$$

invariant under local transformations of G. The introduction of the gauge vector multiplet is completely analogous to the introduction of gauge fields in non supersymmetric gauge theories. To first order, the infinitesimal transformation of V is

$$\delta V = -i(\Lambda - \Lambda^\dagger). \tag{5.9}$$

This corresponds to the transformation (4.99) (with $\phi = -i\Lambda$). The Wess-Zumino gauge, in which the gauge superfields V^a contain only the components v_μ^a, λ^a and D^a is then a choice of the local gauge in a supersymmetric gauge theory.

The chiral superfields W_α and $\overline{W}_{\dot\alpha}$, as defined by Eq. (4.106), are no longer invariant under gauge transformations (5.7), at least when the gauge group is not abelian ($[\Lambda, V] \neq 0$). Since the gauge transformation is given by exponentials, it is natural to look for a new form of W_α containing only e^V. Since $D_\alpha\Lambda^\dagger = 0$, one has

$$\begin{aligned} D_\alpha(e^{i\Lambda^\dagger}e^Ve^{-i\Lambda}) &= e^{i\Lambda^\dagger}D_\alpha(e^Ve^{-i\Lambda}) \\ &= e^{i\Lambda^\dagger}D_\alpha(e^V)e^{-i\Lambda} + e^{i\Lambda^\dagger}e^VD_\alpha(e^{-i\Lambda}). \end{aligned} \tag{5.10}$$

Then, observe that e^{-V} is the inverse of e^V. It transforms according to

$$e^{-V} \to e^{i\Lambda}e^{-V}e^{-i\Lambda^\dagger}. \tag{5.11}$$

Then

$$e^{-V}D_\alpha e^V \to e^{i\Lambda}(e^{-V}D_\alpha e^V)e^{-i\Lambda} + e^{i\Lambda}(D_\alpha e^{-i\Lambda}). \tag{5.12}$$

Using now $\overline{D}_{\dot\alpha}\Lambda = 0$, one finds

$$\overline{D}\overline{D}e^{-V}D_\alpha e^V \to e^{i\Lambda}\left(\overline{D}\overline{D}e^{-V}D_\alpha e^V\right)e^{-i\Lambda} + e^{i\Lambda}\left(\overline{D}\overline{D}D_\alpha e^{-i\Lambda}\right). \tag{5.13}$$

The second term vanishes (see also Eq. 4.108):

$$\begin{aligned}\overline{D}\overline{D}D_\alpha e^{-i\Lambda} &= -\overline{D}^{\dot\alpha}\overline{D}_{\dot\alpha}D_\alpha e^{-i\Lambda}\\ &= -\overline{D}^{\dot\alpha}\{D_\alpha, \overline{D}_{\dot\alpha}\}e^{-i\Lambda}\\ &= 2i(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu\overline{D}^{\dot\alpha}e^{-i\Lambda}\\ &= 0.\end{aligned} \tag{5.14}$$

One can then define

$$\begin{aligned}W_\alpha &= -\frac{1}{4}\overline{D}\overline{D}e^{-V}D_\alpha e^V,\\ \overline{W}_{\dot\alpha} &= -\frac{1}{4}DDe^{-V}\overline{D}_{\dot\alpha}e^V,\end{aligned} \tag{5.15}$$

with the transformations

$$\begin{aligned}W_\alpha &\to e^{i\Lambda}W_\alpha e^{-i\Lambda},\\ \overline{W}_{\dot\alpha} &\to e^{i\Lambda}\overline{W}_{\dot\alpha}e^{-i\Lambda}.\end{aligned} \tag{5.16}$$

The invariant kinetic Lagrangian will then be

$$\mathcal{L}_V = \frac{1}{4n}Tr[W^\alpha W_\alpha]_{\theta\theta} + \frac{1}{4n}Tr[\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}]_{\overline{\theta}\overline{\theta}}, \tag{5.17}$$

where the trace is taken over the gauge group indices and the normalization factor n will be determined later.

The full supersymmetric Lagrangian is then

$$\begin{aligned}\mathcal{L} =& [\phi_i^\dagger(e^V)^i{}_j\phi^j]_{\theta\theta\overline{\theta}\overline{\theta}}\\ &+ \frac{1}{4n}Tr[W^\alpha W_\alpha]_{\theta\theta} + \frac{1}{4n}Tr[\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}]_{\overline{\theta}\overline{\theta}}\\ &+ [W(\phi^i)]_{\theta\theta} + [\overline{W}(\phi_i^\dagger)]_{\overline{\theta}\overline{\theta}},\end{aligned} \tag{5.18}$$

with a superpotential $W(\phi^i)$ (see Eq. 5.4) subject to the constraints of invariance under the gauge transformations of G. If this gauge group contains abelian $U(1)$ factors, the $\theta\theta\overline{\theta}\overline{\theta}$ component of the corresponding vector multiplets V_A^a is invariant under the gauge group (according to (4.70), the $\theta\theta\overline{\theta}\overline{\theta}$ component of the chiral superfield Λ is a total derivative) and transforms under supersymmetry with a total derivative. Then, in the presence of abelian vector fields, one can add to the Lagrangian (5.18) pieces linear in the abelian vector multiplets:

$$\mathcal{L}_{FI} = \sum_a \xi^a[V_A^a]_{\theta\theta\overline{\theta}\overline{\theta}}. \tag{5.19}$$

These new terms are called Fayet-Iliopoulos terms. They play an important role in spontaneous breaking of supersymmetry (see next chapter). The parameters ξ^a have dimension $(\text{mass})^2$. They then introduce new scales in the Lagrangian.

We now proceed to compute the component form of the most general supersymmetric gauge theory given by (5.18) and (5.19). Although this Lagrangian looks non renormalizable, the high order interactions introduced by the exponentials contain mainly auxiliary fields C^a, M^a, N^a, χ^a. This is obvious from the fact that e^V is a dimensionless superfield. In the Wess-Zumino gauge, $V^aV^bV^c = 0$, and only terms with up to four fields remain. The Wess-Zumino Lagrangian is then clearly renormalizable. However, the Wess-Zumino gauge is a choice of a supersymmetry gauge: it is not preserved by supersymmetry transformations. The statement of renormalizability must then be supplemented by a proof that renormalization preserves to all order supersymmetry. This proof has now been given (for these questions which go beyond the scope of these notes, see ref. [13]).

We compute the Lagrangian in the Wess-Zumino gauge, for which

$$e^V = 1 + V + \frac{1}{2}V^2. \tag{5.20}$$

In components:

$$\begin{aligned}e^V =& 1 + \theta\sigma^\mu\overline{\theta}v_\mu + i\theta\theta\overline{\theta}\overline{\lambda} - i\overline{\theta}\overline{\theta}\theta\lambda +\\ &+ \frac{1}{2}\theta\theta\overline{\theta}\overline{\theta}(D + \frac{1}{2}v^\mu v_\mu),\end{aligned} \tag{5.21}$$

where

$$v_\mu = v_\mu^a T^a \quad , \quad \lambda = \lambda^a T^a \quad , \quad D = D^a T^a. \tag{5.22}$$

The generators of the representation R of the chiral multiplets satisfy

$$\begin{aligned}Tr(T^aT^b) &= \tau_R\delta^{ab},\\ [T^a, T^b] &= if^{abc}T^c,\end{aligned} \tag{5.23}$$

where the real numbers f^{abc} are the fully antisymmetric structure constants of the gauge group G. One then finds:

$$\begin{aligned}\left[\phi_i^\dagger V^i{}_j\phi^j\right]_{\theta\theta\overline{\theta}\overline{\theta}} =& -\frac{i}{2}(z_i^\dagger T^{ai}{}_j\partial_\mu z^j)v^{a\mu} + \frac{i}{2}(\partial_\mu z_i^\dagger T^{ai}{}_j z^j)v^{a\mu} +\\ &+ \frac{1}{2}(\psi^j\sigma^\mu\overline{\psi}_i)T^{ai}{}_j v_\mu^a + \frac{i}{\sqrt{2}}\overline{\psi}_i T^{ai}{}_j\overline{\lambda}^a z^j -\\ &- \frac{i}{\sqrt{2}}z_i^\dagger T^{ai}{}_j\lambda^a\psi^j + \frac{1}{2}D^a z_i^\dagger T^{ai}{}_j z^j,\end{aligned} \tag{5.24}$$

$$\left[\frac{1}{2}\phi_i^\dagger(V^2)^i{}_j\phi^j\right]_{\theta\theta\overline{\theta}\overline{\theta}} = \frac{1}{4}\left(z_i^\dagger(T^aT^b)^i{}_j z^j\right)v_\mu^a v^{b\mu}, \tag{5.25}$$

and $[\phi_i^\dagger \phi^i]_{\theta\theta\overline{\theta\theta}}$ is given in Eq. (4.85). Then:

$$\begin{aligned}\left[\phi_i^\dagger (e^V)^i{}_j \phi^j\right]_{\theta\theta\overline{\theta\theta}} =&(D_\mu z)_i^\dagger (D^\mu z)^i + \frac{i}{2}(\psi^i \sigma^\mu (D_\mu \overline{\psi})_i - (D_\mu \psi)^i \sigma^\mu \overline{\psi}_i) + f_i^\dagger f^i + \\ &+ \frac{i}{\sqrt{2}}(\overline{\psi}_i \overline{\lambda}^a) T^{ai}{}_j z^j - \frac{i}{\sqrt{2}} z_i^\dagger T^{ai}{}_j (\lambda^a \psi^j) + \frac{1}{2} D^a z_i^\dagger T^{ai}{}_j z^j,\end{aligned} \tag{5.26}$$

where the gauge covariant derivatives are

$$\begin{aligned}(D_\mu z)^i &= \partial_\mu z^i + \frac{i}{2} T^{ai}{}_j z^j v_\mu^a, \\ (D_\mu \psi)^i &= \partial_\mu \psi^i + \frac{i}{2} T^{ai}{}_j \psi^j v_\mu^a.\end{aligned} \tag{5.27}$$

The superpotential terms are the same as in Eq. (4.95). In the Wess-Zumino gauge, the chiral superfields W_α and $\overline{W}_{\dot\alpha}$ have the expansion

$$\begin{aligned}W_\alpha &= -\frac{1}{4}\overline{DD} e^{-V} D_\alpha e^V \\ &= -\frac{1}{4}\overline{DD} D_\alpha V + \frac{1}{4}\overline{DD} V D_\alpha V - \frac{1}{8}\overline{DD} D_\alpha V^2 \\ &= -\frac{1}{4}\overline{DD} D_\alpha V + \frac{1}{8}\overline{DD}[V, D_\alpha V], \\ \overline{W}_{\dot\alpha} &= -\frac{1}{4} DD e^{-V} \overline{D}_{\dot\alpha} e^V \\ &= -\frac{1}{4} DD\overline{D}_{\dot\alpha} V + \frac{1}{4} DDV\overline{D}_{\dot\alpha} V - \frac{1}{8} DD\overline{D}_{\dot\alpha} V^2 \\ &= -\frac{1}{4} DD\overline{D}_{\dot\alpha} V + \frac{1}{8} DD[V, \overline{D}_{\dot\alpha} V],\end{aligned} \tag{5.28}$$

since $D_\alpha V^2 = (D_\alpha V)V + V(D_\alpha V)$. The first terms $-\frac{1}{4}\overline{DD} D_\alpha V$ and $-\frac{1}{4} DD\overline{D}_{\dot\alpha} V$ are the same as in Eqs. (4.109-111). The new term reads

$$\begin{aligned}\frac{1}{8}\overline{DD}[V, D_\alpha V] =& -\frac{i}{2}\theta\theta[v_\mu, (\sigma^\mu \overline{\lambda})_\alpha] - \\ &- \frac{1}{4}(\theta\sigma^\mu\overline{\sigma}^\nu)_\alpha \left([v_\mu, v_\nu] - i\theta\sigma^\rho\overline{\theta}\partial_\rho [v_\mu, v_\nu]\right).\end{aligned} \tag{5.29}$$

This expression is certainly a chiral superfield and can be written only in terms of the variables y and θ:

$$\frac{1}{8}\overline{DD}[V, D_\alpha V] = -\frac{i}{2}\theta\theta[v_\mu(y), (\sigma^\mu \overline{\lambda}(y))\alpha] - \frac{1}{4}(\theta\sigma^\mu\overline{\sigma}^\nu)_\alpha [v_\mu(y), v_\nu(y)]. \tag{5.30}$$

The superfield W_α is then

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D(y) + \frac{i}{2}(\theta\sigma^\mu\overline{\sigma}^\nu)_\alpha F_{\mu\nu}(y) - \theta\theta(\sigma^\mu D_\mu \overline{\lambda}(y))_\alpha, \tag{5.31}$$

with

$$\begin{aligned}F_{\mu\nu} &= \partial_\mu v_\nu - \partial_\nu v_\mu + \frac{i}{2}[v_\mu, v_\nu], \\ D_\mu \overline{\lambda}^{\dot\alpha} &= \partial_\mu \overline{\lambda}^{\dot\alpha} + \frac{i}{2}[v_\mu, \overline{\lambda}^{\dot\alpha}].\end{aligned} \tag{5.32}$$

In components and using (5.23), one has:

$$\begin{aligned}F_{\mu\nu}^a &= \partial_\mu v_\nu^a - \partial_\nu v_\mu^a - \frac{1}{2} f^{abc} v_\mu^b v_\nu^c, \\ D_\mu \overline{\lambda}^a &= \partial_\mu \overline{\lambda}^a - \frac{1}{2} f^{abc} v_\mu^b \overline{\lambda}^c.\end{aligned} \tag{5.33}$$

It then follows that

$$\begin{aligned}[W^\alpha W_\alpha]_{\theta\theta} =& 2i\lambda\sigma^\mu (D_\mu \overline{\lambda}) + D^2 - \frac{1}{2} F^{\mu\nu} F_{\mu\nu} - \frac{i}{4}\epsilon^{\mu\nu\rho\chi} F_{\mu\nu} F_{\rho\chi}, \\ [\overline{W}_{\dot\alpha} \overline{W}^{\dot\alpha}]_{\overline{\theta\theta}} =& -2i(D_\mu \lambda)\sigma^\mu \overline{\lambda} + D^2 - \frac{1}{2} F^{\mu\nu} F_{\mu\nu} + \frac{i}{4}\epsilon^{\mu\nu\rho\chi} F_{\mu\nu} F_{\rho\chi}.\end{aligned} \tag{5.34}$$

Taking the trace gives

$$\begin{aligned}Tr[W^\alpha W_\alpha]_{\theta\theta} =& \tau_R\big(2i\lambda^a \sigma^\mu (D_\mu \overline{\lambda})^a + D^a D^a - \frac{1}{2} F_{\mu\nu}^a F^{a\mu\nu} - \frac{i}{4}\epsilon^{\mu\nu\rho\chi} F_{\mu\nu}^a F_{\rho\chi}^a\big), \\ Tr[\overline{W}_{\dot\alpha} \overline{W}^{\dot\alpha}]_{\overline{\theta\theta}} =& \tau_R\big(-2i(D_\mu \lambda)^a \sigma^\mu \overline{\lambda}^a + D^a D^a - \frac{1}{2} F_{\mu\nu}^a F^{a\mu\nu} + \frac{i}{4}\epsilon^{\mu\nu\rho\chi} F_{\mu\nu}^a F_{\rho\chi}^a\big).\end{aligned} \tag{5.35}$$

We finally need to introduce the gauge coupling constants g. This is easily done by redefining the fields λ^a and v_μ^a according to

$$\begin{aligned}\lambda^a &\to 2g\lambda^a, \\ v_\mu^a &\to 2g v_\mu^a, \\ D^a &\to 2g D^a,\end{aligned} \tag{5.36}$$

or, for the superfield V, $V \to 2gV$. We then have

$$\begin{aligned}Tr[W^\alpha W_\alpha]_{\theta\theta} =& 4g^2 \tau_R\big(2i\lambda^a \sigma^\mu (D_\mu \overline{\lambda})^a + D^a D^a - \\ &- \frac{1}{2} F_{\mu\nu}^a F^{a\mu\nu} - \frac{i}{4}\epsilon^{\mu\nu\rho\chi} F_{\mu\nu}^a F_{\rho\chi}^a\big), \\ Tr[\overline{W}_{\dot\alpha} \overline{W}^{\dot\alpha}]_{\overline{\theta\theta}} =& 4g^2 \tau_R\big(-2i(D_\mu \lambda)^a \sigma^\mu \overline{\lambda}^a + D^a D^a - \\ &- \frac{1}{2} F_{\mu\nu}^a F^{a\mu\nu} + \frac{i}{4}\epsilon^{\mu\nu\rho\chi} F_{\mu\nu}^a F_{\rho\chi}^a\big).\end{aligned} \tag{5.37}$$

The normalization factor n in the Lagrangian (5.17) is then given by

$$n = 4g^2 \tau_R,$$

so that

$$\begin{aligned}\mathcal{L}_V &= \frac{1}{16g^2\tau_R}Tr[W^\alpha W_\alpha]_{\theta\theta} + \frac{1}{16g^2\tau_R}Tr[\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}]_{\bar\theta\bar\theta} \\ &= \frac{i}{2}\lambda^a\sigma^\mu(D_\mu\bar\lambda)^a - \frac{i}{2}(D_\mu\lambda)^a\sigma^\mu\bar\lambda^a + \frac{1}{2}D^aD^a - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}.\end{aligned} \tag{5.38}$$

To summarize, the most general supersymmetric gauge theory Lagrangian is, in superfields,

$$\begin{aligned}\mathcal{L} = &\left[\phi_i^\dagger(e^{2gV})^i{}_j\phi^j\right]_{\theta\theta\bar\theta\bar\theta} + \\ &+\left[W(\phi^i) + \frac{1}{16g^2\tau_R}Tr(W^\alpha W_\alpha)\right]_{\theta\theta} + \\ &+\left[\overline{W}(\phi_i^\dagger) + \frac{1}{16g^2\tau_R}Tr(\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha})\right]_{\bar\theta\bar\theta} + \\ &+\sum_a g\xi^a V^a,\end{aligned} \tag{5.39}$$

where:

$$W(\phi^i) = a_i\phi^i + \frac{1}{2}m_{ij}\phi^i\phi^j + \frac{1}{3}\lambda_{ijk}\phi^i\phi^j\phi^k. \tag{5.40}$$

The component Lagrangian contains the covariant derivatives

$$\begin{aligned}(D_\mu z)^i &= \partial_\mu z^i + igv^a_\mu(T^a)^i{}_j z^j, \\ (D_\mu\psi)^i &= \partial_\mu\psi^i + igv^a_\mu(T^a)^i{}_j\psi^j, \\ (D_\mu\lambda)^a &= \partial_\mu\lambda^a - gf^{abc}v^b_\mu\lambda^c,\end{aligned} \tag{5.41}$$

and the gauge field strengths

$$F^a_{\mu\nu} = \partial_\mu v^a_\nu - \partial_\nu v^a_\mu - gf^{abc}v^b_\mu v^c_\nu. \tag{5.42}$$

The complete component Lagrangian is

$$\begin{aligned}\mathcal{L} =&(D_\mu z)^\dagger_i(D^\mu z)^i + \frac{i}{2}\psi^i\sigma^\mu(D_\mu\bar\psi)_i - \frac{i}{2}(D_\mu\psi)^i\sigma^\mu\bar\psi_i - \\ &-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{i}{2}\lambda^a\sigma^\mu(D_\mu\bar\lambda)^a - \frac{i}{2}(D_\mu\lambda)^a\sigma^\mu\bar\lambda^a + \\ &+f_i^\dagger f^i + \frac{1}{2}D^aD^a + \\ &+\sqrt{2}ig(\bar\psi_i\bar\lambda^a)T^{ai}{}_j z^j - \sqrt{2}igz_i^\dagger T^{ai}{}_j(\lambda^a\psi^j) - \\ &-\frac{1}{2}\frac{d^2W(z^k)}{dz^i dz^j}\psi^i\psi^j - \frac{1}{2}\frac{d^2\overline{W}(z_k^\dagger)}{dz_i^\dagger dz_j^\dagger}\bar\psi_i\bar\psi_j + \\ &+gD^a(z_i^\dagger T^{ai}{}_j z^j) - \frac{dW(z^k)}{dz^i}f^i - \frac{d\overline{W}(z_k^\dagger)}{dz_i^\dagger}f_i^\dagger + \\ &+\sum_a g\xi^a D^a.\end{aligned} \tag{5.43}$$

The equations of motion of the auxiliary fields are

$$\begin{aligned}f_i^\dagger &= \frac{dW(z^k)}{dz^i}, \\ D^a &= -g(z_i^\dagger T^{ai}{}_j z^j) - g\xi^a.\end{aligned} \tag{5.44}$$

Once substituted into the Lagrangian, one finds:

$$\begin{aligned}\mathcal{L} =&(D_\mu z)^\dagger_i(D^\mu z)^i + \frac{i}{2}\psi^i\sigma^\mu(D_\mu\bar\psi)_i - \frac{i}{2}(D_\mu\psi)^i\sigma^\mu\bar\psi_i - \\ &-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{i}{2}\lambda^a\sigma^\mu(D_\mu\bar\lambda)^a - \frac{i}{2}(D_\mu\lambda)^a\sigma^\mu\bar\lambda^a + \\ &+\sqrt{2}ig(\bar\psi_i\bar\lambda^a)T^{ai}{}_j z^j - \sqrt{2}igz_i^\dagger T^{ai}{}_j(\psi^j\lambda^a) - \\ &-\frac{1}{2}\frac{d^2W(z^k)}{dz^i dz^j}(\psi^i\psi^j) - \frac{1}{2}\frac{d^2\overline{W}(z_k^\dagger)}{dz_i^\dagger dz_j^\dagger}(\bar\psi_i\bar\psi_j) - V(z^i, z_j^\dagger),\end{aligned} \tag{5.45}$$

where the scalar potential is

$$\begin{aligned}V(z^i, z_j^\dagger) &= \sum_i\left|\frac{dW}{dz^i}\right|^2 + \frac{1}{2}g^2\sum_a\left(z_i^\dagger T^{ai}{}_j z^j + \xi^a\right)^2 \\ &= \sum_i |f^i|^2 + \frac{1}{2}\sum_a(D^a)^2.\end{aligned} \tag{5.46}$$

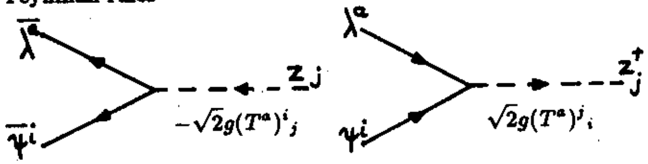
The scalar potential is a sum of positive terms. This is a reflexion of the positivity of energy for supersymmetric multiplets.

Some comments on the Lagrangian (5.45) are in order. The kinetic terms (the two first lines of 5.45) have nothing particular. Supersymmetry only imposes that the fermions transform in the same way as their bosonic supersymmetric partners. The representation of the chiral multiplets is arbitrary, but the 'gauginos' λ^a have to belong to the adjoint representation of G: they are the supersymmetric partners of the gauge fields v^a_μ.

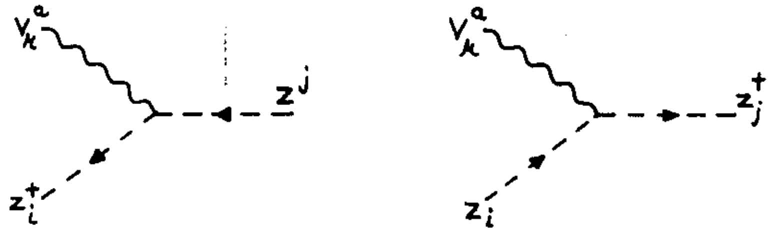
The other interaction terms are fully dictated by supersymmetry. The Yukawa interactions

$$\sqrt{2}ig(\bar\psi_i\bar\lambda^a)T^{ai}{}_j z^j - \sqrt{2}igz_i^\dagger T^{ai}{}_j(\lambda^a\psi^j)$$

with Feynman rules



have a strength given by the gauge coupling constant. The presence of these interactions is not due to gauge invariance, but only to supersymmetry. They arise when supersymmetry transformations are applied to the vertices



replacing v_μ^a and z^i by λ^a and ψ^i respectively.

Chapter 6

Spontaneous breaking of supersymmetry

The supersymmetric gauge theories obtained in ch. 5 lead to complicated non linear equations of motion for the various fields they contain. The usual way to study this dynamics is to use perturbation theory. To be sensible, perturbation theory should be performed around a stable configuration, which is by itself a solution of these equations of motion. The standard choice for this background (or vacuum) configuration is to allow for constant values of Lorentz invariant fields. Thus only scalar fields z^i are allowed to have non zero vacuum expectation values (v.e.v.), denoted by $< z^i >$. For this configuration, the theory reduces to the scalar potential V ($\partial_\mu < z^i >=< \psi^i >=< \lambda^a >=< v_\mu^a >= 0$). The equations of motion are then simply

$$\left.\frac{\partial V}{\partial z^i}\right|_{<z^i>=z^i} = 0. \qquad (6.1)$$

Moreover, the criterion of stability corresponds to the requirement that the vacuum $< z^i >$ minimizes the energy of the system, and cannot evolve towards an energetically more favourable vacuum state. There are two stability criteria. Firstly, local (or classical) stability corresponds to the condition that $< z^i >$ is a local minimum of the potential V. It can be any kind of minimum, not necessarily a global minimum of V. The condition of local stability is then that the matrix

$$\begin{pmatrix} \frac{\partial^2 V}{\partial z^i \partial z_j^\dagger} & \frac{\partial^2 V}{\partial z^i \partial z^j} \\ \frac{\partial^2 V}{\partial z_i^\dagger \partial z_j^\dagger} & \frac{\partial^2 V}{\partial z_i^\dagger \partial z^j} \end{pmatrix}$$

has only positive or zero eigenvalues. This matrix is, in the language of field theory, the (mass)^2 matrix of scalar fields z^i. Local stability is then equivalent to the absence of scalar fields with negative mass squared (absence of tachyonic states).

Classically, a local minimum is stable. At the quantum level, one can have transitions by tunnel effect towards another local minimum with lower energy. For constant configurations, the energy (the Hamiltonian) is the scalar potential. Global stability then means

$$V(< z^i >) \leq V(z^i) \quad \text{for all } z^i. \qquad (6.2)$$

Such a tunneling process is characterized by its lifetime describing the decay rate of the false vacuum. Depending on dynamics and on the shape of the potential, this decay can be slow or fast. False vacua could even be sensible vacua for a time comparable to the time scale of the evolution of the Universe.

In the case of supersymmetric gauge theories, the scalar potential is

$$V(z^i, z_i^\dagger) = f^i f_i^* + \frac{1}{2} D^a D^a, \tag{6.3}$$

with

$$f_i^\dagger = \frac{dW(z^j)}{dz^i}, \tag{6.4}$$

and

$$D^a = -g^a \left(z_i^\dagger (T^a)^i{}_j z^j + \xi^a \right). \tag{6.5}$$

The potential is semi-positive. If the equations

$$f^i(< z_j^\dagger >) = D^a(< z^i >, < z_j^\dagger >) = 0 \tag{6.6}$$

have a solution, then the corresponding vacuum configuration is a global minimum of V. It is then a stable vacuum. Notice that

$$\frac{dV}{dz^i} = f^j \frac{d^2W}{dz^i dz^j} - g D^a \left(z_j^\dagger (T^a)^j{}_i \right) = 0 \tag{6.7}$$

does not imply that all minima satisfy Eq. (6.6). There are also in general local minima, satisfying (6.1) but not (6.6). The corresponding energy will always be positive (see Eq. 6.3), and these local minima will all be metastable, except if the Lagrangian is such that (6.6) has no solution. The minimum equation (6.7) can be written in a peculiar but useful way:

$$\left\langle \left(f^i \quad D^a \right) \begin{pmatrix} \frac{d^2W}{dz^i dz^j} & -g z_j^\dagger (T^a)^j{}_i \\ -g z_i^\dagger (T^a)^i{}_j & 0 \end{pmatrix} \right\rangle = (0 \quad 0) \tag{6.8}$$

(the brackets $< \ldots >$ indicate v.e.v.). The second condition introduced in eq. (6.8),

$$z_j^\dagger (T^a)^j{}_i f^i = 0, \tag{6.9}$$

is only the statement of gauge invariance of the superpotential: since

$$f^i = \frac{d\overline{W}}{dz_i^\dagger}, \tag{6.10}$$

Eq. (6.9) reads (using $\delta z_i^\dagger = -i\epsilon^a (T^a)_i{}^j z_j^\dagger = \epsilon^a \delta_a z_i^\dagger$):

$$\frac{d\overline{W}}{dz_i^\dagger} (T^a)^j{}_i z_j^\dagger = i \frac{d\overline{W}(z_i^\dagger)}{dz_i^\dagger} \delta_a z_i^\dagger = 0. \tag{6.11}$$

The form (6.8) is useful since the matrix it contains is related to the fermion mass matrix. The non derivative bilinear fermion terms in Lagrangian (5.45) are

$$-\frac{1}{2} \frac{d^2W}{dz^i dz^j} \psi^i \psi^j - \sqrt{2} i g z_i^\dagger (T^a)^i{}_j \psi^j \lambda^a$$

and can be rewritten as

$$-\frac{1}{2} \left(\psi^i \quad \sqrt{2} i \lambda^b \right) \begin{pmatrix} \frac{d^2W}{dz^i dz^j} & -g z_j^\dagger (T^a)^j{}_i \\ -g z_i^\dagger (T^b)^i{}_j & 0 \end{pmatrix} \begin{pmatrix} \psi^j \\ \sqrt{2} i g \lambda^a \end{pmatrix}, \tag{6.12}$$

with the same matrix as in (6.8). The consequence is the Goldstone theorem for supersymmetry: a minimum of the scalar potential for which

$$< D^a > \neq 0 \quad \text{or} \quad < f^i > \neq 0 \quad \text{for some } i, a \tag{6.13}$$

generates a massless spin 1/2 state defined by

$$\lambda_G = < f_i > \psi^i - \frac{i}{\sqrt{2}} g < D^a > \lambda^a, \tag{6.14}$$

up to an irrelevant normalization factor. λ_G is called the Goldstone spinor, or the Goldstino. The vanishing of its mass is due to the fact that the minimum equation (6.8) implies that the eigenvalue in the direction ($< f^i > \quad < D^a >$) vanishes.

This result is fully analogous to the usual Goldstone theorem of field theory: starting from a potential V invariant under symmetries generated by T^a:

$$\delta V = \frac{dV(\phi_i)}{d\phi_i} \delta\phi_i = i\epsilon^a \frac{dV}{d\phi_i} (T^a)_i{}^j \phi_j = 0, \tag{6.15}$$

one assumes that this potential is minimized at $\phi_i = < \phi_i >$:

$$\left. \frac{dV}{d\phi_i} \right|_{\phi_i = <\phi_i>} = 0. \tag{6.16}$$

Differentiating (6.15) leads to (ϵ^a are arbitrary):

$$0 = i\epsilon^a \left(\frac{dV}{d\phi_i} (T^a)_i{}^j + \frac{d^2V}{d\phi_i d\phi_j} (T^a)_i{}^k \phi_k \right)_{\phi_i = <\phi_i>}. \tag{6.17}$$

The first term vanishes and the second term indicates that for each direction $< \phi_i >$ for which $(T^a)_i{}^j < \phi_j > \neq 0$, the scalar mass matrix $< \frac{d^2V}{d\phi_i d\phi_j} >$ has a zero eigenvalue, corresponding to a massless Goldstone scalar boson.

The existence of a Goldstone fermion is a signal that supersymmetry is spontaneously broken. This is also confirmed by the fact that a vacuum with non vanishing auxiliary fields $< f^i >$ and/or $< D^a >$ corresponds to a positive energy, since

$$< V >=< f^i f_i^\dagger > + \frac{1}{2} < D^a D^a > . \tag{6.18}$$

We have seen in ch. 3 that the energy in a supersymmetric multiplet is always positive. One can in fact show that a vacuum state $|\Omega >$ which preserves supersymmetry has always zero vacuum energy. The vacuum energy can be written

$$< \Omega|P^0|\Omega >= \frac{1}{8} < \Omega|Q_\alpha Q_\alpha^\dagger + Q_\alpha^\dagger Q_\alpha|\Omega > \tag{6.19}$$

according to Eq. (3.38). Then

$$< \Omega|P^0|\Omega >= \frac{1}{8}||Q_\alpha|\Omega > ||^2 + \frac{1}{8}||Q_\alpha^\dagger|\Omega > ||^2, \tag{6.20}$$

where $||\ldots||$ indicates the (Hilbert space) norm of the state. The vacuum state $|\Omega >$ is invariant if

$$Q_\alpha|\Omega >= 0, \tag{6.21}$$

which implies that a supersymmetric vacuum has always zero energy. Thus, vacua satisfying (6.6) are supersymmetric while vacua with (6.13) break (spontaneously) supersymmetry. This last statement can be made more explicit. Consider a field ϕ with an infinitesimal supersymmetry transformation $\delta\phi$. The vacuum expectation values $< \phi >$ of ϕ preserves supersymmetry only if

$$< \delta\phi >= 0 =< i\bar{\epsilon}Q\phi > \tag{6.22}$$

(we use four-component notation here). Then, to break supersymmetry, one must have a set of v.e.v.'s such that there exists a field ϕ with $< \delta\phi >\neq 0$. Considering now the transformations of the chiral multiplet (see Eq. 4.81), we see that if

$$< f^i >\neq 0, \tag{6.23}$$

then

$$< \delta\psi^i >= -\sqrt{2} < f^i > \epsilon \neq 0. \tag{6.24}$$

In the case of the vector multiplet (Eq. 4.116) in the Wess-Zumino gauge, one finds that if

$$< D^a >\neq 0, \tag{6.25}$$

then

$$< \delta\lambda^a >= i < D^a > \epsilon \neq 0. \tag{6.26}$$

We conclude that a vacuum breaking supersymmetry is characterized by a positive vacuum energy ($< V >\geq 0$) and, equivalently, by a non zero vacuum expectation value of auxiliary fields ($< f^i >$ and/or $< D^a >$). The consequence is the existence of a massless Goldstone fermion, with a supersymmetry transformation containing a constant term.

A supersymmetric gauge theory will break spontaneously if it possesses only vacua breaking supersymmetry, i.e. if equations (6.6) have no solution. It is a matter of choosing the form of the superpotential and of the D^a auxiliary fields in an appropriate way. These two choices give rise to two mechanisms of spontaneous supersymmetry breaking called O'Raifeartaigh and Fayet-Iliopoulos mechanisms. This will be the subject of section 6.2.

An important aspect of spontaneous supersymmetry breaking is the existence of mass formula. These relations have no counterpart in non supersymmetric theories. They would play an important role in constructing realistic supersymmetric gauge theories, containing the Glashow–Salam–Weinberg model of strong and electroweak interactions. In such models, each quark and lepton would have a scalar supersymmetric partner (scalar quarks or 'squarks' and scalar leptons or 'sleptons') and each gauge boson would have a spin 1/2 partner ('photino', 'gluinos', 'winos' and 'zino'). Since no such new particle has been detected, we have from experiment lower limits on the masses of supersymmetric particles. In any case, mass inequalities like

$$\begin{aligned} m_{squarks} &> m_{quarks}, \\ m_{sleptons} &> m_{leptons}, \\ m_{photino} &> m_{photon} = 0, \\ m_{gluino} &> m_{gluon} = 0, \end{aligned} \tag{6.27}$$

must be obtained in a realistic model. Supersymmetry has then to be broken, and in a very specific way. Inequalities (6.27) give strong constraints on the choice of the mechanism of supersymmetry breaking.

6.1. The mass formula

We have seen in previous section that a realistic, supersymmetric model of particle interactions must produce mass inequalities able to justify the absence of any supersymmetric partner of quarks, leptons and gauge bosons in the energy range accessible to present day experiments. This must be achieved by the mechanism breaking supersymmetry.

A very particular feature of supersymmetric theories is the existence of a mass formula valid for all possible vacua, breaking spontaneously or preserving supersymmetry, relating the masses of all fields present in the theory. This mass formula is very convenient when discussing realistic models. We will now proceed to derive it.

We know already a mass formula valid when supersymmetry is not broken: all states belonging to a given supermultiplet have the same mass. This result has for consequence the following sum rule. Consider the 'supertrace' of the mass matrices squared of all states:

$$STr\,\mathcal{M}^2 \equiv 3Tr\,\mathcal{M}_1^2 - 2Tr\,\mathcal{M}_{1/2}^2 + Tr\,\mathcal{M}_0^2, \tag{6.28}$$

where $\mathcal{M}_1^2$, $\mathcal{M}_{1/2}^2$ and $\mathcal{M}_0^2$ are respectively the mass matrices squared of the spin 1, 1/2 (two-component spinors) and 0 (real scalars) states of the theory. For a supersymmetric multiplet of mass M, $STr\,\mathcal{M}^2$ is defined so that

$$\begin{aligned} STr\,\mathcal{M}^2 &= M^2 \times (\text{number of bosons} - \text{number of fermions}) \\ &= 0. \end{aligned} \tag{6.29}$$

This is an example of a 'spin sum rule' characteristic of supersymmetry. Of course, $STr\,\mathcal{M}^2 = 0$ for a supersymmetric theory is much weaker than the statement of the equality of all masses within a supermultiplet. A formula for $STr\,\mathcal{M}^2$ can however be generalized to arbitrary vacua, including those breaking supersymmetry.

In order to calculate $STr\,\mathcal{M}^2$, we need the explicit form of the three mass matrices in Eq. (6.28). We start by considering $\mathcal{M}_1^2$. When scalar fields z_i acquire a vacuum expectation value, some gauge bosons will become massive in general. This is related to the presence in $(D_\mu z_i^\dagger)(D^\mu z^i)$ of a term of the form

$$g^2 v_\mu^a v^{b\mu} < z_i^\dagger (T^a)^i{}_j (T^b)^j{}_k z^k > . \tag{6.30}$$

The part of the Lagrangian bilinear in the vector fields is then

$$-\frac{1}{2}\left[(\partial_\mu v_\nu^a)(\partial^\mu v^{\nu a} - \partial^\nu v^{\mu a}) - 2g^2 < z^\dagger T^a T^b z > v_\mu^a v^{b\mu}\right], \tag{6.31}$$

omitting the indices of scalar fields. This expression means that the mass matrix (squared) of spin 1 particles is

$$(\mathcal{M}_1^2)^{ab} = 2g^2 < z^\dagger T^a T^b z >, \tag{6.32}$$

or, introducing the notations

$$\begin{aligned} D_i^a &\equiv \frac{\partial D^a}{\partial z^i} = -g z_j^\dagger T^{aj}{}_i, \\ D^{ai} &\equiv \frac{\partial D^a}{\partial z_i^\dagger} = -g T^{ai}{}_j z^j, \end{aligned} \tag{6.33}$$

one gets

$$(\mathcal{M}_1^2)^{ab} = 2 < D_i^a D^{bi} > . \tag{6.34}$$

Then,

$$3Tr\,\mathcal{M}_1^2 = 6 < D_i^a D^{ai} > . \tag{6.35}$$

Turning to the spin 1/2 mass matrix, we collect all terms bilinear in fermion fields with possible vacuum expectation values $< z^i >$. They read:

$$\begin{aligned} &\frac{1}{2}[i\psi^i\sigma^\mu\partial_\mu\overline{\psi}_i - i\partial_\mu\psi_i\sigma^\mu\overline{\psi}^i - < f_{ij} > \psi^i\psi^j - < \overline{f}^{ij} > \overline{\psi}_i\overline{\psi}_j \\ &+ i\lambda^a\sigma^\mu\partial_\mu\overline{\lambda}^a - i\partial_\mu\lambda^a\sigma^\mu\overline{\lambda}^a - 2\sqrt{2}i < D^{ai} > \overline{\psi}_i\overline{\lambda}^a + 2\sqrt{2}i < D_i^a > \psi^i\lambda^a], \end{aligned} \tag{6.36}$$

where

$$f_{ij} \equiv \frac{d^2W}{dz^i dz^j}; \qquad \overline{f}^{ij} \equiv \frac{d^2\overline{W}}{dz_i^\dagger dz_j^\dagger}. \tag{6.37}$$

The mass terms can be written in a matrix form:

$$\begin{aligned} -\frac{1}{2}\begin{pmatrix}\psi^i & \lambda^a\end{pmatrix}\begin{pmatrix} < f_{ij} > & \sqrt{2}i < D_i^b > \\ \sqrt{2}i < D_j^a > & 0 \end{pmatrix}\begin{pmatrix}\psi^i \\ \lambda^b\end{pmatrix} \\ + \text{hermitian conjugate.} \end{aligned} \tag{6.38}$$

The mass matrix is then

$$\mathcal{M}_{1/2} = \begin{pmatrix} < f_{ij} > & \sqrt{2}i < D_i^b > \\ \sqrt{2}i < D_j^a > & 0 \end{pmatrix}, \tag{6.39}$$

and then

$$-2Tr\,\mathcal{M}_{1/2}\mathcal{M}_{1/2}^\dagger = -2 < \overline{f}^{ij} f_{ij} > -8 < D_i^a D^{ai} > . \tag{6.40}$$

Notice that in (6.38) we have taken out a factor 1/2. This is appropriate for two component spinor notation where the kinetic part of the Dirac Lagrangian contains the same factor (see 6.36).

The last thing we need is the scalar mass matrix (squared). The scalar Lagrangian has the form

$$\mathcal{L}_S = (\partial_\mu z^i)(\partial^\mu z_i^\dagger) - V(z^i, z_i^\dagger), \tag{6.41}$$

the scalar potential V being given in Eq. (6.3). The bilinear terms are

$$(\partial^\mu z^i)(\partial_\mu z_i^\dagger) - \left\langle \frac{\partial^2 V}{\partial z^i \partial z_j^\dagger} \right\rangle z^i z_j^\dagger - \frac{1}{2}\left\langle \frac{\partial^2 V}{\partial z^i \partial z^j} \right\rangle z^i z^j - \frac{1}{2}\left\langle \frac{\partial^2 V}{\partial z_i^\dagger \partial z_j^\dagger} \right\rangle z_i^\dagger z_j^\dagger, \tag{6.42}$$

which can be written as

$$(\partial_\mu z^i)(\partial^\mu z_i^\dagger) - \frac{1}{2}\begin{pmatrix} z^i & z_j^\dagger \end{pmatrix}(\mathcal{M}_0^2)\begin{pmatrix} z_i^\dagger \\ z^j \end{pmatrix}, \tag{6.43}$$

in a matrix form. The scalar $(\text{mass})^2$ matrix $\mathcal{M}_0^2$ reads

$$\mathcal{M}_0^2 = \begin{pmatrix} \left\langle \frac{\partial^2 V}{\partial z^i \partial z_k^\dagger} \right\rangle & \left\langle \frac{\partial^2 V}{\partial z^i \partial z^l} \right\rangle \\ \left\langle \frac{\partial^2 V}{\partial z_j^\dagger \partial z_k^\dagger} \right\rangle & \left\langle \frac{\partial^2 V}{\partial z_j^\dagger \partial z^l} \right\rangle \end{pmatrix}. \tag{6.44}$$

It is hermitian. Then:

$$Tr\, \mathcal{M}_0^2 = 2 \frac{\partial^2 V}{\partial z^i \partial z_i^\dagger}. \tag{6.45}$$

From the scalar potential (6.3), one has

$$\begin{aligned} \frac{\partial^2 V}{\partial z_i^\dagger \partial z^j} &\equiv V_j^i = f_{jk}\overline{f}^{ik} + D_j^a D^{ai} + D^a D_j^{ai}, \\ \frac{\partial^2 V}{\partial z_i^\dagger \partial z_j^\dagger} &\equiv V^{ij} = \overline{f}^{ijk} f_k + D^{ai} D^{aj}, \\ \frac{\partial^2 V}{\partial z^i \partial z^j} &\equiv V_{ij} = f_{ijk}\overline{f}^k + D_i^a D_j^a, \end{aligned} \tag{6.46}$$

where:

$$\begin{aligned} f_{ijk} &\equiv \frac{\partial^3 W}{\partial z^i \partial z^j \partial z^k}; \qquad \overline{f}^{ijk} \equiv \frac{\partial^3 \overline{W}}{\partial z_i^\dagger \partial z_j^\dagger \partial z_k^\dagger}; \\ D_j^{ai} &\equiv \frac{\partial^2 D^a}{\partial z^j \partial z_i^\dagger} = -g_a (T^a)_j^i. \end{aligned} \tag{6.47}$$

Then:

$$Tr\, \mathcal{M}_0^2 = 2 < f_{ij}\overline{f}^{ij} > +2 < D_i^a D^{ai} > +2 < D^a D_i^{ai} > . \tag{6.48}$$

Finally, collecting results (6.35), (6.40) and (6.48) leads to the supertrace mass formula

$$STr\, \mathcal{M}^2 = 2 \sum_a < D^a >< D_i^{ai} >= -2 \sum_a g^a < D^a > Tr(T^a), \tag{6.49}$$

which is valid for arbitrary vacuum expectation values $< z^i >$.

We are now ready to consider the implications of supersymmetry breaking for the mass spectrum of the theory. We will consider separately the possible vacuum expectation values $< f^i > \neq 0$ and $< D^a > \neq 0$. In explicit cases, we will quite often encounter simultaneously both and then need simply to add both effects.

Consider first the case $< f^i > \neq 0, < D^a >= 0$. Then, from (6.49), $STr\, \mathcal{M}^2 = 0$. More informations can be obtained from the scalar mass (squared) matrix (Eqs. 6.44

and 6.46), which in our case reads:

$$\begin{aligned} \mathcal{M}_0^0 &= \mathcal{M}_{SUSY}^2 + \mathcal{M}_{BREAK}^2, \\ \mathcal{M}_{SUSY}^2 &= \begin{pmatrix} < f_{ik}\overline{f}^{jk} + D_i^a D^{aj} > & < D_i^a D_k^a > \\ < D^{aj} D^{al} > & < f_{ki}\overline{f}^{il} + D_k^a D^{al} > \end{pmatrix}, \\ \mathcal{M}_{BREAK}^2 &= \begin{pmatrix} 0 & < f_{ikl}\overline{f}^l > \\ < \overline{f}^{jli} f_i > & 0 \end{pmatrix}. \end{aligned} \tag{6.50}$$

We have divided $\mathcal{M}_0^2$ into a supersymmetric part $\mathcal{M}_{SUSY}^2$, independent of $< f_i >$, and a supersymmetry breaking part $\mathcal{M}_{BREAK}^2$. The new mass terms, induced by supersymmetry breaking are then

$$\frac{1}{2} < \overline{f}^{ijk} f_k > z_i^\dagger z_j^\dagger + \frac{1}{2} < f_{ijk}\overline{f}^k > z^i z^j. \tag{6.51}$$

Notice that f_{ijk} is proportional to the Yukawa couplings (see Eq. 5.40):

$$f_{ijk} =< f_{ijk} >= 2\lambda_{ijk}. \tag{6.52}$$

Choosing for simplicity $\lambda_{ijk} < \overline{f}^k >$ real, the supersymmetry breaking mass terms are

$$\begin{aligned} &\sum_{i,j,k} \lambda_{ijk} < \overline{f}^k > \left(z^i z^j + z_i^\dagger z_j^\dagger \right) = \\ &= \frac{1}{2} \sum_{i,j,k} \lambda_{ijk} < \overline{f}^k > \left((z^i + z_i^\dagger)(z^j + z_j^\dagger) - i(z^i - z_i^\dagger) i(z^j - z_j^\dagger) \right). \end{aligned} \tag{6.53}$$

Defining the real fields

$$\begin{aligned} \frac{1}{\sqrt{2}} A_i &= Re\, z^i, \\ \frac{1}{\sqrt{2}} B_i &= Im\, z^i, \end{aligned} \tag{6.54}$$

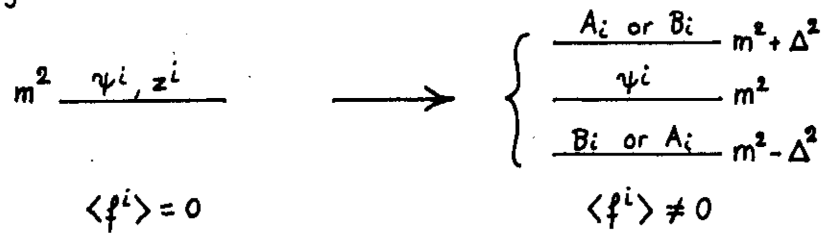
one finds the supersymmetry breaking mass terms

$$\frac{1}{2} \sum_{i,j,k} \lambda_{ijk} < \overline{f}^k > (A_i A_j - B_i B_j). \tag{6.55}$$

The effect of supersymmetry breaking due to $< f_i > \neq 0$ is to split the complex scalars z^i into two real fields A_i and B_i. The masses of A_i and B_i are shifted with respect to the mass of the fermion ψ^i by an equal and opposite amount. The spectrum is depicted in Figure 6.1.

The quantity Δ^2 is proportional to $< f_i >$. Clearly, inequalities (6.27) are never satisfied, and supersymmetry breaking with $< f_i > \neq 0, < D^a > = 0$ does not give realistic mass relations, at least at the classical level (tree graphs).

Fig. 6.1:



We now turn to the case $< f_i > = 0, < D^a > \neq 0$. Then, according to (6.49), $STr\, \mathcal{M}^2 \neq 0$ provided the associate generator of the gauge group T^a is not traceless. This is only possible for a generator of an abelian $U(1)$ group, characterized by the charges q_i^a of the fields z^i:

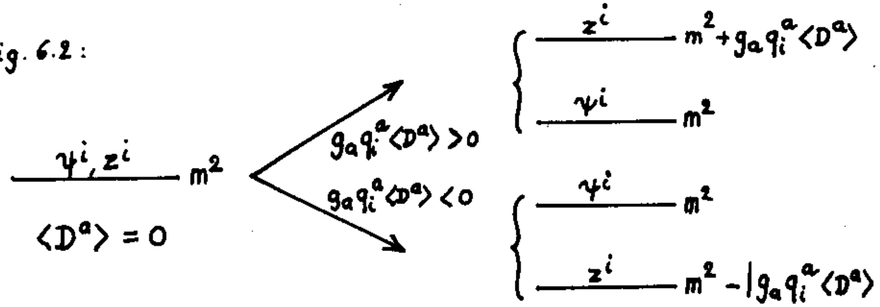
$$\sum_j (T^a)^i{}_j z^j = q_i^a z^i, \qquad \sum_j (T^a)^j{}_i z_j^\dagger = -\, q_i^a z_i^\dagger \tag{6.56}$$

(no sum on i). The supersymmetry breaking mass terms are now

$$-\sum_{a,i,j} g_a (T^a)^i{}_j < D^a > z_i^\dagger z^j = -\sum_{a,i} g_a q_i^a < D^a > (z_i^\dagger z^i). \tag{6.57}$$

For every non zero $< D^a >$, the spectrum is depicted in Figure 6.2.

Fig. 6.2:



In principle, the phenomenological inequalities (6.27) can be satisfied as long as

$$-\sum_a g_a q_i^a < D^a > > 0 \tag{6.58}$$

for indices i corresponding to quark and lepton multiplets. The weak hypercharge Y of the standard model does not satisfy the requirement, since

$Y = -1/2$ for lepton doublets,
$Y = 1/3$ for charge 1/3 antiquarks,
$Y = -2/3$ for charge $-2/3$ antiquarks,
$Y = 1$ for (charged) antileptons,
$Y = 1/6$ for quark doublets.

Both signs are present and (6.58) is violated. The way out would be to introduce a new $U(1)'$ gauge group with appropriate charges q_i'. One encounters several difficulties at this construction, due to the fact that condition (6.58) hardly coexists with the other requirements one must impose on $U(1)'$:

Absence of chiral anomalies,
Broken supersymmetry, i.e. non existence of a minimum of V with $< D^a > = < f_i > = 0$.

Also, we will see in the next chapter that quadratic divergences will be present in the theory, except if the new $U(1)'$ is traceless, $\sum_i q_i' = 0$. This condition is in fact highly desirable, since it also corresponds to the absence of gravitational anomalies.

Many attempts to construct realistic models using additional $U(1)$ groups have been performed. They have quite generally failed and this idea has now been abandoned, even though there is no 'no-go' theorem closing the subject. (For reviews on the construction of realistic supersymmetric modelsand their phenomenology, see refs. [14, 15]).

From the previous discussion, it seems that spontaneous breaking of supersymmetry never produces realistic mass relations. This statement is however only true at the tree-level since loop corrections will in general lead to different mass relations. One can construct satisfactory models with the help of the following strategy. One introduces some new chiral multiplets (other than quarks, leptons and Higgses), used to break supersymmetry with $0 \neq < f_i > \simeq \Delta^2$, $< D^a > = 0$. At tree-level, only these new fields receive non supersymmetric mass terms. Quantum corrections will however propagate supersymmetry breaking to the sector of quark, lepton, Higgs and gauge multiplets, provided there exists some coupling of the new multiplets to this sector. At some order (depending on the model), all scalar partners of fermions and gauginos will have received positive mass corrections, allowing to satisfy inequalities (6.27). Since these mass corrections are obtained at loop level only, they will be large enough to survive the experimental bounds on supersymmetric partners of quarks, leptons and gauge bosons only if Δ is much larger than these bounds (which are now in the range of 100 GeV). Even though elegance is not their main quality, models following this line of reasoning have been successfully constructed (see [14] and references therein).

The difficulties of obtaining a realistic spectrum in spontaneously broken globally supersymmetric models seem to indicate that global supersymmetry is not the fundamental symmetry of a unified theory. It turns out to be much easier and natural to obtain a correct spectrum for the supersymmetric particles if one consider supergravity theories (i. e. theories with local supersymmetry), with a spontaneously broken local supersymmetry (the so-called super-Higgs phenomenon). (For reviews, see [14], [16] and also [17] for supergravity).

We will conclude this chapter by a discussion of the explicit realization of spontaneous supersymmetry breaking in the two cases of O'Raifeartaigh and Fayet-Iliopoulos mechanisms.

6.2. O'Raifeartaigh and Fayet-Iliopoulos mechanisms

These two mechanisms correspond to the two fundamental methods to induce spontaneous supersymmetry breaking. It is clear from the preceding discussion that spontaneous breaking of supersymmetry is possible only if the equations

$$\begin{aligned} f_i^\dagger &= \frac{dW}{dz^i} = 0, \\ D^a &= -g(z_i^\dagger (T^a)^i{}_j z^j + \xi^a) = 0 \end{aligned} \tag{6.59}$$

have no solution. In order to forbid the solution $< z^i >= 0$ (all i's), one must have either a linear term $a_i z^i$ in the superpotential or at least one non zero parameter ξ^a.

The first possibility leads to the O'Raifeartaigh mechanism. The superpotential contains the terms

$$W = a_i z^i + \frac{1}{2} m_{ij} z^i z^j + \frac{1}{3} \lambda_{ijk} z^i z^j z^k.$$

It must be chosen so that there is no solution to $\frac{dW}{dz^i} = 0$. Since we need a linear term, the gauge invariance of W implies that at least one gauge singlet field, say Y, is present. The idea is then to use Y to force another field, say X, to have a vacuum expectation value. A third field, transforming in the simplest case in the same real representation of the gauge group as X, forces then the v.e.v. of X to vanish. For instance, one takes

$$W = Y(M^2 - X^2) + \mu Z X + w(X, z^i), \tag{6.60}$$

where z^i denotes all the other fields. Then:

$$\begin{aligned} \frac{\partial W}{\partial Y} &= M^2 - X^2, \\ \frac{\partial W}{\partial Z} &= \mu X, \end{aligned} \tag{6.61}$$

and both equations cannot vanish simultaneously. Several variants of this superpotential are possible. They have in common the singlet field Y and the presence of the linear term YM^2 in the superpotential W. For instance, if X transforms in a complex representation $\mathbf{R}$ of the gauge group, one introduces two fields Z_1 and Z_2 transforming in the conjugate representation $\mathbf{R}^*$. One can take

$$W = Y(M^2 - XZ_1) + \mu(XZ_2) + w(X, z^i), \tag{6.62}$$

so that

$$\begin{aligned} 0 = \frac{\partial W}{\partial Y} &= M^2 - XZ_1, \\ 0 = \frac{\partial W}{\partial Z_1} &= XY, \\ 0 = \frac{\partial W}{\partial Z_2} &= \mu X \end{aligned} \tag{6.63}$$

have no solutions. The O'Raifeartaigh mechanism requires always at least three fields.

We now compute the supersymmetry breaking mass terms for the simplest superpotential (6.60). We first need to determine the minimum of the scalar potential

$$V = |M^2 - X^2|^2 + \mu^2 |X|^2 + |\mu Z - 2XY + \frac{\partial w}{\partial X}|^2 + \sum_i \left|\frac{\partial w}{\partial z^i}\right|^2 + \frac{1}{2}\sum_a (D^a)^2. \tag{6.64}$$

Assuming that the equations

$$< \frac{\partial w}{\partial z^i} >=< D^a >= 0 \tag{6.65}$$

have solutions, the minimum of V corresponds to the vacuum expectation values

$$\begin{aligned} < X^2 > &= M^2 - \frac{1}{2}\mu^2, \\ < \mu Z - 2XY + \frac{\partial w}{\partial X} > &= 0. \end{aligned} \tag{6.66}$$

where M^2 and μ^2 have been chosen real by a choice of the phases of X, Y and Z. The auxiliary fields $f_Y^\dagger$ and $f_Z^\dagger$ receive a vacuum expectation value

$$\begin{aligned} < f_Y^\dagger > &= \frac{1}{2}\mu^2, \\ < f_Z^\dagger > &= \mu\sqrt{M^2 - \frac{1}{2}\mu^2}. \end{aligned} \tag{6.67}$$

All others are zero. At the minimum

$$< V >= \mu^2 \left(\frac{1}{4}\mu^2 + \left| M^2 - \frac{1}{2}\mu^2 \right| \right) > 0. \tag{6.68}$$

The non supersymmetric mass terms are (see Eq. 6.51):

$$\frac{1}{2} < f_Y^\dagger > f^{YXX} X^2 + \frac{1}{2} < f^Y > f_{YXX}^\dagger (X^\dagger)^2 = \frac{1}{2}\mu^2 (X^2 + X^{\dagger 2}). \tag{6.69}$$

Only the chiral multiplet X has then a non supersymmetric spectrum.

We now turn to the case of the Fayet-Iliopoulos mechanism, requiring a non zero ξ^a. Since the Lagrangian $[\xi^a D^a]_{\theta\theta\bar\theta\bar\theta}$ is only gauge invariant for abelian multiplets, the gauge group must contain one or several $U(1)$ factors. This is however not sufficient to break supersymmetry since the absence of chiral anomaly for a $U(1)$ group with charges q_i of the chiral multiplets imposes

$$\sum_i q_i^3 = 0, \tag{6.70}$$

which implies that charges of both signs are always present. Then,

$$0 = D^a = -g \left(\sum_i q_i |z^i|^2 + \xi^a \right) \tag{6.71}$$

has always solutions and supersymmetry cannot be broken. One needs further constraints arising from the superpotential, or from other $U(1)$ groups with their own ξ^a.

Broken supersymmetry arises in the following $U(1)$ supersymmetric gauge theory. The chiral multiplets and their $U(1)$ charges Q are

$$E \quad \text{with} \quad Q = -1,$$
$$\overline{E} \quad \text{with} \quad Q = 1.$$

This model corresponds to supersymmetric electrodynamics of an electron E, $\overline{E}$ being the positron. The D term is

$$D = -e(z_{\overline{E}} z_{\overline{E}}^\dagger - z_E z_E^\dagger + \xi), \tag{6.72}$$

and can vanish for non zero scalar fields. We however take the superpotential

$$W = mE\overline{E}, \tag{6.73}$$

m being the electron mass. The equations

$$\begin{aligned} \frac{\partial W}{\partial z_E} &= m z_{\overline{E}} = 0, \\ \frac{\partial W}{\partial z_{\overline{E}}} &= m z_E = 0 \end{aligned} \tag{6.74}$$

are then incompatible with $D = 0$ and supersymmetry is broken, but only for $m \neq 0$. This example shows that a conspiracy of the superpotential and a D-term, but not a single D, can break supersymmetry.

6.3. The Higgs mechanism in supersymmetric gauge theories

The vacuum expectation values of scalar fields will also induce spontaneous gauge symmetry breaking, for fields with non trivial gauge transformations. Even though the Higgs mechanism in supersymmetric theories can in principle be treated in the same way as in non supersymmetric theories, the particular structure of the scalar potential (6.3) leads to interesting features. The potential contains two terms, with different gauge invariance. The f-terms of the potential are derived from the superpotential W, which is a gauge invariant polynomial in the complex fields z^i. Gauge invariance is the statement

$$0 = \frac{dW}{dz^j}\delta z^j = \frac{dW}{dz^j} i\omega^a (T^a)^j{}_k z^k, \tag{6.75}$$

with *real* infinitesimal transformation parameters ω^a, and hermitian generators T^a of the gauge group for the representation of the chiral multiplets. It is however clear that if W is gauge invariant, in the sense of eq. (6.75), it is also invariant under the *complexified* form of the gauge group, with transformations obtained by promoting ω^a to *complex* parameters. The superpotential is then invariant under a group with (real) dimension twice bigger as for the gauge group. Applying now the Goldstone theorem, as explained in (6.15–17), each broken gauge symmetry corresponds to two massless real Goldstone bosons (or one complex Goldstone scalar).

For a minimum with unbroken supersymmetry, $< f_i >= 0$, this can also be seen in the following way. Gauge invariance of the f-terms of the potential reads

$$\overline{f}^i f_{ij} D^{aj} - \overline{f}^{ij} f_i D_j^a = 0. \tag{6.76}$$

Differentiating, and using $< f_i >= 0$ leads to

$$< \overline{f}^{ij} f_{ik} D_j^a >=< \overline{f}^{ik} f_{ij} D^{aj} >= 0. \tag{6.77}$$

Since $<\overline{f}^{ij} f_{ik}>$ is the scalar mass matrix, one sees that for broken symmetries, both directions $< D_i^a >$ and the complex conjugate $< D^{ai} >$ are massless: for each broken symmetry, one gets two real massless Goldstone states.

However, the second term of the scalar potential is certainly not invariant under the complexified gauge group: under a transformation (6.75) with complex parameters, (omitting scalar indices $i, j, k, \ldots$)

$$
\begin{aligned}
\delta\left(\frac{1}{2}D^a D^a\right) =& D^a\, \delta D^a \\
=& i\sum_{a,b} D^a g^a \left[\omega^{b*}(z^\dagger T^b T^a z) - \omega^b (z^\dagger T^a T^b z)\right] \\
=& \frac{1}{2} i \sum_{a,b} g^a D^a \left[(\omega^{b*} - \omega^b) z^\dagger \{T^a, T^b\} z - (\omega^b + \omega^{b*}) z^\dagger [T^a, T^b] z\right] \\
=& \sum_{a,b} g^a D^a Im\, \omega^b\, z^\dagger \{T^a, T^b\} z - \sum_{a,b} f^{abc} Re\, \omega^b D^a D^c \\
=& \sum_{a,b} g^a D^a Im\, \omega^b z^\dagger \{T^a, T^b\} z.
\end{aligned}
\tag{6.78}
$$

(To get the fourth equality, one uses that a Fayet-Iliopoulos term can be present in D^a only for abelian vector multiplets, for which $[T^a, T^b] = 0$). As it should, only the usual gauge group with real parameters is an invariance of the full potential. Because of the D-terms in the potential, the unwanted massless (Goldstone) scalars arising in the f-terms receive a mass. This can be explicitly seen in the scalar mass matrix, eqs. (6.44) and (6.46). Consider a minimum of the potential with unbroken supersymmetry, satisfying (6.6). The scalar mass matrix (6.44) simplifies then to

$$
\mathcal{M}_0^2 = \begin{pmatrix} < f_{im}\overline{f}^{mk} > + < D_i^a D^{ak} > & < D_i^a D_l^a > \\ < D^{aj} D^{ak} > & < f_{lm}\overline{f}^{mj} > + < D_l^a D^{aj} > \end{pmatrix}. \tag{6.79}
$$

Using (6.77), one finds that

$$
\mathcal{M}_0^2 \begin{pmatrix} < D_k^b > \\ \pm < D^{bl} > \end{pmatrix} = < D^{ak} D_k^b \pm D^{bk} D_k^a > \begin{pmatrix} < D_i^a > \\ < D^{aj} > \end{pmatrix}. \tag{6.80}
$$

But one can also repeat (6.76–77) for the D-terms of the potential. Gauge invariance of $D^a D^a/2$ reads

$$
D^a D_i^a D^{bi} - D^a D_i^b D^{ai} = 0, \tag{6.81}
$$

and differentiating with respect to z^j or $z_j^\dagger$ implies

$$
\begin{aligned}
< D_j^a D_i^a D^{bi} > &= < D_j^a D_i^b D^{ai} >, \\
< D^{aj} D_i^a D^{bi} > &= < D^{aj} D_i^b D^{ai} >,
\end{aligned}
\tag{6.82}
$$

for supersymmetric vacua, $< D^a > = 0$. Returning to (6.80), one finds that the states corresponding to the directions

$$
\frac{1}{\sqrt{2}}\left(< D_k^b > - < D^{bk} >\right) = Im\ < D_k^b > \tag{6.83}
$$

are the massless Goldstone bosons, while directions

$$
\frac{1}{\sqrt{2}}\left(< D_k^b > + < D^{bk} >\right) = Re\ < D_k^b > \tag{6.84}
$$

have a mass matrix given by $2 < D_i^a D^{ai} > = 2g^2 < z^\dagger T^a T^b z >$: These states have the same mass as the corresponding gauge bosons (see the spin 1 mass matrix, eq. 6.32).

Analogously, acting with the spin 1/2 mass matrix, and using $< f_{ij} D^{bi} > = 0$, one finds immediately that the fermion

$$
\begin{pmatrix} < D_i^b > \psi^i \\ \lambda^b \end{pmatrix}
$$

is a Dirac fermion with the same mass matrix as the gauge bosons.

With unbroken supersymmetry, the Higgs mechanism operates then in the following way. To give a mass to a vector multiplet, one uses a full chiral multiplet in which the complex scalar splits into a massless real Goldstone state giving the longitudinal polarisation of the gauge field in the unitary gauge, and a real physical scalar, with the same mass as the gauge boson. In the fermionic sector, the gaugino and the chiral spinor form a Dirac fermion with again the same mass as the gauge boson. A massive vector multiplet in then constructed, with four bosonic and four fermionic degrees of freedom (This supermultiplet was already described in chapter 3).

If supersymmetry is not broken, the Higgs mechanism can be completely expressed in superfield language. The vacuum expectation values $< z^i >$ of the scalar fields contained in the chiral superfields ϕ^i, are themselves superfields:

$$
< \phi^i > = < z^i > . \tag{6.85}
$$

One can then define new superfields, with vanishing expectation values, by the shift

$$
\varphi^i = \phi^i - < \phi^i >, \tag{6.86}
$$

and substitute these new superfields in the original superfield Lagrangian. One can also directly define the unitary gauge in the following way: the chiral superfields ϕ^i are parametrized by

$$
\phi^i = e^{i\phi_G^a T^a}(\phi_P^i + < \phi^i >). \tag{6.87}
$$

In this superfield equation, the chiral left-handed superfields ϕ_G^a and ϕ_P^i denote respectively the Goldstone and physical superfields. There are as many ϕ_G^a as broken generators, $T^a < \phi^i > \neq 0$, and the superfields ϕ_P^i contain all remaining, physical chiral multiplets. The unitary gauge is obtained when the gauge transformation

$$\begin{aligned} \phi^i &\longrightarrow e^{-i\phi_G^a T^a}\phi^i, \\ e^{2gV} &\longrightarrow e^{-i(\phi_G^a)^\dagger T^a} e^{2gV} e^{i\phi_G^a T^a}, \end{aligned} \tag{6.88}$$

is applied. The resulting Lagrangian is expressed in terms of the physical superfields ϕ_P^i, $< \phi^i >$ and V. There is a mass term for the vector superfields of the form

$$(\mathcal{M}^2)^{ab}[V^a V^b]_{\theta\theta\overline{\theta\theta}} = 2g^2 < \phi_i^\dagger > (T^a T^b)^i{}_j < \phi^j > [V^a V^b]_{\theta\theta\overline{\theta\theta}}, \tag{6.89}$$

which corresponds to the spin 1 mass matrix also obtained in (6.32).

Notice however that the unitary gauge must be chosen in a supersymmetric way, by a gauge transformation with full superfield parameters. One cannot simultaneously take the unitary and the Wess-Zumino gauge. The superfield Lagrangian in the unitary gauge must then be computed with the complete vector superfields, with components $C, M, N, D, v_\mu, \chi, \lambda$, as in eq. (4.98).

Chapter 7

Non renormalization theorems and softly broken supersymmetry

We have seen in the preceding chapter that the necessary breaking of supersymmetry is a source of difficulties when constructing realistic supersymmetric models. The Fayet-Iliopoulos mechanism is inadequate and unesthetic. It would require a larger gauge group and a terribly complicated structure in the matter multiplet sector. The O'Raifeartaigh mechanism, because of its tree-level mass relations, can be used only with the help of loop corrections.

It is when invoking quantum corrections that supersymmetric gauge theories differ in the most spectacular way from normal gauge theories. Supersymmetry is able to cancel many divergences usually present in field theory, leading to 'non-renormalization theorems'. These theorems have important implications for supersymmetric unified theories. It turns out to be much easier to obtain independent scale parameters m_i, corresponding either to masses of scalar particles or to scales of symmetry breaking, with very large ratios m_i/m_j. In normal field theories, loop corrections have the effect of bringing these ratios back to $m_i/m_j \sim 1$. This is no more the case with supersymmetry. Thus, supersymmetric theories allow hierarchies of scales which are necessary to unify strong and electroweak interactions (and also gravity). Even though supersymmetry does not help in understanding why scales of unification are so different (the electroweak scale is $\sim 10^2 GeV$, the gravity scale is $M_P \sim 10^{19} GeV$), it is at least technically possible with supersymmetry to construct a field theory where vastly different scales coexist at the quantum level.

7.1. The non renormalization theorems

Supersymmetric gauge theories are renormalizable field theories, and renormalization can be performed to all orders in perturbation theory without breaking supersymmetry [13]. This means that the counterterm Lagrangian to be added to the classical Lagrangian discussed in chapter 5:

$$\mathcal{L} = \mathcal{L}_{classical} + \mathcal{L}_{counterterm}, \tag{7.1}$$

can also be expressed in the superfield formalism and contains the same terms as the classical Lagrangian. Divergences are absorbed by redefining the parameters of the classical Lagrangian (gauge couplings g^a, parameters of the superpotential W, and Fayet-Iliopoulos terms ξ^a), and by rescaling the superfields.

Counterterms will fall into two classes. Those who have the form of a $\theta\theta$ component of a chiral superfield,

$$[\phi_1 \dots \phi_n]_{\theta\theta} + [\overline{\phi}_1 \dots \overline{\phi}_n]_{\overline{\theta\theta}} \quad ; \qquad \overline{D}_{\dot{\alpha}} \phi_i = 0,$$

will be called F-counterterms, and those who are obtained by taking the $\theta\theta\overline{\theta\theta}$ of a vector superfield

$$[\dots]_{\theta\theta\overline{\theta\theta}}$$

will be called D-counterterms. Clearly, F-counterterms will be added to renormalize the superpotential and the gauge kinetic terms.

The interest of this classification is due to the fact that loop diagrams can be fully analysed in the superfield formalism. In general, a loop Feynman diagram will have supersymmetric partners, with the same external states, but with fields on the loops replaced by their supersymmetric partners. For instance, a diagram containing a loop with a scalar field belonging to a chiral multiplet will have a partner diagram with the chiral fermion on the loop. It is however possible to include all the different diagrams in a superfield formalism for Feynman diagrams. One defines superfield Feynman rules (superfield propagators and superfield vertices), and loop integrations are now taken over all superspace, $d^4k d^2\theta d^2\overline{\theta}$. We will not discuss here the superfield formalism for quantum corrections (see for instance [17]). The divergent parts of diagrams, compensated by the counterterms can then be studied in superfield formalism and one can prove (to all orders) the following non renormalization theorem:

> Any perturbative quantum contribution to the effective action must be expressible as one integral over the whole superspace.

This means that loop corrections are always of the form $[\dots]_{\theta\theta\overline{\theta\theta}}$, they are only '$D$-contributions'.

As a consequence, the parameters of the superpotential are not renormalized. There is no need for F-counterterms. Notice that the non renormalization theorem means that the parameters of the superpotential do not receive *any* quantum corrections (even finite) as long as supersymmetry is preserved.

The only necessary renormalization constants are then $(Z_\phi)^i{}_j$ to renormalize the wave functions of chiral multiplets ϕ^i:

$$\phi^i_{(r)} = (Z_\phi^{-1/2})^i{}_j \phi^j, \tag{7.2}$$

where (r) stands for renormalized, and $(Z_V)^{ab}$, to renormalize the vector multiplets

$$V^a_{(r)} = (Z_V^{-1/2})^{ab} V^b. \tag{7.3}$$

There is then one renormalization constant for each superfield. The renormalization of the gauge coupling constant is related to the one of V^a by Ward identities. The renormalizations of the trilinear Yukawa couplings λ_{ijk} and of the mass parameters m_{ij} present in the superpotential are given by

$$\begin{aligned} \lambda_{(r)ijk} &= (Z_\phi^{1/2})_i{}^{i'} (Z_\phi^{1/2})_j{}^{j'} (Z_\phi^{1/2})_k{}^{k'} (Z_\lambda)_{i'j'k'}^{lmn} \lambda_{lmn}, \\ m_{(r)ij} &= (Z_\phi^{1/2})_i{}^{i'} (Z_\phi^{1/2})_j{}^{j'} (Z_m)_{i'j'}^{lm} m_{lm} + \delta m_{ij}, \end{aligned} \tag{7.4}$$

with, as a consequence of the non-renormalization theorem,

$$\begin{aligned} (Z_\lambda)_{ijk}^{lmn} &= \delta_i^l \delta_j^m \delta_k^n, \\ (Z_m)_{ij}^{lm} &= \delta_i^l \delta_j^m, \\ \delta m_{ij} &= 0. \end{aligned} \tag{7.5}$$

One then finds

$$\begin{aligned} (\lambda_{ijk} \phi^i \phi^j \phi^k)_{(r)} &= \lambda_{ijk} \phi^i \phi^j \phi^k, \\ (m_{ij} \phi^i \phi^j)_{(r)} &= m_{ij} \phi^i \phi^j, \end{aligned} \tag{7.6}$$

and the superpotential is not renormalized at all, as stated by the theorem.

Notice that Z_ϕ and Z_V cannot contain quadratic divergences. As long as there is no Fayet-Iliopoulos term $[\xi^a V^a]_{\theta\theta\overline{\theta\theta}}$, a supersymmetric theory contains only logarithmic divergences. The renormalization of the Fayet-Iliopoulos term is controlled by another non renormalization theorem:

> If ξ^a is associated with a traceless generator T^a ($Tr\, T^a = 0$), there is no quadratic divergence and ξ^a is only multiplicatively renormalized.

The non renormalization theorems mean for instance that if one imposes a scalar field to be massless at tree-level, then it will remain massless to all orders. Also, a tree-level vanishing ξ^a will remain zero to all orders, if $Tr\, T^a = 0$. This is a sharp difference with normal gauge theories, where quadratic divergences generate quantum corrections to a zero tree-level scalar mass.

7.2. One-loop renormalization

In perturbation theory, at a given loop order, the non renormalization theorems correspond to cancellations of terms arising from different Feynman diagrams. For instance, in a Wess-Zumino model, the Yukawa couplings and the scalar self-couplings are related: both are obtained from the superpotential $W(\phi_i)$ of the theory. These relations between couplings, which do not exist in non supersymmetric field theories, are preserved by renormalization to all orders [13], and they lead to related contributions from Feynman diagrams which would be completely independent in the absence of supersymmetry. Notice that the cancellations can always be obtained before performing loop integrals: there is in general no need for a regularization (or any cut-off). After summing (with correct combinatorial and statistical factors!) the Feynman diagrams contributing to a Green's function, the integrant of loop integrals is found to vanish as a consequence of the non renormalization theorems.

There are actually two mechanisms of cancellation. They can be simply illustrated at the one-loop level in a Wess-Zumino model, with chiral multiplets only. Firstly, the scalar fields $z^i = (A^i + iB^i)/\sqrt{2}$ contain real scalar fields A^i and real pseudoscalars B^i. The couplings of the pseudoscalars B_i carry additional i factors which will cancel contributions of the scalars A_i. This scalar–pseudoscalar cancellation mechanism is for instance at work in the vanishing of the one loop corrections to chiral fermion masses in the Wess-Zumino model. The Yukawa couplings contained in

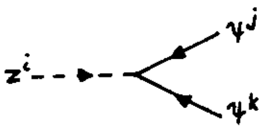
$$-\frac{1}{2}\frac{d^W}{dz^i dz^j}\psi^i\psi^j - \frac{1}{2}\frac{d^2\overline{W}}{dz_i^\dagger dz_j^\dagger}\overline{\psi}_i\overline{\psi}_j$$

read

$$-\lambda_{ijk}z^i\psi^j\psi^k - \lambda^{\dagger ijk}z_i^\dagger\overline{\psi}_j\overline{\psi}_k = -\frac{1}{\sqrt{2}}\left(\lambda_{ijk}A^i\psi^j\psi^k + i\lambda_{ijk}B^i\psi^j\psi^k\right) - h.c. \qquad (7.7)$$

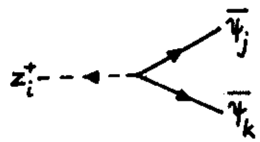
The additional i factor for pseudoscalars B_i brings a minus sign in diagrams where this Yukawa coupling appears twice. This can in fact be summarized by depicting scalars by oriented lines corresponding to the vertices

$\lambda_{ijk}\, z^i\psi^j\psi^k$



Yukawa couplings:

$\lambda^{\dagger\, ijk}\, z_i^\dagger\, \overline{\psi}_j\overline{\psi}_k$



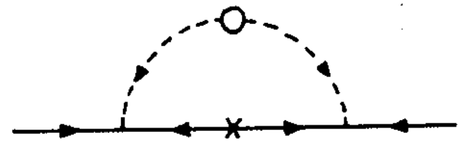
Propagator:

Consider now the one-loop contribution to the mass of chiral fermions ψ^i. Since the mass term is

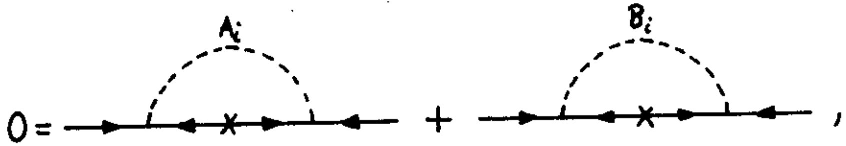
$$\frac{1}{2}m_{ij}\psi^i\psi^j + \frac{1}{2}m^{\dagger ij}\overline{\psi}_i\overline{\psi}_j,$$

it is depicted by

where the cross represents the tree-level contribution and the black dot the one-loop graph, which would be

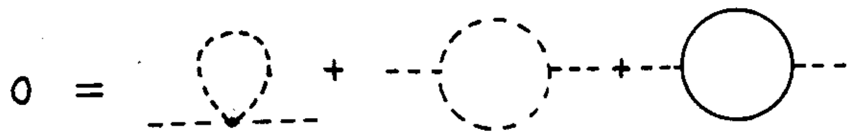


The circle requires the reversing of the scalar arrows, but there is no term $(z^i)^2 + (z_i^\dagger)^2$ in the Lagrangian. Then, the diagram does not exist. This rule of oriented scalar lines summarizes that the diagrams with scalars A_i and pseudoscalars B_i cancel,



because of the additional i^2 factor in the contribution of the pseudoscalars B_i. In general, the scalar–pseudoscalar cancellation mechanism operates along fermion lines traversing a Feynman diagram.

The second cancellation mechanism is well known: fermion loops carry a minus sign due to Fermi-Dirac statistics, and are able to cancel scalar loops provided masses and couplings of scalars and fermions are related as implied by supersymmetry. In some sense this cancellation mechanism is the supersymmetric partner of the A–B cancellation described above: the vanishing of one-loop corrections to scalar masses is due to the minus sign of the fermion loop diagram. It is simple to verify in a (massive) Wess-Zumino model that



As required by non renormalization theorems, the sum of these diagrams is already zero before loop integration.

The use of the oriented lines for scalars and fermions is very useful to identify the vanishing diagrams where a cancellation between A and B has taken place. It is of little help when the cancellation arises between boson and fermion loops.

7.3. The hierarchy problem and supersymmetry

Unified theories of particle interactions contain at least two scales. The lightest scale corresponds to the spontaneous breaking of $SU(2)\times U(1)$ into $U(1)_{em}$, giving a mass to the $W^{\pm}$ and Z^{0} gauge bosons and also to quarks and leptons. The order of magnitude of the electroweak scale is then 10^2 GeV. It is obtained from the scalar potential for the $SU(2)$ doublet of Higgs scalars H,

$$V=-\mu^2 HH^{\dagger}+\frac{1}{2}\lambda(HH^{\dagger})^2, \tag{7.8}$$

which gives a vacuum expectation to H at its minimum:

$$<HH^{\dagger}>=\mu^2/\lambda. \tag{7.9}$$

To get $<HH^{\dagger}>\sim 10^4$ GeV^2, since λ cannot be arbitrarily small, μ^2 should not be very far from the same order of magnitude.

One must however take into account the other scales relevant to the unification of particle interactions. Firstly, one can have a 'grand unification' of strong and electroweak interactions, $SU(3)\times SU(2)\times U(1)$ being embedded into a larger simple gauge group like $SU(5)$, $SO(10)$ or E_6 for example. The scale of this unififcation, which corresponds to the scale at which the unified group is spontaneously broken into $SU(3)\times SU(2)\times U(1)$, cannot be smaller than

$$M_{GUT}\sim 10^{14-16}GeV.$$

Also, gravitational corrections should be taken into account. The scale of (quantum) gravity is the Planck scale,

$$M_P\sim 10^{19}GeV.$$

At these energies, gravitation is of comparable strength with the other forces and quantum gravity corrections become sizeable.

The question of the smallness of the weak interaction scale compared with the other scales then arises. At tree-level, one can easily choose a form of the potential

such that the mass term, with parameter $-\mu^2$, of the Higgs doublet has the right magnitude. However quantum corrections will not in general preserve this choice. The renormalized parameter $\mu^2_{(r)}$ will be related, at one loop, to the bare parameter μ^2, chosen at the adequate scale, by

$$\mu^2_{(r)}=C\alpha^2M^2+\mu^2, \tag{7.10}$$

where C is a number of order $10^{0\pm1}$, α is some coupling constant and M is the large scale of the theory, i.e. $M=M_{GUT}$ or $M=M_P$. These quantum corrections are for instance due to diagrams where the loop contains a heavy (mass $\sim M$) gauge field of the unified group. In this case, the coupling constant α is a gauge coupling, related in general to the strong or electroweak coupling constant at low energies. Then, α cannot be chosen arbitrarily small and the quantum corrections to μ^2 are at least 10^{30} orders of magnitude too large. The only way out in this situation is to tune μ^2 so that $\mu^2_{(r)}$ has the right order of magnitude, i.e.

$$C\alpha^2M^2+\mu^2\sim O(10^4GeV^2).$$

This tuning condition must be modified at every order in perturbation theory, making the appearance of a small scale completely unnatural. Quantum field theory always favours a unique mass scale. This problem is called the hierarchy problem.

Supersymmetry however changes completely the picture. The corrections of order M^2 in (7.10) arise only if the theory has quadratic divergences. Realistic supersymmetric models are always free of quadratic divergences, so that μ^2 is at most logarithmically renormalized. At one loop,

$$\mu^2_{(r)}=\mu^2\left(1+C\alpha^2 ln(M^2/\mu^2)\right), \tag{7.11}$$

so that $\mu^2_{(r)}$ is not very different from μ^2. Supersymmetry then solves the hierarchy problem, at least in the technical sense: a tree-level hierarchy is stable under quantum corrections. There remains the physical question: why is there a hierarchy of scale in nature?

This solution to the hierarchy problem is one of the main motivation to introduce supersymmetry in unified theories of particle interactions. There is no other known mechanism powerful enough to obtain the same result. This is related to the fact that only supersymmetric theories (with unbroken supersymmetry or with spontaneously broken supersymmetry) possess non renormalization theorems, which then have deep physical implications. These theorems are however too strong in the sense that they remove quadratic divergences, but also some logarithmic and even finite quantum contributions which are not at all incompatible with a hierarchy, and are even useful in realistic applications. Requiring only the absence of quadratic divergences, in view of the hierarchy problem, leads to softly broken supersymmetric theories.

7.4. Soft breaking of supersymmetry

The hierarchy problem is solved, at least in the technical sense, if the theory is free of quadratic divergences. This is the case of supersymmetric gauge theories, even if supersymmetry is spontaneously broken, if all gauge symmetry generators are traceless. We have however seen that spontaneous breaking of supersymmetry hardly gives realistic models.

The absence of quadratic divergences does not however imply that the theory must be supersymmetric. One can add some terms to the supersymmetric Lagrangian, breaking supersymmetry, but introducing only new logarithmic divergences.

These new terms are in fact present in the low-energy effective Lagrangian of spontaneously broken supergravity theories [14, 16]. They are then natural ingredients of supergravity unified theories.

These new terms, which break supersymmetry without introducing quadratic divergences, are called ***soft breaking terms***. Their enumeration is particularly simple at one loop because of the following argument. We have seen that in supersymmetric theories, all divergences are found in the wave function renormalization of chiral multiplets and in the renormalization of gauge coupling constants g^a (equivalent to wave function renormalization of vector multiplets). It is then sufficient to consider the one-loop renormalization of the scalar potential, which contains a gauge part $(\frac{1}{2}D^aD^a)$ depending on g^a, to determine all renormalization constants of the theory. The one-loop divergent contributions to the scalar potential are given by [18, 19]

$$\delta V = \frac{\Lambda^2}{32\pi^2} STr\,\mathcal{M}^2(z) + \frac{1}{64\pi^2} STr\,\mathcal{M}^4(z) ln\left(\frac{\mathcal{M}^2(z)}{\Lambda^2}\right), \tag{7.12}$$

where Λ is a cut-off mass parameter. The first term contains the quadratic divergences. $STr\,\mathcal{M}^2(z)$ is the quantity calculated in section 6.1, but now considered as a function of the scalar fields z^i and not only of their vacuum expectation values $< z^i >$. $STr\,\mathcal{M}^4(z)$ is the analogous supertrace for the fourth power of the mass matrices. The second term contains all logarithmic one-loop divergences.

The problem of finding soft breaking terms corresponds to add new gauge invariant terms to the Lagrangian such that $STr\,\mathcal{M}^2(z)$ does not receive any new field dependent contributions, which would correspond to new quadratic divergences. A

constant term of the form (constant)Λ^2 is of course irrelevant in the Lagrangian. This simple method allows to investigate easily one-loop soft breaking terms. One can then show that these terms are also soft to all orders [20], provided they do not give any field dependent contribution to the trace of $\mathcal{M}^2(z)$ for all states of a given spin (i.e. no contribution to $\mathcal{M}_0^2(z)$, $\mathcal{M}_{1/2}^2(z)$ and $\mathcal{M}_1^2(z)$). Since a mass term for gauge bosons is not gauge invariant, we only need to consider $\mathcal{M}_0^2(z)$ and $\mathcal{M}_{1/2}^2(z)$.

The spin zero mass matrix squared is given by equations (6.44–48). It is clear that

$$\delta\mathcal{L}_{SOFT}(z, z^\dagger) = \eta(z) + \eta^\dagger(z^\dagger) + M_i{}^j z^i z_j^\dagger, \tag{7.13}$$

where $\eta(z)$ is an arbitrary gauge invariant cubic function of z^i:

$$\eta(z) = \eta_{(1)i} z^i + \frac{1}{2}\eta_{(2)ij} z^i z^j + \frac{1}{3}\eta_{(3)ijk} z^i z^j z^k \tag{7.14}$$

generates a new contribution to $\mathcal{M}_0^2(z)$ of the form

$$\mathcal{M}_{0\,SOFT}^2 = \frac{1}{2}\begin{pmatrix} M_i{}^j & \eta_{ik} \\ \eta^{\dagger lj} & M^l{}_k \end{pmatrix}, \tag{7.15}$$

where

$$\eta_{ij} = \frac{d^2\eta}{dz^i dz^j} \quad , \quad \eta^{\dagger ij} = \frac{d^2\eta^\dagger}{dz_i^\dagger dz_j^\dagger}. \tag{7.16}$$

Since

$$Tr\,\mathcal{M}_{0\,SOFT}^2 = M_i^i = constant, \tag{7.17}$$

the terms in $\delta\mathcal{L}_{SOFT}$ are soft breaking terms. Eq. (7.13) contains the most general scalar soft breaking terms. They contain scalar mass terms of both forms $zz^\dagger$ and $z^2 + z^{\dagger 2}$, and trilinear 'analytic' interactions $z^3 + z^{\dagger 3}$.

The spin 1/2 mass matrix is given in Eq. (6.39). Clearly, a gaugino mass term

$$-\frac{1}{2}\Delta^{ab}\lambda^a\lambda^b - \frac{1}{2}\Delta^{ab}\overline{\lambda}^a\overline{\lambda}^b \tag{7.18}$$

will only give a constant contribution $\Delta^{ab}\Delta^{ba}$ to $Tr\,\mathcal{M}_{1/2}^2$. It is then a soft term. All other additions to $\mathcal{M}_{1/2}$ are not soft in general. For instance, a mass term

$$-\frac{1}{2}\delta_{ij}\psi^i\psi^j - \frac{1}{2}\delta^{\dagger ij}\overline{\psi}_i\overline{\psi}_j \tag{7.19}$$

gives a new contribution to $Tr\,\mathcal{M}_{1/2}^2$ which reads

$$f_{ij}\delta^{\dagger ij} + \overline{f}^{ij}\delta_{ij} + \delta_{ij}\delta^{\dagger ij}, \tag{7.20}$$

in the notation of chapter 6. This is field dependent except if δ_{ij} is non zero only when f_{ij} is a constant. Such a choice however leads only to one-loop soft breaking terms. In general higher order contributions spoil the softness.

The soft breaking terms (7.13) and (7.18) contain all what we need to satisfy the mass inequalities (6.27) for an acceptable spectrum of supersymmetric particles. We have scalar mass soft terms to rise the masses of scalar quarks and leptons, and we also have gaugino mass terms to rise the mass of gluinos and of the photino.

7.5. One-loop renormalization group equations

We have already discussed briefly the different possibilitites of incorporating supersymmetry in realistic models of strong and electroweak interactions. We have seen that the main difficulty lies in the mechanism used to break supersymmetry, which should give large enough masses to the scalar partners of quarks and leptons. Our conclusion was that only the addition of soft breaking terms is a satisfactory option. The question is now to discuss the origin of these soft breaking terms.

The important observation is that soft breaking terms arise naturally in the effective low-energy theory of spontaneously broken supergravity theories coupled to matter multiplets (see [14, 16] for reviews). These theories possess a *local* supersymmetry which can be spontaneously broken using the *super-Higgs* mechanism. At energies much lower than the Planck scale, M_P, gravitation decouples and the effective gauge theory (which is obtained in the limit $M_P \to \infty$) is globally supersymmetric except for a set of soft breaking terms characteristic of the super-Higgs mechanism. In general, one could construct the sector which induces the super-Higgs mechanism in such a way that any possible soft breaking terms could be generated. However, most of the phenomenologically interesting theories lead to strong constraints on the structure of the soft breaking terms. For instance, all mass terms of the form $(m^2)^i_j z^\dagger_i z^j$ are equal: $(m^2)^i_j = m^2\delta^i_j$. These constraints are very useful to reduce the arbitrariness of the soft terms, but they are subject to important renormalizations from the Planck scale at which they hold strictly, to energies close to the masses of the weak gauge bosons where supersymmetry is phenomenologically relevant. These renormalization effects are controlled by the renormalization group equations of the theory, which we will now derive at the one-loop level.

We consider the most general supersymmetric Lagrangian with arbitrary soft terms, gauge group G and representation of the chiral multiplets R. We will only assume that there is no Fayet-Iliopoulos terms, and all formula will be given for a simple gauge groups, with a unique gauge coupling constant g. The generalization

to non simple groups is straightforward. The scalar potential is then

$$V = f^a f_a + \frac{1}{2}D^A D^A + (m^2)^a_b z^\dagger_a z^b + \eta(z) + \overline{\eta}(z^\dagger), \qquad (7.21)$$

where $f(z^a)$ is the superpotential and

$$f_a = \frac{df}{dz^a}, \qquad f^a = \frac{df^\dagger}{dz^\dagger_a},$$

the D-terms read

$$D^A = -g z^\dagger_a T^{A\,a}{}_b z^b \qquad (7.22)$$

in terms of the generators T^A of the gauge group for the representation of the chiral multiplets, and finally $\eta(z)$ is an arbitrary gauge invariant polynomial of third degree in the fields z^a. The superpotential will be written

$$f = \frac{1}{6}f_{abc}z^a z^b z^c + \frac{1}{2}\mu_{ab}z^a z^b + \alpha_a z^a, \qquad (7.23)$$

with

$$f_{abc} = \frac{d^3 f}{dz^a dz^b dz^c}. \qquad (7.24)$$

We will use for η the decomposition given in Eq. (7.14).

We further need to add gaugino terms of the form

$$-\frac{1}{2}\Delta^{AB}\lambda^A\lambda^B - \frac{1}{2}\Delta^{AB}\overline{\lambda}^A\overline{\lambda}^B. \qquad (7.25)$$

The gaugino masses can always be chosen real (as it was done in 7.25) by a phase redefinition of the gaugino spinors, and we will also diagonalize the matrix Δ in the form $\Delta^{AB} = \Delta^A\delta^{AB}$.

With the exception of the gaugino mass parameters, one can obtain all renormalization constants by only considering the scalar potential. The gauge coupling constant renormalization is present in the gauge potential and all Yukawa couplings are either proportional to g or directly given by f_{abc}. Since all soft breaking terms are at most of dimension three, Yukawa coulings are not affected by them and their renormalization is the same as in the fully supersymmetric case. It can then be obtained from the scalar sector. We will now compute all renormalization constants and the correponding renormalization group (RG) equations to one loop. The one-loop contributions to the scalar potential are given in Eq. (7.12). Since by definition of soft breaking terms $STr\,\mathcal{M}^2(z)$ does not receive any field dependent contribution, and since we have assumed there is no Fayet-Iliopoulos term, our theory is free of quadratic divergences. All divergences are then logarithmic, and they are given by

$$\delta V = -\frac{1}{32\pi^2}STr\,\mathcal{M}^4(z)\,ln\Lambda. \qquad (7.26)$$

The renormalization constants are then obtained by the condition

$$V(\underline{z}^a, \underline{z}_b^\dagger, \underline{parameters}) = V(z^a, z_b^\dagger, parameters) + \delta V(z^a, z_b^\dagger, parameters), \quad (7.27)$$

where *parameters* stands for all coupling constants and mass parameters and underlined quantities are renormalized.

The main task is then to compute the supertrace of the quartic mass matrix for arbitrary values of the scalar fields z^a. We first write the mass matrices for every spin, including the arbitrary soft breaking terms. Spin one states are not affected by soft breaking terms. Their mass matrix squared is then the same as in Eq. (6.34):

$$(\mathcal{M}_1^2)^{AB} = D_a^A D^{Ba} + D_a^B D^{Aa}, \quad (7.28)$$

so that

$$3Tr\, \mathcal{M}_1^4 = 6(D_a^A D^{Ba})(D^{Ab} D_b^B) + 6(D_a^A D^{Ba})(D_b^A D^{Bb}). \quad (7.29)$$

Compared with (6.39), the fermion mass matrix will now include the gaugino mass terms and become

$$\mathcal{M}_{1/2} = \begin{pmatrix} f_{ab} & \sqrt{2}iD_a^B \\ \sqrt{2}iD_b^A & \Delta^{AB} \end{pmatrix}. \quad (7.30)$$

One then obtains

$$\begin{aligned} -2Tr\, \mathcal{M}_{1/2}^4 = & -2f_{ab}f^{bc}f_{cd}f^{da} - 16f_{ab}f^{ac}D_c^A D^{Ab} - 16(D_a^A D^{Ab})(D^{Ba}D_b^B) \\ & + 8f_{ab}D^{Aa}\Delta^{AB}D^{Bb} + 8f^{ab}D_a^A\overline{\Delta}^{AB}D_b^B - 16\overline{\Delta}^{AC}\Delta^{CB}D_a^A D^{Ba} \\ & - 2\Delta^{AB}\overline{\Delta}^{BC}\Delta^{CD}\overline{\Delta}^{DA}. \end{aligned} \quad (7.31)$$

The mass matrix squared of scalar fields can be expressed in terms of second derivatives of the scalar potential:

$$\mathcal{M}_0^2 = \begin{pmatrix} V_b^a & V^{ac} \\ V_{db} & V_d^c \end{pmatrix}, \quad (7.32)$$

where

$$\begin{aligned} V_{ab} &= \frac{d^2V}{dz^a dz^b} = f_{abc}f^c + D_a^A D_b^A + \eta_{ab}, \\ V^{ab} &= \frac{d^2V}{dz_a^\dagger dz_b^\dagger} = f^{abc}f_c + D^{Aa}D^{Ab} + \overline{\eta}^{ab}, \\ V_b^a &= \frac{d^2V}{dz_a^\dagger dz^b} = f^{ac}f_{bc} + D^{A}{}_b^a D^A + D^{Aa}D_b^A + (m^2)_b^a. \end{aligned} \quad (7.33)$$

Then, since $STr\, \mathcal{M}^4$ is invariant under the transformations of the gauge group, it can be reexpressed only in terms of group invariants, without any use of explicit group

generators. This is achieved by exploiting the group invariance of the superpotential and of the gauge potential. We have

$$f_a D^{Aa} = -g\frac{df}{dz^a}T^{A}{}_b^a z^b = 0, \quad (7.34)$$

which, when differentiated, also implies

$$\begin{aligned} f_{ab}D^{Aa} &= -f_a D^{A}{}_b^a, \\ f_{abc}D^{Aa} &= -f_{ab}D^{A}{}_c^a - f_{ac}D^{A}{}_b^a. \end{aligned} \quad (7.35)$$

The invariance of the gauge potential reads

$$D^A D_a^A D^{Ba} - D^A D^{Aa} D_a^B = 0. \quad (7.36)$$

Analogously, the polynomial η is group invariant:

$$\eta_a D^{Aa} = 0, \quad (7.37)$$

which can be used in the same way as for the superpotential. These identities can be used to reexpress $STr\, \mathcal{M}^4$ in terms of the group invariants

$$\begin{aligned} T(R)\delta^{AB} &= Tr(T^A T^B) = T^{A}{}_b^a T^{B}{}_a^b, \\ C(R)\delta_b^a &= T^{A}{}_c^a T^{A}{}_b^c, \end{aligned} \quad (7.38)$$

and also $C(G)$ (the quadratic Casimir invariant of the gauge group) which is the same as $C(R)$ when R is the adjoint representation of G, and can be obtained from the structure constants f_G^{ABC} by

$$C(G)\delta^{AB} = f_G^{ACD}f_G^{BCD}. \quad (7.39)$$

Notice the obvious relation

$$T(R)\, dim\, G = C(R)\, dim\, R. \quad (7.40)$$

One can as well use the gauge invariance of the mass terms $(m^2)_b^a z_a^\dagger z^b$, which reads

$$(m^2)_b^a z_a^\dagger D^{Ab} - (m^2)_b^a z^b D_a^A = 0. \quad (7.41)$$

By combining these various identities due to gauge invariance, one gets new expressions for many of the terms present in $STr\,\mathcal{M}^4$, like for instance:

$$\begin{aligned}
2f^{ac}f_{bc}{D^A}^b_a &= -f_{abc}f^{ab}D^{Ac} = -f^{abc}f_{ab}D^A_c,\\
f^{ac}f_{bc}D^A_a D^{Ab} &= g^2C(R)f_a f^a,\\
f^{abc}f_c D^A_a D^A_b &= g^2C(R)f_a f^a - g^2C(R)z^\dagger_a f_b f^{ab},\\
f_{abc}f^c D^{Aa}D^{Ab} &= g^2C(R)f_a f^a - g^2C(R)z^a f^b f_{ab},\\
f^{ab}D^A_a D^A_b &= -g^2C(R)f^a z^\dagger_a,\\
f_{ab}D^{Aa}D^{Ab} &= -g^2C(R)f_a z^a,\\
\eta_{ab}D^{Aa}D^{Ab} &= -g^2C(R)\eta_a z^a,\\
\overline{\eta}^{ab}D^A_a D^A_b &= -g^2C(R)\overline{\eta}^a z^\dagger_a,\\
(m^2)^a_b D^A_a D^{Ab} &= g^2C(R)(m^2)^a_b z^\dagger_a z^b,\\
D^A(D^A_a {D^B}^a_b D^{Bb}) &= g^2C(R)D^A D^A,\\
D^A(D^B_a {D^A}^a_b D^{Bb}) &= g^2\left[C(R) - \frac{1}{2}C(G)\right]D^A D^A,\\
D^A D^B({D^A}^a_b {D^B}^b_a) &= g^2T(R)D^A D^A.
\end{aligned} \tag{7.42}$$

Finally one finds

$$\begin{aligned}
STr\,\mathcal{M}^4 = &2g^2\left[T(R) + 2C(R) - 3C(G)\right]D^A D^A - 8g^2C(R)f^a f_a\\
&+ 4f^{ac}f_{bc}{D^A}^b_a D^A + 2f^{abc}f_{abd}f^d f_c - 2g^2C(R)\left[z^a f^b f_{ab} + z^\dagger_a f_b f^{ab}\right]\\
&- 8g^2C(R)\sum_A \Delta^A(f_a z^a + f^a z^\dagger_a) - 16g^2C(R)\sum_A \Delta^2_A(z^a z^\dagger_a) - 2\sum_A \Delta^4_A\\
&+ 2f^{abc}f_c\eta_{ab} + 2f_{abc}f^c\overline{\eta}^{ab}\\
&- 2g^2C(R)(\overline{\eta}^a z^\dagger_a + \eta_a z^a) + 2\eta_{ab}\overline{\eta}^{ab}\\
&+ 4f^{ab}f_{ac}(m^2)^c_b + 4g^2C(R)(m^2)^a_b z^\dagger_a z^b\\
&+ 2(m^2)^a_b(m^2)^b_a + 4(m^2)^a_b(D^A)^b_a D^A.
\end{aligned} \tag{7.43}$$

Notice that the last term vanishes as a consequence of Schur's lemma for traceless generators of semi-simple groups. This is not generally true for possible $U(1)$ factors present in the gauge group.

One can easily rewrite this expression in terms of the scalar fields, the coupling constants and the mass parameters by inserting the expansions on the superpotential f (Eq. 7.23) and of η (Eq. 7.14). The following terms are then present in $STr\,\mathcal{M}^4$:

Terms of order $zzz^\dagger z^\dagger$:

$$\begin{aligned}
&2g^2[T(R) - 3C(G)]D^A D^A + 4g^2C(R)D^A D^A - gz^\dagger_a\{X, T^A\}^a_b z^b D^A\\
&-4g^2C(R)f^{abc}f_{ade}z^\dagger_b z^\dagger_c z^d z^e + \frac{1}{2}X^a_b f_{acd}f^{bc'd'}z^c z^d z^\dagger_{c'} z^\dagger_{d'}.
\end{aligned} \tag{7.44}$$

Terms of order $zzz^\dagger + h.c.$:

$$\begin{aligned}
&-7g^2C(R)f_{abc}\overline{\mu}^{ad}z^b z^c z^\dagger_d + h.c.\\
&+X^a_b f_{acd}\overline{\mu}^{be}z^\dagger_e z^c z^d + h.c.
\end{aligned} \tag{7.45}$$

Terms of order $z^3 + h.c.$:

$$\begin{aligned}
&-4g^2C(R)\sum_A \Delta^A f_{abc}z^a z^b z^c + h.c.\\
&+2f^{abc}f_{cde}\eta_{3abd'}z^d z^e z^{d'} + h.c.\\
&-2g^2C(R)\eta_{3abc}z^a z^b z^c + h.c.
\end{aligned} \tag{7.46}$$

Terms of order $zz^\dagger$:

$$\begin{aligned}
&-12g^2C(R)\mu_{ab}\overline{\mu}^{bc}z^a z^\dagger_c + 2X^a_b\overline{\mu}^{bc}\mu_{ad}z^\dagger_c z^d\\
&-16g^2C(R)\sum_A(\Delta^A)^2(z^\dagger_a z^a) + 4(m^2)^a_b f_{acd}f^{bce}z^\dagger_e z^d\\
&+4g^2C(R)(m^2)^a_b z^\dagger_a z^b + 8\eta_{3abc}\overline{\eta}_3^{abd}z^c z^\dagger_d.
\end{aligned} \tag{7.47}$$

Terms of order zz:

$$\begin{aligned}
&-6g^2C(R)f_{abc}\overline{\alpha}^a z^b z^c + h.c.\\
&+X^a_b\overline{\alpha}^b f_{acd}z^c z^d + h.c.\\
&-8g^2C(R)\sum_A \Delta^A\mu_{ab}z^a z^b + h.c.\\
&+f^{abc}f_{cde}\eta_{2ab}z^d z^e + h.c.\\
&+4f^{abc}\mu_{cd}\eta_{3abe}z^d z^e + h.c.\\
&-2g^2C(R)\eta_{2ab}z^a z^b + h.c.
\end{aligned} \tag{7.48}$$

Terms of order z:

$$\begin{aligned}
&-10g^2C(R)\mu_{ab}\overline{\alpha}^a z^b + h.c.\\
&+2X^a_b\overline{\alpha}^b\mu_{ac}z^c + h.c.\\
&+4f_{bcd}\overline{\mu}^{ac}(m^2)^b_a z^d + h.c.\\
&-8g^2C(R)\sum_A \Delta^A\alpha_a z^a + h.c.\\
&+2f^{abc}\mu_{cd}\eta_{2ab}z^d + h.c.\\
&+4f^{abc}\alpha_c\eta_{3abd}z^d + h.c.\\
&-2g^2C(R)\eta_{1a}z^a + h.c.\\
&+4\eta_{3abc}\overline{\eta}_2^{ab}z^c + h.c.
\end{aligned} \tag{7.49}$$

The notation

$$X^a_b = f^{acd}f_{bcd}$$

has been used. As already discussed, the dimension four terms of order $zzz^\dagger z^\dagger$ are not affected by soft terms. The renormalization of g and of the chiral Yukawa couplings f_{abc} are then the same as with unbroken supersymmetry. One then defines the following renormalization constants:

$$\begin{aligned} \underline{z}^a &= \left(\delta^a_b - \frac{1}{2}\epsilon^a_b\right) z^b, \\ \underline{g} &= (1+\rho)g, \\ \underline{f}_{abc} &= f_{abc} + \left(\gamma^{a'b'c'}_{abc} + \frac{1}{2}\epsilon^{a'}_a \delta^{b'}_b \delta^{c'}_c + \frac{1}{2}\delta^{a'}_a \epsilon^{b'}_b \delta^{c'}_c + \frac{1}{2}\delta^{a'}_a \delta^{b'}_b \epsilon^{c'}_c\right) f_{a'b'c'}. \end{aligned} \tag{7.50}$$

These renormalization constants can then be obtained with the help of the renormalization condition (7.27) applied to the terms of order $zzz^\dagger z^\dagger$. One finds

$$\begin{aligned} \rho &= -2kg^2[T(R) - 3C(G)], \\ \epsilon^a_b &= 4kg^2 C(R)\delta^a_b - 2kX^a_b, \\ \gamma^{abc}_{a'b'c'} &= 2kg^2\left[C(A)+C(B)+C(C)\right]\delta^a_{a'}\delta^b_{b'}\delta^c_{c'}, \end{aligned} \tag{7.51}$$

with the notation

$$k = \frac{1}{32\pi^2} ln\,\Lambda. \tag{7.52}$$

The expression $C(A)+C(B)+C(C)$ corresponds to the sum of the group invariants $C(R)$ computed for the representations R of the three indices a, b and c. The renormalization of the Yukawa couplings is then given by

$$\begin{aligned} \underline{f}_{abc} =& f_{abc} + 4kg^2[C(A)+C(B)+C(C)]f_{abc} \\ &- k\left(X^{a'}_a f_{a'bc} + X^{b'}_b f_{ab'c} + X^{c'}_c f_{abc'}\right), \end{aligned} \tag{7.53}$$

which can also be written

$$\underline{f}_{abc} = (Z^{1/2})^{a'}_a (Z^{1/2})^{b'}_b (Z^{1/2})^{c'}_c f_{a'b'c'}, \tag{7.54}$$

with

$$(Z^{1/2})^a_b = \delta^a_b + 4kg^2 C(A)\delta^a_b - kX^a_b. \tag{7.55}$$

This renormalization constant Z corresponds to the wave function renormalization applied to chiral superfields in the case of unbroken supersymmetry.

The other renormalization constants can be obtained in an analogous way. The terms of order $zzz^\dagger$ do not depend on the soft terms. This implies that also the mass parameters μ_{ab} will have the same renormalization as in the case of unbroken supersymmetry. One defines

$$\underline{\mu}_{ab} = \mu_{ab} + \left(\mu^{a'b'}_{ab} + \frac{1}{2}\epsilon^{a'}_a \delta^{b'}_b + \frac{1}{2}\delta^{a'}_a \epsilon^{b'}_b\right)\mu_{a'b'}. \tag{7.56}$$

Comparison with one-loop terms of order $zzz^\dagger$ leads to

$$\mu^{a'b'}_{ab} = 2kg^2[C(A)+C(B)]\delta^{a'}_a \delta^{b'}_b \tag{7.57}$$

and, as expected from supersymmetry,

$$\begin{aligned} \underline{\mu}_{ab} &= \mu_{ab} + 4kg^2[C(A)+C(B)]\mu_{ab} - k\left(X^{a'}_a \mu_{a'b} + X^{b'}_b \mu_{ab'}\right) \\ &= (Z^{1/2})^{a'}_a (Z^{1/2})^{b'}_b \mu_{a'b'}. \end{aligned} \tag{7.58}$$

All other parameters are influenced by the soft breaking terms. In general, these parameters will not be multiplicatively renormalised. For instance, gaugino masses will generate at one loop terms of order $zz^\dagger$, irrespective of the other couplings, so that zero values of $(m^2)^a_b$ are not stable at one loop. This is also the case of the parameters in η, since there is at one loop a term proportional to the product of gaugino masses by the superpotential. The missing renormalization constants can then be obtained in the following way: the terms of order $zz^\dagger$ will define $(\underline{m}^2)^a_b$, those of order z^3 determine $\underline{\eta}_{3abc}$. There is then an ambiguity for terms of order z^2 and z related to the fact that these terms (already at tree-level) arise from the superpotential part $f^a f_a$ in presence of parameters α_a as well as in the soft terms η. One can then define $\underline{\alpha}_a$ by the requirement that the structure of supersymmetric theories is preserved, leading to the requirement

$$\begin{aligned} \underline{\alpha}_a &= (Z^{1/2})^b_a \alpha_b \\ &= \alpha_a + 4kg^2 C(A)\alpha_a - kX^b_a \alpha_b. \end{aligned} \tag{7.59}$$

One then gets for the remaining parameters

$$\begin{aligned} (\underline{m}^2)^a_b =& (m^2)^a_b - kX^a_c (m^2)^c_b - kX^c_b (m^2)^a_c - 4kf_{ace}f^{bde}(m^2)^c_d \\ &- 8k\eta_{3acd}\overline{\eta}_3^{bcd} + 8kg^2[C(A)\Delta^2_A + C(B)\Delta^2_B]\delta^a_b, \end{aligned} \tag{7.60}$$

$$\begin{aligned} \underline{\eta}_{3abc} =& \eta_{3abc} + 4kg^2[C(A)+C(B)+C(C)]\eta_{3abc} \\ &- 2k[\delta^{a'}_a f_{bcd}f^{b'c'd} + \delta^{b'}_b f_{acd}f^{a'c'd} + \delta^{c'}_c f_{abd}f^{a'b'd}]\eta_{3a'b'c'} \\ &- k[X^{a'}_a \eta_{3a'bc} + X^{b'}_b \eta_{3ab'c} + X^{c'}_c \eta_{3abc'}] \\ &+ 4kg^2[C(A)\Delta_A + C(B)\Delta_B + C(C)\Delta_C]f_{abc}, \end{aligned} \tag{7.61}$$

$$\begin{aligned} \underline{\eta}_{2ab} =& \eta_{2ab} + 4kg^2[C(A)+C(B)]\eta_{2ab} - k[X^{a'}_a \eta_{2a'b} + X^{b'}_b \eta_{2ab'}] \\ &- 2kf_{abc}f^{cde}\eta_{2de} - 4kf^{cde}[\eta_{3acd}\mu_{eb} + \eta_{3bcd}\mu_{ea}] \\ &+ 8kg^2[C(A)\Delta_A + C(B)\Delta_B]\mu_{ab}, \end{aligned} \tag{7.62}$$

$$\begin{aligned} \underline{\eta}_{1a} =& \eta_{1a} + 4kg^2 C(A)\eta_{1a} - kX^b_a \eta_{1b} \\ &- 2kf^{bcd}\mu_{ab}\eta_{2cd} - 4kf^{bcd}\alpha_b \eta_{3cda} - 4k\eta_{3abc}\overline{\eta}_2^{bc} \\ &- 4kf_{abc}\overline{\mu}^{cd}(m^2)^b_d + 8kg^2 C(A)\Delta_A \alpha_a, \end{aligned} \tag{7.63}$$

omitting from now on the summations over gaugino indices A.

The renormalization group equations are then straightforward to obtain. Renormalization is specified at an energy Q. Then the quantity k entering the one loop expressions should become

$$k = \frac{1}{32\pi^2} ln(\Lambda/Q).$$

RG equations control the freedom we have in choosing Q. For instance,

$$\begin{aligned} Q\frac{d}{dQ}g &= -\Lambda\frac{d}{d\Lambda}g \\ &= -\frac{g^3}{16\pi^2}[3C(G) - T(R)] \end{aligned}$$

which gives the standard one-loop beta function for gauge couplings in $N = 1$ supersymmetric gauge theories. One then gets the full set of renormalization group equations which read:

$$Q\frac{d}{dQ}g = -\frac{g^3}{16\pi^2}[3C(G) - T(R)], \tag{7.64}$$

$$\begin{aligned} Q\frac{d}{dQ}f_{abc} = &-\frac{1}{32\pi^2}\left[4g^2[C(A) + C(B) + C(C)]f_{abc}\right. \\ &\left.-(X_a^{a'}f_{a'bc} + X_b^{b'}f_{ab'c} + X_c^{c'}f_{abc'})\right], \end{aligned} \tag{7.65}$$

$$Q\frac{d}{dQ}\mu_{ab} = -\frac{1}{32\pi^2}\left[4g^2[C(A) + C(B)]\mu_{ab} - (X_a^{a'}\mu_{a'b} + X_b^{b'}\mu_{ab'})\right], \tag{7.66}$$

$$Q\frac{d}{dQ}\alpha_a = -\frac{1}{32\pi^2}\left[4g^2C(A)\alpha_a - X_a^b\alpha_b\right], \tag{7.67}$$

$$\begin{aligned} Q\frac{d}{dQ}(m^2)_a^b = &-\frac{1}{32\pi^2}\left[8g^2[C(A)\Delta_A^2 + C(B)\Delta_B^2]\delta_a^b - X_c^b(m^2)_a^c - X_a^c(m^2)_c^b\right. \\ &\left.-4f_{ace}f^{bde}(m^2)_d^c - 8\eta_{3acd}\overline{\eta}_3^{bcd}\right], \end{aligned} \tag{7.68}$$

$$\begin{aligned} Q\frac{d}{dQ}\eta_{3abc} = &-\frac{1}{32\pi^2}\left[4g^2[C(A) + C(B) + C(C)]\eta_{3abc}\right. \\ &-2(\eta_{3ab'c'}f^{b'c'd}f_{bcd} + \eta_{3a'bc'}f^{a'c'd}f_{acd} + \eta_{3a'b'c}f^{a'b'd}f_{abd}) \\ &-X_a^{a'}\eta_{3a'bc} - X_b^{b'}\eta_{3ab'c} - X_c^{c'}\eta_{3abc'} \\ &\left.+4g^2[C(A)\Delta_A + C(B)\Delta_B + C(C)\Delta_C]f_{abc}\right], \end{aligned} \tag{7.69}$$

$$\begin{aligned} Q\frac{d}{dQ}\eta_{2ab} = &-\frac{1}{32\pi^2}\left[4g^2[C(A) + C(B)]\eta_{2ab} - X_a^{a'}\eta_{2a'b} - X_b^{b'}\eta_{2ab'}\right. \\ &-4(\eta_{3acd}f^{cde}\mu_{eb} + \eta_{3bcd}f^{cde}\mu_{ea}) - 2f_{abc}f^{cde}\eta_{2de} \\ &\left.+8g^2[C(A)\Delta_A + C(B)\Delta_B]\mu_{ab}\right], \end{aligned} \tag{7.70}$$

$$\begin{aligned} Q\frac{d}{dQ}\eta_{1a} = &-\frac{1}{32\pi^2}\left[4g^2C(A)\eta_{1a} - X_a^b\eta_{1b} - 4\eta_{3acd}f^{cde}\alpha_e\right. \\ &-4\eta_{3abc}\overline{\eta}_2^{bc} - 4f_{abc}\overline{\mu}^{cd}(m^2)_d^b - 2f^{bcd}\mu_{ab}\eta_{2cd} \\ &\left.+8g^2C(A)\Delta_A\alpha_a\right]. \end{aligned} \tag{7.71}$$

The generalization of these equations to non simple gauge groups is very simple. Combinations like $g^2[C(A)+C(B)]$ only need to be replaced by a sum over all factors in the gauge groups (all simple components and $U(1)$ factors) of the same combination with the gauge coupling constant and the group invariant $C(R)$ corresponding to each factor group.

Chapter 8

The minimal supersymmetric Standard Model

The minimal supersymmetric Standard Model is obtained by supersymmetrizing the Lagrangian of the Standard Model in the simplest possible way. The spin 0, 1/2 and 1 fields of the Standard Model are supplemented by their supersymmetric partners, with spin 1/2, 0 and 1/2 respectively. It turns out that the Higgs sector must be doubled: the supersymmetric Standard Model contains at least two complex Higgs doublets (and their supersymmetric partners called Higgsinos).

As stressed in ch. 6, the model can only be realistic if the supersymmetric particles are always heavier than the known quarks, leptons and massless gauge bosons: no such state has been detected (yet ?). Electron and proton colliders provide precise lower limits on the masses of the supersymmetric partners, but for our purposes it will be sufficient to require a supersymmetry breaking mechanism which increases the mass of all supersymmetric particles, without specifying numbers. However, the minimal supersymmetric Standard Model does not break supersymmetry spontaneously and we must include soft breaking terms to make it realistic. This theory can then only be physically justified as the effective theory resulting at lower energies from a more unified theory, which plausibly includes gravitation and gives a justification for the introduction of soft terms. At present, the best candidate for this 'superunified' theory is a superstring theory. The soft breaking terms are thought to be the remnants at low energy of spontaneous breaking of local supersymmetry (the 'super-Higgs' mechanism) in the superunified theory.

The minimal supersymmetric extension of the Standard Model is a perfectly consistent and well defined theory. It can be experimentally tested and its phenomenological aspects have been extensively studied (see for instance [15]). We will only outline here its construction, and discuss the structure of its scalar potential, with the characteristic mass relations satisfied by the physical scalar states.

8.1. The supersymmetric Lagrangian

The first step in the construction of the minimal supersymmetric version of the Standard Model is to introduce the spin 1/2 partners of the gauge bosons of $SU(3)_c \times SU(2)_L \times U(1)_Y$ as well as the scalar partners of quarks and leptons:

gluons → gluinos,

$W^\pm$ → winos,

Z^0 → zino,

photon → photino,

quarks → scalar quarks or squarks,

leptons → scalar leptons or sleptons.

The chiral multiplets containing the quark and lepton generations will have the usual $SU(3)_c \times SU(2)_L \times U(1)_Y$ quantum numbers:

$Q^a : (3, 2, 1/6)$: left-handed quarks,

$U_c^a : (\overline{3}, 1, -2/3)$: left-handed antiquarks, charge $-2/3$,

$D_c^a : (\overline{3}, 1, 1/3)$: left-handed antiquarks, charge $1/3$,

$L^a : (1, 2, -1/2)$: left-handed leptons,

$E_c^a : (1, 1, 1)$: left-handed charged antileptons.

The index a runs over the quark-lepton generations. One could also add a left-handed antineutrino, with quantum numbers $(1, 1, 0)$ for each generation, but this is not necessary in the minimal model. The left-handed antineutrino is anyway a singlet of the gauge group: as long as neutrino masses need not be included, its presence would not influence the following discussion.

With these chiral multiplets, one can construct a superpotential of the form

$$W = \alpha^{abc} U_c^a D_c^b D_c^c + \beta^{abc} Q^a D_c^b L^c + \gamma^{abc} L^a L^b E_c^c. \tag{8.1}$$

(In this expression, $SU(3)_c$ and $SU(2)_L$ indices are omitted. It is understood that gauge invariant combinations are constructed. For instance, $U_c^a D_c^b D_c^c$ means $\epsilon_{ijk} (U_c^a)^i (D_c^b)^j (D_c^c)^k$, with colour indices $i, j, k = 1, 2, 3$). This superpotential will induce Yukawa and scalar interactions. The phenomenological implications of the latter cannot be very problematic. Scalar quarks and leptons must have a large enough mass to make their phenomenology consistent with the absence of any experimental evidence for their existence, and this turns out to be rather easy and natural, with the use of soft breaking terms. The Yukawa interactions on the contrary would have dramatic physical implications. Clearly, the superpotential (8.1) violates baryon number B and lepton number L conservations. The first term has $B = -1$, $L = 0$ while the two other terms have $B = 0$ and $L = 1$. The exchange of a scalar partner of D_c^a induces a four fermion interaction of the form

$$\alpha^{abc}(\beta^{ade})^* \frac{1}{m^2(D_c^a)} U_c^b D_c^c \overline{Q}^d \overline{L}^e \tag{8.2}$$

where $m^2(D_c^a)$ is the mass (squared) of the scalar partner of the antiquarks D_c^a. We will see later that the scalar masses find their origin in the soft breaking terms which are used to break supersymmetry, and simultaneously to break $SU(2)_L \times U(1)_Y$ into $U(1)_{em}$. These scalar masses are then naturally of order $10^2\,GeV$. The four fermion interaction (8.2) generates nucleon decay via the processes like $u + d \to \overline{u} + \overline{e}$ or $u + d \to \overline{d} + \overline{\nu}$ at an unacceptably fast rate. These interactions must then be suppressed: the easiest way out is then to assume that the dangerous terms in the superpotential are not present at all.

Also, since the couplings α^{abc}, β^{abc} and γ^{abc} are naturally non diagonal in the generations, these couplings generate flavour changing neutral current interactions at unacceptable rates.

It is then a common assumption which we will also adopt that the interactions described by the superpotential (8.1) do not exist. Because of the non renomalization theorems (even with soft breaking terms) assuming the vanishing of Yukawa couplings is natural since radiative corrections will not generate them.

The construction of the supersymmetric Standard Model becomes slightly more involved in the Higgs sector. In the Standard Model, quarks and leptons receive their mass through their Yukawa couplings to a unique (in the minimal model) Higgs doublet H, which breaks $SU(2)_L \times U(1)_Y$. H has quantum numbers $(1,2,-1/2)$. Quarks with charge $-1/3$ and charged leptons couple to H, while charge $2/3$ quarks couple to $H^\dagger$. In the supersymmetric case however, Yukawa couplings arise from the superpotential, which is a function of the chiral superfields, but not of their conjugate. It is then clear that one cannot use a single Higgs chiral multiplet to give a mass to all quarks and charged leptons. To obtain massive charge $2/3$ quarks, we need a second, independent Higgs chiral multiplet

$$\overline{H} : (1,2,+1/2)$$

with a superpotential

$$W_{\overline{H}} = \lambda_U^{ab} Q^a U_c^b \overline{H}. \tag{8.3}$$

Notice that even though the quantum numbers of H are identical to those of L^a, one cannot use L^a as a Higgs multiplet. But the scalars of L^a can only give masses to quarks and leptons via the last term of the superpotential (8.1). This would however

leave the quarks with charge $2/3$ massless. To give them a mass, we need a multiplet $\overline{H}$, which contains also chiral fermions, the Higgsinos. If the theory only contains the chiral multiplets for $\overline{H}$ and the generations, the chiral Higgsinos remain massless and the theory has a chiral anomaly, destroying its renormalizability. This is cured by the necessary introduction of the chiral multiplet H.

Taking into account a minimal Higgs sector H, $\overline{H}$, the complete superpotential of the minimal supersymmetric Standard Model is

$$W = \lambda_E^{ab} L^a E_c^b H + \lambda_D^{ab} Q^a D_c^b H + \lambda_U^{ab} Q^a U_c^b \overline{H} + mH\overline{H}, \tag{8.4}$$

with summations on the generation indices a and b. The superpotential for the Higgs multiplets introduces a mass parameter m. The matrices λ_E, λ_D and λ_U contain the (complex) Yukawa couplings. Quark and lepton mass matrices will be generated when both Higgs multiplets $H : (1,2,-1/2)$ and $\overline{H} : (1,2,+1/2)$ acquire vacuum expectation values $< H >$ and $< \overline{H} >$. The fermion mass matrices will then arise from

$$\lambda_E^{ab} < H > \quad , \quad \lambda_D^{ab} < H > \quad , \quad \lambda_U^{ab} < \overline{H} >,$$

exactly as in the (non supersymmetric) Standard Model. The vacuum expectation values correspond to the minimum of the scalar potential obtained from the complete superpotential and from the D-terms associated to the gauge group.

As in the non-supersymmetric Standard Model, the fermion masses and the mixing parameters arising from the Yukawa couplings are all free parameters. Supersymmetry does not predict any new relation on these parameters.

To obtain the complete Lagrangian, we first write the gauge transformations of the various chiral superfields. Under $SU(3)_c$, we have:

$$\begin{aligned} Q^a &\to e^{i\Lambda_3} Q^a, \\ U_c^a &\to e^{-i\Lambda_3} U_c^a, \\ D_c^a &\to e^{-i\Lambda_3} D_c^a, \\ L^a, E_c^a, H, \overline{H} &\to L^a, E_c^a, H, \overline{H}, \end{aligned} \tag{8.5}$$

where $\Lambda_3 = \sum_{A=1}^{8} \Lambda_3^A \lambda^A/2$ parametrizes an $SU(3)_c$ transformation: the λ^A are Gell-Mann matrices, and Λ_3^A are the eight left-handed chiral superfields used as parameters. For $SU(2)_L$,

$$\begin{aligned} Q^a &\to e^{i\Lambda_2} Q^a, \\ L^a &\to e^{i\Lambda_2} L^a, \\ H &\to e^{i\Lambda_2} H, \\ \overline{H} &\to e^{i\Lambda_2} \overline{H}, \\ U_c^a, D_c^a, E_c^a &\to U_c^a, D_c^a, E_c^a, \end{aligned} \tag{8.6}$$

and $\Lambda_2 = \sum_{i=1}^{3} \Lambda_2^i \sigma^i/2$, in terms of the Pauli matrices σ^i. Finally, $U(1)_Y$ involves a single left-handed chiral superfield Λ_1 as parameter and the transformations are:

$$\begin{aligned} Q^a &\to e^{\frac{1}{6}i\Lambda_1} Q^a, \\ U_c^a &\to e^{-\frac{2}{3}i\Lambda_1} U_c^a, \\ D_c^a &\to e^{\frac{1}{3}i\Lambda_1} D_c^a, \\ L^a &\to e^{-\frac{1}{2}i\Lambda_1} L^a, \\ E_c^a &\to e^{i\Lambda_1} E_c^a, \\ H &\to e^{-\frac{1}{2}i\Lambda_1} H, \\ \overline{H} &\to e^{\frac{1}{2}i\Lambda_1} \overline{H}. \end{aligned} \tag{8.7}$$

To write the complete, gauge invariant Lagrangian, we introduce the following vector multiplets:

$$\begin{aligned} V_3 &= \sum_{A=1}^{8} V_3^A \lambda^A/2 \qquad \text{for } SU(3)_c, \\ V_2 &= \sum_{i=1}^{3} V_2^i \sigma^i/2 \qquad \text{for } SU(2)_L, \\ V_1 & \qquad\qquad\qquad\quad \text{for } U(1)_Y. \end{aligned} \tag{8.8}$$

The vector multiplets V_3^A, $A = 1, \ldots, 8$, V_2^i, $i = 1,2,3$ and V_1 contain respectively the $SU(3)_c$, $SU(2)_L$ and $U(1)_Y$ gauge bosons and their fermionic gaugino partners as physical degrees of freedom. The complete Lagrangian for the supersymmetric Standard Model is then defined by the expression

$$\begin{aligned} \mathcal{L}_{susy} = &[(Q^a)^\dagger e^{V_3} e^{V_2} e^{\frac{1}{6}V_1} Q^a + (U_c^a)^\dagger e^{-V_3} e^{-\frac{2}{3}V_1} U_c^a \\ &+ (D_c^a)^\dagger e^{-V_3} e^{\frac{1}{3}V_1} D_c^a + (L^a)^\dagger e^{V_2} e^{-\frac{1}{2}V_1} L^a \\ &+ (E_c^a)^\dagger e^{V_1} E_c^a + H^\dagger e^{V_2} e^{-\frac{1}{2}V_1} H + \overline{H}^\dagger e^{V_2} e^{\frac{1}{2}V_1} \overline{H}]_{\theta\theta\overline{\theta}\overline{\theta}} \\ &+ [W]_{\theta\theta} + [W^\dagger]_{\overline{\theta}\overline{\theta}} \\ &+ \frac{1}{8g_3^2} \left(Tr[W_3^\alpha W_{3\alpha}]_{\theta\theta} + Tr[\overline{W}_{3\dot\alpha} \overline{W}_3^{\dot\alpha}]_{\overline{\theta}\overline{\theta}} \right) \\ &+ \frac{1}{8g_2^2} \left(Tr[W_2^\alpha W_{2\alpha}]_{\theta\theta} + Tr[\overline{W}_{2\dot\alpha} \overline{W}_2^{\dot\alpha}]_{\overline{\theta}\overline{\theta}} \right) \\ &+ \frac{1}{16g_1^2} \left([W_1^\alpha W_{1\alpha}]_{\theta\theta} + [\overline{W}_{1\dot\alpha} \overline{W}_1^{\dot\alpha}]_{\overline{\theta}\overline{\theta}} \right). \end{aligned} \tag{8.9}$$

The kinetic Lagrangian for the gauge supermultiplets is built with the chiral multiplets W_3^α for $SU(3)_c$, W_2^α for $SU(2)_L$ and W_1^α for $U(1)_Y$, as defined in eqs. (5.15), (5.31) and (5.34). We could have introduced a Fayet-Iliopoulos term

$$[\xi V_1]_{\theta\theta\overline{\theta}\overline{\theta}}$$

for the abelian vector superfield of the weak hypercharge. The parameter ξ has dimension $(mass)^2$ and corresponds to a second free mass scale in the theory. The Fayet-Iliopoulos term does not however induce a spontaneous breaking of supersymmetry: the equation of motion of the associated D_1 auxiliary field is

$$\begin{aligned} D_1 = -g_1 \Big(&\frac{1}{6}(Q^a)^\dagger Q^a - \frac{2}{3}(U_c^a)^\dagger U_c^a + \frac{1}{3}(D_c^a)^\dagger (D_c^a) - \frac{1}{2}(L^a)^\dagger L^a \\ &+ (E_c^a)^\dagger E_c^a - \frac{1}{2}H^\dagger H + \frac{1}{2}\overline{H}^\dagger \overline{H} + \xi \Big). \end{aligned} \tag{8.10}$$

One easily finds solutions to the equations for a supersymmetric minimum of the potential:

$$\begin{aligned} &D_3^A = D_2^i = D_1 = 0, \\ &\frac{\partial W}{\partial Q^a} = \frac{\partial W}{\partial U_c^a} = \frac{\partial W}{\partial D_c^a} = \frac{\partial W}{\partial L^a} = \frac{\partial W}{\partial E_c^a} = \frac{\partial W}{\partial H} = \frac{\partial W}{\partial \overline{H}} = 0. \end{aligned} \tag{8.11}$$

In general, these solutions lead to unrealistic gauge symmetry breakings of $SU(3)_c \times U(1)_Y$, and supersymmetry is anyway not broken. We then just omit the Fayet-Iliopoulos term. It is a non-renormalization theorem that since the weak hypercharge is traceless, no Fayet-Iliopoulos term can be generated by higher order loop corrections to the theory.

8.2. The soft breaking terms

Since supersymmetry is not broken in the Lagrangian $\mathcal{L}_{susy}$, we must include soft breaking terms, as discussed in section 7.4. In the framework of the minimal supersymmetric Standard Model, one just assumes their existence. These terms should ultimately find an explanation when a completely unified theory, including gravitation, is available. Their form and magnitude will be related to the mechanism of (spontaneous) breaking of supersymmetry in this 'superunified' theory. The prototype of this generation of soft breaking terms is the super-Higgs mechanism breaking spontaneously local supersymmetry in $N = 1$ supergravity theories.

The possible soft breaking terms have been described in section 7.4. They involve mass terms of the form $z_i^\dagger z^i$ for all scalar fields, and analytic scalar terms, as given in (7.14). Notice however that only those terms which are already present in the superpotential W of the theory (as functions of the chiral superfields) will in general be generated as soft terms (as functions of the scalar fields only, like in (7.14), and with different coefficients as in W). For the supersymmetric Standard Model, the complete Lagrangian with soft breaking terms is then:

$$\mathcal{L} = \mathcal{L}_{susy} + \mathcal{L}_{soft}, \tag{8.12}$$

where $\mathcal{L}_{susy}$ is defined in (8.9) and

$$\begin{aligned}\mathcal{L}_{soft} = & \sum_a [(m^a_Q)^2|z^a_Q|^2 + (m^a_{U_c})^2|z^a_{U_c}|^2 + (m^a_{D_c})^2|z^a_{D_c}|^2 \\ & + (m^a_L)^2|z^a_L|^2 + (m^a_{E_c})^2|z^a_{E_c}|^2] + m^2_H|z_H|^2 + m^2_{\overline{H}}|z_{\overline{H}}|^2 \\ & + \sum_{ab} (A^{ab}_U z^a_Q z^b_{U_c} z_{\overline{H}} + A^{ab}_D z^a_Q z^b_{D_c} z_H + A^{ab}_L z^a_L z^b_{E_c} z_H + \text{c.c.}) \\ & + \frac{1}{2} M_3 \sum_{A=1}^{8} (\lambda^A_3 \lambda^A_3 + \overline{\lambda}^A_3 \overline{\lambda}^A_3) \\ & + \frac{1}{2} M_2 \sum_{i=1}^{3} (\lambda^i_2 \lambda^i_2 + \overline{\lambda}^i_2 \overline{\lambda}^i_2) \\ & + \frac{1}{2} M_1 (\lambda_1 \lambda_1 + \overline{\lambda}_1 \overline{\lambda}_1)\end{aligned} \tag{8.13}$$

Indices a, b run through the generations of fermions, z denote scalar fields and λ denote the gauginos. The last three lines contain the gaugino mass terms for $SU(3)_c$, $SU(2)_L$ and $U(1)_Y$ respectively.

If the origin of the soft term is to be explained by a spontaneous breaking of supersymmetry in a supergravity theory, all scalar soft breaking terms are of the same scale, the gravitino mass $m_{3/2}$. But gaugino soft masses M_3, M_2 and M_1 can in general correspond to rather different scales.

8.3. The scalar potential and the scalar fields

The scalar potential V of the supersymmetric Standard Model contains 'supersymmetric' terms arising from the superpotential W and from the auxiliary fields D of the vector mulitiplets. It also receives contributions from the soft breaking terms (8.13). The vacuum of the theory is given by the minimum of this potential. In general, the magnitude of the soft breaking terms is such that the vacuum expectation values of all scalar quarks and leptons naturally vanish. Only z_H and/or $z_{\overline{H}}$ can be non zero, inducing the correct symmetry breaking to $SU(3)_c \times U(1)_{em}$.

All equations for the minimum of V for scalar quarks and leptons,

$$\frac{\partial V}{\partial z^i} = 0, \qquad z^i = z^a_Q, z^a_{U_c}, z^a_{D_c}, z^a_L, z^a_{E_c},$$

are satisfied when scalar quarks and leptons have zero vacuum expectation values, for arbitrary values of z_H and $z_{\overline{H}}$. We can then limit our study of the potential to those terms of V which only contain z_H and $z_{\overline{H}}$, which we will from now on denote by H and $\overline{H}$ respectively. The scalar potential for these fields reads

$$V = V_{SUSY} + V_{SOFT}, \tag{8.14}$$

with

$$V_{SUSY} = m^2(H^\dagger H + \overline{H}^\dagger \overline{H}) + \frac{1}{8} g^2 \sum_a (H^\dagger \sigma^a H + \overline{H}^\dagger \sigma^a \overline{H})^2 + \frac{1}{8} g'^2 (H^\dagger H - \overline{H}^\dagger \overline{H})^2, \tag{8.15}$$

$$V_{SOFT} = \mu_1^2 H^\dagger H + \mu_2^2 \overline{H}^\dagger \overline{H} - \mu_3^2 (H\overline{H} + H^\dagger \overline{H}^\dagger). \tag{8.16}$$

The quadratic terms in the supersymmetric part of the potential, V_{SUSY}, are obtained using the superpotential

$$W = mH\overline{H}, \tag{8.17}$$

where the notation $H\overline{H}$ means $\epsilon_{ij} H^i \overline{H}^j$. The parameter m is in general complex, but since it appears in the potential only via $|m^2|$, one can choose it real. The quartic terms arise from the D-terms for $SU(2)$:

$$D^a = -\frac{1}{2} g (H^\dagger \sigma^a H + \overline{H}^\dagger \sigma^a \overline{H}), \tag{8.18}$$

where g is the weak coupling constant and σ^a are Pauli matrices, and from the D-term for the weak hypercharge:

$$D_Y = -\frac{1}{2} g' (-H^\dagger H + \overline{H}^\dagger \overline{H}), \tag{8.19}$$

where $H^\dagger H$ denotes $H_i^\dagger H^i$, and g' is the coupling constant of weak hypercharge.

The most general $SU(2) \times U(1)$ invariant soft breaking terms are contained in V_{SOFT}. In Eq. (8.16), reality of the Lagrangian implies that the parameters μ_1^2 and μ_2^2 are real. Notice that they can be negative. Reality does not imply that μ_3^2 must be real. One can however redefine H (or $\overline{H}$) and absorb a possible phase of μ_3^2. We will then choose μ_3^2 real and positive.

The scalar doublets H and $\overline{H}$ contain the components

$$H = \begin{pmatrix} H_0 \\ H_- \end{pmatrix} \quad , \quad \overline{H} = \begin{pmatrix} \overline{H}_+ \\ \overline{H}_0 \end{pmatrix}. \tag{8.20}$$

where the subscripts indicate the electric charge. One can then easily compute the component form of the potential. The rearrangement formula

$$\sum_a (\sigma^a)^i_j (\sigma^a)^k_l = \delta^i_j \delta^k_l - 2\epsilon^{ik}\epsilon_{jl}, \tag{8.21}$$

(with $\epsilon^{12} = \epsilon_{12} = 1$) allows to rewrite the quartic terms in the form

$$\begin{aligned} & \sum_a (H^\dagger \sigma^a H)(H^\dagger \sigma^a H) = (H^\dagger H)^2, \\ & \sum_a (H^\dagger \sigma^a H)(\overline{H}^\dagger \sigma^a \overline{H}) = (H^\dagger H)(\overline{H}^\dagger \overline{H}) - 2(H^\dagger \overline{H}^\dagger)(H\overline{H}), \\ & \sum_a (\overline{H}^\dagger \sigma^a \overline{H})(\overline{H}^\dagger \sigma^a \overline{H}) = (\overline{H}^\dagger \overline{H})^2. \end{aligned} \tag{8.22}$$

In components, the quartic terms of the potential are then

$$\frac{1}{2}D^aD^a+\frac{1}{2}D_YD_Y=\frac{1}{8}(g^2+g'^2)\left[(H_0H_0^\dagger-\overline{H}_0\overline{H}_0^\dagger)^2+(H_-H_-^\dagger-\overline{H}_+\overline{H}_+^\dagger)^2\right]$$
$$+\frac{1}{4}g^2(H_0H_0^\dagger+\overline{H}_0\overline{H}_0^\dagger)(H_-H_-^\dagger+\overline{H}_+\overline{H}_+^\dagger)$$
$$+\frac{1}{4}g'^2(H_0H_0^\dagger-\overline{H}_0\overline{H}_0^\dagger)(H_-H_-^\dagger-\overline{H}_+\overline{H}_+^\dagger)$$
$$+\frac{1}{2}g^2\left[(H_0H_-^\dagger)(\overline{H}_0\overline{H}_+^\dagger)+(H_0^\dagger H_-)(\overline{H}_0^\dagger\overline{H}_+)\right]. \tag{8.23}$$

The quadratic terms are easily obtained:

$$(m^2+\mu_1^2)H^\dagger H+(m^2+\mu_2^2)\overline{H}^\dagger\overline{H}-\mu_3^2(H\overline{H}+H^\dagger\overline{H}^\dagger)=$$
$$(m^2+\mu_1^2)(H_0H_0^\dagger+H_-H_-^\dagger)+(m^2+\mu_2^2)(\overline{H}_0\overline{H}_0^\dagger+\overline{H}_+\overline{H}_+^\dagger) \tag{8.24}$$
$$-\mu_3^2(H_0\overline{H}_0+H_0^\dagger\overline{H}_0^\dagger-H_-\overline{H}_+-H_-^\dagger\overline{H}_+^\dagger).$$

The supersymmetric part of the potential, V_{SUSY}, is a sum of positive terms. It is then bounded below. This is however not true in general when soft breaking terms are introduced. The quartic terms possess directions where they vanish, corresponding to the solutions of equations

$$D^a=0, \tag{8.25.a}$$
$$D_Y=0. \tag{8.25.b}$$

Conditions (8.25.a) are straightforward to analyse using the method of Ref. [21]. The only $SU(2)$-invariant one can construct with H and $\overline{H}$ is $I=H\overline{H}$ (a power of I is irrelevant). Then, (8.25.a) is equivalent to the condition

$$\frac{dI}{dz^i}=C(z^i)^\dagger \tag{8.26}$$

where z^i denotes all fields of H and $\overline{H}$ and C is an arbitrary complex constant. The solution is

$$H_0=e^{i\alpha}\overline{H}_0^\dagger,\qquad H_-=-e^{i\alpha}\overline{H}_+^\dagger, \tag{8.27}$$

and it also solves (8.25.b). For these directions, the full scalar potential reduces to

$$V=(2m^2+\mu_1^2+\mu_2^2-2\cos\alpha\,\mu_3^2)(H_0H_0^\dagger+H_-H_-^\dagger). \tag{8.28}$$

The potential is then bounded below only if

$$2m^2+\mu_1^2+\mu_2^2>|2\mu_3^2|. \tag{8.29}$$

Since the parameters of the potential are subject to the renormalization group equations studied in the previous chapter, they depend explicitly on the energy scale. Condition (8.29) must then hold at all scales in order to have a sensible theory.

Minimization of the potential:

To be acceptable, the global minimum of the potential should have non zero vacuum expectation values for both H_0 and $\overline{H}_0$. This is necessary to give masses to all quarks and charged leptons. The parameter μ_3^2 plays a crucial role in obtaining such a minimum: without it, one always has either $<H_0>=0$ or $<\overline{H}_0>=0$.

We are interested in finding vacuum expectation values which leave $U(1)_{em}$ invariant. Then, no charged scalars will have a v.e.v. and, since there is no term linear in charged fields, it is enough to study the minimum of the potential with only neutral fields.

The potential for neutral fields only reads

$$V=V_{SUSY}+V_{SOFT}, \tag{8.30}$$

with

$$V_{SUSY}=m^2(H_0^\dagger H_0+\overline{H}_0^\dagger\overline{H}_0)+\frac{1}{8}(g^2+g'^2)(|H_0|^2-|\overline{H}_0|^2)^2, \tag{8.31}$$
$$V_{SOFT}=\mu_1^2|H_0|^2+\mu_2^2|\overline{H}_0|^2-\mu_3^2(H_0\overline{H}_0+H_0^\dagger\overline{H}_0^\dagger). \tag{8.32}$$

Let us first consider the supersymmetric potential V_{SUSY}. The minimum equations are

$$\frac{\partial V}{\partial H_0}=H_0^\dagger\left(m^2+\frac{1}{4}(g^2+g'^2)(|H_0|^2-|\overline{H}_0|^2)\right)=0,$$
$$\frac{\partial V}{\partial \overline{H}_0}=\overline{H}_0^\dagger\left(m^2-\frac{1}{4}(g^2+g'^2)(|H_0|^2-|\overline{H}_0|^2)\right)=0. \tag{8.33}$$

For $m^2\neq0$, the only solution is obviously $<H_0>=<\overline{H}_0>=0$. For $m=0$, the only constraint comes from the D-terms which give $<|H_0|^2>=<|\overline{H}_0|^2>$. This is not sufficient since the states of the Higgs chiral multiplet will remain massless, and supersymmetry is not broken.

Adding now the soft terms, we have the following minimum equations:

$$\frac{\partial V}{\partial H_0}=H_0^\dagger\left((m^2+\mu_1^2)+\frac{1}{4}(g^2+g'^2)(|H_0|^2-|\overline{H}_0|^2)\right)-\mu_3^2\overline{H}_0=0,$$
$$\frac{\partial V}{\partial \overline{H}_0}=\overline{H}_0^\dagger\left((m^2+\mu_2^2)-\frac{1}{4}(g^2+g'^2)(|H_0|^2-|\overline{H}_0|^2)\right)-\mu_3^2H_0=0. \tag{8.34}$$

Apart from the obvious solution $< H_0 >=< \overline{H}_0 >= 0$, for which $< V >= 0$, one shows easily that one can only have $< H_0 >\neq 0 \neq< \overline{H}_0 >$ when $\mu_3^2 \neq 0$. If however $\mu_3^2 = 0$, this possibility is completely unnatural. Assuming $v \equiv< H_0 >\neq 0$ and $\overline{v} \equiv< \overline{H}_0 >\neq 0$, one finds

$$0 = m^2 + \mu_1^2 + \frac{1}{4}(g^2 + g'^2)(|H_0|^2 - |\overline{H}_0|^2) - \mu_3^2 \frac{\overline{v}}{v}, \qquad (8.35.a)$$

$$0 = m^2 + \mu_2^2 - \frac{1}{4}(g^2 + g'^2)(|H_0|^2 - |\overline{H}_0|^2) - \mu_3^2 \frac{v}{\overline{v}}. \qquad (8.35.b)$$

Summing these two equations leads to

$$0 = 2m^2 + \mu_1^2 + \mu_2^2 - \mu_3^2 \frac{v^2 + \overline{v}^2}{v\overline{v}}. \qquad (8.36)$$

Clearly, this solution is compatible with $\mu_3^2 = 0$ only if $2m^2 + \mu_1^2 + \mu_2^2 = 0$ which is completely unnatural. The case $\mu_3^2 = 0$ has then for solutions

$$\begin{aligned}
&1) \quad < H_0 > = 0, && < \overline{H}_0 > = 0, \quad < V > = 0, \\
&2) \quad < H_0^2 > = -4\frac{m^2 + \mu_1^2}{g^2 + g'^2}, && < \overline{H}_0 > = 0, \quad < V > = -2\frac{(m^2 + \mu_1^2)^2}{g^2 + g'^2}, \\
&3) \quad < \overline{H}_0^2 > = -4\frac{m^2 + \mu_2^2}{g^2 + g'^2}, && < H_0 > = 0 \quad < V > = -2\frac{(m^2 + \mu_2^2)^2}{g^2 + g'^2}.
\end{aligned} \qquad (8.37)$$

Solutions 2) and 3) exist only if respectively $m^2 + \mu_1^2 < 0$ and $m^2 + \mu_2^2 < 0$. If one of these two conditions is met, $SU(2) \times U(1)$ is broken into $U(1)_{em}$, but in case 2), all charge 2/3 quarks are massless and in case 3), all charge $-1/3$ quarks and all leptons are massless.

We finally consider the general case, with $\mu_3^2 \neq 0$. Then, the non trivial minimum has always $v \neq 0 \neq \overline{v}$ and the corresponding minimum equations are given by Eqs. (8.35). The solution is best parametrized by an angle θ defined by

$$\begin{aligned}
cos\theta &= \frac{v}{\sqrt{v^2 + \overline{v}^2}}, & sin\theta &= \frac{\overline{v}}{\sqrt{v^2 + \overline{v}^2}}, \\
sin(2\theta) &= 2\frac{v\overline{v}}{v^2 + \overline{v}^2}, & cos(2\theta) &= \frac{v^2 - \overline{v}^2}{v^2 + \overline{v}^2}.
\end{aligned} \qquad (8.38)$$

The value of θ is given by (8.36), which indicates that

$$sin(2\theta) = \frac{2\mu_3^2}{2m^2 + \mu_1^2 + \mu_2^2}. \qquad (8.39)$$

Notice that the condition of boundedness of the potential corresponds to $|sin(2\theta)| < 1$. It is then straightforward to combine (8.35.a) and (8.35.b) in order to obtain a

condition on $v^2 - \overline{v}^2$. Subtracting (8.35.b) multiplied by $\overline{v}^2$ from (8.35.a) multiplied by v^2 gives:

$$v^2 + \overline{v}^2 = \frac{4(m^2 + \mu_2^2)\overline{v}^2 - 4(m^2 + \mu_1^2)v^2}{(g^2 + g'^2)(v^2 - \overline{v}^2)}, \qquad (8.40)$$

or

$$v^2 + \overline{v}^2 = \frac{4}{g^2 + g'^2}\frac{(m^2 + \mu_2^2)sin^2\theta - (m^2 + \mu_1^2)cos^2\theta}{cos(2\theta)}. \qquad (8.41)$$

A more convenient writing of Eq. (8.41) is

$$\frac{1}{2}(g^2 + g'^2)(v^2 + \overline{v}^2) = -\left(2m^2 + \mu_1^2 + \mu_2^2 + \frac{\mu_1^2 - \mu_2^2}{cos(2\theta)}\right). \qquad (8.42)$$

The quantity $v^2 + \overline{v}^2$ is related to the weak boson masses:

$$M_W^2 = \frac{1}{2}g^2(v^2 + \overline{v}^2), \qquad (8.43.a)$$

$$M_Z^2 = \frac{1}{2}(g^2 + g'^2)(v^2 + \overline{v}^2), \qquad (8.43.b)$$

so that (8.42) gives directly the mass of Z in terms of the parameters of the potential. The two minimum equations (8.35) are then equivalent to

$$sin(2\theta) = \frac{2\mu_3^2}{2m^2 + \mu_1^2 + \mu_2^2}, \qquad (8.44.a)$$

$$M_Z^2 = -\left(2m^2 + \mu_1^2 + \mu_2^2 + \frac{\mu_1^2 - \mu_2^2}{cos(2\theta)}\right). \qquad (8.44.b)$$

A natural situation is then to have all mass parameters in the potential at the scale of the gauge bosons Z^0.

Masses of physical scalars

The spectrum of the spin 0 fields is easily obtained from the scalar potential. It shows a structure characteristic of models with two Higgses. Notice however that compared to a generic two Higgs model, supersymmetry implies that all quartic terms of the potential include only one coupling constant, $g^2 + g'^2$. The spin zero fields in H and $\overline{H}$ correspond to four charged and four neutral states. Two charged and one (pseudoscalar) neutral states are however the Goldstone bosons of the Higgs mechanism: they provide the longitudinal polarizations of the gauge bosons $W^\pm$ and Z^0. The spectrum of physical spin 0 fields contains then two scalar and one pseudoscalar neutral states, and a pair of conjugate charged scalar fields. It turns out that the mass spectrum is completely characterized by the mass of the pseudoscalar (neutral) state which reads

$$m_P^2 = \mu_3^2\frac{v^2 + \overline{v}^2}{v\overline{v}} = 2m^2 + \mu_1^2 + \mu_2^2 = \frac{2\mu_3^2}{sin(2\theta)}. \qquad (8.45)$$

The charged scalar has then a mass given by

$$m_{ch}^2 = M_W^2 + m_P^2, \tag{8.46}$$

and the two neutral scalars satisfy the sum rule

$$\begin{aligned} m_{S_+}^2 + m_{S_-}^2 &= M_Z^2 + m_P^2 \\ &= M_Z^2 - M_W^2 + M_{ch}^2, \end{aligned} \tag{8.47}$$

where S_+ (S_-) denotes the heaviest (lightest) scalar state. The complete formula for the masses of scalar bosons is

$$m_{S_\pm}^2 = \frac{1}{2}\left[M_Z^2 + m_P^2 \pm \sqrt{(M_Z^2 + m_P^2)^{-}4M_Z^2 m_P^2 cos^2(2\theta)}\right]. \tag{8.48}$$

Notice that the lightest scalar state S_- is always lighter that the gauge boson Z_0.

Appendix A: Conventions and spinors in four dimensions

The metric for flat space-time is

$$\eta^{\mu\nu} = diag(+1, -1, -1, -1). \tag{A1}$$

For the Dirac algebra, gamma matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \qquad (\mu, \nu = 0, 1, 2, 3). \tag{A2}$$

We then have

$$\begin{aligned}
(\gamma^0)^2 &= -(\gamma^i)^2 = I_4, \qquad (i = 1, 2, 3)\\
(\gamma^0)^\dagger &= \gamma^0,\\
(\gamma^i)^\dagger &= -\gamma^i,\\
\gamma^0\gamma^\mu\gamma^0 &= (\gamma^\mu)^\dagger,\\
\gamma^0 &= \gamma_0,\\
\gamma^i &= -\gamma_i,
\end{aligned} \tag{A3}$$

where I_4 is the 4×4 identity matrix, and † denotes the hermitean conjugate. Defining

$$\begin{aligned}
\gamma_5 &= i\gamma^0\gamma^1\gamma^2\gamma^3,\\
{\gamma_5}^\dagger &= \gamma_5,\\
(\gamma_5)^2 &= I_4,
\end{aligned} \tag{A4}$$

the helicity projectors are given by

$$\begin{aligned}
L &= \frac{1}{2}(I_4 + \gamma_5),\\
R &= \frac{1}{2}(I_4 - \gamma_5),\\
L^2 &= L, \qquad R^2 = R, \qquad LR = RL = 0.
\end{aligned} \tag{A5}$$

where L and R mean respectively *left* and *right*. Weyl spinors are subject to the constaint

$$\begin{aligned}
R\psi_L &= 0 \qquad (\psi_L = L\psi),\\
L\psi_R &= 0 \qquad (\psi_R = R\psi).
\end{aligned} \tag{A6}$$

They contain only two independent components. The (Dirac) conjugate spinors are given by

$$\begin{aligned}
\overline{\psi_L} &= (L\psi)^\dagger\gamma^0 = \overline{\psi}R,\\
\overline{\psi_R} &= \overline{\psi}L.
\end{aligned} \tag{A7}$$

Then

$$\overline{\psi_L}\psi_L = \overline{\psi_L}\gamma_5\psi_L = 0, \qquad (A8)$$
$$\overline{\psi_L}\gamma^\mu\gamma^\nu\psi_L = 0.$$

It is convenient to use a representation of the γ matrices such that Weyl spinors take a simple form, and can be written as two-component objects only:

$$\gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}, \qquad i = 1,2,3, \qquad (A9)$$

in terms of the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$${\sigma_i}^2 = I_2, \qquad {\sigma_i}^\dagger = \sigma_i,$$
$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}, \qquad [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k, \qquad (i,j,k = 1,2,3).$$

With this choice of γ matrices,

$$\gamma_5 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \qquad L = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 0 \\ 0 & I_2 \end{pmatrix}, \qquad (A10)$$

and a Dirac spinor reads

$$\psi = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} \qquad (A11)$$

in terms of two-component left-handed (right-handed) Weyl spinors χ_L and χ_R.

Majorana spinors satisfy the condition

$$\lambda = \lambda_C = C(\overline{\lambda})^\tau, \qquad (A12)$$

where the charge conjugation matrix C is such that

$$(C\gamma^0)\gamma^{\mu *}(C\gamma^0)^{-1} = -\gamma^\mu. \qquad (A13)$$

In our representation, $\gamma^0\gamma^{\mu *}\gamma^0 = \gamma^{\mu\tau}$, so that

$$C\gamma^{\mu\tau}C^{-1} = -\gamma^\mu, \qquad (A14)$$
$$C^{-1}\gamma^\mu C = -\gamma^{\mu\tau}.$$

One can choose

$$C = i\gamma^2\gamma^0, \qquad (A15)$$
$$C^2 = -I_4, \qquad C = -C^\dagger = -C^\tau = C^*,$$

so that charge conjugate spinors are given by

$$\lambda_C = C\overline{\lambda}^\tau = C\gamma^0\lambda^* = i\gamma^2\lambda^*. \qquad (A16)$$

A Majorana spinor can then be written

$$\lambda = \begin{pmatrix} \chi_L \\ i\sigma_2{\chi_L}^* \end{pmatrix}. \qquad (A17)$$

For two anticommuting Majorana spinors ϵ and λ, using that

$$\lambda = C\overline{\lambda}^\tau = C\gamma^0(\lambda^\dagger)^\tau, \qquad \overline{\epsilon} = \epsilon^\tau C$$

leads to the following properties

$$\begin{aligned}
\overline{\epsilon}\lambda = \overline{\lambda}\epsilon \quad &= (\overline{\epsilon}\lambda)^\dagger, \\
\overline{\epsilon}\gamma_5\lambda = \overline{\lambda}\gamma_5\epsilon \quad &= -(\overline{\epsilon}\gamma_5\lambda)^\dagger, \\
\overline{\epsilon}\gamma^\mu\lambda = -\overline{\lambda}\gamma^\mu\epsilon \quad &= -(\overline{\epsilon}\gamma^\mu\lambda)^\dagger, \\
\overline{\epsilon}\gamma^\mu\gamma_5\lambda = \overline{\lambda}\gamma^\mu\gamma_5\epsilon \quad &= (\overline{\epsilon}\gamma^\mu\gamma_5\lambda)^\dagger, \\
\overline{\epsilon}\gamma^\mu\gamma^\nu\lambda = -\overline{\lambda}\gamma^\mu\gamma^\nu\epsilon &= (\overline{\epsilon}\gamma^\mu\gamma^\nu\lambda)^\dagger, \qquad (\mu \neq \nu).
\end{aligned} \qquad (A18)$$

The hermitian combinations are then

$$\begin{aligned}
&\overline{\epsilon}\lambda, \\
&i\,\overline{\epsilon}\gamma_5\lambda, \\
&i\,\overline{\epsilon}\gamma^\mu\lambda, \\
&\overline{\epsilon}\gamma^\mu\gamma_5\lambda, \\
&\overline{\epsilon}\gamma^\mu\gamma^\nu\lambda, \qquad (\mu \neq \nu).
\end{aligned} \qquad (A19)$$

Appendix B: Identities in two-component notation, and for Grassmann variables

In two-component notation, a four-dimensional Majorana spinor is written

$$\psi = \begin{pmatrix} \psi_\alpha \\ \overline{\psi}^{\dot\alpha} \end{pmatrix}, \qquad \overline{\psi} = (\psi^\alpha \quad \overline{\psi}_{\dot\alpha}). \tag{B1}$$

The gamma matrices are

$$\gamma^\mu = \begin{pmatrix} 0 & (\sigma^\mu)_{\alpha\dot\alpha} \\ (\overline{\sigma}^\mu)^{\dot\alpha\alpha} & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{B2}$$

with $(\overline{\sigma}^\mu)^{\dot\alpha\alpha} = \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}(\sigma^\mu)_{\beta\dot\beta}$, $\sigma^\mu = (1, -\sigma^i)$, $\overline{\sigma}^\mu = (1, \sigma^i)$. Then, for two anticommuting spinors ψ and χ:

$$\begin{aligned}
\overline{\psi}\chi &= \psi\chi + \overline{\psi}\overline{\chi} = \psi^\alpha\chi_\alpha + \overline{\psi}_{\dot\alpha}\overline{\chi}^{\dot\alpha}, \\
\overline{\psi}\gamma_5\chi &= \psi\chi - \overline{\psi}\overline{\chi} = \psi^\alpha\chi_\alpha - \overline{\psi}_{\dot\alpha}\overline{\chi}^{\dot\alpha}, \\
\overline{\psi}\gamma^\mu\chi &= \psi\sigma^\mu\overline{\chi} - \chi\sigma^\mu\overline{\psi} = \psi^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\overline{\chi}^{\dot\alpha} - \chi^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\overline{\psi}^{\dot\alpha}, \\
\overline{\psi}\gamma^\mu\gamma_5\chi &= -\psi\sigma^\mu\overline{\chi} - \chi\sigma^\mu\overline{\psi} = -\psi^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\overline{\chi}^{\dot\alpha} - \chi^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\overline{\psi}^{\dot\alpha}, \\
\overline{\psi}\gamma^\mu\gamma^\nu\chi &= \psi\sigma^\mu\overline{\sigma}^\nu\chi + \overline{\psi}\overline{\sigma}^\mu\sigma^\nu\overline{\chi} = \psi^\alpha(\sigma^\mu)_{\alpha\dot\alpha}(\overline{\sigma}^\nu)^{\dot\alpha\beta}\chi_\beta + \overline{\psi}_{\dot\alpha}(\overline{\sigma}^\mu)^{\dot\alpha\alpha}(\sigma^\nu)_{\alpha\dot\beta}\overline{\chi}^{\dot\beta}.
\end{aligned} \tag{B3}$$

For anticommuting Grassmann spinors,

$$\theta^\alpha \quad , \quad \overline{\theta}_{\dot\alpha} \quad , \quad \alpha = 1,2, \quad \dot\alpha = \dot1, \dot2,$$

one finds

$$\begin{aligned}
\theta^\alpha\theta^\beta &= -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta^\gamma\theta_\gamma, \\
\overline{\theta}^{\dot\alpha}\overline{\theta}^{\dot\beta} &= +\frac{1}{2}\epsilon^{\dot\alpha\dot\beta}\overline{\theta\theta} = +\frac{1}{2}\epsilon^{\dot\alpha\dot\beta}\overline{\theta}_{\dot\gamma}\overline{\theta}^{\dot\gamma}, \\
\theta_\alpha\theta_\beta &= +\frac{1}{2}\epsilon_{\alpha\beta}\theta\theta = +\frac{1}{2}\epsilon_{\alpha\beta}\theta^\gamma\theta_\gamma, \\
\overline{\theta}_{\dot\alpha}\overline{\theta}_{\dot\beta} &= -\frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\overline{\theta\theta} = -\frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\overline{\theta}_{\dot\gamma}\overline{\theta}^{\dot\gamma},
\end{aligned} \tag{B4}$$

recalling that

$$\epsilon^{12} = \epsilon^{\dot1\dot2} = -\epsilon^{21} = -\epsilon^{\dot2\dot1} = -\epsilon_{12} = -\epsilon_{\dot1\dot2} = \epsilon_{21} = \epsilon_{\dot2\dot1} = 1.$$

The equations (B4) use

$$\begin{aligned}
\theta\theta &= -2\theta^1\theta^2 = -2\theta_1\theta_2, \\
\overline{\theta\theta} &= +2\overline{\theta}^{\dot1}\overline{\theta}^{\dot2} = +2\overline{\theta}_{\dot1}\overline{\theta}_{\dot2}.
\end{aligned} \tag{B5}$$

Then:

$$\begin{aligned}
(\theta\sigma^\mu\overline{\theta})\theta^\alpha &= -\frac{1}{2}(\theta\theta)\epsilon^{\alpha\beta}(\sigma^\mu)_{\beta\dot\beta}\overline{\theta}^{\dot\beta} = -\frac{1}{2}\theta\theta(\sigma^\mu\overline{\theta})^\alpha, \\
(\theta\sigma^\mu\overline{\theta})\overline{\theta}^{\dot\alpha} &= +\frac{1}{2}(\overline{\theta\theta})\theta^\alpha(\sigma^\mu)_{\alpha\dot\beta}\epsilon^{\dot\beta\dot\alpha} = -\frac{1}{2}\overline{\theta\theta}(\theta\sigma^\mu)^{\dot\alpha}.
\end{aligned} \tag{B6}$$

(Obviously, $\theta^\alpha\theta^\beta\theta^\gamma = \overline{\theta}^{\dot\alpha}\overline{\theta}^{\dot\beta}\overline{\theta}^{\dot\gamma} = 0$). One also finds

$$(\theta\sigma^\mu\overline{\theta})(\theta\sigma^\nu\overline{\theta}) = \frac{1}{2}\eta^{\mu\nu}(\theta\theta)(\overline{\theta\theta}). \tag{B7}$$

This last result uses

$$(\sigma^\mu)_{\alpha\dot\alpha}(\overline{\sigma}^\nu)^{\dot\alpha\alpha} = Tr(\sigma^\mu\overline{\sigma}^\nu) = 2\eta^{\mu\nu}. \tag{B8}$$

The Fierz rearrangement formula read

$$\begin{aligned}
(\theta\phi)(\theta\psi) &= -\frac{1}{2}(\theta\theta)(\phi\psi), \\
(\overline{\theta\phi})(\overline{\theta\psi}) &= -\frac{1}{2}(\overline{\theta\theta})(\overline{\phi\psi}).
\end{aligned} \tag{B9}$$

Derivatives:

$$\begin{aligned}
\frac{d}{d\theta^\alpha}(\theta\theta) &= \left(\frac{d}{d\theta^\alpha}\theta^\beta\right)\theta_\beta - \theta^\beta\left(\frac{d}{d\theta^\alpha}\theta_\beta\right) = 2\theta_\alpha, \\
\frac{d}{d\theta_\alpha}(\theta\theta) &= \left(\frac{d}{d\theta_\alpha}\theta^\beta\right)\theta_\beta - \theta^\beta\left(\frac{d}{d\theta_\alpha}\theta_\beta\right) = -2\theta^\alpha, \\
\frac{d}{d\overline{\theta}^{\dot\alpha}}(\overline{\theta\theta}) &= -2\overline{\theta}_{\dot\alpha}, \\
\frac{d}{d\overline{\theta}_{\dot\alpha}}(\overline{\theta\theta}) &= 2\overline{\theta}^{\dot\alpha}, \\
\epsilon^{\alpha\beta}\frac{d}{d\theta^\alpha}\frac{d}{d\theta^\beta}(\theta\theta) &= \left(2\frac{d}{d\theta^1}\frac{d}{d\theta^2}\right)(-2\theta^1\theta^2) = +4, \\
\epsilon^{\dot\alpha\dot\beta}\frac{d}{d\overline{\theta}^{\dot\alpha}}\frac{d}{d\overline{\theta}^{\dot\beta}}(\overline{\theta\theta}) &= \left(2\frac{d}{d\overline{\theta}^{\dot1}}\frac{d}{d\overline{\theta}^{\dot2}}\right)(2\overline{\theta}^{\dot1}\overline{\theta}^{\dot2}) = -4.
\end{aligned} \tag{B10}$$

Notice also that

$$\begin{aligned}
\epsilon^{\alpha\beta}\frac{d}{d\theta^\beta} &= -\frac{d}{d\theta_\alpha}; \qquad \epsilon_{\alpha\beta}\frac{d}{d\theta_\beta} = -\frac{d}{d\theta^\alpha}; \\
\epsilon^{\dot\alpha\dot\beta}\frac{d}{d\overline{\theta}^{\dot\beta}} &= -\frac{d}{d\overline{\theta}_{\dot\alpha}}; \qquad \epsilon_{\dot\alpha\dot\beta}\frac{d}{d\overline{\theta}_{\dot\beta}} = -\frac{d}{d\overline{\theta}^{\dot\alpha}}.
\end{aligned} \tag{B11}$$

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