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# DUALITIES IN STRING THEORY

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# DUALITIES IN THEORIES WITH 32 AND 16 SUPERSYMMETRIES. (LECTURES 1, 2, 3).

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## I (A) 32 SUPERSYMMETRIES (11 AND 10 D)

- The maximal number of supersymmetries, in components, if we require spins $\leq 2$, is 32. In 4 dimensions this would correspond to 8 Majorana supercharges, or $N = 8$.

- The maximal spacetime dimension which admits 32 supersymmetries is $d = 11$. Therefore we will start with that, and go down. Indeed, in 11 dimensions there is a unique supersymmetric theory. It is called "M-theory".

- In 10 dimensions there are two distinct realizations of 32 supersymmetries: the type IIA and IIB theories.

  IIA: vectorlike (parity conserving)
  IIB: chiral (parity violating)

We will say all that we can about these three theories: M, IIA, IIB, before going to lower dimensions.

We will encounter the following interesting facts along the way:

(i) M-theory is a theory of massless gravitons and other massless particles, and it also contains 2-branes and 5-branes in its spectrum. It is not a string theory, but (as far as we understand it) it is a consistent theory of quantum gravity in 11 dimensions.

(ii) IIA and IIB theories can both be obtained from M-theory. These are also consistent theories of quantum gravity, in 10 dimensions. In their spectrum they contain, besides massless particles, various kinds of p-branes. Among these are 1-branes or strings. The IIA and IIB theories can be phrased as string theories, with a consistent perturbation series.

⊙ <u>M-theory</u> The unique theory of particles of spin $\leq 2$ in 11 dimensions, is given by a Lagrangian for a collection of massless particles:

$g_{MN}$ : metric
$C_{MNP}$ : 3-form potential
$\Psi_{M\alpha}$ : "spin-$\frac{3}{2}$" Majorana fermion.

On-shell components:

$$g_{MN} \; ; \quad \frac{9 \times 10}{2} - 1 = 44$$

$$C_{MNP} \; ; \quad \frac{9 \times 8 \times 7}{6} = 84$$

$$\psi_{M\alpha} \; ; \quad 9 \times 16 - 1 \times 16 = 128$$

So, degrees of freedom match between bosons and fermions.

Let $e = \sqrt{-\det g}$ (and all other conventions are as in Green-Schwarz-Witten). The classical Lagrangian is

$$\mathcal{L} = \mathcal{L}_B + \mathcal{L}_F$$

$$\mathcal{L}_B = -\frac{1}{2\kappa^2} e R - \frac{1}{48} e\, G_{MNPQ} G^{MNPQ}$$

$$- \frac{\sqrt{2}\,\kappa}{3456} \epsilon^{MNPQRSTUVWX} C_{MNP}\, G_{QRST}\, G_{UVWX}$$

$$\mathcal{L}_F = -\frac{1}{2} e\, \bar{\psi}_M \Gamma^{MNP} D_N \psi_P$$

$$- \frac{\sqrt{2}\,\kappa}{192} \left( \bar{\psi}_M \Gamma^{MNPQRS} \psi_S + 12 \bar{\psi}^N \Gamma^{PQ} \psi^R \right) G_{NPQR}$$

$$+ \text{4-fermi terms.}$$

(here $\Gamma^{MN} = \frac{1}{2} (\Gamma^M \Gamma^N - \Gamma^N \Gamma^M)$ and similarly for $\Gamma^{M \cdots N}$.)

$$G_{MNPQ} = \frac{1}{4} \left[ \partial_M C_{NPQ} + 3 \text{ terms} \right]$$

This action is invariant under the supersymmetry transformations

$$\delta e_M^{\ A} = \frac{\kappa}{2} \bar{\eta} \Gamma^A \Psi_M$$

$$\delta C_{MNP} = -\frac{\sqrt{2}}{8} \bar{\eta} \Gamma_{[MN} \Psi_{P]}$$

$$\delta \Psi_{M\alpha} = \frac{1}{\kappa} (D_M \eta)_\alpha$$

$$+ \frac{\sqrt{2}}{288} \left[ \left( \Gamma_M^{\ PQRS} - 8 \delta_M^P \Gamma^{QRS} \right) \eta \right]_\alpha G_{PQRS}$$

$$+ \text{ 3-fermi terms.}$$

A useful tip about dealing with complicated supergravity Lagrangians (this is one of the simplest!) is that one usually needs only: (i) the bosonic part $\mathcal{L}_B$ of the Lagrangian, (ii) the fermionic variation $\delta \Psi_{M\alpha}$ in ~~of~~ the supersymmetry transformation.

So far we have only written down a classical Lagrangian. It is not clear that there is a corresponding quantum theory. Evidence for the latter will slowly emerge.

Continuing with the classical theory, we look for stable solitonic solutions of the equations of motion. These would be important if the theory can be quantized, as they would be nonperturbatively stable quantum states.

Generically, solitons correspond to unstable states in quantum theory unless they carry a quantized charge. Solitons carrying a single unit of this charge cannot decay by charge conservation (or at least there should be some stable soliton of charge 1)

Particles naturally carry electric charge with respect to the electromagnetic field $A_\mu$, manifested by the possibility of including a term $e\oint A_\mu dx^\mu$ on the world-line of the particle. A magnetic monopole will instead couple to $\hat{A}_\mu$ via $\oint \hat{A}_\mu dx^\mu$ where $\partial_{[\mu}\hat{A}_{\nu]} = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}\,\partial^{[\lambda}A^{\rho]}$.

This concept generalizes to higher-rank tensor fields in arbitrary dimensions. For our purposes, an electric object must have $\int C_{MNP}\, dx^M dx^N dx^P$ on its world-volume, while a magnetic object will have $\int \tilde{C}_{MNPQRS}\, dx^M \cdots dx^S$ where

$$\partial_{[T}\tilde{C}_{MNPQRS]} = \frac{1}{7!}\underbrace{\epsilon_{TMNPQRSABCD}}_{11 \text{ indices}}\,\partial^{[A}C^{BCD]}$$

It follows that in M-theory, stable electrically charged objects are 2-branes (with (2+1) dimensional worldvolume) while stable magnetically charged objects are 5-branes (with (5+1) dimensional world volume)

Indeed the solutions do exist and are given by

<u>2-brane:</u>

$$ds^2 = \left(1 + \frac{k}{r^6}\right)^{-2/3} dx^\mu dx^\nu \eta_{\mu\nu} + \left(1 + \frac{k}{r^6}\right)^{-1/3} dy^m dy^n \delta_{mn}$$

$$C_{\mu\nu\lambda} = \epsilon_{\mu\nu\lambda}\left(1 + \frac{k}{r^6}\right)^{-1} \qquad \left(\mu, \nu = 0, 1, 2\ ;\ r = \sqrt{y^m y^m}\right)$$

(other components zero)

<u>**5-brane**:</u>

$$ds^2 = \left(1 + \frac{k}{r^3}\right)^{-1/3} dx^\mu dx^\nu \eta_{\mu\nu} + \left(1 + \frac{k}{r^3}\right)^{2/3} dy^m dy^n \delta_{mn}$$

$$G_{mnpq} = 3k\, \epsilon_{mnpqs} \frac{y^s}{r^5}$$

$$\left(\mu, \nu = 0, 1, \ldots 5\ ;\ r = \sqrt{y^2}\right)$$

One may try to assume that there is a quantum theory with massless point particles corresponding to the fields in the classical Lagrangian, and also quantum states corresponding to stable 2-branes and 5-branes. This is the conjecture that M-theory exists. We will uncover evidence for this conjecture.

Two important facts that we will not prove (but will probably be discussed in other lectures at this school) are:

(i) The 2-brane and 5-brane solutions are annihilated by half the supersymmetries - the remaining half are broken by the branes. In other words, of the 32 independent supersymmetry variations $\delta \Psi_{M\alpha}$ (for 32 independent constant spinors $\eta$), 16 are zero when the brane solution for $G_{MNPQ}$ is inserted on the RHS.

(ii) This fact is closely related to the fact that the most general supersymmetry algebra in 11 d is:

$$\{Q_\alpha, \bar{Q}_\beta\} = (\Gamma^M)_{\alpha\beta} P_M + (\Gamma_{MN})_{\alpha\beta} Z^{MN}_{(2)}$$

$$+ (\Gamma_{MNPQR})_{\alpha\beta} Z^{MNPQR}_{(5)}$$

where $Z_{(2)}$ and $Z_{(5)}$ are "central charges" which are non-vanishing precisely on the 2 and 5-branes respectively.
(On the infinitely extended branes we have found, they actually take infinite values - but the values are proportional to both the charge and the total volume of the brane.)

The branes ~~have~~ satisfy an important relation between their mass density (measured by $P_0$) and their charge density (measured by $Z_{(2)}$, $Z_{(5)}$).

Schematically, half the supercharges satisfy

$$\{Q_\alpha^{(1)}, Q_\beta^{(1)}\} \sim P + Z$$

the other half,

$$\{Q_\alpha^{(2)}, Q_\beta^{(2)}\} \sim P - Z$$

So if the branes satisfy $P - Z = 0$ or $P + Z = 0$ the they can preserve half the supersymmetries (otherwise they break all)

This is true of the extremal, stable 2-brane and 5-brane of M-theory hence they are called "BPS-saturated".

Note that the total number of independent ~~susy~~ charges (including momenta) occurring on the RHS is 11 (momenta) + 55 ($Z_{(2)}^{MN}$) + 462 ($Z_{(5)}^{MNPQR}$) = 528.

⊙ The <u>type IIA</u> and IIB theories can be discovered in a rather analogous way. Just by studying the realizations of 32 supersymmetries in 10 dimensions, one finds that there are two <u>distinct</u> multiplets of massless fields with spins $\leq 2$ :

($\alpha, \beta, \dots \mu, \nu, \dots$ go from 0 to 9)

<u>IIA</u> :

$$\underset{\text{metric}}{g_{\mu\nu}}, \quad \underset{\substack{\text{scalar}\\(\text{0-form})}}{\varphi}, \quad \underset{\text{1-form}}{A_\mu}, \quad \underset{\text{2-form}}{B_{\mu\nu}}, \quad \underset{\text{3-form}}{C_{\mu\nu\rho}} \qquad (\text{Bose})$$

$$\underset{\substack{\text{Left-handed}\\ \text{Majorana-Weyl}\\ \text{spinor}}}{\psi^{(1)}_{\mu\alpha}}, \quad \underset{\substack{\text{Right-handed}\\ \text{MW spinor}}}{\psi^{(2)}_{\mu\dot\alpha}} \; ; \; \lambda^{(1)}_{\alpha}, \; \lambda^{(2)}_{\dot\alpha} \qquad (\text{Fermi})$$

<u>IIB</u> :

$$\underset{\text{metric}}{g_{\mu\nu}}, \quad \underbrace{\varphi, \tilde\varphi}_{\text{0-forms}}, \quad \underbrace{B_{\mu\nu}, \tilde B_{\mu\nu}}_{\text{2-forms}}, \quad \underset{\substack{\text{"self-dual"}\\ \text{4-form}}}{\tilde D^{+}_{\mu\nu\rho\sigma}} \qquad (\text{Bose})$$

$D^+$ satisfies : $\quad d\tilde D^+ = {}^* d\tilde D^+$

or $\quad \partial_{[\alpha} D^+_{\beta\gamma\delta\epsilon]} = \frac{1}{5!} \epsilon_{\alpha\beta\gamma\delta\epsilon\rho\sigma\lambda\mu\nu} \partial^{[\rho} D^{+\sigma\lambda\mu\nu]}$

$$\underbrace{\psi^{(1)}_{\mu\alpha}, \quad \psi^{(2)}_{\mu\alpha}}_{} \; ; \; \lambda^{(1)}_{\dot\alpha}, \; \lambda^{(2)}_{\dot\alpha} \qquad (\text{Fermi})$$

<u>Both</u> left-handed MW spinors.

The IIA fields are related to those of M-theory in a very interesting way. Suppose M-theory is compactified on a circle (call this dimension 11). ~~Borrowing~~ The resulting spectrum from the 10d point of view consists of some massless fields (coming from 11d fields independent of the 11 direction) and massive fields from Fourier modes excited in the 11 direction.

Consider first just the massless modes.

$$g_{MN} \; (d=11) \quad \rightarrow \quad \begin{array}{lll} g_{\mu\nu} & (d=10) & \\ g_{\mu\, 11} & (d=10) & = A_\mu \\ g_{11\, 11} & (d=10) & = \varphi \end{array}$$

$$C_{MNP} \; (d=11) \quad \rightarrow \quad \begin{array}{lll} C_{\mu\nu\rho} & (d=10) & \\ C_{\mu\nu\, 11} & (d=10) & = B_{\mu\nu} \end{array}$$

Thus the Bose fields of IIA are exactly reproduced (the same is true of the Fermi fields)

What about the Lagrangian? It is clear that taking the M-theory Lagrangian to be independent of $x^{11}$ will lead to a Lagrangian in 10d — which will necessarily have the right supersymmetries, so it must be a valid way of writing the type IIA Lagrangian.

From now on, we will ignore numerical coefficients in front of individual terms in supergravity Lagrangians. Thus we write the bosonic 11d Lagrangian as:

$$\mathcal{L}_B^{(11)} = \frac{1}{\kappa^2}\left[ eR + e|G|^2 + C\wedge G\wedge G \right]$$

(we have used $C \to \frac{C}{\kappa}$).

To compactify on a circle, we replace $g_{MN}$ by a matrix like

$$g_{MN} \sim \begin{bmatrix} g_{\mu\nu} & A_\mu \\ A_\nu & e^{\alpha\varphi} \end{bmatrix}$$

where $\alpha$ is an arbitrary constant to be chosen later.

This form exhibits the Kaluza-Klein scalar $\varphi$ and gauge field $A_\mu$. However, it needs modification since $\sqrt{g^{(11)}}$ depends on $A_\mu$. An improved ansatz is

$$g_{MN} \sim \begin{bmatrix} g_{\mu\nu} + e^{\alpha\varphi}A_\mu A_\nu & e^{\alpha\varphi}A_\mu \\ e^{\alpha\varphi}A_\nu & e^{\alpha\varphi} \end{bmatrix}$$

so that $\sqrt{g^{(11)}} = e^{\frac{\alpha}{2}\varphi}\sqrt{g^{(10)}}$.

Thus if we treat $x^{11}$ as an angle-valued coordinate (0 to $2\pi$), the radius of the compactification circle is $R_{11} = e^{\frac{\alpha}{2}\varphi}$. Note that $\varphi(x)$ is a scalar field, so we really mean $R_{11} = e^{\frac{\alpha}{2}\langle\varphi(x)\rangle} = e^{\frac{\alpha}{2}\varphi}$ where $\varphi$ is the constant part, or VEV, of $\varphi(x)$.

Now we have: $\sqrt{g^{(11)}} \sim e^{\frac{\alpha}{2}\varphi}\sqrt{g^{(10)}}$

$$R^{(11)} = g^{MN} R^{(11)}_{MN} \sim g^{\mu\nu}\Big[ R^{(10)}_{\mu\nu}$$

$$+ \partial_\mu\varphi\, \partial_\nu\varphi + e^{-\alpha\varphi}|e^{\alpha\varphi}F|^2 + \dots\Big]$$

$$= g^{\mu\nu}\Big[ R^{(10)}_{\mu\nu} + \partial_\mu\varphi\,\partial_\nu\varphi + e^{\alpha\varphi}|F|^2\Big]$$

So $\sqrt{g^{(11)}}R^{(11)} \sim \sqrt{g^{(10)}}\Big[ e^{\frac{\alpha}{2}\varphi}R^{(10)}$

$$+ e^{\frac{\alpha}{2}\varphi}|d\varphi|^2 + e^{\frac{3}{2}\alpha\varphi}|F|^2\Big]$$

Similarly, $\sqrt{g^{(11)}}\,|G^{(11)}|^2 \sim e^{\frac{\alpha\varphi}{2}}\sqrt{g^{(10)}}\,|G^{(10)}|^2$

$$+ e^{\frac{\alpha}{2}\varphi}\sqrt{g^{(10)}}\,.\, e^{-\alpha\varphi}|H^{(10)}|^2$$

$$= \sqrt{g^{(10)}}\Big[ e^{\frac{\alpha}{2}\varphi}|G^{(10)}|^2 + e^{-\frac{\alpha}{2}\varphi}|H^{(10)}|^2\Big]$$

As a result the 10d bosonic Lagrangian is (dropping the (10) superscript and ignoring $C\wedge G\wedge G$ terms)

$$\mathcal{L}^{(10)}_B = \sqrt{g^{(10)}}\Big[ e^{\frac{\alpha}{2}\varphi}R + e^{\frac{\alpha}{2}\varphi}|d\varphi|^2$$
$$+ e^{\alpha\frac{\varphi}{2}}|G|^2 + e^{\frac{3}{2}\alpha\varphi}|F|^2 + e^{-\frac{\alpha}{2}\varphi}|H|^2\Big]$$

where $F = dA$ ; 2-form field strength
$H = dB$ ; 3-form " "
$G = dC$ : 4-form " "

It is convenient to make a Weyl rescaling of $g^{(10)}_{\mu\nu}$ :

$$g^{(10)}_{\mu\nu} \to e^{\beta\varphi} g^{(10)}_{\mu\nu}$$

Then $\sqrt{g^{(10)}} \to e^{5\beta\varphi} \sqrt{g^{(10)}}$

$$R_{\mu\nu} \to R_{\mu\nu} + \cdots$$

$$g^{\mu\nu} R_{\mu\nu} = R \to e^{-\beta\varphi} R \; + $$

$$|d\varphi|^2 \to e^{-\beta\varphi} |d\varphi|^2$$
$$|F|^2 \to e^{-2\beta\varphi} |F|^2$$
$$|H|^2 \to e^{-3\beta\varphi} |H|^2$$
$$|G|^2 \to e^{-4\beta\varphi} |G|^2$$

Hence
$$\mathcal{L}^{(10)}_B = \sqrt{g^{(10)}} \Big[ e^{(4\beta + \frac{\alpha}{2})\varphi} R$$
$$\cancel{+ e^{(2\beta - \frac{\alpha}{2})} |H|^2} + e^{(4\beta + \frac{\alpha}{2})\varphi} |d\varphi|^2$$
$$+ e^{(3\beta + \frac{3}{2}\alpha)\varphi} |F|^2 + e^{(2\beta - \frac{\alpha}{2})\varphi} |H|^2$$
$$+ e^{(\beta + \frac{\alpha}{2})\varphi} |G|^2 \Big]$$

Now notice that by taking $\beta + \frac{\alpha}{2} = 0$, we make the $e^{\varphi}$ terms disappear in front of <u>both</u> $|F|^2$ and $|G|^2$.

Also the factors in front of the other three terms $R$, $|d\varphi|^2$, & $|H|^2$ become equal:

$$e^{(4\beta + \frac{\alpha}{2})\varphi} = e^{-\frac{3}{2}\alpha\varphi} \quad \text{if} \quad \beta + \frac{\alpha}{2}$$

$$e^{(2\beta - \frac{\alpha}{2})\varphi} = e^{-\frac{3}{2}\alpha\varphi} \quad \text{"} \quad \text{"} .$$

Thus with this choice,

$$\mathcal{L}_B^{(10)} = \sqrt{g}\left[ e^{-\frac{3}{2}\alpha\varphi}\left(R + |d\varphi|^2 + |H|^2\right) + \left(|F|^2 + |G|^2\right)\right]$$

It is conventional to choose $\alpha = \frac{4}{3}$, so we finally get

$$\mathcal{L}_B^{(10)} = \sqrt{g}\left[ e^{-2\varphi}\left(R + |d\varphi|^2 + |H|^2\right) + |F|^2 + |G|^2 \right]$$

The constant part of $e^{-2\varphi}$ is like $\frac{1}{\lambda^2}$, where $\lambda$ is the coupling constant

for the metric, $\varphi$ and $B$ fields. The other two fields appear in a "non standard" way, with no coupling factors in front.

We have $\lambda = e^{\varphi}$, while $R_{(11)} = e^{\frac{2}{3}\varphi}$

so:

$$\boxed{R_{(11)} = \lambda^{2/3}}$$

All this has a physical interpretation. In M-theory there is no scalar field, hence no parameter like $\langle \varphi \rangle$ which can play the role of an adjustable, dimensionless coupling. M-theory only has a dimensionful constant $\epsilon$, the 11-dimensional Planck scale. Thus there is nothing analogous to a weak-coupling limit in M-theory.

In type IIA theory, obtained by compactification on a circle, there is such a scalar — and its VEV is (upto a power) just the radius of the 11 th dimension. Thus type IIA may admit a weak coupling expansion.

Before investigating this expansion, let us examine the spectrum of stable branes in the type IIA theory. The gauge fields which can stabilize a brane are:

$A_\mu$ : 0-brane (electric)
$B_{\mu\nu}$ . 1-brane ( " )
$C_{\mu\nu\lambda}$ : 2-brane ( " )

$\hat{A}_{\mu\nu\lambda\rho\sigma\tau\alpha}$ is the dual of $A_\mu$ via $d\hat{A} = *dA$. A 6-brane couples electrically to this, or magnetically to $A_\mu$. Thus we also have.

$A_\mu$: 6 - brane (magnetic)
$B_{\mu\nu}$: 5 - brane (magnetic)
$C_{\mu\nu\lambda}$: 4 - brane (magnetic)

Thus IIA theory is expected to have stable 0, 1, 2, 4, 5, 6 - branes. All the corresponding soliton solutions are known.

Since IIA theory arose by compactifying M-theory, all its branes should have explanations in M-theory. Indeed we can find them as follows

| M-theory | IIA |
|---|---|
| 2-brane (reduced) | 2-brane |
| 2-brane (wrapped on $S^1$) | 1-brane |
| 5-brane (reduced) | 5-brane |
| 5-brane (wrapped on $S^1$) | 4-brane. |

That leaves the 0-brane and 6-brane of type IIA, which cannot arise from branes in M-theory. One way to see this is that the gauge field $A_\mu$, whose electric charge is carried by the 0-brane, is a Kaluza-Klein gauge field arising from the 11d metric. So the Kaluza-Klein mechanism should be responsible for creating it. Indeed, this is the case.

On the circle $x^{11}$, we can make a mode expansion of the 11 d massless fields:

here $0 \leq x^{11} \leq 2\pi R$

$$g^{(11)}_{MN}(x^1 \ldots x^{10}; x^{11}) = \sum_{n\in\mathbb{Z}} e^{i\frac{x^{11}}{R}n}\, g^{(10)(n)}_{MN}(x^1; x^{10})$$

where the fields on the RHS have to be reduced into 10d multiplets.

Note that; (i) a translation $x^{11} \to x^{11} + \epsilon R$ of the eleventh direction sends

$$g^{(10)(n)}_{MN} \to e^{i\frac{n}{R}\epsilon R}\, g^{(10)(n)}_{MN}$$

(ii) the local version of this transformation, $x^{11} \to x^{11} + \epsilon(x^1, \ldots x^{10})$ is just a local $u(1)$ gauge transformation for which $A_\mu$ is the gauge field.

Thus the $g^{(10)(n)}_{MN}$ are fields of charge $n$ under $A_\mu$.

Moreover, these fields (except for $n=0$, which we already considered) are all <u>massive</u>: if

$$\Box^{(11)} g^{(11)}_{MN} = 0 \quad , \text{ then}$$

$$\Box^{(10)} g^{(10)(n)}_{MN} - \left(\frac{n}{R}\right)^2 g^{(10)(n)}_{MN} = 0.$$

It follows that the KK fields have mass ~~∝~~ charge. In other words, the Kaluza-Klein mechanism automatically gives BPS states! (Actually in string units, mass $= \frac{1}{\lambda}$ (charge))

The field of lowest nonzero charge will be stable by reasons we have stated. So we can identify the multiplet of particles corresponding to $g^{(10)(1)}_{MN}$, $C^{(10)(4)}_{MNP}$ with the multiplet of states of a unit-charge 0-brane.

By a similar token the "Kaluza-Klein monopole", magnetically charged under $A_\mu$, is the 6-brane of IIA theory.

So far we have not needed any detailed information about either M-theory or IIA theory, especially about how to quantize them. But the supersymmetry / BPS formulae give us a lot of information if we merely assume that some quantum theory exists.

At this stage, however, we can argue that a quantum type IIA theory exists. Among the branes of the IIA action, is a 1-brane. One can work out the world-sheet action for this 1-brane (string). Then one quantizes the type IIA string.

It turns out that the string theory so obtained is perturbatively well-defined, consistent (even finite) and unitary. The perturbation series is given, in powers of $\lambda^2 = e^{2\phi}$, by an expansion in Riemann surfaces. The low-energy spectrum of the string reproduces the IIA supergravity ~~Lagrangian~~ fields, and string interactions reproduce the type of Lagrangian we have written down, with calculable corrections.

This shows that "IIA theory" (the theory of massless particles and various branes) does correspond to a well-defined quantum theory, at weak coupling. If we simply assume that the theory exists also at strong coupling, then it follows that this theory is 11-dimensional and is a definition of M-theory.

⊛ COMMENT ON NS-NS and R-R FIELDS.

⊙ We may now ask whether type IIB theory also arises from M-theory. The answer is yes, but in a rather surprising way.

Recall that type IIB theory has two 2-form fields $B_{\mu\nu}$ and $\tilde{B}_{\mu\nu}$, and a self-dual 4-form field $D^+_{\mu\nu\lambda\rho}$.

Following our previous arguments, we expect to find the following branes:

i) $B_{\mu\nu}$ : 1 - brane (electric)
ii) : 5 - brane (magnetic)
(iii) $\tilde{B}_{\mu\nu}$ : 1 - brane (electric)
(iv) 5 - brane (magnetic)
(v) $\tilde{D}^+_{\mu\nu\lambda\rho}$ : 3 - brane (self-dual, i.e. simultaneously electric & magnetic).

(We have so far been hiding some "high branes": in addition to the above, type IIB theory has a 7-brane and a 9-brane, while type IIB has an 8-brane. They will be introduced later, when needed).

The electric 1-brane charged under $B_{\mu\nu}$ is very similar to that in type IIA theory. Quantizing this string leads to "type IIB string theory".

There are two distinct string theories in 10 dimensions, but on compactifying on a circle to 9 dimensions, they coincide through an operation called "T-duality".

T-duality is seen as follows: in the quantization of strings propagating on a circle, we find momentum modes quantized in units of $\frac{1}{R}$ (as for particles) and winding modes quantized in units of $R$. The interchange $R \to \frac{1}{R}$ interchanges these two types of modes. Since they appear symmetrically in the formalism (spectrum, interactions) T duality would appear to be a symmetry of string theory – but for one subtlety. The operation $R \to \frac{1}{R}$ changes the spacetime chirality of half the fermions, so it interchanges the type IIA fermions $(\psi^{(1)}_{\mu\alpha}, \psi^{(2)}_{\mu\dot\alpha})$ with type IIB fermions $(\psi^{(1)}_{\mu\alpha}, \psi^{(2)}_{\mu\alpha})$.

Indeed, it can be checked that under a single T-duality $R \to \frac{1}{R}$, the type IIA and IIB strings are exchanged. As a result, IIA on a circle of radius $R$ is the <u>same</u> as IIB on a circle of radius $1/R$.

It follows that ~~to~~ if we take type IIA in 10d, compactify on a circle of radius $R$ and take the limit $R \to 0$, we recover type IIB in 10 dimensions! It is important to realize that from the type IIA point of view, as $R \to 0$ the <u>winding</u> modes of the $x^{10}$ direction together with <u>momentum</u>

modes in the remaining (noncompact) directions ~~to~~ give 10-dimensional Lorentz invariance — a string miracle which would be hard to understand without knowing T-duality.

It appears to follow that IIB theory is obtained by compactifying M-theory on a 2-torus ~~and~~ $(R_{11}, R_{10})$ and taking $R_{10} \to 0$. This is not quite correct. Radii of circles as measured in M-theory and string theory are different because of the Weyl rescaling that we made. So let us examine the issue more closely.

Let $R_{11}, R_{10}$ be the radii in the M-theory metric. We have found that

$$R_{11} \sim \lambda^{2/3} .$$

Let $R_{10}^{(IIA)}$ be the radius of $x^{10}$ in the type IIA metric. Since

$$g^{(10)}_{MN}(M) = e^{\beta\varphi}\, g^{(10)}_{MN}(IIA) \quad \text{with } \beta = -2/3 .$$

Hence $$R_{10} = e^{-\frac{1}{3}\varphi} R_{10}^{(IIA)} = \frac{R_{10}^{(IIA)}}{\lambda^{1/3}}$$

$$\boxed{R_{10} = \frac{R_{10}^{(IIA)}}{\lambda^{1/3}}}$$

Now T-duality takes place in 9 dimensions and keeps the 9 dimensional coupling invariant.

For the IIA string in 9 dimensions, the 9d coupling is

$$\frac{1}{\lambda_9^2} = \frac{R_{10}^{(IIA)}}{\lambda_A^2} = \frac{R_{10}^{(IIB)}}{\lambda_B^2}$$

and $$R_{10}^{(IIB)} = \frac{1}{R_{10}^{(IIA)}}$$

So $$\lambda_B = \frac{\lambda_A}{R_{10}^{A}} = \frac{R_{11}^{3/2}}{R_{10}\,\lambda^{1/3}} = \frac{R_{11}}{R_{10}}$$

while $$R_{10}^{(IIB)} = \frac{1}{R_{10}^{(IIA)}} = \frac{1}{R_{10}\,\lambda^{1/3}} = \frac{1}{R_{10}\,R_{11}^{1/2}}$$

Since we want the limit $R_{10}^{(IIB)} \to \infty$ with $\lambda_B$ fixed, we must take $R_{10}$, $R_{11}$ to zero together, with the ratio $\frac{R_{11}}{R_{10}}$ fixed.

But this implies ~~that~~ an important property of type IIB theory: since the interchange $R_{10} \leftrightarrow R_{11}$ is a symmetry of M-theory, the type IIB theory must have a symmetry $\lambda_B \to \frac{1}{\lambda_B}$.

One can repeat the above calculation for the case of a slanted 2-torus in the 10-11 directions. In this case one finds that $i\lambda_B$ is replaced by a complex quantity $\tau$, ~~and~~ the modular parameter labelling possible complex structures of the torus.

In this case the symmetry $\lambda_B \to \frac{1}{\lambda_B}$ is replaced by $\tau \to -\frac{1}{\tau}$, but (as is well known) ~~the~~ global diffeomorphisms of a torus are given by

$$\tau = \left\{ \frac{a\tau + b}{c\tau + d} \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}). \right.$$

Thus we get a prediction that, if M-theory has 11d Lorentz invariance, type IIB theory has $SL(2,\mathbb{Z})$ symmetry! This includes a strong-weak coupling duality $\lambda_B \to \frac{1}{\lambda_B}$.

Since $\lambda_B = e^{\varphi}$ ($\varphi$: the dilaton of type IIB theory.)

it must be that $\tau$ is a complex scalar field in type IIB, whose imaginary part is $e^{\varphi}$. Indeed there is another scalar field $\tilde{\varphi}$ in IIB supergravity, so $\tau = \tilde{\varphi} + i e^{-\varphi}$.

At this point it is useful to write down the lowest-order (in derivatives) action for the massless fields in IIA and IIB theories.

$$\mathcal{L}^{(IIA)}_{Bose} = \sqrt{g} \left[ e^{-2\varphi} \left( R + |d\varphi|^2 + |H|^2 \right) \right.$$

$$\left. + |F|^2 + |G|^2 \right] + B \wedge G \wedge G$$

where $G = dC + B \wedge F$

$$\mathcal{L}^{(IIB)}_{Bose} = \sqrt{g}\left[ e^{-2\varphi}\left( R + |d\varphi|^2 + |H|^2 \right]\right.$$
$$\left. + |d\tilde{\varphi}|^2 + |\tilde{H} - \tilde{\varphi} H|^2 + |I|^2 \right]$$
$$+ D^+ \wedge H \wedge \tilde{H}$$

where : $H = dB$, $\tilde{H} = d\tilde{B}$, $I = dD^+$

Note that $|I|^2 = 0$ by virtue of self-duality: $|I|^2 = I \wedge {}^*I = I \wedge I = 0$ since $I$ is a 5-form. We adopt the procedure of relaxing the self-duality condition in the action, and then imposing it after obtaining equations of motion.

A Weyl transformation is convenient to exhibit the $SL(2,\mathbb{Z})$ symmetry of this action that must follow from our considerations above.

Let $g_{\mu\nu} \rightarrow e^{\frac{1}{2}\varphi} g_{\mu\nu}$

$\sqrt{g} \rightarrow e^{\frac{5}{2}\varphi}\sqrt{g}$, $R \rightarrow e^{-\frac{1}{2}\varphi} R$
$|d\varphi|^2 \rightarrow e^{-\frac{1}{2}\varphi}|d\varphi|^2$, $|H|^2 \rightarrow e^{-\frac{3}{2}\varphi}|H|^2$
$|I|^2 \rightarrow e^{-\frac{5}{2}\varphi}|I|^2$ etc.

So. $$\mathcal{L}^{(IIB)}_{Bose} = \sqrt{g}\left[ R + \left(|d\varphi|^2 + e^{2\varphi}|d\tilde{\varphi}|^2\right)\right.$$
$$\left. + e^{-\varphi}|H|^2 + |I|^2 + e^{\varphi}|\tilde{H} - \tilde{\varphi} H|^2 \right] + D^+ \wedge H \wedge \tilde{H}$$

Now we see that if $\tau = \varphi + ie^{-\varphi}$, then

$$|d\varphi|^2 + e^{2\varphi}|d\tilde{\varphi}|^2 = \frac{|d\tau|^2}{(\mathrm{Im}\,\tau)^2}$$

It is easy to check that under $\tau \to \frac{a\tau+b}{c\tau+d}$

with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,R)$, this term is invariant.

(Indeed the massless action is invariant under all of $SL(2,R)$ and not just $SL(2,Z)$, but the other states — strings and branes — which carry various charges, can only be invariant under $SL(2,Z)$ because of charge quantization).

Also, if $H, \tilde{H}$ transform as

$$(H, -\tilde{H}) \to (H, -\tilde{H})\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

under $SL(2,Z)$, then $e^{\varphi}|\tilde{H} - \tilde{\varphi}H|^2 + e^{-\varphi}|H|^2$ is invariant.

Thus, the low-energy action of type IIB theory supports the presence of an $SL(2,Z)$ <u>duality</u> symmetry which includes strong-weak coupling duality.

⊙ <u>Moduli Space</u> At this point it is useful to introduce the concept of moduli space for the theories we are discussing. This is just the parameter space of the theory, modulo global identifications.

Moduli space can be assigned a topology and a metric. The idea is that an infinitesimal vector tangent to moduli space shifts a theory in one background to a theory in a neighbouring background. In string theory, a background is described by a conformal field theory (CFT) and a deformation by a marginal operator. If the collection of all marginal operators is denoted $\Phi_i(z,\bar{z})$ then the CFT can be perturbed by shifting the (2d) action:

$$S \to S + \sum_i g_i \int d^2z \; \Phi_i(z,\bar{z}) = S + \delta S.$$

Now the correlation function on the 2-sphere

$$\langle \Phi_i(z,\bar{z}) \, \Phi_j(w,\bar{w}) \, e^{-\delta S} \rangle = \frac{g_{ij}(g_i)}{|z-w|^4}$$

defines a metric $g_{ij}(g)$ on the parameter space.

Looked at in this way, the moduli space of the IIB theory is the space of the $\tau = \tilde{\varphi} + ie^{-\varphi}$ (since their VEVs is undetermined) modulo the identification $\tau \to \frac{a\tau + b}{c\tau + d}$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$

(since that is a symmetry). Thus the moduli space is $(UHP)/SL(2,\mathbb{Z})$, which is just the moduli space (of complex structures) of the 2-torus.
(For IIA theory the moduli space is $\mathbb{R}^+$, the VEV of $e^{\varphi}$.)