UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL ATOMIC ENERGY AGENCY
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

SMR.996 - 6

II

SUMMER SCHOOL IN HIGH ENERGY PHYSICS AND COSMOLOGY

2 June - 4 July 1997

RECENT DEVELOPMENTS IN SUPERGRAVITY p-BRANES

K. STELLE
The Blackett Laboratory
Imperial College
Prince Consort Road
London SW7 2BZ
UK

Please note: These are preliminary notes intended for internal distribution only.

MAIN BUILDING STRADA COSTIERA, 11 TEL. 2240111 TELEFAX 224163 TELEX 460392 ADRIATICO GUEST HOUSE VIA GRIGNANO, 9 TEL. 224241 TELEFAX 224531 TELEX 460449
MICROPROCESSOR LAB. VIA BEIRUT, 31 TEL. 2249911 TELEFAX 224600 TELEX 460392 GALILEO GUEST HOUSE VIA BEIRUT, 7 TEL. 2240311 TELEFAX 2240310 TELEX 460392

Recent Developments in Supergravity p-branes

Trieste lectures 1997

K.S. Stelle

Outline

1. Maximal Supergravity and its charges
2. Kaluza-Klein reduction and consistent truncation
3. The p-brane ansatz ; single-charge p-branes
4. The origin of the harmonic functions:
   null geodesics on non-compact σ-models
5. The cast of p-branes (and friends) in $D=11$
6. Supersymmetry preservation
7. No-force conditions
8. Intersecting p-branes

Maximal Supergravity and its charges

Let us begin[1] with the action of $D=11$ supergravity, restricted to its bosonic sector only:

$$I_{11} = \int d^{11}x \left\{ \sqrt{-g}\left(R - \tfrac{1}{48} F_{[4]}^2\right) + \tfrac{1}{6} F_{[4]} \wedge F_{[4]} \wedge A_{[3]} \right\}.$$

When restricted to purely bosonic fields in this way, there is no particular reason to combine the kinetic term for the $A_{[3]}$ gauge potential and the FFA "Chern-Simons" term, but when one extends the above action to include the spin 3/2 field Ψ_M, $D=11$ supersymmetry requires the above combination. We shall not consider in detail this supersymmetric extension, but it will be important to note the $D=11$ supersymmetry algebra:

$$\{Q_\alpha, Q_\beta\} = (C\Gamma^M)_{\alpha\beta} P_M + \tfrac{1}{2}(C\Gamma^{MN})_{\alpha\beta} U_{MN} + \tfrac{1}{5!}(C\Gamma^{M_1\cdots M_5})_{\alpha\beta} V_{M_1\cdots M_5}.$$

The RHS is the most general expansion of the symmetric product of two 32-component $D=11$ Majorana spinors. Counting components, we see that the reducible $SO(1,10)$ representation carried by $\{Q_\alpha, Q_\beta\}$ has $\frac{32\cdot 33}{2} = 528$ components; under $SO(1,10)$ these break up as follows: $\underset{\sim}{528} = \underset{\sim}{11} + \underset{\sim}{55} + \underset{\sim}{462}$. Clearly, the $\underset{\sim}{11}$ is carried by the 11-momentum P_M; the antisymmetric tensor charges $U_{[2]}$ and $V_{[5]}$ carry the $\underset{\sim}{55}$ and the $\underset{\sim}{462}$, respectively.

The 2-form and 5-form charges in the $D=11$ supersymmetry algebra carry $SO(1,10)$ representations, but are "central" with respect to the $D=11$ supersymmetry, as $[Q_\alpha, U_{MN}] = 0$; $[Q_\alpha, V_{M_1\cdots M_5}] = 0$ (and similarly, $[Q_\alpha, P_M] = 0$). Thus, these charges play rôles similar to those of the familiar central charges of the $D=4$ supersymmetry algebra. The study of supergravity p-brane solutions is essentially the study of extremal solutions carrying these charges. We shall see what "extremal" means in due course.

In some more detail, the existence of conserved bosonic 2-form and

5-form charges in the D=11 supergravity theory follows from the field equations and Bianchi identity for $A_{[3]}$:

$$\partial_M(\sqrt{-g}F^{MUVW}) + \frac{1}{2(4!)^2}\epsilon^{UVW\cdots xxxx}F_{\cdots}F_{xxxx} = 0 \; ; \qquad (\text{fe})$$

$$\partial_{[N}F_{MUVW]} \equiv 0 \, . \qquad (\text{Bianchi})$$

The latter clearly reflects the gauge invariance of the theory under $\delta A_{MNP} = \partial_M\Lambda_{NP} + \partial_N\Lambda_{PM} + \partial_P\Lambda_{MN}$. In form notation, these equations become $d({}^*F_{[4]} + \frac{1}{2}A_{[3]}\wedge F_{[4]}) = 0$; $dF_{[4]} \equiv 0$, noting that ${}^*F_{[4]}$ is a 7-form and $F_{[4]} = dA_{[3]}$. These equations lead directly to the Gauss' law surface-integral expression

$$U_{ab} = \frac{1}{4\Omega_7}\int_{\partial M_8^{ab}}({}^*F_{[4]} + \tfrac{1}{2}A_{[3]}\wedge F_{[4]}) \qquad \text{"electric" Page charge}$$

and to the analogous surface-integral form for $V_{A_1\cdots A_5}$:

$$V_{a_1\cdots a_5} = \frac{1}{4\Omega_4}\int_{\partial M_5^{a_1\cdots a_5}} F_{[4]} \, . \qquad \text{"magnetic" Page charge}$$

The surface integrals in both of these expressions are to be thought of as being taken over the boundaries at spatial infinity of some space-like hypersurfaces M_8^{ab} and $M_5^{a_1\cdots a_5}$. For the $D = 1+10$ theory, there are clearly an infinite number of such hypersurfaces that can be embedded into the $D-1 = 10$ dimensional space; let M_8^{ab} and $M_5^{a_1\cdots a_5}$ be a basis for this choice of spacelike hyperplanes.

The other bosonic "charge" that one can construct is the ADM energy-momentum. Picking an asymptotic coordinate system where the spatial momentum vanishes, the remaining non-vanishing component is the energy, given by

$$P^0 = E = \frac{1}{4\Omega_9}\int_{\partial M_{10}} d^9\Sigma^a(\partial^b h_{ab} - \partial_a h^b{}_b) \qquad \text{where } a = 1\cdots 10 \text{ are spatial indices;}$$

$h_{ab} = g_{ab} - \eta_{ab}$ is the fluctuation of the gravitational field away from the asymptotically flat space.

Just as one can consider boosted spacetimes, in which the spatial ADM momentum is non-vanishing and joins together with the energy to form the D=11 vector P_M, so can one extend the charges U_{ab} and $V_{a_1 \cdots a_5}$, with spatial indices only, to charges U_{MN} and $V_{M_1 \cdots M_5}$ carrying 11-dimensional indices. In practice, however, it will be more convenient to stay in the center-of-mass rest frame whenever possible (with an exception for lightlike P_M in some cases).

<u>Kaluza-Klein reduction and consistent truncations</u> [2]

In the following, we shall be concerned with the general pattern of p-brane solutions to supergravity theories, and for this we shall want to consider all dimensions $D \leq 11$. The essence of modern Kaluza-Klein theory is to make <u>consistent truncations</u> to the lower-dimensional cases. A consistent truncation is one that commutes with the variation of the action to produce field equations, i.e. a restriction of the fields such that solutions to the restricted theory are also specific kinds of solutions to the unrestricted theory. Truncation to the zero-charge sector with respect to some symmetry is always a guarantee of consistency (Setting charged fields to zero while retaining chargeless fields is consistent because the chargeless fields can never act as sources that might excite the charged fields.) To perform a Kaluza-Klein dimensional reduction, one demands that the fields be invariant under some translational symmetry. In a curved-space context, this means requiring that there be a translational <u>Killing vector</u> generating an isometry of the metric and with respect to which the other fields have vanishing Lie derivatives. Choosing an <u>adapted coordinate</u> z with respect to this isometry, the truncation is simply effected by setting ∂_z(any field) = 0. For the indices, let $x^M = \{x^m, z\}$.

Suppose we wish to reduce from $(D+1)$ to D dimensions. The standard Kaluza-Klein ansatz for the metric is

$$ds^2_{D+1} = e^{2\alpha\varphi} ds^2_D + e^{-2(D-2)\alpha\varphi}(dz + \mathcal{A}_{[1]})^2 ,$$

where $\mathcal{A}_{[1]} = \mathcal{A}_m dx^m$. The exponential prefactor of the second term has been chosen to ensure that a pure Einstein-Hilbert action $(eR)_{D+1}$ reduces to give a pure Einstein-Hilbert term without

e^{φ} factors in D dimensions. Specifically, one obtains

$$(eR)_{D+1} \longrightarrow (eR)_D - \tfrac{1}{4} e\, e^{-2(D-1)\alpha\varphi} \mathcal{F}_{[2]}^2 - \tfrac{1}{2} e (\partial\varphi)^2 , \quad \left\{ \begin{array}{l}\text{all fields on}\\ \text{RHS independent}\\ \text{of } z\end{array}\right.$$

where $\mathcal{F}_{[2]} = d\mathcal{A}_{[1]}$ and the canonical normalization of the kinetic term for φ is obtained by setting $\alpha^{-2} = 2(D-1)(D-2)$.

The reduction of gauge potentials is effected by setting $A_{[n]}(x,z) \longrightarrow A_{[n]}(x) + A_{[n-1]}(x) \wedge dz$. Noting that the dz differential in the reduction of ds^2_{D+1} occurs in the combination $dz + \mathcal{A}_{[1]}$, one finds it convenient to reduce the field strength $F_{[n]} = dA_{[n-1]}$ for the gauge potential $A_{[n-1]}$ as follows

$$\begin{aligned} F_{[n]} &\longrightarrow dA_{[n-1]} + dA_{[n-2]} \wedge dz \\ &= dA_{[n-1]} - dA_{[n-2]} \wedge \mathcal{A}_{[1]} + dA_{[n-2]} \wedge (dz + \mathcal{A}_{[1]}). \end{aligned}$$

Thus, it is natural to define $F_{[n-1]} = dA_{[n-2]}$, but for $F_{[n]}$ in the reduced theory, it is natural to take $F_{[n]} = dA_{[n-1]} - dA_{[n-2]} \wedge \mathcal{A}_{[1]}$. The kinetic term for $A_{[n-1]}$ then reduces as follows:

$$-\frac{1}{2n!} e F_{[n]}^2 \longrightarrow -\frac{1}{2n!} e\, e^{-2(n-1)\alpha\varphi} F_{[n]}^2 - \frac{1}{2(n-1)!} e\, e^{2(D-n)\alpha\varphi} F_{[n-1]}^2 .$$

Repeating this 1-step reduction procedure 11-D times, one works one's way down from 11-dimensional supergravity to D-dimensional supergravity. Here is the inventory of fields obtained in the lower-dimensional theory:

$$\begin{aligned} g_{MN} &\longrightarrow g_{MN},\ \vec{\Phi},\ \mathcal{A}^{i}_{[1]},\ \mathcal{A}^{ij}_{[0]} \qquad \vec{\Phi}: \text{11-D "dilatonic scalars"} \\ A_{[3]} &\longrightarrow A_{[3]},\ A^{i}_{[2]},\ A^{ij}_{[1]},\ A^{ijk}_{[0]}, \end{aligned}$$

where the indices run over the 11-D values corresponding to the reduced dimensions, and multiple i,j,k indices are restricted to values $i<j<k$ in labels like $\mathcal{A}^{ij}_{[0]}$ or $A^{ijk}_{[0]}$. Note that "Straight" $A_{[n]}$ fields emerge from $A_{[3]}$ in D=11, while the "Calligraphic" $\mathcal{A}_{[n=1,0]}$ fields emerge from Kaluza-Klein vectors $\mathcal{A}_{[1]}$ coming out of the metric.

After the dust settles, the bosonic sector of supergravity reduced to D dimensions via a sequence of one-step reductions like that above is given by

$$I_D = \int d^Dx \sqrt{-g} \Big\{ R - \tfrac{1}{2}(\partial\vec{\phi})^2 - \tfrac{1}{48} e^{\vec{a}\cdot\vec{\phi}} F_{[4]}^2 - \tfrac{1}{12}\sum_i e^{\vec{a}_i\cdot\vec{\phi}} (F_{[3]}^i)^2$$
$$- \tfrac{1}{4}\sum_{i<j} e^{\vec{a}_{ij}\cdot\vec{\phi}} (F_{[2]}^{ij})^2 - \tfrac{1}{4}\sum_i e^{\vec{b}_i\cdot\vec{\phi}} (\mathcal{F}_{[2]}^i)^2$$
$$- \tfrac{1}{2}\sum_{i<j<k} e^{\vec{a}_{ijk}\cdot\vec{\phi}} (F_{[1]}^{ijk})^2 - \tfrac{1}{2}\sum_{i<j} e^{\vec{b}_{ij}\cdot\vec{\phi}} (\mathcal{F}_{[1]}^{ij})^2 \Big\} + \mathcal{L}_{FFA}$$

The 11-D dilatonic scalars $\vec{\phi}$ appearing in the exponential prefactors here have coupling coefficients that have been called "dilaton vectors." The dilaton vector $\vec{a}$ for $F_{[4]}$ and the 11-D dilaton vectors $\vec{a}_i$ for $F_{[3]}^i$ determine the rest:

$$\vec{a}_{ij} = \vec{a}_i + \vec{a}_j - \vec{a} \qquad \vec{b}_i = -\vec{a}_i + \vec{a}$$
$$\vec{a}_{ijk} = \vec{a}_i + \vec{a}_j + \vec{a}_k - 2\vec{a} \qquad \vec{b}_{ij} = -\vec{a}_i + \vec{a}_j .$$

The independent dilaton vectors $\vec{a}$, $\vec{a}_i$ have the following dot products: $\vec{a}\cdot\vec{a} = \frac{2(11-D)}{D-2}$ $\quad \vec{a}\cdot\vec{a}_i = \frac{2(8-D)}{D-2}$ $\quad \vec{a}_i\cdot\vec{a}_j = 2\delta_{ij} + \frac{2(6-D)}{D-2}$.

An indication of the nonlinearly realized duality symmetries found by Cremmer and Julia may already be seen here: after rescaling, these become the same dot product relations as those of the fundamental weights of the Cremmer-Julia symmetry groups:

D	9	8	7	6	5	4	3
G	$GL(2,\mathbb{R})$	$SL(3,\mathbb{R})\times SL(2,\mathbb{R})$	$SL(5,\mathbb{R})$	$SO(5,5)$	$E_{6(+6)}$	$E_{7(+7)}$	$E_{8(+8)}$

We shall see the importance of these groups for p-brane solutions to supergravity theories in what follows.

The dimensionally-reduced action I_D gives a rather complicated-looking set of equations of motion. To make some headway, we shall now make a further radical, but consistent, truncation. We truncate to the single charge sector, keeping only the metric g_{MN}, one n-form field strength combination $F_{[n]}$ and one dilatonic scalar combination ϕ. To ensure the consistency of this truncation, consider the field equations for the dilatonic scalars that are to be truncated out. Letting $\vec{\Phi} = \vec{n}\phi + \vec{\Phi}_\perp$, $\vec{n}\cdot\vec{\Phi}_\perp = 0$, and letting the retained active field strengths be $F_{\alpha[n]}$, $\alpha = 1,\dots,N$, one finds the field equation for $\vec{\Phi}_\perp$:

$$\Box\vec{\Phi}_\perp - \sum_\alpha \Pi_\perp \cdot \vec{a}_\alpha e^{\vec{a}_\alpha\cdot\vec{\Phi}}(F_{\alpha[n]})^2 = 0 \quad , \Pi_\perp = \text{projector perpendicular to } \vec{n}.$$

Setting $\vec{\Phi}_\perp = 0$, so as to retain only the single dilatonic scalar ϕ, requires that the second term in the above equation vanish. This may be achieved by requiring the retained field strengths $F_{\alpha[n]}(x)$ all to be proportional to each other and also by requiring the $e^{\vec{a}_\alpha\cdot\vec{\Phi}}$ prefactors to be the same, demanding $\vec{a}_\alpha\cdot\vec{n} = a$ for all $\alpha = 1,\dots,N$. One must still then satisfy $\Pi_\perp \cdot \sum_\alpha \vec{a}_\alpha \cdot (F_{\alpha[n]})^2 = 0$, i.e. one requires that $\sum_\alpha \vec{a}_\alpha (F_{\alpha[n]})^2$ be parallel to $\vec{n}$ in dilaton-vector space. Remembering that $\vec{a}_\alpha\cdot\vec{n} = a$, one has $\sum_\alpha \vec{a}_\alpha (F_{\alpha[n]})^2 = a\vec{n}\sum_\alpha (F_{\alpha[n]})^2$. Letting the dot product of the dilaton vectors for the retained field strengths $F_{\alpha[n]}$ be $M_{\alpha\beta} = \vec{a}_\alpha\cdot\vec{a}_\beta$, one then has $\sum_\alpha M_{\beta\alpha}(F_{\alpha[n]})^2 = a^2\sum_\alpha (F_{\alpha[n]})^2$. It turns out to be sufficient to consider the cases with $M_{\alpha\beta}$ invertible in order to get the general pattern, and then one finds $\vec{a}^2 = \sum_{\alpha,\beta} M^{-1}_{\alpha\beta}$ and, introducing a common-factor field strength $F_{[n]}$, things are properly normalized if $(F_{\alpha[n]})^2 = a^2\sum_\beta M^{-1}_{\alpha\beta}(F_{[n]})^2$.

The resulting simplified system is just

$$I^{\text{single charge}} = \int d^D x\sqrt{-g}\,\{R - \tfrac{1}{2}\nabla_M\phi\nabla^M\phi - \tfrac{1}{2n!}e^{a\phi}F^2_{[n]}\}.$$

The class of such single-charge systems that will be of most interest to us will be that where $M_{\alpha\beta} = 4\delta_{\alpha\beta} - \frac{2(n-1)(D-n-1)}{(D-2)}$, as this class will possess the p-brane solutions with some amount of unbroken supersymmetry. Recalling that the number of retained field strengths, i.e. the size of the $M_{\alpha\beta}$ matrix, is N, and letting $\Delta = 4/N$, one finds $a^2 = \Delta - \frac{2(n-1)(D-n-1)}{(D-2)}$.

The importance of the quantity Δ is that it is preserved under further dimensional reductions of $I^{\text{single charge}}$, even though the dilaton coupling parameter a changes with dimension D and with the rank n of the "descendant" field strengths.

The equations of motion following from $I^{\text{single charge}}$ are

$$R_{MN} = \tfrac{1}{2}\partial_M\phi\,\partial_N\phi + S_{MN}$$
$$S_{MN} = \frac{1}{2(n-1)!}e^{a\phi}\left(F_{M\cdots}F_N{}^{\cdots} - \frac{n-1}{n(D-2)}F^2 g_{MN}\right)$$

$$\nabla_{M_1}\left(e^{a\phi}F^{M_1\cdots M_n}\right) = 0$$
$$\Box\phi = \frac{a}{2n!}e^{a\phi}F^2.$$

The p-brane ansatz; single-charge p-branes

Although vastly simplified with respect to the equations following from the full reduced maximal supergravity in D dimensions, these equations still have a fairly complicated structure, as always is the case in General Relativity and its extensions.

To make further progress, we now shall restrict our attention to solutions consistent with an ansatz. Noting the analogy of the $A_{[n-1]}$ gauge potential for $F_{[n]}$ to the classic Maxwell gauge field which couples naturally to a $d=1$ dimensional worldline for a particle, one may suppose that $A_{[n-1]}$ couples to the worldvolume of an extended object with $d=n-1$ worldvolume dimensions. Letting $p=d-1$ be the spatial dimensional of this extended object, we have $p=n-2$. To make things as simple as possible, we shall suppose that this extended stress-tensor distribution in spacetime yields a metric that is stationary and with translational symmetries in the p spatial worldvolume directions — in fact, we shall require that the metric become flat when restricted to the worldvolume. In the directions transverse to the worldvolume, we shall initially suppose the metric is isotropic, although we shall relax this later on. Thus, we take $ds^2 = e^{2A(r)}dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B(r)}dy^m dy^n \delta_{mn}$

where $r=\sqrt{y^m y^m}$; $\mu=0,1,\cdots,p$ and $m=p+1\cdots D-1$

Similarly, the dilaton will be taken to be $\phi=\phi(r)$. Overall, the ansatz has $(\text{Poincaré})_d \times SO(D-d)$ isometries.

For the antisymmetric tensor gauge potential, the corresponding ansatz is

$$A_{\mu_1\cdots\mu_{n-1}} = \epsilon_{\mu_1\cdots\mu_{n-1}} e^{C(r)} \quad ; \text{ others zero.}$$

The resulting field strength is

$$F^{(el.)}_{m\mu_1\cdots\mu_{n-1}} = \epsilon_{\mu_1\cdots\mu_{n-1}} \partial_m e^{C(r)} \quad ; \text{ others zero.} \quad \text{[electric ansatz]}$$

Now, there is a second form of ansatz related to the above by duality. The dualized field strength ${}^*F_{[n]}$ is a $D-n$ form. When the equations of motion for $F_{[n]}$ are re-written in terms of ${}^*F_{[n]}$, there is an interchange of rôles between equations of motion and Bianchi identities — the original field equation for $F_{[n]}$ may be interpreted as a Bianchi identity allowing ${}^*F_{[n]}$ to be derived, at least in some local neighborhood, from a dual potential $\tilde{A}_{[D-n-1]}$. Although we shall not pursue further this dual potential formulation, it allows us to see that there should be a "dual" type of ansatz with worldvolume dimension $d \to \tilde{d} = D-n-1$, i.e. $\tilde{d} = D-d-2$. For the metric and the dilaton, this is simply implemented by letting the worldvolume indices μ range over $\tilde{d}$ values, and the transverse indices m range over the remaining $D-\tilde{d} = n+1$ values. Instead of writing the ansatz in terms of the dual potential $\tilde{A}_{[D-n-1]}$, it is more convenient to write it directly in terms of our original field strength:

$$F_{m_1\cdots m_n} = \lambda \epsilon_{m_1\cdots m_n p} \frac{y^p}{r^{n+1}} \quad ; \text{ others zero.} \quad \text{[magnetic ansatz]}$$

The power of r in this ansatz is determined by requiring $F_{[n]}$ to satisfy the Bianchi identity $\partial_{[m_{n+1}} F_{m_1\cdots m_n]} = 0$. The magnetic charge parameter λ is a constant of integration.

Let us now jump directly to the solution to the field equations under these ansätze. The solution may be succinctly written as follows:

$$ds^2 = H^{\frac{-4\tilde{d}}{\Delta(D-2)}} dx^\mu dx^\nu \eta_{\mu\nu} + H^{\frac{4d}{\Delta(D-2)}} dy^m dy^m$$

$$e^\phi = H^{\frac{2a}{\varsigma\Delta}} \qquad \varsigma = \begin{cases} +1 & \text{electric} \\ -1 & \text{magnetic} \end{cases}$$

[Note that d here is the dimension of the solution's worldvolume, whether electric or magnetic. These are related by $d_{mag} = \tilde{d}_{el}$]

and in the electric case, the function $C(r)$ determining the gauge potential $A_{[n-1]}$ is given by

$$e^C = \frac{2}{\sqrt{\Delta}} H^{-1}.$$

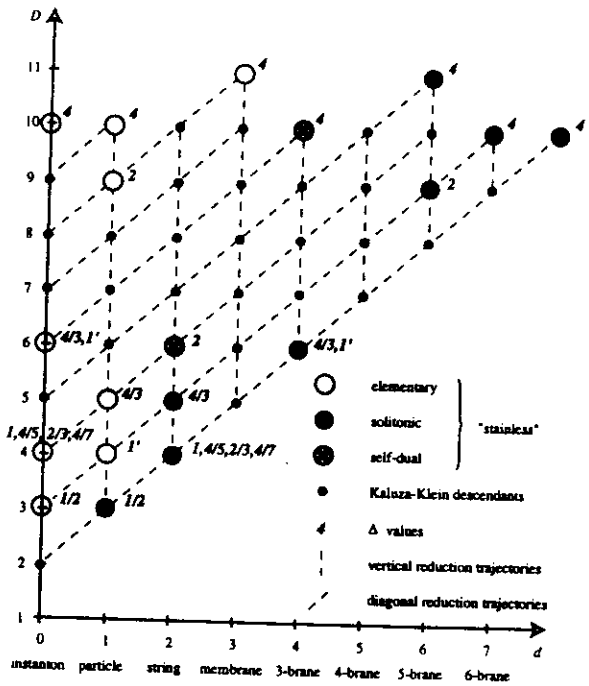
The function H above is a <u>harmonic function</u> on the transverse space. For the current transverse-isotropic ansatz, it should be taken to be centered at the origin,

$H(y) = 1 + \frac{k}{r^{\tilde{d}}}$ — the integration constant 1 has been chosen to make the metric approach flat space everywhere at transverse infinity, and to make the dilaton tend to the asymptotic value 1. The power $\tilde{d}$ of r in this function is that correct for a harmonic function in $D-d = \tilde{d}+2$ dimensions. When considering the magnetic case, note that $\tilde{\tilde{d}} = d$ — so if the worldvolume dimension in that case is taken to be $d_m = \tilde{d}$, then the power of r in H is $\tilde{\tilde{d}} = d$. In the magnetic case, the constant of integration k appearing in H is related to the magnetic charge parameter λ by $k = \frac{\sqrt{\Delta}}{2\tilde{d}}\lambda = \frac{\sqrt{\Delta}}{2(n-1)}\lambda$. In the electric case, the analogous formula, $k = \frac{\sqrt{\Delta}}{2\tilde{d}}\lambda = \frac{\sqrt{\Delta}}{2(D-n-1)}\lambda$, may be taken to <u>define</u> the parameter λ.

One may summarise the pattern of isotropic p-brane solutions in a plot of spacetime dimension D versus worldvolume dimension d. On this "brane scan" plot, one finds dimensional reduction trajectories passing <u>diagonally</u> from (D,d) to $(D-1, d-1)$, and so on. The p-brane solutions automatically satisfy the basic requirement for interpretation as solutions in a Kaluza-Klein reduced theory: they are completely independent of the x^μ coordinates, so reduction on these coordinates is trivial.

The only thing that needs to be done in order to fit a p-brane solution into the Kaluza-Klein reduction ansatz is an overall Weyl rescaling by a factor $H^{\frac{-4(D-d-2)}{\Delta(D-2)(D-3)}}$, needed to keep the Einstein-Hilbert action in the standard form, clean of e^{φ} dilaton factors in the lower $(D-1)$ dimensional theory.

Here is the brane-scan plot:



Brane-scan of supergravity p-brane solutions ($p \leq (D-3)$)

There are <u>vertical</u> reduction trajectories as well as the diagonal ones discussed above. In order do understand these, we shall have to discuss multicenter solutions, which will follow in due course. Another feature of the brane scan is the set of special, or "stainless" solutions shown. Once one has reduced a p-brane diagonally to a lower dimension, one can clearly reverse the process, i.e. "oxidise" it. But this oxidation process must stop at some point – <u>either</u> because one runs out of supergravity theories to oxidise into, <u>or</u> because the solution ceases to look like a single-charge isotropic p-brane. These have all been called "stainless" – oxidation into isotropic p-branes stops.

A more recent viewpoint would make a distinction between the two kinds of "stainless" solutions. Those for which oxidation stops because one runs out of supergravity theories are the $p=2$ and $p=5$ branes in $D=11$ (and perhaps the self-dual 3-brane in $D=10$, which belongs to the type IIB theory — although there is hot debate about whether this actually can in some sense be obtained by dimensional reduction). The other "heads" of single-charge reduction trajectories can be oxidized into higher dimensions, and are quite interesting — we shall see these can be interpreted as more generalized kinds of p-branes. By being "over-oxidized," they lose the Poincaré (worldsheet) $\otimes$ isotropic (transverse) symmetry of the single-charge p-branes. But this fact is now seen mainly as the harbinger of an interesting extended class of solutions that are either "boosted" or involve "intersections".

<u>The origin of the harmonic functions:</u>
<u>null geodesics on non-compact σ-models</u> [3]

One of the more striking features of the class of p-brane solutions considered so far is their dependence on simple harmonic functions on the transverse space. Let us now try to understand the origin of these by performing, for every p-brane solution on the brane-scan, the ultimate diagonal dimensional reduction: all the way to the lower left of the scan, to the (-1) branes, or "instantons". Actually "instanton" is something of a misnomer — proper instantons are solutions to an Euclideanized theory, involving making analytic continuations of solutions through complex times. What one obtains by pushing diagonal dimensional reduction to its limit, reducing even on the worldvolume time coordinate (possible because the p-brane solutions are static) is a somewhat unconventional type of theory involving Euclidean gravity coupling to an indefinite-signature nonlinear σ-model.

Before we explain the structure of these indefinite-signature σ-models, let's recall the σ-model structure of ordinary Minkowskian supergravities. We have already noted the sequence of Cremmer-Julia symmetry groups occurring in maximal supergravities as one descends through the spacetime dimensions D. The scalar sector of a given theory comprises the dilatonic scalars $\vec{\phi}$, the various zero-form potentials $A^{ijk}_{[0]}$, $\mathcal{A}^{ij}_{[0]}$, and the results of any dualisations to scalars from higher-order-form gauge potentials. For example, in $D=5$, $F_{[4]}$ dualises to a 1-form ${}^*F_{[4]}$, and the field equations for this describe the same dynamics as a massless scalar. Likewise, in $D=4$, the 3-form field strengths $F^i_{[3]}$ dualise to 1-forms ${}^*F^i_{[3]}$, and these once again describe the dynamics of som massless scalars; in $D=3$, it is the 2-form field strengths that dualise to 1-forms and effectively give scalar dynamics. It is only after performing these dualisations that the Cremmer-Julia symmetry groups unveil themselves. The occurrence of the exceptional groups E_6, E_7, E_8 in $D=5,4,3$ is related to the dualisations to scalars possible in these lower dimensions.

Taking all the scalars from whatever origin together, one finds a sequence of nonlinear σ-models coupled to gravity and the other (undualizable) gauge potentials. For ordinary Minkowskian supergravities, these σ-models all have scalar "target" manifolds G/H, where G is the corresponding Cremmer-Julia group and H is its maximal compact subgroup:

D	9	8	7	6	5	4	3
G	$GL(2,\mathbb{R})$	$SL(3,\mathbb{R})\times SL(2,\mathbb{R})$	$SL(5,\mathbb{R})$	$SO(5,5)$	$E_{6(+6)}$	$E_{7(+7)}$	$E_{8(+8)}$
H	$SO(2)$	$SO(3)\times SO(2)$	$SO(5)$	$SO(5)\times SO(5)$	$USp(8)$	$SU(8)$	$SO(16)$

The subgroups H are always linearly realized, both on the bosonic and on the fermionic supergravity fields, and the supercharges of the theory also carry H representations. Thus, H (together with the D-dimensional Lorentz group) forms part of the outer automorphism group of the standard supersymmetry algebra (standard: the one with "central" charges set to zero).

Now return to the "Kaluza-Klein instantons". The difference between reduction involving a time-like as opposed to a space-like Killing vector is seen in the signs of kinetic terms in the reduced theory. These signs depend on the number of temporal indices that are carried by a given term in the action. Thus, for example, the Kaluza-Klein vector's kinetic term appearing in a temporal reduction has an odd number of contractions of temporal indices, so the sign of such a term is opposite to that of a standard-sign kinetic term. On the other hand, the kinetic term for the Kaluza-Klein scalar appearing out of the metric in a temporal reduction has an even number of contractions of temporal indices. In contrast yet again, however, scalars appearing out of normal-sign vectors under a temporal reduction have odd number of such contractions, so acquire non-standard signs.

We are particularly concerned with the σ-model sector of the temporally-reduced theory. From the above considerations, one sees that the net change in this σ-model is to flip the signs of the kinetic terms for scalars appearing out of vectors in the temporal reduction from one dimension higher. These sign flips affect the structure of the linearly-realised subgroup H' occurring in the temporarily-reduced theory, but the non-linearly realized Cremmer-Julia symmetry G remains as in the Minkowskian theory at the same dimension D. Thus, reduction on a temporal coordinate instead of a spatial coordinate produces a sigma model G/H' instead of G/H. Here are the H' groups for the various dimensions:

D	9	8	7	6	5	4	3
H	$SO(2)$	$SO(3)\times SO(2)$	$SO(5)$	$SO(5)\times SO(5)$	$USp(8)$	$SU(8)$	SO
H'	$SO(1,1)$	$SO(2,1)\times SO(1,1)$	$SO(3,2)$	$SO(3,2)\times SO(4,1)$	$USp(4,4)$	$SU^*(8)$	SO^*

As one can see, the H' groups are always noncompact versions of the standard Minkowskian supergravity H groups. The non-compactness of the H' groups gives σ-model target manifolds G/H' with metrics of <u>indefinite signature</u>.

Now let's see how this explains the origin of the harmonic functions in p-brane solutions. The $p=-1$ Kaluza-Klein instantons will be supported by $n=p+2=1$ field strengths, for which the "gauge potentials" are zero-forms (hence the quotation marks — a zero-form "gauge potential" actually has no gauge symmetry). Thus, to obtain the Kaluza-Klein instantons, we need keep only gravity and the non-linear σ model resulting from the dimensional reduction. The general form of the action when restricted to this sector is

$$I_\sigma = \int (R - \tfrac{1}{2} G_{AB}(\phi)\partial_i \phi^A \partial_j \phi^B g^{ij})\sqrt{g} \quad , \quad \text{where } g_{ij}$$

is now an Euclidean-signature spacetime metric, while $G_{AB}(\phi)$ is the indefinite-signature σ-model metric for G/H'. The equations of motion for the ϕ^A are

$$\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\, g^{ij} G_{AB}(\phi)\partial_j \phi^B) = 0$$

while the Euclidean Einstein equations read

$$R_{ij} = \tfrac{1}{2} G_{AB}(\phi)\partial_i\phi^A \partial_j \phi^B .$$

Here come the harmonic maps: the whole class of p-brane solutions discussed so far corresponds to taking the spatial metric to be <u>flat</u>, $g_{ij} = \delta_{ij}$, and then solving

the equation of motion for ϕ^A and the constraint (from the Einstein eq.)
$G_{AB}(\phi)\,\partial_i\phi^A\partial_j\phi^B = 0$; this latter certainly has solutions since $G_{AB}(\phi)$ is of indefinite signature.

The ϕ^A equation with $g_{ij} = \delta_{ij}$ is just

$$\partial_i\left(G_{AB}(\phi)\,\partial_i\phi^B\right) = 0 .$$

Now suppose that $\phi^A(x)$ depend on the Euclidean space coordinates x^i through some scalar function $\sigma(x)$, i.e. $\phi^A(x) = \phi^A(\sigma(x))$. Then the ϕ^A equation becomes

$$\nabla^2\sigma\; G_{AB}(\phi)\frac{d}{d\sigma}\phi^B + (\partial_i\sigma)(\partial_i\sigma)\left[\frac{d^2\phi^A}{d\sigma^2} + \Gamma^A_{BC}(G)\frac{d\phi^B}{d\sigma}\frac{d\phi^C}{d\sigma}\right] = 0 .$$

Thus, if one takes $\sigma(x)$ to be a harmonic function, $\nabla^2\sigma = \partial_i\partial_i\sigma = 0$, then the ϕ^A equation is satisfied if

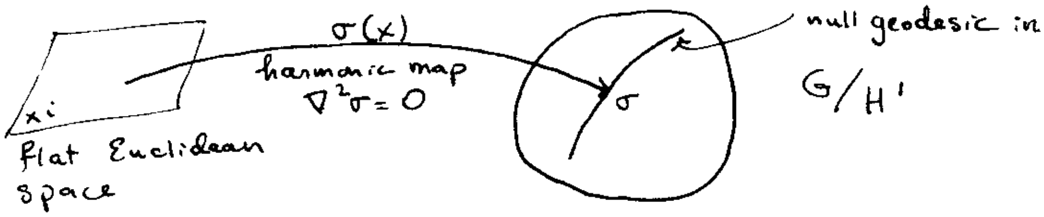
$$\frac{d^2\phi^A}{d\sigma^2} + \Gamma^A_{BC}(G)\frac{d\phi^B}{d\sigma}\frac{d\phi^C}{d\sigma} = 0 ,$$

i.e. if the <u>geodesic equation</u> on G/H' is satisfied. Moreover, the constraint $G_{AB}\,\partial_i\phi^A\partial_j\phi^B = 0$ becomes now

$$\partial_i\sigma\,\partial_j\sigma\; G_{AB}(\phi)\frac{d\phi^A}{d\sigma}\frac{d\phi^B}{d\sigma} = 0 ,$$ which requires

$$G_{AB}(\phi)\frac{d\phi^A}{d\sigma}\frac{d\phi^B}{d\sigma} = 0 ,$$ i.e. the geodesic on G/H' must be <u>null</u>. If this condition is satisfied at some point on the geodesic, then the geodesic equation will maintain it along its full length.

The geometrical picture of what is happening is like this:



Let us now consider a simple explicit example of the above construction. $D=10$ type IIA supergravity contains the following fields: the metric, one dilaton, one vector potential, one 2-form potential and one 3-form potential. Upon reduction to $D=9$, one obtains a metric, two dilatons and an axion (i.e. a zero-form potential), three vectors, two 2-form potentials and one 3-form potential. Under ordinary reduction from 10 to 9 dimensions on a spatial dimension, all these fields have standard-sign kinetic terms. Under a temporal reduction from 10 to 9, however, the axion coming from the $D=10$ vector field, plus the Kaluza-Klein vector emerging from the $D=10$ metric, the vector descending from the $D=10$ 2-form and the $D=9$ 2-form descending from the $D=10$ 3-form all have non-standard signs. Of course, after the temporal reduction there is no more "time" in the theory, so these non-standard signs for "kinetic" terms are not physically disastrous. However, the above pattern of signs for the kinetic terms agrees with the $SO(1,1)$ linearly-realized H' symmetry that we have anticipated.

Under a normal spatial reduction from 10 to 9, the gravity–σ-model sector of the theory is

$$I_{\sigma 9}^{\text{spatial red.}} = \int\left(eR - \tfrac{e}{2}(\partial\phi_1)^2 - \tfrac{e}{2}(\partial\phi_2)^2 - \tfrac{e}{2}e^{-2\phi_1}(\partial\chi)^2\right),$$

in which the scalar fields ϕ_1, ϕ_2, χ take their values in a $GL(2,R)/SO(2)$ coset space. Under a temporal $10\to 9$ reduction, on the other hand, the σ-model – gravity action is

$$I_{\sigma 9}^{\text{temporal red.}} = \int\left(eR - \tfrac{e}{2}(\partial\phi_1)^2 - \tfrac{e}{2}(\partial\phi_2)^2 + \tfrac{e}{2}e^{-2\phi_1}(\partial\chi)^2\right),$$

in which the scalars take their values in $GL(2,R)/SO(1,1)$.

Now let us see how we may satisfy the conditions required for Kaluza-Klein instantons. The group $SL(2,\mathbb{R})$ is isomorphic to $SO(2,1)$. For the normal spatial reduction, one can express the $SO(2,1)/SO(2)$ target space as a constrained surface embedded in a 3-dimensional space (X, Y, Z)

with the $SO(2,1)$-invariant constraint $X^2-Y^2-Z^2=1$. The $I_{\sigma q}^{\text{spatial}}$ σ-model is then obtained by coordinatizing this constrained surface by

$$X = \cosh\phi_1 + \tfrac{1}{2}\chi^2 e^{-\phi_1}$$
$$Y = \sinh\phi_1 + \tfrac{1}{2}\chi^2 e^{-\phi_1}$$
$$Z = \chi e^{-\phi_1}.$$

Of course, the second dilaton ϕ_2 is free in this σ-model, so it does not participate in the above geometry – in other words, we treat $GL(2,\mathbb{R})/SO(2)$ as $\mathbb{R}\times SO(2,1)/SO(2)$, with ϕ_2 being the coordinate on $\mathbb{R}$.

Under a temporal $10\to 9$ reduction, the appropriate $SO(2,1)$-invariant constraint for the $SO(2,1)/SO(1,1)$ surface embedded in 3 dimensions is $X^2-Y^2-Z^2=-1$, i.e. the sign of the invariant is flipped. On this $SO(2,1)/SO(1,1)$ surface, the appropriate coordinates are

$$X = \sinh\phi_1 - \tfrac{1}{2}\chi^2 e^{-\phi_1}$$
$$Y = \cosh\phi_1 - \tfrac{1}{2}\chi^2 e^{-\phi_1}$$
$$Z = \chi e^{-\phi_1}.$$

One may verify that substitution into the $SO(2,1)$-invariant action $I^{SO(2,1)} = \int \left(\frac{e}{2}(\partial X)^2 - \frac{e}{2}(\partial Y)^2 - \frac{e}{2}(\partial Z)^2 \right)$ now yields the $SO(2,1)/SO(1,1)$ σ-model $I_{\sigma q}^{\text{temporal}}$.

In order to identify the appropriate null geodesic needed for generating a p-brane solution, it is more convenient to use the X,Y,Z coordinates with Z eliminated via $X^2-Y^2-Z^2=-1$. The σ-model action becomes

$$\int e\left(1+x^2-y^2\right)^{-1}\left((1-y^2)(\partial x)^2 - (1+x^2)(\partial y)^2 + 2xy\,\partial x\,\partial y\right).$$

Now, one may directly verify that imposing the constraint $X=Z$, so $Y=\pm 1$, causes this action to vanish. Hence, the stress tensor for the X,Y fields vanishes for $X=Z$; this is our null geodesic in $SO(2,1)/SO(1,1)$.

Now consider the field equation for the Y field, varying the action <u>before</u> setting $Y=\pm 1$ [note – the X eqn. is satisfied trivially for $Y=$ const.]

$$2\nabla^i\left(\frac{(1+x^2)\partial_i y}{1+x^2-y^2} - \frac{xy\,\partial_i x}{1+x^2-y^2}\right) + \frac{2x\,\partial^i x\partial_i y - 2y(\partial x)^2}{1+x^2-y^2}$$
$$+ (1+x^2-y^2)^{-2}\,2y\left((1-y^2)(\partial x)^2 - (1+x^2)(\partial y)^2 + 2xy\,\partial^i x\,\partial_i y\right)$$

One may verify that imposing the constraint $y=\pm 1$ in this equation yields $\frac{\nabla x}{x^2} = 0$. Thus, the promised harmonic function in this case is just the coordinate x, $\sigma(x^i)=x(x^i)$. In terms of (χ,ϕ_1), the null geodesic $x=z$ is given by $\chi = e^{\phi}-1$.

The above construction gives a Kaluza-Klein instanton in $D=9$. If the harmonic function $x(x^i)=\sigma(\ldots)$ is taken to be isotropic, it will have the form $x = \frac{-k}{r^7} \Rightarrow e^{\phi_1} = (1+\frac{k}{r^7})^{-1}$. The solution so obtained takes the form of our general solution, specialized to the case $D=9$, $d=0$: Since there is no "worldvolume" for the instanton, one has simply $ds^2 = H^0 dy^m dy^m = dy^m dy^m$, flat, as claimed. Clearly, however, the general instanton construction permits extensions that would insert a more general Ricci-flat metric in the place of the flat $g_{mn}=\delta_{mn}$.

The above construction of harmonic maps into null geodesics on G/H' can also be explicitly carried out using matrix realizations. For the supergravity cases at hand, these are symmetric spaces. Picking an appropriate coset representative matrix $M(\phi^A)$, the target space metric is $d\ell^2 = G_{AB}\,d\phi^A d\phi^B = -\frac{1}{2}\,\mathrm{tr}(dM dM^{-1})$ and the geodesic equation becomes
$$\frac{d}{d\sigma}\left(M^{-1}\frac{dM}{d\sigma}\right) = 0.$$

This may be solved by taking $M = Ae^{B\sigma}$, where A and B are constant with $A\in G/H'$ giving some initial point on the geodesic (corresponding to $\sigma=0$) and $B\in \mathfrak{Lie}(G)$ being chosen such that $M(\sigma)$ remains within the set of matrices parametrizing G/H'. Obviously, the details of how this is to be done depend on the G/H' parametrization chosen

The constraint on the geodesic coming from the Einstein equation $R_{ij} = \frac{1}{4} \mathrm{tr}(B^2)\, \partial_i \sigma\, \partial_j \sigma$ becomes, for a Ricci-flat metric, just $\mathrm{tr}(B^2) = 0$. This ensures that the geodesic $M(\sigma)$ remains <u>null</u>, since $d\ell^2 = \frac{1}{4}\mathrm{tr}(B^2) = 0$.

The matrix realization of G/H' also allows us to extend the construction to σ-model solutions involving several harmonic functions $\sigma^a(x^i)$, $\nabla^2 \sigma^a = 0$. One can consider $M = A \exp(\sum_a B_a \sigma_a)$. Now the σ-model field equations are $\nabla^i (M^{-1} \partial_i M) = 0$ when written in terms of M. With several harmonic functions present, one finds that for these to be satisfied, one needs to impose the additional requirement $[[B_a, B_b], B_c] = 0$, as this allows one to rewrite $M = A \exp(-\frac{1}{2} \sum_{c>b} \sum_b [B_b, B_c] \sigma_b \sigma_c) \prod_a e^{B_a \sigma_a}$, where the first factor then commutes with B_a. Then, the matrix current becomes

$$M^{-1}\partial_i M = \sum_a B_a \partial_i \sigma_a - \frac{1}{2} \sum_{c>b} \sum_b [B_b, B_c](\sigma_b \partial_i \sigma_c - \sigma_c \partial_i \sigma_b)$$

and this is then conserved providing the σ_a are harmonic.

The Einstein equations for solutions with multiple harmonic functions become

$$R_{ij} = \frac{1}{4} \sum_a \sum_b \mathrm{tr}(B_a B_b)\, \partial_i \sigma_a\, \partial_j \sigma_b ,$$

so for Ricci-flat solutions such as the trully flat $g_{ij} = \delta_{ij}$ class we have considered, one requires

$$\mathrm{tr}(B_a B_b) = 0 ,$$

i.e. the geodesics parametrized by varying the various σ_a must be <u>orthogonal</u> and <u>null</u>. The general stationary p-brane solution is thus found by identifying the set of totally null, totally geodesic submanifolds of G/H' such that the velocity vectors B_a satisfy $[[B_a, B_b], B_c] =$

The cast of p-branes (and friends) in D=11 [4]

Let us start from the instanton we found in $D=9$ and follow it up as it oxidizes into $D=11$. The first step of oxidation to $D=10$ is straightforward — one obtains the solution diagonally up to the right on the brane-scan, a $p=0$ brane, or supersymmetric black hole, in $D=10$. This solution is supported by the vector field of type IIA supergravity, which has a dilaton coupling given by $\Delta=4$. But now we have reached the top of one of the trajectories of the standard single-charge p-branes. Indeed, if one were to be able to oxidise the $p=0$ brane in $D=10$ up to a $p=1$ brane in $D=11$, there would have to be a 2-form gauge potential in $D=11$ to support it. But the $D=11$ theory has no such 2-form gauge field. So something else must happen.

What happens upon oxidation of the $D=10$ black hole (particle) solution can be guessed by noting where the supporting gauge field in $D=10$ came from: this is the Kaluza-Klein vector $\mathcal{A}_{[1]}$ emerging out of the $D=11$ metric. Here is what one gets by oxidising the $D=10$ particle:

$$ds^2 = -dt^2 + d\rho^2 + (H-1)(dt+d\rho)^2 + ds^2(\mathbb{E}^9)$$

$$A_{[3]} = 0 \qquad\qquad [\text{wave}]$$

where $ds^2(\mathbb{E}^9)$ is the flat Euclidean 9-space carried up from our $D=9$ instanton (with no conformal factor — this having disappeared in the Weyl rescaling part of the oxidation procedure), and H is the harmonic function on $\mathbb{E}^9$ from our $D=9$ instanton discussion. In $D=11$, there is no dilaton, and the $A_{[3]}$ gauge potential plays no rôle in the present solution, so is zero (or pure gauge). Thus, the above solution is a solution of pure $D=11$ General Relativity

This is a type of solution discovered by Brinkman in 1925. Such solutions are known as pp waves, where "pp" stands for "plane-fronted waves with parallel rays". Thus, the p-brane family has some cousins — solutions lacking the (Poincaré)d × (isotropic) structure of single-charge isotropic p-branes, but obtainable from them by oxidation.

Now let's proceed to the proper $D=11$ p-branes, of which there are two: $p=2$, and its dual, $p=5$. Although we have written out the general single-charge p-branes already, let's rewrite these $\Delta=4$ solutions in a different style:

membrane, $p=2$:

$$ds_{11}^2 = H^{1/3}(y)\left[H^{-1}(y)(-dt^2+dx_1^2+dx_2^2) + dy^m dy^m\right]$$

$$A_{[3]} = H^{-1}(y)\, dt\wedge dx_1\wedge dx_2 \qquad \nabla^2 H = 0.$$

5-brane:

$$ds_{11}^2 = H^{2/3}(y)\left[H^{-1}(y)(-dt^2+dx_1^2+\cdots+dx_5^2) + dy^m dy^m\right]$$

$$F_{[4]} = \ast dH \qquad \nabla^2 H = 0\,,$$

where the dual used in constructing $F_{[4]}$ is taken in the transverse 5-dimensional space.

Clearly, the 2-brane and the 5-brane form a dual pair, in the sense of the electric and magnetic ansätze discussed earlier. One would be entitled to ask if the pp wave solution has an associated "dual" solution. Since this solution arose from dimensional oxidation in our discussion, and not from the p-brane ansatz itself, there is not a direct answer to this question in $D=11$. However, if we reduce the wave back down one dimension to $D=10$, we obtain the $p=0$ brane, particle, solution. In $D=10$, this dualises to a solution with $\tilde{d} = 10-1-2 = 7$, i.e.

a 6-brane, the magnetic dual of the $D=10$ particle
Now, as one can see from the brane scan, the $D=10$ 6-brane is also at the top of a reduction/oxidation trajectory. If it were to oxidise to a 7-brane in $D=11$ there would once again have to be a 2-form gauge potential to support it via the magnetic ansatz. But, as we have noted, there is no such 2-form gauge field.

Instead, the $D=10$ 6-brane oxidises to the following $D=11$ solution:

$$ds_{11}^2 = -dt^2 + dx_1^2 + \dots + dx_6^2 + ds^2_{TN}$$

$$A_{[3]} = 0 \qquad \text{Taub-NUT (Also known as the "Kaluza-Klein monopole")}$$

where ds^2_{TN} is the Euclidean $D=4$ Taub-NUT solution:

$$ds^2_{TN} = H(y^i)dy^i dy^i + H^{-1}(y)(d\psi + V_i^{(y)}dy^i)^2 \qquad i=1,2,3$$

$$\nabla^2 H = \partial_i\partial_i H = 0 \text{ (harmonic in 3 dims)}$$

where $\vec{\nabla}\times\vec{V} = \nabla H$. Strictly speaking, the true Taub-NUT solution is obtained by taking the harmonic function in the y^i coordinates to be single-centered, $H = 1 + \frac{k}{r}$, for which $V_{[1]} = k\cos\theta\, d\phi$ in spherical polar coordinates on $\mathbb{E}^3$.

The Taub-NUT metric appears to be singular at $r=0$, but this in fact turns out to be just a coordinate singularity, provided one chooses ψ to be a periodic coordinate with period $4\pi k$. This sort of quantisation is characteristic of non-singular magnetic solitons.

The $D=11$ solutions all benefit from the absence of a dilaton — this makes them non-singular at $r=0$, which is just a coordinate singularity in all the fundamental $D=11$ cases discussed. Upon reduction to lower dimensions, dilatonic scalars emerge and at the same time the group of general coordinate transformations shrinks. Singularities generically appear at $r=0$ in these dilatonic scalars, as well as in the corresponding Einstein-frame metrics.

<u>Supersymmetry preservation</u> [4]

The p-brane solutions and the associated and Taub-NUT solutions that we have considered all leave some portion of the original supersymmetry of the theory unbroken. For an example of this, consider the p = 2, membrane, case in D = 11. Preservation of supersymmetry for a purely bosonic solution means that setting all the spinors of a supergravity theory to zero is consistent with at least some portion of the original supersymmetry of the theory. Of course, it is the inhomogeneously-transforming gravitino that poses the greatest difficulty with this. Out of the original local supersymmetry of the theory, at best a finite-dimensional rigid supersymmetry can survive. The solution with maximal unbroken supersymmetry is just empty flat space, which for D = 11 supergravity has an unbroken 32-component rigid supersymmetry, with constant transformation parameter ϵ. This sets the standard against which the amounts of unbroken supersymmetry in other bosonic solutions are measured — so "half preservation" means 16 components of rigid supersymmetry survive, etc. These fractions of surviving supersymmetry are determined by solving $\delta\psi_A|_{\psi=0} = \tilde{D}_A\epsilon(x) = 0$, where $\psi_A = e_A{}^M\psi_M$ is the gravitino with indices referred to a local orthonormal frame.

The derivative operator appearing in $\delta\psi_A|_{\psi=0}$ for D = 11 supergravity is given by

$$\tilde{D}_A\epsilon = D_A\epsilon - \frac{1}{288}(\Gamma_A{}^{BCDE} - 8\,\delta_A{}^B\Gamma^{CDE})F_{BCDE}\,\epsilon,$$

with $D_A\epsilon = (\partial_A + \frac{1}{4}\omega_A{}^{BC}\Gamma_{BC})\epsilon$, and all Γ matrices here are antisymmetrized products of Γ_A with "strength one", i.e. $\Gamma_{AB} = \frac{1}{2}(\Gamma_A\Gamma_B - \Gamma_B\Gamma_A)$, etc. In order to have any amount of unbroken supersymmetry, one must find solutions to $\tilde{D}_A\epsilon = 0$.

For the $p=2$ isotropic membrane background, with $SO(2,1)\times SO(8)$ unbroken remainder of the $D=11$ Lorentz symmetry, we pick an adapted representation for the Dirac Γ matrices: $\Gamma_A = (\gamma_\mu \otimes \Sigma_9\,,\ \mathbb{1}_{(2)} \otimes \Sigma_{m-3})$, where the γ_μ and $\mathbb{1}_{(2)}$ are 2×2 $SO(2,1)$ matrices, while Σ_9 and Σ_{m-3} are 16×16 $SO(8)$ matrices; $\Sigma_9 = \Sigma_1\Sigma_2\cdots\Sigma_8$, so $\Sigma_9^{\,2} = \mathbb{1}_{(16)}$. Now, the most general spinor field consistent with the $(\text{Poincaré})_3\times SO(8)$ symmetry of the membrane background is of the form $\epsilon(x,y) = \epsilon_{(2)} \otimes \eta_{(16)}(r)$ where $\epsilon_{(2)}$ is a <u>constant</u> $SO(2,1)$ spinor and $\eta_{(16)}(r)$ is an $SO(8)$ spinor depending only on the radial coordinate $r=\sqrt{y^m y^m}$ of the transverse space. Note that a 16-dimensional spinor is a <u>reducible</u> representation of $SO(8)$, however – it may be decomposed into Σ_9 eigenstates by use of the projectors $\frac{1}{2}(\mathbb{1} \pm \Sigma_9)$.

Analysis of the differential equation $\tilde{D}_A \epsilon = 0$ shows that the solutions must have $\eta(r) = H^{-1/6}(r)\,\eta_0$, where η_0 is a constant $SO(8)$ spinor that furthermore must be <u>chiral</u>: $(\mathbb{1} + \Sigma_9)\eta_0 = 0$. (Note that the selection of $SO(8)$ chirality here is correlated with the sign of $A_{[3]}$ in the p-brane ansatz. This is not inconsistent with parity invariance of the $D=11$ theory, because the FFA term requires $A_{[3]}$ to flip sign under a parity transformation, which would also flip the $SO(8)$ chirality of η_0.) Thus, the surviving supersymmetry is described by the product $\epsilon_{(2)} \otimes \eta_0$, which has $2\times 8 = 16$ independent components, or <u>half</u> that of $D=11$ flat space.

Now let's return to the bosonic charges we encountered at the beginning of this course. Recalling the definition $\lambda = \frac{2\tilde{d}}{\sqrt{\Delta}}k$ ($= \tilde{d}k$ for $\Delta=4$) of the charge parameter in the electric case, and orienting the M_8 submanifold occurring in the specification of the charge $\mathcal{U}$ to coincide with the membrane's transverse space, one finds

$$\mathcal{U} = \frac{1}{4\Omega_7} \int_{\partial M_{7}} d\Sigma^{M}_{(7)}\, F_{M012} = \frac{\lambda}{4}\,.$$

Let's compare this with the energy. Now, in fact, the definition that we gave earlier for the total energy, in terms of a $d^9\Sigma$ integral over the boundary ∂M_{10} of the entire spatial submanifold of $D=11$ spacetime, diverges since the membrane solution is independent of the two spatial worldvolume coordinates. So the total quantity E for this $(\text{Poincaré})_3$ invariant membrane solution is not the most meaningful quantity to consider. Instead, one should consider an energy/unit p-vol. $\mathcal{E} = \frac{1}{4\Omega_{D-d-1}} \int_{\partial M_T} d^{D-d-1}\Sigma^a(\partial^b h_{ab} - \partial_a h^b{}_b)$, which is finite. For the membrane solution, this gives $\mathcal{E} = \frac{k\tilde{d}}{\Delta}$ and since $\Delta = 4$ for this solution, we have $\mathcal{E} = \frac{\lambda}{2\sqrt{\Delta}}$, i.e. $\mathcal{E} = \lambda/4$. Thus, the energy density $\mathcal{E}$ and the charge (density) $\mathcal{U}$ agree. In general, one can show, starting from the supersymmetry algebra, that $\mathcal{E} \geq \frac{2}{\sqrt{\Delta}}\mathcal{U}$. The saturation of this bound (for $\Delta=4$) and the preservation of ½ unbroken supersymmetry go hand-in-hand. This is the Bogomol'ny - Prasad - Sommerfield bound, so solutions saturating it are known as BPS states.

Now let's return to the issue of supersymmetry preservation, and see if we can re-state the issue more telegraphically. Consider the supersymmetry algebra equation divided by the (infinite) volume of the spatial worldsheet coordinates, specialized to the membrane solution case: note: $P_0(\text{2-vol.}) = -\mathcal{E}$

$$(\text{2-vol.})^{-1}\{Q_\alpha, Q_\beta\} = -(C\Gamma^0)_{\alpha\beta}\,\mathcal{E} + (C\Gamma^{12}\mathcal{U}_{12}),$$

letting the membrane be oriented along the 12 plane in the $D=10$ spatial submanifold. The charge conjugation matrix C is just $C = \Gamma^0$ in the Dirac algebra representation that we have been using. Noting that $\mathcal{E} = \mathcal{U}_{12}$ for the membrane solution, we may write the algebra in the membrane background in the form

$$\frac{1}{(\text{2-vol})}\{Q_\alpha, Q_\beta\} = 2\mathcal{E}\left(\frac{\mathbb{1} + \Gamma^{012}}{2}\right) = 2\mathcal{E}\,P_{012},$$

where P_{012} is a projector, i.e. $P_{012}^2 = P_{012}$. Written out in the $SO(2,1)\times SO(8)$ Dirac-algebra basis that we have been using, this projector just becomes $\frac{1}{2}\mathbb{1}_{(2)}\otimes(\mathbb{1}_{(16)} + \Sigma_9)$, so the chirality condition on the surviving parameter of unbroken supersymmetry is just $P_{012}\,\epsilon = 0$. Moreover, we see directly from this that the fraction of surviving generators is $1/2$ because $\operatorname{tr}\Gamma^{012} = 0$, so we have $\operatorname{tr} P_{012} = \frac{1}{2}\cdot 32 = 16$, and, noting that a projector (satisfyi. $P^2 = P$) can only have eigenvalues 0 or 1, this means that P_{012} projects into precisely $1/2$ the space of $D=11$ Majorana spinor parameters ϵ.

Now let's consider the other solutions in our fundamental $D=11$ set. The 5-brane goes similarly to the 2-brane, except that it carries the V, magnetic, type of charge. Orienting the 5-hypersurface whose boundary ∂M_5 is the surface of integration for V so as to coincide with the 5-brane's transverse space M_{5T}, one has $V = \frac{1}{4\Omega_4}\int_{\partial M_{5T}} d\Sigma^m_{(4)}\,\epsilon_{mnpqr}F^{npqr} = \frac{\lambda}{4}$. Now the relevant quantity to compare this with is the energy (unit 5-vol), which once again takes the value $\mathcal{E} = \lambda/4$, so once again we have $\mathcal{E} = V$, saturating the $\Delta = 4$ Bogomol'ny bound, i.e. we again have a BPS state. Now consider the corresponding supersymmetry projector. Let the 5-bran lie in the 12345 spatial hyperplane. Then from the supersymmetry algebra in this background, we find

$$\frac{1}{(5\text{-vol.})}\{Q_\alpha, \bar{Q}_\beta\} = 2\mathcal{E}\left(\frac{\mathbb{1} + \Gamma^{012345}}{2}\right) = 2\mathcal{E}\,P_{012345}.$$

Similarly to the $p=2$ case, $P_{012345}^2 = P_{012345}$, so this is a projector; moreover, $\operatorname{tr}\Gamma^{012345} = 0$, so $\operatorname{tr} P_{012345} = \frac{32}{2} = $ so supersymmetry is $1/2$ preserved once again.

Now look at the pp wave. Here, there is neither U charge nor V charge. However, while this solution is stationary, it cannot be described as static, since there is non-vanishing spatial momentum present — there has to be, since this solution describes plane waves propagating at the speed of light. The energy and momentum expressions should be integrated over $\partial\mathbb{E}^9$, where $\mathbb{E}^9$ is the transverse space that the harmonic function H depends on in this case — i.e. we let $\mathcal{E}$ = energy/length = momentum/length, and obtain $\frac{1}{\text{length}}\{Q_\alpha, Q_\beta\} = 2\mathcal{E}\left(\frac{\mathbb{1}+\Gamma^{01}}{2}\right) = 2\mathcal{E}P_{01}$,

where we have let the p direction along which the wave is travelling be the 1 direction. Again, P_{01} is a projector, $P_{01}^2 = P_{01}$ and $\operatorname{tr}\Gamma^{01} = 0$, so $\operatorname{tr}P_{01} = \frac{32}{2}$, so supersymmetry is 1/2 preserved.

Finally to the Taub-NUT solution, also known as the Kaluza-Klein monopole. Here we have a charge V_{01234} (with a time direction), equal in magnitude to the energy/(unit 6-vol.) $\mathcal{E}$, and one finds

$$\frac{1}{(6\text{-vol.})}\{Q_\alpha, Q_\beta\} = 2\mathcal{E}(\mathbb{1}-\Gamma^{1234}) = 2\mathcal{E}P_{1234};$$

this is again a projector, $P_{1234}^2 = P_{1234}$, and $\operatorname{tr}P_{1234} = \frac{32}{2}$, so supersymmetry is 1/2 preserved.

No-force conditions [4,5]

The single-charge p-brane solutions that we have seen so far depend on a harmonic function H that takes the form $H(y) = 1 + \frac{k}{r^{\tilde{d}}}$ for our single-center, isotropic solutions. The requirement of $SO(D-d)$ isotropic symmetry merely characterises the simplest such solutions, however, and is not required by the equations of motion, which admit any harmonic function on the transverse space y^m. So, in particular, one could take a multi-center solution with $H = 1 + \sum_{\alpha=1}^{k} \frac{k_\alpha}{|\vec{y}-\vec{y}_\alpha|^{\tilde{d}}}$, which gives a solution that may be described as l p-branes of the same type with parallel worldvolumes in the overall D-dimensional spacetime, but with charge centers located at the separate positions $\vec{y}_\alpha$ in the transverse directions. It is clear from the fact that stationary solutions of this sort solve the equations of motion that all forces between the individual charged hyperplanes must cancel. A physical interpretation of how this can happen is that the <u>attractive</u> gravitational and scalar forces <u>cancel</u> the <u>repulsive</u> $(p+1)$-form antisymmetric-tensor forces. One way to model this for widely separated p-branes is to consider consider a heavy p-brane situated at the origin (e.g. let k_1 be large, with $\vec{y}_1 = 0$), and take a light <u>test brane</u> in the background created by the heavy one, ignoring back-reaction effects of the test brane on the underlying solution.

For example, in $D=11$ one can have a test membrane in the background of a heavy membrane. The motion of such a test brane may be described by the Supermembrane action, which is a generalization of the Nambu-Goto action for the superstring. If one chooses the test membrane's worldvolume coordinates to be in "static gauge", where the 3 worldvolume coordinates align with the coordinates of the background spacetime, $\xi^\mu = X^\mu$, $\mu = 0,1,2$ then the test membrane's action becomes

$$I_{test} = -T \int d^3\xi \left[\sqrt{-\det\left(e^{2A}\eta_{ij} + e^{2B}\partial_i y^m(\xi)\partial_j y^m(\xi)\right)} - e^C\right]$$

where the e^C term comes from a coupling of $A_{[3]}$ to the worldvolume $-\frac{1}{3!}\epsilon^{ijk}\partial_i X^M \partial_j X^N \partial_k X^P A_{MNP}$, generalizing the analogous coupling of the 2-form gauge potential to a string worldsheet. Expanding the exponential, one finds a potential for the test brane $V = T(e^{3A} - e^C)$. This combination, however vanishes for the membrane background, and this is in fact a condition necessary for the background to preserve some unbroken supersymmetry.

<u>Intersecting p-branes</u> [5,6]

There are several ways to see that the class of single-charge p-branes that we have mainly focussed on can be extended. An exhaustive way to do this would be to carry out the classification of the "Kaluza-Klein instanton" σ-model solutions involving several harmonic functions. In a number of cases, it is transparent that one may satisfy the conditions required for multiple harmonic functions — e.g. in $D=8$, where the G/H' σ-model is $\dfrac{SL(3,\mathbb{R})\times SL(2,\mathbb{R})}{SO(2,1)\times SO(1,1)}$, with two mutually commuting sectors.

Another way to easily see the possibilities for generalizations of the single-charge p-branes is to consider a test brane corresponding to one field strength moving in a background p-brane solution supported by a different field strength, looking for situations where the test brane's potential vanishes, generalizing the $D=11$ example for parallel membranes that we have just seen.

Yet another way to see when more generalized solutions are possible is to require some degree of unbroken supersymmetry, since this will ensure a "no-force" condition. This possibility may be checked easily using the supersymmetry projectors that we have introduced.

Let's consider an example: let's see if a solution combining two $D=11$ membranes could preserve some supersymmetry, not with the membranes parallel as before, but <u>intersecting</u>. Let one membrane worldvolume be in the 012 subspace, and let the other be in the 034 subspace. Now, one may check that $[P^{\pm}_{012}, P^{\pm}_{034}] = 0$, so it would not

(where $P^{\pm} = \frac{1}{2}(1\!\!1 \pm \Gamma^{012})$, ...)

be inconsistent to have a surviving supersymmetry parameter ϵ that is a simultaneous non-trivial eigenstate under both of these projectors. Moreover, $\mathrm{tr}(P^{\pm}_{012} P^{\pm}_{034}) = \frac{1}{4} \cdot 32 = 8$, so this simultaneous projection would preserve 1/4 supersymmetry. Given this positive indication, one is emboldened to look for an actual solution. Here it is:

$$ds^2 = (H_1 H_2)^{1/3}\left[-(H_1H_2)^{-1}dt^2 + H_1^{-1}(dx_1^2+dx_2^2) + H_2^{-1}(dx_3^2+dx_4^2) + dy^m dy^m\right]$$

where $m = 5, \ldots, 10$. The harmonic functions $H_1(y)$ and $H_2(y)$ depend only on the 5 y^m coordinates, which may be considered "overall transverse." The field-strength components supporting this solution are $F_{012m} = -c_1 \partial_m (H_1^{-1})$, $F_{034m} = -c_2 \partial_m (H_2^{-1})$,

where $c_{1,2} = \pm 1$. The surviving supersymmetry parameter ϵ takes the form $\epsilon = (H_1H_2)^{-1/6}\epsilon_0$, where ϵ_0 satisfies the projection conditions $P^{c_1}_{012}\epsilon = P^{c_2}_{034}\epsilon = 0$.

The structure of the metric in this example illustrates what has been called the "<u>harmonic function rule</u>". For each constituent brane, there is a harmonic function H_α that depends only on the overall transverse coordinates. This harmonic function appears in the metric as H_α^{-1} multiplying the directions belonging to the α^{th} brane's worldvolume and also in the overall conformal factor, with a power corresponding to the p-value of the α^{th} component.

In the above example, the two 2-branes share just the time direction. This is described by saying that the two 2-branes "intersect over a point." This may be denoted $2\perp2(0)$.

Here's another example: consider a solution with a 2-brane in the 012 directions and a 5-brane in the 013456 directions. First the supersymmetry check, to see if this possibility has a chance: $[P^{\pm}_{012}, P^{\pm}_{013456}] = 0$, and $\mathrm{tr}(P_{012}P_{013456}) = \frac{1}{4}\cdot 32$, so it looks good for a

1/4 supersymmetry-preserving solution. Here is the solution for this situation, denoted $2\perp5(1)$:

$$ds^2 = H_1^{1/3} H_2^{2/3}\left[H_1^{-1}H_2^{-1}(-dt^2+dx_1^2) + H_1^{-1}(dx_2^2) + H_2^{-1}(dx_3^2+dx_4^2+dx_5^2+dx_6^2) + dy^m dy^m\right]$$

$$F_{012m} = -c_1\partial_m(H_1^{-1}), \quad F_{1mnp} = -c_2\,\epsilon_{mnpq}\partial_q H_2,$$

where $H_1(y)$ and $H_2(y)$ depend, once again, only on the overall transverse directions. The surviving supersymmetry parameter ϵ must satisfy the projection conditions $P^{c_1}_{012}\epsilon = P^{c_2}_{013456}\epsilon = 0$, cutting the surviving supersymmetry down to 1/4 that of flat space. The functional form of the surviving supersymmetry parameter inherits a factor from each of the solution's components: as we have seen, for a membrane one has $\epsilon(y) = H^{-1/6}\epsilon_0$; for a 5-brane the corresponding form is $\epsilon(y) = H^{-1/12}\epsilon_0$ — so for the $2\perp5(1)$ solution, one has $\epsilon(y) = H_1^{-1/6}H_2^{-1/12}\epsilon_0$.

This kind of analysis can be carried out for various intersecting p-brane situations. It is too early to state the general situation — for example, non-orthogonal intersecting p-branes have only recently been found. But for p-branes supported by field strengths corresponding to orthogonal charges, the picture is already fairly clear. In addition to the $2\perp2(0)$ and $2\perp5(1)$ intersections described above, there is a $5\perp5(3)$ intersection, with a functional form following the above pattern of the harmonic function rule. One may also have 3 or more intersecting p-branes. Examples are $2\perp2\perp2(0)$, $2\perp5\perp5(1)$, $5\perp5\perp5(1)$ and $5\perp5\perp5(2)$. In some of these cases, an extra p-brane can be added without breaking supersymmetry further than the 1/8 degree of preservation characteristic of the 3 p-brane intersections. For example, $2\perp2\perp5\perp5(0)$ and $5\perp5\perp5\perp5(1)$ are possible — an orientation of the fourth component existing such that the $\{Q_\alpha\, Q_\beta\}$ projector corresponding to this fourth component has the property that supersymmetry is not broken beyond the $1/8$ th fraction corresponding to the 3-component solution.

Since the intersecting p-brane solutions depend only on the overall transverse dimensions, they are ripe for dimensional reduction on any of the other coordinates — from the "common worldvolume" or any of the "relative transverse" sectors. As one carries out these reductions in the various ways possible, one will encounter situations that have lost their "intersecting" p-brane character, but instead look like arrays of p-branes of a single type. For example, if one reduces the $2 \perp 5(1)$ solution on the 5 relative transverse dimensions, one obtains a solution in $D=6$ with only the $d=2$ common worldvolume and overall transverse dimensions left. This solution still depends on the independent harmonic functions H_1 and H_2. Depending on where one chooses to locate the charge centers determined by H_1 and H_2, the solutions will appear as various kinds of array of $D=6$ strings, geometrically parallel but carrying charges under two orthogonal field strengths in charge space. Since one still has the possibility of arbitrary placement of the charge centers in the Laplace equation solutions H_1 and H_2, these strings will still experience a zero-force condition.

One specific configuration of the $D=6$ reduction of the $2 \perp 5(1)$ solution will be coincident $D=6$ strings, each coupling to a different orthogonal field strength; moreover, the magnitudes of the respective charges may be taken to be the same. Here we meet a solution of the $D=11$ theory that may also be viewed as one of the "single-charge" class that we first encountered in these lectures. The coincident-string solution in $D=6$ may also be viewed as a solution of one of our simplified actions with just a metric, one linear combination of field strengths (in this case, $F_{[3]}$ field strengths) and one combination of dilatonic scalars. In this particular case, the resulting value of the dilaton coupling parameter a happens to vanish, corresponding to $\Delta = 2$, in agreement with $\Delta = 4/N$ for $N=2$ field strengths. This solution preserves 1/4 supersymmetry — not the 1/2 preservation for which the Bogomol'ny bound gives a mass density/charge ratio of 1, but $\mathcal{E}/\mathcal{U} = \frac{2}{\sqrt{\Delta}} = \sqrt{2}$ instead. This ratio fits perfectly with the composition of charges and masses, for the orthogonal charges of the constituent H_1 and H_2 strings combine vectorially, while the masses add in a scalar fashion. Thus, in terms of the charge = mass density $\mathcal{U}$ of the constituent strings, the coincident configuration has $\mathcal{E}_{\text{coincident}} = 2\mathcal{U}$ but the magnitude of the combined charge is $\sqrt{2}\,\mathcal{U}$. Hence $\mathcal{E}_{\text{coincident}}/\mathcal{U}_{\text{coincident}} = \sqrt{2}$, exactly saturating the Bogomol'ny bound for $\Delta = 2$.

The above picture generalises to all of the $\Delta = 4/N$ "single-charge" p-branes with $N>1$ on the brane scan given earlier — they may all be viewed as coincident p-branes carrying orthogonal charges, and subject to a no-force condition because they leave unbroken some portion of the supersymmetry. This surviving portion is determined by the simultaneous application of the projection operators for the various constituents. Since these solutions may be "pulled apart" with zero force into separated constituents, they have been called "bound states at threshold." The individual constituents, that remain irreducible, are then always $\Delta = 4$ solutions.

The picture that emerges from the study of the intersecting p-branes is one with four basic "elements" in the $D=11$ theory: the pp-wave, the membrane and the 5-brane, and the Taub-NUT/Kaluza-Klein monopole solution. All of these "elemental" D=11 solutions need to be understood in a generalised sense with arbitrary harmonic functions $H(y)$ — and should not be restricted to the original isotropic solution that we first considered. This rather classical picture of four basic elements in the spectrum of stable supergravity solutions is one of the very attractive features of D=11 supergravity — the king of supergravity theories.

References

1. Some previous reviews:
- K.S. Stelle, "Lectures on Supergravity p-branes," hep-th/9701088.
- H. Lü and C.N. Pope, "p-brane Taxonomy," hep-th/9702086;
- M.J. Duff, "Supermembranes," hep-th/9611203.
- C.G. Callan, J.A Harvey and A. Strominger, Nucl. Phys. B359 (1991) 611; B367 (1991) 60

2. p-brane ansatz and solution of the field equations; Kaluza-Klein dimensional reduction:
- H. Lü, C.N. Pope, E. Sezgin and K.S. Stelle, "Stainless Super p-branes," Nucl. Phys. B456 (1995) 669;
- H. Lü and C.N. Pope, "An approach to the classification of p-brane solitons," hep-th/9601089.
- H. Lu, C.N. Pope and K.S. Stelle, "An approach to the classification of p-brane Solitons."
- M.J. Duff, P.S. Howe, T. Inami and K.S. Stelle, Phys. Lett. B191 (1987) 70.

3. The origin of the harmonic functions, null geodesics on non-compact σ-models:
- G. Neugebaur and D. Kramer, Ann. der Physik (Leipzig) 24 (1969) 62.
- P. Breitenlohner, D. Maison and G. Gibbons, Comm. Math. Phys. 120 (1988) 253.
- G. Clement and D. Gal'tsov, "Stationary BPS solutions to Dilaton-Axion Gravity," hep-th/9607043

4. The D=11 elementary solutions:
- M.J. Duff and K.S. Stelle, "Multi-membrane solutions of D=11 supergravity," Phys. Lett. B253 (1991) 113;
- R. Güven, "Black p-brane solitons of D=11 Supergravity Theory," Phys. Lett. B276 (1992) 49.

All of the litterature on explicit p-brane solutions uses techniques developed in the original paper on the elementary string solution in D=10 type II supergravity:
- A. Dabholkar, G. Gibbons, J.A. Harvey and F. Ruiz Ruiz, "Superstrings and Solitons," Nucl. Phys. B340 (1990) 33.

5. No-force conditions — this is a direct consequence of the preservation of some unbroken supersymmetry, and so figure prominently in the papers listed in [4]; for application to intersecting p-branes, see
- A.A. Tseytlin, "No-force condition and BPS combinations of p-branes in 11 and 10 dimensions," hep-th/9609212.

6. Intersecting p-branes:
- G. Papadopoulos and P.K. Townsend, "Intersecting M-branes," Phys. Lett. B380 (1996) 273;
- A.A. Tseytlin, Nucl. Phys. B475 (1996) 149;
- J.P. Gauntlett, D.A. Kastor and J. Traschen, Nucl. Phys. B478 (1996) 544;
- J.P. Gauntlett, "Intersecting Branes," hep-th/9705011;

These various intersecting-brane solutions are related by string duality symmetries — for a review, see
- J.G. Russo, "BPS bound states, supermembranes and T-duality in M-theory," hep-th/9703118.

For the description of $\Delta = 4/N$ solutions with $N>1$ as "bound states at threshold," see
- N. Khviengia, Z. Khviengia, H. Lü and C.N. Pope, "Intersecting M-branes and bound states," Phys. Lett. B388 (1996) 21;
- M.J. Duff and J. Rahmfeld, "Bound States of Black Holes and other p-branes," hep-th/9605085