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SUMMER SCHOOL IN HIGH ENERGY PHYSICS AND COSMOLOGY

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ORIENTIFOLDS AND DUALITY

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Please note: These are preliminary notes intended for internal distribution only.

Orientifolds & Duality

Some Useful References:

I) D-branes and modern orientifold construction.

1) J. Polchinski, S. Chaudhuri & C. Johnson
hep-th/9602052

2) J. Polchinski
hep-th/9611050

3) E. Gimon & J. Polchinski
hep-th/9611050.

II) Orbifolds

1) L. Dixon, J. Harvey, C. Vafa, E. Witten.
Nucl. Phys. B261 (85) 678, B274 (86) 285

2) M. Walton
Phys. Rev. D37 (88) 377

III) F-theory, ~~and~~ orientifolds and duality

1) C. Vafa
hep-th/9602022

2) A. Sen
hepth/9605150, hep-th/9609176, hep-th/9604070

3) B. Greene, A. Shapere, C. Vafa, & S.T. Yau
Nucl. Phys. B337 (90) 1

IV) Six-dimensional models, ~~phase transitions~~

1) A. Dabholkar & J. Park
hepth/9602030, hep-th/9604178

2) N. Seiberg & E. Witten
hep-th/9603003

3) M. Berkooz + five more authors.
hep-th/9605184

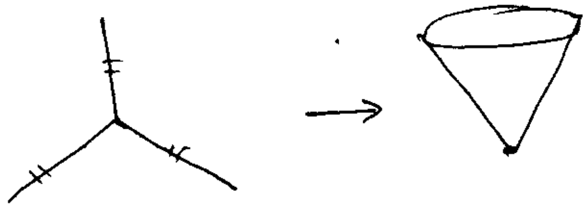
Orientifolds and ~~String~~ Duality

1) Orientifold construction: general remarks
2) F-theory
3) Orientifolds and duality.
4) Some Models
   a) 16 supersymmetries: Type-I, Type-I'
   b) 8 supersymmetries: $D=6$, $N=1$

Orientifolds are a generalization of orbifolds.
Let's first review a few basic facts about orbifolds.
Consider a theory A with a discrete symmetry group G. One can construct a new theory A' = orbifold of A by G

$$A' = A/G.$$

In point particle theory, we simply take the Hilbert space of A and keep only those states that are invariant under G, to obtain the Hilbert space of A'.
In closed string theory we must also add the "twisted sectors": sectors in which the string is closed only upto an action by an element of the group. ~~For example~~:
Let $X(\sigma)$ be the coordinates of the string and let g be some geometric symmetry of space.
For example; consider a plane, g = Rotation by $\frac{2\pi}{3}$.
$G = \{1, g, g^2 = g^{-1}\}$



1) Untwisted sector:

$X(\sigma+2\pi) = X(\sigma)$ Free to move around.

2) Twisted sector, for each element of G

a) $X(\sigma+2\pi) = g\,X(\sigma)$

b) $X(\sigma+2\pi) = g^{-1}X(\sigma)$.

strings closed up to rotation by 120°. Stuck at the <u>fixed points</u> of the symmetry group i.e. the tip of the cone. In general <u>fixed planes</u>

Keep only G-invariant states. This can be achieved by projection

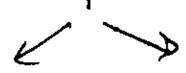
$$\hat{P} = \frac{1}{3}\left(1 + R(g) + R^2(g)\right)$$

<u>Untwisted sectors:</u>

$$|\psi'\rangle = \frac{1}{3}\left(|\psi\rangle + R|\psi\rangle + R^2|\psi\rangle\right)$$

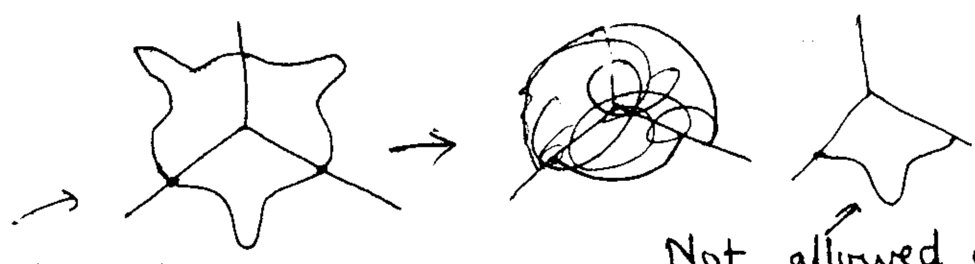
e.g. Let $|\psi\rangle$ be a momentum eigenstate with momentum p

↑ $p = p_1 + ip_2$



$$|p'\rangle = \frac{1}{3}\left(|p\rangle + |e^{\frac{2\pi i}{3}}p\rangle + |e^{-\frac{2\pi i}{3}}p\rangle\right)$$

<u>Twisted sector</u>

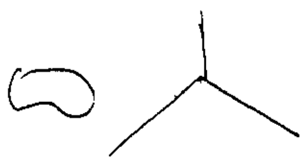


Allowed on the plane ~~both~~ <u>and</u> on the cone.

Not allowed on the plane, but on the cone.

In conformal field theory, there is a well-defined procedure for constructing orbifolds, and the twisted sectors. Twisted sectors are necessary for modular invariance.

Physically: <u>Unitarity requires</u> twisted sectors.
Even if you excluded twisted states at tree level, Once you include interactions, they will appear in loops.



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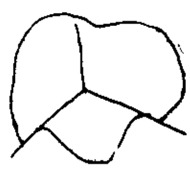


← twisted by g^{-1}

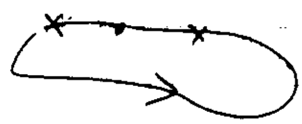
$U \rightarrow T_1 + T_2$

↑ twisted by g

Consistency $\Rightarrow$ twisted sector.

So far we considered a geometric symmetry of space that acts on the spacetime coordinates. Theories of closed, oriented strings have an additional symmetry on the worldsheet, viz. worldsheet parity or orientation reversal.

$\Omega: \quad X(\sigma) \rightarrow X(2\pi - \sigma)$



So in general, we can have a symmetry operation that is now a combination of spacetime symmetry and orientation reversal on the worldsheet.

$$G = (G_1 + \Omega G_2)$$

$A' = A/G$ is now called an "Orientifold"

(1) Untwisted sector can be obtained as before, by keeping states that are invariant under G.

~~correction~~ Typically, starting with oriented ~~string~~ closed strings one gets unoriented ~~str~~ closed strings.

(2) Is there an analog of "twisted sectors" at the fixed planes: ? Does consistency require the a addition of extra sectors as in the case of orbifolds? ~~A~~

A proper understanding of this question has become possible only after the remarkable ~~recent~~ work of Polchinski on D-branes.

Fixed planes of Orientifold group G is called an orientifold plane. Note that it is not an object ~~beast~~ like D-brane but a plane. It has negative RR charge and negative mass.

To cancel the charge requires the addition of D-branes. which introduces the open string sector.

Open string sector in orientifolds are analogous to but not exactly the same as the twisted sectors in orbifolds

Before discussing the details of the orientifold construction let me say a few worlds about ~~why we con~~ what orientifolds are good for:

$$\left[\begin{array}{ll} dH_n = *J_{9-n} & d*H_n = *J_{n-1} \\ \int_{C_k} *J_{10-k} = 0 & \forall \text{ closed curves } C_k. \end{array}\right]$$ consistency requirement.

Motivation:

1) New dualities from old.

$$\begin{tikzcd}
A \text{ on } K_A \arrow[r, leftrightarrow] & B \text{ on } K_B
\end{tikzcd}$$

$$\begin{tikzcd}
A \text{ on } K_A \times M \arrow[r, leftrightarrow] \arrow[d, "G_A"'] & B \text{ on } K_B \times M \arrow[d, "G_B"] \\
A/G_A \arrow[r, leftrightarrow] & B/G_B .
\end{tikzcd}$$

M = smooth manifold.

Under suitable conditions, we can obtain dual pairs with lower supersymmetry using orbifolding & orientifolding. $32 \to 16 \to 8 \to 4$ etc.

a) IIA on K3 $\leftrightarrow$ Het on T^4

b) F on K3 $\leftrightarrow$ Het on T^2

2) New compactifications

Orientifolds give novel string compactifications that were not accessible before.

We'll see some examples in $d = 6$ $N = 1$.

3) Connection with F-theory and its duals.

We'll discuss this after discussing F-theory.

Type-IIB string:

In the lightcone gauge, the group of ~~transverse~~ rotations in the transverse plane is $SO(8) \simeq Spin(8)$ is simply connected covering group.

$8_v \quad X^i \qquad \{\Gamma^i, \Gamma^j\} = 2\delta^{ij}$

$8_s \quad S^a \qquad 2^{d/2} = 2^4 = 16 = 8_s + 8_c$

$8_c \quad S^{\dot a}$

$SO(8)$ triality parity $x_1 \to -x_1$: $8_s \to 8_c$

additional automorphism of the weight lattice $\quad 8_s \leftrightarrow 8_v$, $8_c \leftrightarrow 8_s$.

$\Gamma^i \in 8_v$ gives $8_s \oplus 8_c \qquad \{\Gamma^i, \Gamma^j\} = 2\delta^{ij}$

$\Gamma^a \in 8_s$ gives $8_v \oplus 8_c \qquad \{\Gamma^a, \Gamma^b\} = 2\delta^{ab}$

$\Gamma^{\dot a} \in 8_c$ gives $8_s \oplus 8_v$. $\{\Gamma^{\dot a}, \Gamma^{\dot b}\} = 2\delta^{\dot a \dot b}$

$$S_{l.c.} = \frac{-1}{2\pi}\int d^2\sigma \; \partial_a X^i \partial^a X^i - i S^a_- \partial_+ S^a_- - i \tilde{S}^a_+ \partial_- \tilde{S}^a_+$$

$\alpha' = \frac{1}{2}$

Both left-moving and right-moving spinors transform as 8_s gives IIB

S^a	$\tilde{S}^a$	IIB	spacetime chiral	worldsheet nonchiral
S^a	$\tilde{S}^{\dot a}$	IIA ·	spacetime nonchiral	worldsheet chiral.

~~Pertur~~ Spectrum

$S^a(\sigma,\tau) = \sum S^a_n e^{-in(\tau-\sigma)}$ etc.

$\{S_0^a\ S_0^b\} = \delta^{ab}$ gives $(8_v \oplus 8_c)$

from both right-movers & left movers:

$(|i\rangle \oplus |\dot{a}\rangle) \otimes (|j\rangle \oplus |\dot{b}\rangle)$

Bosons: $|i\rangle \otimes |j\rangle$

Irreps: g_{ij}, B_{ij}, ϕ metric 2-form, dilaton.

NS–NS sector

$|\dot{a}\rangle \otimes |\dot{b}\rangle$ $\lambda_1^{\dot{a}} \lambda_2^{\dot{b}} \sim \bar{\lambda}_1\lambda_2 \quad \bar{\lambda}_1\Gamma^{ij}\lambda_2$

$\bar{\lambda}_1 \Gamma^{ijkl}\lambda_2$

Irreps D_{ijkl}, B'_{ij}, a $\quad {}^*D = D$ self dual.

R–R sector

Perturbative Symmetries:

1) Ω = Orientation reversal interchange left-right. $\quad S^a \leftrightarrow \tilde{S}^a \quad |\dot{a}\rangle \leftrightarrow |\dot{b}\rangle$

$|i\rangle \leftrightarrow |j\rangle$

NSNS sector g_{ij}, ϕ even

B_{ij} odd.

RR sector: D_{ijkl} & a odd

B'_{ij} even.

2) $(-1)^{F_L}$ $\quad \tilde{S}^a \to -\tilde{S}^a$

NSNS sector is even $\quad g, B, \phi$ even

RR sector is odd. $\quad D, B', a$ odd.

Nonperturbative symmetry

$SL(2,\mathbb{Z})$ $\qquad\qquad e^{\phi} = \lambda =$

$\lambda = \quad a + i e^{-\phi}$ $\qquad \begin{pmatrix} B \\ B' \end{pmatrix}$ $\qquad \begin{array}{l} D \to D \\ g \to g \end{array}$

$\lambda \to \dfrac{p\lambda + q}{r\lambda + s}$ $\qquad \begin{pmatrix} B \\ B' \end{pmatrix} \to \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} B \\ B' \end{pmatrix}$

$pq - rs = 1$ all integers.

$T: \quad \lambda \to \lambda + 1$ $\qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$S: \quad \lambda \to -1/\lambda$ $\qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$R: \quad \lambda \to \lambda$ $\qquad \begin{pmatrix} B \\ B' \end{pmatrix} \to \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} B \\ B' \end{pmatrix}$

~~Fact~~: Useful observations:

① R is a perturbative symmetry

$R = (-1)^{F_L}\, \Omega$ $\qquad (B, B')$ odd

everything else even

Check, from massless spectrum.

② $S (-1)^{F_L} S^{-1} = \Omega$

$\phi, a\ g, D$ have same action under $(-1)^{F_L}$ & Ω

$B\ B'$ have opposite action.

This will be important later when we discuss F-theory.

Example of an Orbifold: $\mathbb{Z}_2$ orbifold.

$$\mathrm{IIB}/\{1, (-1)^{F_L}\}$$

Untwisted

Removes all RR fields and half the fermions

left with g_{ij}, B_{ij}, ϕ | $(|i\rangle \oplus |\dot{a}\rangle) \otimes |j\rangle$

twisted

$\tilde{S}^a(\sigma + 2\pi) = -\tilde{S}^a(\sigma)$ } half-integer moded.

$$\left.\begin{aligned} S^a(\sigma+2\pi) &= S^a(\sigma) \\ X^i(\sigma+2\pi) &= X^i(\sigma) \end{aligned}\right\} \text{integer moded as before}$$

$\cancel{|i\rangle\otimes}$ $\{S_0^a, S_0^b\} = \delta^{ab}$ gives $|i\rangle \oplus |\dot{a}\rangle$

$\cancel{|0\rangle}$ tachyonic odd under $(-1)^{F_L}$

$\tilde{S}^b_{-\frac{1}{2}}|0\rangle \to a$ so we set $(|i\rangle \oplus |\dot{a}\rangle) \otimes |b\rangle$

This is actually like the NS sector of RNS string

Thus, we have obtained Type-IIA.

$$\boxed{\mathrm{IIA} = \text{orbifold of IIB}}$$

How about the orientifold?

$$\mathrm{IIB}/\{1, \Omega\} = \text{Type-I}.$$

To repeat: (If necessary)

spectrum

IIB: S^a right moving spinor $\{S_0^a, S_0^b\} = \delta^{ab}$ $\Rightarrow$ $(8_v \oplus 8_c)$

$\tilde{S}^a$ left-moving spinor $\Rightarrow$ $(8_v \oplus 8_c)$

Spectrum: $(8_v \oplus 8_c)_R \otimes (8_v \oplus 8_c)_L$

IIA S^a right-moving spinor $(8_v \oplus 8_c)$

$\tilde{S}^{\dot{a}}$ left-moving conjugate spinor $(8_v \oplus 8_s)$

Spectrum $(8_v \oplus 8_c) \otimes (8_v \oplus 8_s)$

Orbifold

From IIB $(-1)^{F_L}$ projects out ~~$8_c \otimes 8_v$~~

$(8_v \oplus 8_c)_R \otimes (8_c)_L$ because 8_c is odd. leaving $(8_v \oplus 8_c) \otimes 8_v$

From the twisted sector, we get

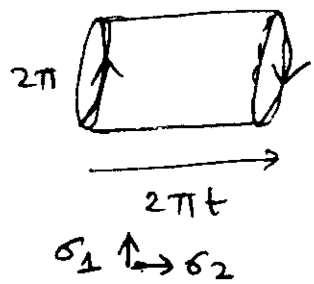
$(8_v \oplus 8_c)_R \otimes 8_s$.

Altogether $(8_v \oplus 8_c) \otimes (8_v \oplus 8_s)$ giving us IIA.

Closed string sector:

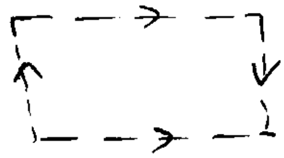
g, B', ϕ are even under Ω and thus survive the orientifold projection $\frac{1+\Omega}{2}$. To see which states survive we can look at one-loop partition function

$$\mathrm{Tr}_c \frac{(1+\Omega)}{2} \exp^{-2\pi t (L_0 + \tilde{L}_0)}$$



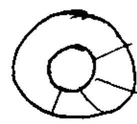
$\mathrm{Tr}\, \frac{1}{2}$ torus

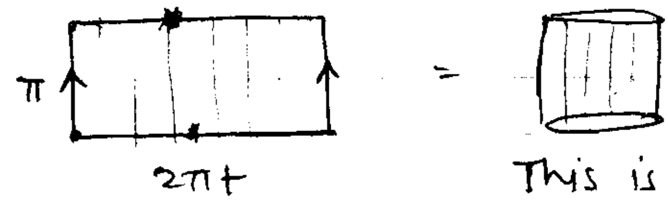
$\mathrm{Tr}\, \frac{\Omega}{2}$: klein bottle



Here we are time slicing in the σ_2 direction to get the loop channel. If we time slice in the σ_1 direction, we'll see a single closed string propagating in the tree-channel.

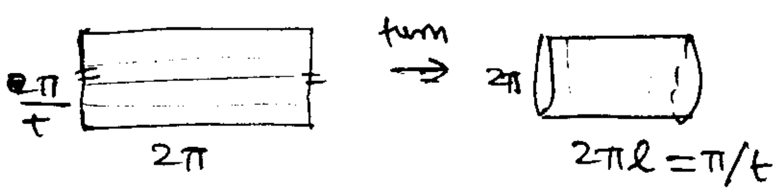
Simpler case to consider is cylinder.





This is open-string loop channel.

but by rescaling, we can view it as,



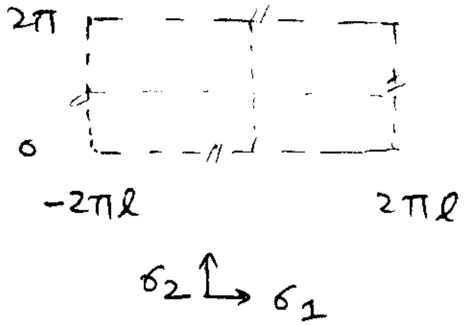
$l = \frac{1}{2t}$ $\boxed{t = \frac{1}{2l}}$

Closed-string tree channel

s-t duality
= " old duality "

For the klein bottle, to see this, we consider the double cover of the bottle namely a torus.

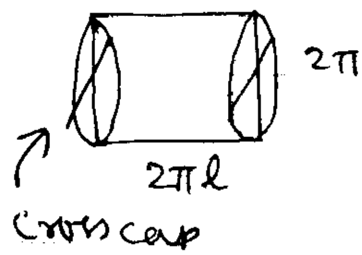
$$K\cdot B = \frac{\text{Torus}}{Z_2}$$

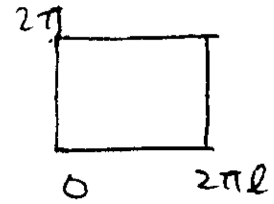


we have a further identification

$$(\sigma_1, \sigma_2) = (-\sigma_1, \sigma_2 + \pi)$$

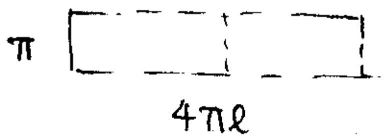
We can take two different fundamental regions.





Tree-channel

or



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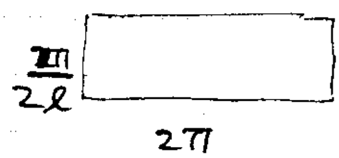
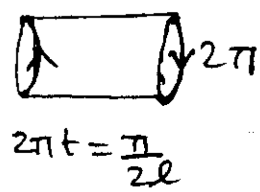


$\overset{\text{turn}}{=}$



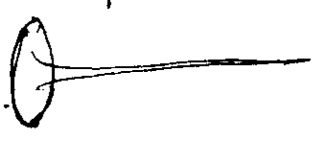
$2\pi t = \frac{\pi}{2l}$

$$\boxed{t = \frac{1}{4l}}$$

= loop channel

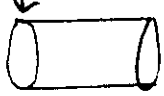
$l \to \infty$ limit gives the tadpoles. $(t \to 0)$.
This is similar to the calculation of the D-brane tension. One should really compute for a D-brane

x = graviton or RR-field vertex operator.



disc stuck at the D-brane.

But instead we compute



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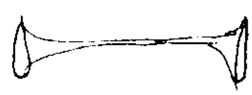
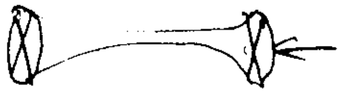


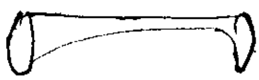
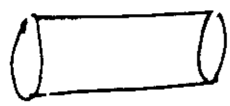
which is much simpler: without any vertex operator, to read off the tadpole.
Here ~~to form~~ the klein bottle, one gets,



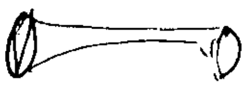
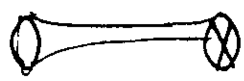
Crosscap stuck at the orientifold plane.

From here one obtains, that the cross cap has charge -32 in D-brane units, i.e. the K.B amplitute goes as $(32)^2$.

We can add N-D branes, and there will be D-brane – D-brane terms from the cylinder

 $\sim N^2$  cylinder

D-brane - Orientifold plane

 or   Mobius strip.

$\sim -2\cdot 32\, N$.

$\sim (-32+N)^2$

$\Rightarrow$ Tadpole will be cancelled if we put exactly 32 9-branes.

Note: the crosscap is localized near one of the orientifold fixed planes.

Remark: Euler character with boundaries (b) & crosscaps (c) and handles (h)

$$\chi = 2-2h-b-c$$

 $\mathbb{RP}2$  disk  KB  Mobius strip etc.

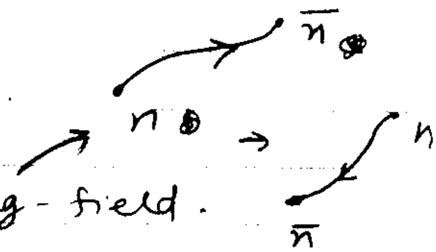
Let us work out one example explicitly.
$SO(32)$ Type-I string as an orientifold of IIB.
I'll illustrate how to get the open-string sector i.e. the number of D-branes.
The orientifold plane is a 9-plane. The klein-bottle therefore has potential tadpoles, because the orientifold plane would be charged w.r.t. a 10-form potential. So let's add N 9-branes.

D-branes and Chan-Paton factors:

$p+1$ longitudinal coordinates Neumann.
$10-(p+1)$ transverse coordinates Dirichlet.
D-brane label $(i,j) \in (1 \cdots N)$ Chan-Paton index.

Denote a general open-string state by.
$|\Psi\ ij\rangle \lambda_{ij}$ $\quad \Psi =$ state of worldsheet oscillators.
$\qquad\qquad\qquad ij =$ chan paton label

$\lambda_{ij} =$ Chan-Paton wave function.



$\lambda^{+} = \lambda$ by reality of the string-field.
$\qquad$ = Lie algebra of $U(n)$ theory.

How does the symmetry-group G act on the D-brane sector.

$$g: \ |\Psi\ ij\rangle \lambda_{ij} \to |g\cdot\Psi\ ij\rangle \lambda'_{ij} \qquad \lambda \to \gamma_g^{-1} \lambda \gamma_g$$

$$\Omega h \ |\Psi\ ij\rangle \lambda_{ij} \longrightarrow |\Omega h \cdot \Psi\ i'j'\rangle \lambda'_{ij} \qquad \lambda \to \gamma_{\Omega h}^{-1} \lambda^{T} \gamma_{\Omega h}$$

In order to gauge the group G, it should furnish a representation (and not just a projective ~~repre~~representation) in the open-string sector.

$$g^2 : \quad \lambda \to (\gamma_g^2)^{-1} \lambda (\gamma_g^2)^{-1}$$

$$(\Omega h)^2 \quad \lambda \to (\gamma_{\Omega h}^{-1} \gamma_{\Omega h}^T) \lambda (\gamma_{\Omega h}^{-1} \gamma_{\Omega h}^T)^{-1}$$

$$\Rightarrow \gamma_g^2 = \pm 1 \qquad \Big| \qquad \gamma_{\Omega h}^{-1} \gamma_{\Omega h}^T = \pm 1$$

$$\text{i.e} \quad \boxed{\gamma_{\Omega h}^T = \pm \gamma_{\Omega h}}$$

<u>Consider the simplest case</u> $G = \{1, \Omega\}$ i.e. Type-I as the IIB orientifold.

I'll now briefly illustrate the salient features of the tadpole calculation.

Calculate $\frac{1}{2} \mathrm{Tr}_{NSNS+RR} \, \Omega \, \frac{(1+(-1)^f)}{2} \, e^{-2\pi t (L_0 + \tilde{L}_0)}$ KB

$\frac{1}{2} \mathrm{Tr}_{NS-R} \, \frac{(1+(-1)^f)}{2} \, e^{-2\pi t L_0}$ cylinder

$\frac{1}{2} \mathrm{Tr}_{NS-R} \, \Omega \left(\frac{1+(-1)^f}{2}\right) e^{-2\pi t L_0}$ Möbius

Traces i.e. loop channels. Then factorize ~~Factor~~ in the tree channel.

$$\left.\begin{aligned} L_0 &= \sum \alpha_{-n}^i \alpha_{-n}^i + p^2/8 \\ \tilde{L}_0 &= \sum \tilde{\alpha}_{-n}^i \tilde{\alpha}_{-n}^i + p^2/8 \end{aligned}\right\} \text{closed.} \qquad \begin{aligned} &+ \sum \psi_{-r}^i \psi_r^i \\ &+ \sum \tilde{\psi}_{-r}^i \tilde{\psi}_r^i \end{aligned}$$

$$L_0 = \sum \alpha^i_{-n}\alpha^i_{+n} + \sum \psi^i_{-r}\psi^i_{-r} + \frac{p^2}{2}\cdot 2\alpha' = \alpha' p^2 + N$$

Normal ordering constant is $-\frac{1}{2}$ in NS sector, 0 in R sector.

Let's see now that the charge of the orientifold plane is 32 in units of D-brane charge.

consider the cylinder:

 $$= \int_0^\infty \frac{dt}{t}\, \mathrm{Tr}_{NS-R} \left(\frac{1+(-1)^F}{2}\right) e^{-2\pi t\alpha' L_0}$$

This one-loop cosmological constant = sum of vacuum energie

$$\sum_{\text{Bosons}} \frac{\hbar\omega}{2} - \sum_{\text{Fermions}} \frac{\hbar\omega}{2} = \sum \frac{1}{2}\,\mathrm{Tr}\log(p^2+m_i^2)(-1)^{F_i}$$

$$= \frac{1}{2}\int_0^\infty \frac{dt}{t} \sum_{\substack{i\\ NS-R}} \exp^{-2\pi t(\alpha'(p^2+m_i^2))} \left(\frac{1+(-1)^F}{2}\right). \qquad \boxed{m_i^2 = \frac{1}{\alpha'}N}$$

Let's calculate $\mathrm{Tr}(-1)^F$ = periodic boundary condition in the time direction = RR exchange in the tree-channel. $\mathrm{Tr}_R(-1)^F = 0$ because of fermion zero mode.

$$\mathrm{Tr}_{NS}\, e^{-2\pi t N}(-1)^F \sim \frac{\prod_n \left(1 - e^{-2\pi t(n+\frac{1}{2})}\right)^8}{\prod \left(1 - e^{-2\pi t n}\right)^8}$$

$$\sim \frac{f_4^8(e^{-\pi t})}{f_1^8(e^{-\pi t})}$$

We also have, the momentum integral

$$\int e^{-2\pi t\alpha' p^2}\, d^{10}p.$$

$$f_1(q) = q^{\frac{1}{12}} \prod_{n=1}^{\infty} (1-q^{2n}) \qquad f_4(q) = q^{-\frac{1}{24}} \prod_1^{\infty} (1-q^{2n-1})$$

$$f_2(q) = \sqrt{2}\, q^{\frac{1}{12}} \prod (1+q^{2n}) \qquad f_3(q) = q^{-\frac{1}{24}} \prod (1+q^{2n-1})$$

$$f_3^8(q) = f_2^8(q) + f_4^8(q) \qquad \text{Jacobi identity.}$$

$$f_1(e^{-\pi s}) = \frac{1}{\sqrt{s}} f_1(e^{-\pi/s})$$

$$f_4(e^{-\pi s}) = f_2(e^{-\pi/s})$$

Let's look at the klein-bottle RR exchange

$$\int \frac{dt}{t} \operatorname{Tr}_{NSNS+RR} (-1)^f e^{-2\pi t(L_0+\tilde{L_0})}\, \Omega$$

NS-R and R-NS go to each other under Ω, hence do not contribute to the trace.

$$\sim \frac{f_4^8(e^{-2\pi t})}{f_1^8(e^{-2\pi t})}$$

The momentum integral is $\int e^{-\pi\alpha' t p^2}\, dp$

KB $\sim 2^5$ cylinder because of momentum integration.

$\sim (\sqrt{2})^8$ in going from $t \to \frac{1}{t}$

$s = 2t$ for KB but t for C.

~ 2 in going from $\frac{1}{t}$ to l.

Parenthetical Remarks

We shall go between NSR and GS formalism frequently. (In the light cone gauge)

Green-Schwarz

$X^i \quad S_+^a \quad \tilde{S}_-^a$ IIB

$S_+^a \quad \tilde{S}_-^{\dot a}$ IIA.

$(S^a, \tilde{S}^a)$ spacetime spinors and worldsheet spinors.

consider only right-movers,

$8v \oplus 8c$ from representations of the zero modes. X^i and S^a $\tilde{S}^a$ all satisfy periodic boundary condition.

Ramond-Neveu-Schwarz

$X^i \quad \psi_+^i \quad \tilde{\psi}_+^i$ For both IIA and IIB.

$(\psi^i, \tilde{\psi}^i)$: spacetime vectors and worldsheet spinors.

Now there are two sectors ~~again~~

ψ^i periodic Ramond sector

ψ^i antiperiodic Neveu schwarz.

$\psi^i(0) = \pm \psi^i(2\pi)$

ψ^i integer moded in Ramond sector ψ_0^i, ψ_1^i

half integer moded in NS sector $\psi_{\frac{1}{2}}^i \quad \psi_{-\frac{1}{2}}^i$

<u>Open strings</u>

$\partial_+ X|_0 = \pm \partial_- X|_0$ Neuman or Dirichlet.

$\partial_+ X = \partial_- X \Rightarrow \frac{\partial X}{\partial \sigma}\Big|_0 = 0 \quad X' = 0$

$\partial_+ X|_\pi = \pm \partial_- X|_\pi$ standing wave:

$\tilde{\psi}_+|_\pi = \pm \tilde{\psi}_-|_\pi$ + by convention.

$\tilde{\psi}_+|_0 = \pm \psi_-|_0$ $+$ = Ramond

$-$ = NS.

Doubling trick: Instead of considering left-moving and right-moving fermions that are related at the boundary, we can consider say just right-moving fermion that goes from 0 to π $\tilde{\psi}(\sigma^1, \sigma^2) = \psi(2\pi - \sigma^1, \sigma^2)$

These right-moving fields are periodic (or antiperiodic) in R (NS) sector.

$$\tilde{\psi}(\pi) = \psi(\pi)$$
$$\tilde{\psi}(0) = \psi(2\pi) = \pm \psi(0)$$

<u>Ramond sector</u>: Sector in which the worldsheet supercurrent is periodic.

In the Ramond sector, bose-fermi vacuum energy cancels on the worldsheet. The ground state is representation of Clifford algebra

$\{\psi_0^i, \psi_0^j\} = \delta^{ij}$ $\qquad (8_s \oplus 8_c)$

In the NS sector, ~~we~~ ~~see~~ the total vacuum energy is $-\frac{1}{2}$ so there is a Tachyon.

			GSO
$\lvert 0\rangle_{NS}$	$-\frac{1}{2}$	scalar	-1
$\psi^i_{-\frac{1}{2}} \lvert 0\rangle_{NS}$	0	8_v.	$+1$

GSO projection: $(-1)^{f_L}$ is a symmetry

Note f_L, f_R are <u>worldsheet</u> fermion number

So use $\frac{1+(-1)^{f_L}}{2}$ to project out the tachyon from NS sector and 8_s from the R sector.

We have two choices for the left-movers now project$_R$ out either$_L$ 8_s or 8_c, giving us

$(8_v \oplus 8_c) \otimes (8_v \oplus 8_c)$ $\qquad$ IIB

$(8_v \oplus 8_c) \otimes (8_v \oplus 8_s)$ $\qquad$ IIA.

~~Tad~~ Tadpole

$$(32)^2 + N^2 \mp 32\cdot 2\cdot N$$

$\downarrow$ KB $\quad$ $\downarrow$ Cylinder $\quad$ $\downarrow$ Mobius strip.

Mobius strip: Interaction between the orientifold plane and the D-brane.

$\mp 32\cdot 2\cdot N$ for $\gamma_\Omega^T = \pm\gamma_\Omega$

Tadpole cancels if $\gamma_\Omega^T = \gamma_\Omega$ and $N=32$.

Lets look at the open string sector.

~~Tach~~ Tachyon projected out by GSO.

$$\psi^i_{-\frac{1}{2}}|0\ ij\rangle\lambda_{ij} \qquad \Omega:\ \lambda = -\gamma_\Omega^{-1}\lambda^T\gamma_\Omega$$

~~eigenvalues~~ ~~to~~

$\gamma_\Omega^T = \gamma_\Omega$

Under a unitary change of basis

$\gamma_\Omega \to U\gamma_\Omega U^T$ so we can make it $+\mathbb{1}$

which gives us $(\lambda = -\lambda^T)$ $\quad \lambda^+ = \lambda$

$SO(N)$ subgroup of $U(N)$

$\Rightarrow SO(32)$.

Application of orientifolds

I) Fiberwise application of duality.

Consider A compactified on K_A that is dual to B compactified on K_B.

A on $K_A \times R_{n,1}$
B on $K_B \times R_{n,1}$

Then consider A and B respectively on E_A and E_B which are obtained by fibering K_A and K_B over M

Then, A on E_A is dual to B on E_B.
This follows from the adiabatic argument of Vafa & Witten. Locally, E_A looks like $K_A \times R_{n,1}$ as long as the moduli of K_A vary slowly as we move on M.
The argument breaks down if the moduli vary rapidly, especially near singular points on M where the fiber degenerates.
But as long as the number of singular points is of "measure zero" the duality will work, even at the singular points, forced by the duality in the bulk.

Special case: orbifolds ($\mathbb{Z}_2$)

~~Take~~

Take a smooth M with some discrete symmetry $\{1, s\}$. If K_A and K_B have some $\mathbb{Z}_2$ symmetry $\{1, h_A\}$ and $\{1, h_B\}$, then take

$$\frac{K_A \times M}{\{1, s h_A\}} \quad \xleftrightarrow{\text{dual}} \quad \frac{K_B \times M}{\{1, s h_B\}}$$

A B

$\parallel$

A compactified on E_A ~~where~~ with base M/s, fiber K_A, and a twist h_A on the fiber as we go around	B compactified on E_B with base M/s, fiber K_B, and twist h_B.

<u>Example</u>: Derivation of duality between Type IIA on K3 surface $\Longleftrightarrow$ Heterotic on T_4 from Type-IIB $SL(2,\mathbb{Z})$ duality and the duality between Type-I and Heterotic $SO(32)$

~~Take~~

$$\begin{tikzcd}
\mathrm{IIB} \arrow[r, leftrightarrow, "S"] \arrow[d, "T_4"] & \mathrm{IIB} \arrow[d, dash, "T_4"] \\
\mathrm{IIB}/\{1, (-1)^{F_L}\} & \mathrm{IIB}/\{1, \Omega\}
\end{tikzcd}$$

IIA on $T^4/\mathbb{Z}_2$ Type-I on T_4.

$M = T_4$ Base $T^4/\mathbb{Z}_2$

X_6, X_7, X_8, X_9 are the coordinates of the torus

I_{6789}: $(X_6, X_7, X_8, X_9) \rightarrow (-X_6, -X_7, -X_8, -X_9)$

This is also accompanied by

$(-1)^{F_L}$ in A and Ω in B.

$$I_4 (-1)^{F_L} \qquad\qquad \Omega\, I_4.$$

$$\downarrow \text{T dualize } X_9 \qquad\qquad \downarrow \text{T dualize } (6,7,8,9)$$

$$I_4 \text{ in IIA} \qquad\qquad \Omega \text{ in IIB}$$

T-duality in Type II

T-duality is a one-sided parity transform.

$$\left.\begin{array}{l} \partial_+ X \rightarrow -\partial_+ X \\ \partial_- X \rightarrow \partial_- X \end{array}\right\}$$ This is T-duality takes momentum modes to winding modes

$dX \rightarrow *dX \sim F \rightarrow *F$

$X_L^9 \rightarrow -X_L^9$

$X_R^9 \rightarrow X_R^9$

For fermions, then we have:

$\tilde{S}^a \rightarrow \Gamma\Gamma^9 \tilde{S}^a$

$S^a \rightarrow \tilde{S}^a$

Now $\Gamma\Gamma^9 \tilde{S}^a$ transforms as a conjugate spinor, i.e. $\tilde{S}^{\dot{a}}$

we thus get $\{S^a \,\&\, \tilde{S}^{\dot{a}}\}$ or Type IIA

$$\text{Type IIB} \xleftrightarrow{T} \text{Type IIA}.$$

Let us see what happens to various symmetries of Type II theory under T-duality.

1) Ω in IIB $\xrightarrow{T_9}$ $I_9 \Omega$ in IIA

i.e. $T_9 \Omega T_9^{-1} = I_9 \Omega$

Basically, $(X_L^9 \; X_R^9) \xrightarrow{T_9^{-1}} (-X_L^9, X_R^9) \xrightarrow{\Omega} (X_R^9, -X_L^9)$

$$I_9 \Omega (X_L^9, X_R^9) = (-X_R^9, -X_L^9) \xleftarrow{T_9}$$

Note that Ω alone is ~~not~~ not a symmetry in IIA it takes S^a which is right-moving to $\tilde{S}^{\dot a}$ which is right-moving with flipped chiral

$(S^a, \tilde{S}^{\dot a}) \xrightarrow{\Omega} (\tilde{S}^{\dot a}, S^a) \xrightarrow{I_9} (S^a \; \tilde{S}^{\dot a})$

we need I_9 to flip the chirality again

so ΩI_9 is a symmetry of IIA.

2) Ω in IIB $\xrightarrow{T_{89}}$ $I_{89} \Omega (-1)^{F_L}$ in IIB

The additional $(-1)^{F_L}$ comes because

$I_{89}^2 = (-1)^F$. $I_{89}^2 =$ rotation by 2π which ~~flips the fermions~~ changes the sign of fermions.

$I_{89} \Omega (-1)^{F_L} I_{89} \Omega (-1)^{F_L} = \Omega (-1)^{F_L} \Omega (-1)^{F_R} = \Omega^2 = 1$

3) Ω in IIB $\xrightarrow{T_{6789}}$ ΩI_{6789} in IIB.

4) $(-1)^{F_L} I_{6789} \xrightarrow{T_9} I_{6789}$
in IIB in IIA.

All this can be checked on the worldsheet or more easily by the action on the massless fields. One important thing to remember is that under T-duality a p brane goes to a (p+1) brane or a (p−1)-brane

$$p\text{-brane} \xrightarrow[\text{along the brane}]{\text{T duality}} (p-1) \text{ brane}$$

$$p\text{-brane} \xrightarrow[\text{transverse to the brane}]{\text{T-duality}} (p+1)\text{-brane.}$$

This is because T-duality is a one-sided parity transform. so $\partial_+ X \to -\partial_+ X$ $\partial_- X \to \partial_- X$ which changes Dirichlet and Neuman conditions which relate $\partial_+ X$ at the boundary to $\pm\partial_- X$ at the boundary. In terms of T-duality of potentials that couple to the brane, this means that you
remove an index if it exist | $D_{\mu\nu\lambda 9} \to C_{\mu\nu\lambda}$
add an index if does not exist | $B_{\mu\nu} \to C_{\mu\nu 9}$

For example $D_{\mu\nu\lambda 9} \xrightarrow{I_{6789}} -D_{\mu\nu\lambda 9} \xrightarrow{(-1)^{F_L}} D_{\mu\nu\lambda 9}$, $B_{\mu\nu} \to -B_{\mu\nu}$
which is the same action as I_{6789} in IIA.

$C_{\mu\nu\lambda} \to C_{\mu\nu\lambda}$

$C_{\mu\nu 9} \to -C_{\mu\nu 9}$

We thus obtain, from $SL(2,\mathbb{Z})$ duality of Type IIB, and T-duality, using the adiabatic argument that,

$$\text{Type IIA on } T^4/\mathbb{Z}_2 \overset{\text{dual}}{\sim} \text{Type IIB on } T^4/\mathbb{Z}_2$$

$\mathbb{Z}_2 = \{1, I_{6789}\}$ $\qquad$ $\mathbb{Z}_2 \{1, \Omega\}$

$=$ orbifold $= K3$ $\qquad$ $=$ orientifold

IIA on K3 $\qquad$ Type I on $T4$

$\sim$ Heterotic on $T4$

with $SO(32)$ gauge group.

We thus obtain the famous string-string duality in six dimensions.

Some facts about K3

"Kummer's third surface" is ubiquitous in the duality literature. So it is useful to recall some of its properties.

It is 4 real dimensional or 2-complex dimensional manifold.

(x_1, x_2, x_3, x_4) or z_1 z_2

$z_1 = x_1 + i x_2$ $z_2 = x_3 + i x_3$ locall

It satisfies Einstein's equations, $R_{ij} = 0$.

It has $SU(2)$ holonomy. To understand what this means, consider a generic 4d real manifold. If you take a vector in the tangent plane at point p, parallel transport it and come back to point p, then, in general, it will be rotated by an $SO(4)$ matrix, i.e. it has $SO(4)$ holono

$$V_i(p) \longrightarrow R_{ij} V_j(p) \qquad R_{ij} \in SO(4)$$

In the case of K3, the holonomy is a subgroup of $SO(4)$, namely $SU(2)$. The smaller the holonomy group, it means more "symmetric" the space.

For example, for a torus, the holonomy group is just identity, which means no rotation at all, because no curvature.

For a K3, there <u>is</u> curvature i.e. the Riemann tensor is nonvanishing, but it is ~~still~~ not completely arbitrary; Ricci tensor $R_{ij} = 0$.

Only other thing that we need to know about K3, is the topological information:

K3 has 22 nontrivial 2-cycles,

so $b_0 = 1$ $\quad b_2 = 22$ $\quad b_4 = 1$ and all other Betti numbers are zero. $b_1 = b_3 = 0$.

Recall, that the number of nontrivial p-cycles equals the number of harmonic p-form.

A harmonic p-form satisfies the analog of Laplace's equation for forms, or equivalently

$dH = 0 \quad d^+ H = 0.$

~~Every~~ A manifold always has a harmonic 0-form ie namel a constant and a harmonic 4-form namely the volume form $dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \sqrt{g}$. , $b_0 = b_4 = 1$

K3 has 22 harmonic 2-forms and no harmonic 1 and 3-forms.

Basically, this is all the information one needs.

Let us see this explicitly for $T^4/\mathbb{Z}_2$

$SO(4) = SU(2)_L \times SU(2)_R$ familiar from $SO(1,3)$ representations

$4 = (2,2)$

~~The~~ consider $SU(2)_L \times U(1)_R$

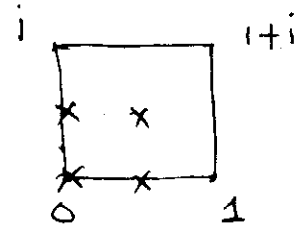
Then, $4 = 2_+ \oplus \bar{2}_-$ $\quad SU(2)$ $2 \sim \bar{2}$ pseudoreal

$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ = doublet of $SU(2)$ $\in 2_+$

$\begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix} \in \bar{2}_-$

$\mathbb{Z}/2: \quad (z_1, z_2) \to (-z_1, -z_2)$

$z_1 \sim z_1 + 1 \sim z_1 + i$



There are 4 fixed points of $z_1 \to -z_1$

Altogether for K3 we have 16 fixed points. Let's look at the cohomology of K3,
First consider T4
harmonic forms

1	1
4	$dx^i \quad i = 1 \ldots 4$
6	$dx^i \wedge dx^j$
4	$dx^i \wedge dx^j \wedge dx^k$
1	$dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$

Under the reflection, only the even-forms survive.

0-form	1		1
1	4		0
2	6	$\rightarrow$	6 + 16 = 22
3	4		0
4	1		1.

In addition, from each fixed point we get one 2-form (harmonic), altogether 16, giving us the cohomology of K3.

It obviously has SU(2) holonomy. ~~Around a fixed~~ Away from the fixed point, a parallelly transported vector goes back to itself, because all the curvature is concentrated at the fixed points.

~~Notice~~ ~~to~~. One more interesting property is that out of the 22 form 19 are antiselfdual and 3 are selfs dual. Basically,

$$dx^i \wedge dx^j = \pm \frac{1}{2} \epsilon^{ijkl}\, dx^k \wedge dx^l$$ so from

~~twisted~~ untwisted sector $6 = 3_a + 3_s$.

$$dx^1 \wedge dx^2 \pm dx^3 \wedge dx^4 \quad \text{etc.}$$

<u>All</u> 2-forms coming from the twisted sector are anti-self-dual, giving us $3_s + 19_a$.

~~TODS bosons to~~

Using this information, one can check explicitly, that the massless spectrum of Type-IIA on K3 matches with heterotic on T^4.

Heterotic on T^4 has 24 ~~vector scalars~~ (1-form that transform as a vector of $SO(4,20)$

$$\begin{array}{cccc} g_{\mu 6} & B_{\mu 6} & A_\mu^a & a \in so(32) \\ 7 & 7 & & \text{or } E_8 \times E_8 \to U(1)^{16} \\ 8 & 8 & & \\ 9 & 9 & & \\ 4 \quad + & 4 & 16 & = 24. \end{array}$$

IIA on K3: There are no odd cycles so we cannot have the analog of $g_{\mu 6}$ etc. All 1-forms come from RR sector.

$$\begin{array}{ll} A_\mu & 1 \\ C_{\mu ij}\ f^{a\,ij} & 19 + 3 \\ \tilde{A}_{\mu 6789} & 1 \\ \hline & 24 \Rightarrow \text{transform as } SO(4,20) \end{array}$$

Exercise check that you get the right number of scalars ~~to~~ on Narain moduli space

$$SO(4,20,\mathbb{Z}) \backslash SO(4,20,\mathbb{R}) / SO(4,\mathbb{R}) \times SO(22,\mathbb{R}).$$