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BACKGROUND MATERIAL FOR LECTURE ON
"Spin diffusion in polarized fermi gases and solids"

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Spin Diffusion in Dilute, Polarized ^{3}He-^{4}He Solutions

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Spin dynamics for arbitrarily polarized and very dilute solutions of ^{3}*He in liquid* ^{4}*He are described. We began at a very fundamental level by deriving a kinetic equation for arbitrarily polarized dilute quantum systems based on a method due to Boercker and Dufty. This approach allows more controlled approximations than our previous derivation based on the Kadanoff–Baym technique. Our previous work is here generalized to include* T*-matrix interactions rather than the Born approximation. Spin hydrodynamic equations are derived. The general equations are valid for both Fermi and Bose systems. By use of a well-known phenomenological potential to describe the* ^{3}*He-*^{3}*He* T*-matrix we calculate longitudinal and transverse spin diffusion coefficients* $D_\perp$ *and* $D_\parallel$ *and the identical-particle spin-rotation parameter* μ. *We confirm that these two diffusion constants differ at low* T *with* $D_\perp$ *approaching a constant as* $T \to 0$, *and* $D_\parallel \sim 1/T^2$. *Estimates of errors made by our approximations are considered in detail. Good agreement is found in comparison with data from both Cornell University and the University of Massachusetts. We find that the s-wave approximation is inadequate and that mean-field corrections are important. Comparison is also made between theory and the recent UMass viscosity measurements.*

1. INTRODUCTION

Recently we presented a theory[1,2] of the spin dynamics of arbitrarily polarized dilute quantum gases that is valid over the entire temperature range from Boltzmann to degenerate statistics. This theory was based on a new kinetic equation[1] for the matrix distribution function needed to describe arbitrarily polarized systems. Transverse and longitudinal spin diffusion, identical-particle spin-rotation effects, including spin waves[3,4] and the Leggett–Rice effect,[5] described previously in the Boltzmann case by Lhuillier and Laloë[3] and by Leggett[5] in the unpolarized degenerate Fermi case, are contained as special cases of the theory. Bose systems have also been treated recently.[6] Among the interesting predictions are that the transverse diffusion constant measured in spin-echo and spin wave experiments will differ from the longitudinal diffusion constant (as originally surmised by Meyerovich[7]) and even approach a temperature-independent value depending on polarization as $T \to 0$.[1,2]

Our previous derivation[1] was based on the Kadanoff–Baym (KB) Green's function approach.[8] While this is a powerful technique, it does lead to some uncontrolled approximations, and is not very easy to extend to include higher order corrections and generalizations. We present here the results of an alternative derivation technique based on work of Boercker and Dufty[9] from some years ago. This approach leads to the same results in the lowest order as the KB method but is much more straightforward in its approach to approximations and generalizations. This new derivation is given in Sec. 2. A limitation on our previous theoretical work[1,2] was its restriction to a Born approximate interaction, with only the s-wave case treated numerically. In the present treatment we report an extension of the theory, easily allowed by the new derivation, to a T-matrix approximation for the interaction.

Spin hydrodynamics based on this kinetic equation are derived in Secs. 3 and 4. It is here that we see how transverse and longitudinal spin diffusion processes are fundamentally different from one another. From the derivation it is evident that our analysis makes several approximations and simplifications, which we attempt to analyze thoroughly in Sec. 6. In Sec. 5 we show how to reduce the results to forms suitable for numerically computation.

Dilute solutions of ^{3}He in liquid ^{4}He provide a nearly ideal system for testing this theory since they allow the experimenter to vary the temperature over both degenerate and Boltzmann regimes and, at low enough concentrations, to polarize the system to very large values of magnetization. Data from two recent experiments, by Candela *et al.*[10] at the University of Massachusetts and Nunes *et al.*[11] at Cornell, have now become available and can be compared with the theory. We approximate the necessary T-matrix elements by using a phenomenological interaction originally introduced by Bardeen *et al.*[12] We use a particularly appropriate form of this interaction due to Ebner[13] in our numerical calculations. Corrections to the s-wave approximation are seen to be substantial in comparisons with both sets of experimental data. These results are presented in Sec. 7. Descrepancies between theory and experiment are within the estimates of the errors of the theory made in Sec. 6, with no adjustments made to the phenomenological interaction

One of the prime results of the present theory, previously reported elsewhere,[2] is the distinction, in a degenerate Fermi system, between transverse and longitudinal spin diffusion processes. Since the theory of this effect, given below, is complicated mathematically we would like to present some heuristic arguments that may give some insight into the difference. As we will see, the main effect is a difference in phase space for the collisions responsible for the spin diffusion. Consider first longitudinal spin diffusion. Mathematically, we can write the magnetization as $\mathbf{m} = m\hat{\mathbf{e}}$ where m is the magnitude and $\hat{\mathbf{e}}$ the direction. Then $\nabla \mathbf{m} = \hat{\mathbf{e}}\nabla m + m\nabla\hat{\mathbf{e}}$. The first term drives a "longitudinal" spin current, which in spin space is parallel to $\mathbf{m}$. The magnetization gradient is in the *magnitude* of the magnetization, giving an uneven picket fence as shown in Fig. 1a. In the case of a polarized degenerate system, the Fermi spheres, shown in momentum space in Fig. 1b, corresponding to two positions at x and $x + dx$, are not quite the same size. The one at x has an up-spin sphere that is a little larger than that at $x + dx$, and the down-spin sphere at x is a little smaller than that at $x + dx$. Consider the diffusion of an up spin from x to $x + dx$. If that spin is in the narrow annular

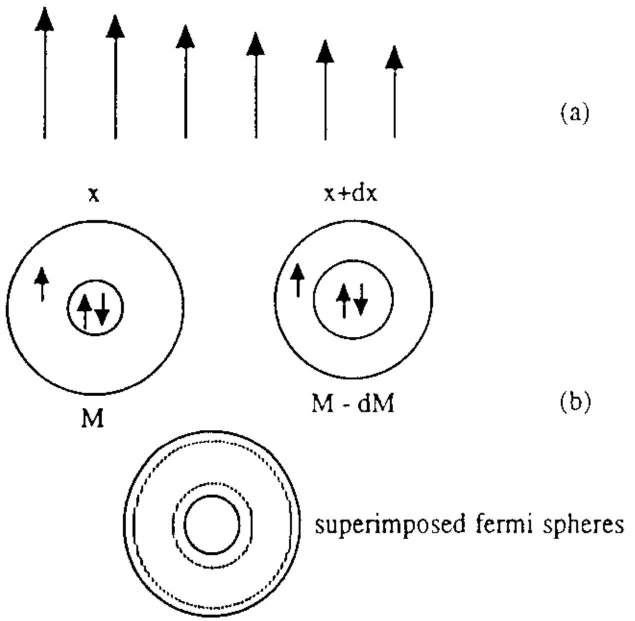

Fig. 1. Longitudinal spin diffusion. The gradient is in the length of the magnetization vector as shown in (a). Thus the fermi spheres of up and down spins are of different sizes at different positions as shown in (b). To restore equilibrium scattering need occur only right at the Fermi surfaces. The spin current is parallel to the local magnetization.

region of up signs that constitutes the difference between the two up-spin Fermi spheres, it is out of equilibrium when it reaches $x+dx$ and must scatter to become equilibrated. Up spins that are farther down in the Fermi sphere may not be able to move from x because their momentum states at $x+dx$ are already occupied; or perhaps such a spin has a large wave packet so that it is really the same spin as in the momentum state at $x+dx$. Thus the scattering occurs just in a little layer around the Fermi sphere and the spin diffusion coefficient will have the characteristic $1/T^2$ factor that arises from scattering limited to the Fermi surface.

On the other hand in a spin-echo experiment the spins are tipped at an angle from the field direction. A gradient field then causes them to precess at differing rates so that the tips of the spins form a spiral as shown in Fig. 2a. The gradient in magnetization then corresponds to the term $m\nabla\hat{\mathbf{e}}$. There is then a "transverse" spin current along $\nabla\hat{\mathbf{e}}$, which is perpendicular to $\mathbf{m}$. As shown in Fig. 2b below, the Fermi spheres are the same size at x and $x+dx$; but they have slightly different *directions* of magnetization. (The different directions of magnetization are greatly exaggerated in the Figure.) Thus a spin migrating from x to $x+dx$ in *any momentum state between the up and down Fermi spheres* is out of equilibrium and must scatter to return to local equilibrium. On the other hand, an up spin or a down spin in a state below the Fermi momentum of the down-spin sphere, comes from a momentum region of zero magnetization into a zero magnetization region and "senses" no lack of equilibrium, does not need to scatter, and indeed cannot scatter. To restore equilibrium in the transverse case, we require scattering throughout the region between the up and down-spin Fermi spheres; we will see that the scattering is proportional to the number of particles in the intermediate momentum region, $(n_{p+}-n_{p-})$, which provides considerably more phase space. The result is that the transverse relaxation time can be considerably shorter than the longitudinal relaxation time. In Fig. 3, we show how these two diffusion constants diverge from one another with $D_\perp$ approaching a constant as $T\to 0$ and $D_\parallel \sim 1/T^2$. The separation occurs in the degenerate Fermi system when the chemical potential difference between up and down spin states exceeds kT. Thus for larger polarization the separation occurs at higher T and the constant approached by $D_\perp$ will

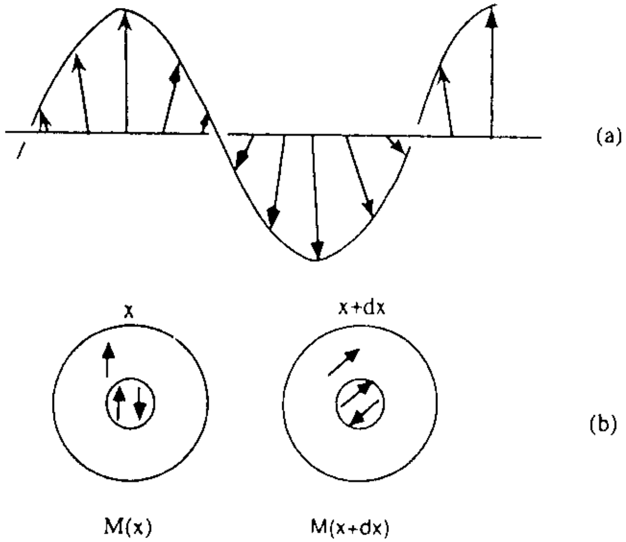

Fig. 2. Transverse spin diffusion. In a spin echo experiment the spins are tipped away from the external field and a gradient field causes a spiral to form as shown in (a). The spin current, which is now driven by a gradient in the direction of the magnetization and not its magnitude, is transverse to the local magnetization. To restore equilibrium all spins between the two Fermi surfaces must scatter.

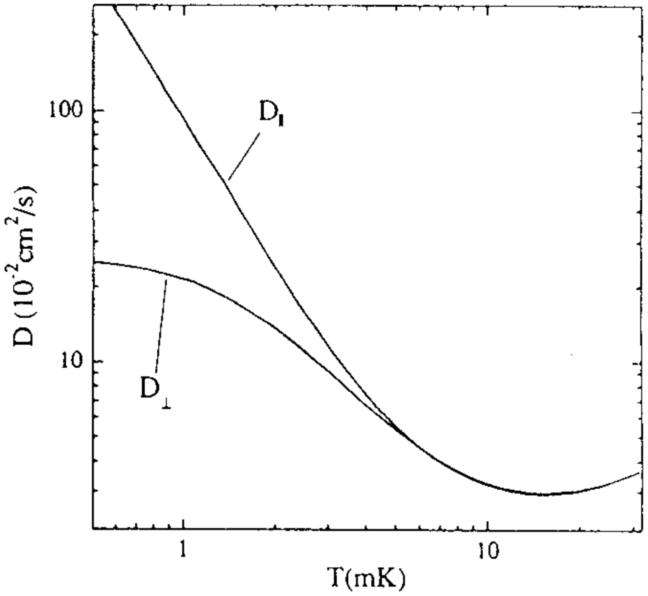

Fig. 3. Comparison of the transverse diffusion coefficient $D_\perp$ and longitudinal diffusion coefficient $D_\parallel$. In the degenerate regime the transverse spin diffusion coefficient goes to a constant at low temperature while the longitudinal coefficient is proportional to T^{-2}. The reason is the much larger density of states for scattering in the transverse case as explained in Fig. 2. The parameters for this calculation are the same as in the experiment described in Fig. 4 in Sec. 7.

be smaller. Our calculations are for dilute solutions, but there seems no reason these general ideas should not be valid for pure liquid ^{3}He as well.

2. DERIVATION OF THE KINETIC EQUATION

In Ref. 1 we presented a derivation of a kinetic equation valid, in the Born approximation, for dilute, degenerate Fermi and Bose gases in arbitrarily polarized arrangements. Thus it was capable of describing transverse spin-waves or spin-echo experiments. Basically such an equation completes a program began by Silin[14] who generalized the Landau equation[15] to arbitrary spin configurations by replacing the up/down spin distribution function by a 2×2 matrix distribution function. However, in his paper Silin actually wrote down only the so-called "drift terms" of the equation—that is, he generalized only the well-known mean-field terms of the Landau equation and did not present the collision integral. Several years later he did discuss the derivation of the collision integral, in the Born approximation, in a textbook[16] that has never been translated from the Russian. Our derivation of the same quantity presented in Ref. 1 reconfirmed his textbook result, corrected some errors in it, and put it in a more compact form much more suitable for analysis and numerical calculations. Some general numerical results without explicit comparison with experiment were presented in Ref. 2. Some comparison with experiments[10,11] was recently presented in preliminary form.[17]

Our basic formalism presented here is a new one based on work by Boercker and Dufty[9] who derived a general kinetic theory formalism for degenerate gases. They never assumed in their derivation that their distribution functions commute, so that their general results may be applied directly to arbitrary spin distribution matrices. We first give a heuristic derivation of their final form; there is no need to repeat their full derivation here. Their result becomes an equation for a distribution function only if one makes a Wigner transform[18] of it. This we do in some detail in this section.

The result of the Wigner transform is a highly nonlocal kinetic equation for the distribution function. One may then make Taylor expansions that assume that the distribution functions vary only slowly in space. The local terms that result are, of course, equivalent to considering the distribution functions to be just functions of momentum and independent of position, i.e., homogeneous functions. The first correction terms are second order in spatial and momentum gradients. Certain of these are equivalent in form to the mean-field drift terms in the Landau-Silin equation. We keep these in our further analysis here in order to estimate the size of such corrections. However, we show that there are also many other gradient terms. Our analysis shows, however, that all of these are smaller than the mean-field terms

we keep with one exception. There are a set of off-energy shell terms that correct the so-called spin rotation terms. These have been discussed previously in the literature[19] and called I_2. They vanish in the Boltzmann limit but in the degenerate case may be the same order as the small mean-field terms that we keep. We have not yet considered these numerically. We make a rather detailed analysis of the order of magnitude of all neglected terms in Sec. 6, as a guide to future work.

We first give a heuristic derivation of the well-known Snider equation,[20] a generalization of the Boltzmann equation. This has validity for only Boltzmann gases; however, the alteration due to degeneracy takes on an understandable form in the representation presented. Readers interested in a more rigorous derivation of the Boercker-Dufty equation can consult their original paper.[9]

The first BBGKY equation[9] for the one-body reduced density matrix $\rho(1)$ is

$$i\hbar \frac{\partial \rho(1)}{\partial t} = [H_I(1), \rho(1)] + \mathrm{Tr}_2\{[V(1,2), \rho_{II}(1,2)]\} \tag{2.1}$$

where H_I is a one-body Hamiltonian, including kinetic energy, Zeeman energy, etc.; $V(1,2)$ is the two-body interaction Hamiltonian; and $\rho_{II}(1,2)$ is the two-body reduced density matrix. The trace is over the coordinates of the second particle. A kinetic equation can result from this if one makes a decoupling approximation for the two-particle reduced density matrix $\rho_{II}(1,2)$. We follow an argument due to Snider.[21] Introduce the Möller operator Ω_{12} "which essentially converts a free-particle wave function into an interacting wave function by tracing freely back into time [before the collision began so the particles were free and not yet interacting] and then forward an equal amount of time according to interacting motion." Thus we assume that $\rho_{II}(1,2) = \Omega_{12}[\rho_{II}(1,2)]_{\text{before}}\Omega_{12}^{\dagger}$ where $[\rho_{II}(1,2)]_{\text{before}}$ is the matrix before the collision. The standard assumption is that before the collision the two-body matrix decouples according to $[\rho_{II}(1,2)]_{\text{before}} = \rho(1)\rho(2)$. This assumption has recently been called into question[22] (and alternative assumptions proposed[21,23]) because of the failure of the theory to reproduce the correct second virial coefficient; but these modifications are too recent for us to incorporate them into our present analysis. So we accept the decoupling approximation $\rho_{II}(1,2) = \Omega_{12}\rho(1)\rho(2)\Omega_{12}^{\dagger}$. The Snider equation thus becomes[20]

$$i\hbar \frac{\partial \rho(1)}{\partial t} = [H_I(1), \rho(1)] + \mathrm{Tr}_2\{[V(1,2), \Omega_{12}\rho(1)\rho(2)\Omega_{12}^{\dagger}]\} \tag{2.2}$$

This equation is valid for a Boltzmann gas of particles, which can have internal degrees of freedom such as spin.

The Boercker–Dufty (BD) equation,[9] valid for degenerate systems, looks very much like the Snider equation but has some important modifications. The most obvious change is the inclusion of a factor $(1+\eta P_{12})$, where P_{12} is a permutation operator interchanging coordinates of pairs of particles, and η is -1 or $+1$ depending on whether the particles are fermions or bosons, respectively. The other change is more subtle because it occurs in the character of the Möller operator definition. If the T-operator is defined as

$$T=V\Omega \tag{2.3}$$

then the matrix elements of T are

$$(p_1'p_2'|T|p_1p_2)=(p_1'p_2'|V|p_1p_2)+(p_1'p_2'|VR(E)T|p_1p_2) \tag{2.4}$$

where the resolvent operator R is defined as

$$R(E)=(E+i\varepsilon-H_0(1,2))^{-1}S(1,2) \tag{2.5}$$

in the limit of ε going to 0^+, with H_0 the two-particle Hamiltonian upon neglect of $V(1,2)$. The above would be the usual Lippmann–Schwinger equation except for the "shielding factor"

$$S(1,2)=\tilde{\rho}(1)\tilde{\rho}(2)-\rho(1)\rho(2) \tag{2.6}$$

where $\tilde{\rho}(i)=1+\eta\rho(i)$. It is S that leads ultimately to the final state factors in the collision integral that, for example, keep fermions from scattering into states that are already occupied.[24] The BD equation is then

$$i\hbar\frac{\partial\rho(1)}{\partial t}=[H_I(1),\rho(1)]+\mathrm{Tr}_2\{[V(1,2),\Omega_{12}\rho(1)\rho(2)\Omega_{12}^{\dagger}](1+\eta P_{12})\} \tag{2.7}$$

with account taken of the screened form of the Möller operator. A rigorous derivation of this result can be seen in Ref. 9. By making use of Eqs. (2.3) and (2.4) in (2.7), we have

$$\begin{aligned}\frac{\partial\rho(1)}{\partial t}=\frac{1}{i\hbar}[H_I(1),\rho_1]&+\frac{1}{i\hbar}\mathrm{Tr}_2\{(T_{12}\rho_1\rho_2-\rho_1\rho_2T_{12}^{\dagger})(1+\eta P_{12})\}\\&+\frac{1}{i\hbar}\mathrm{Tr}_2\{(T_{12}\rho_1\rho_2T_{12}^{\dagger}R_{12}^{\dagger}-R_{12}T_{12}\rho_1\rho_2T_{12}^{\dagger})(1+\eta P_{12})\}\end{aligned} \tag{2.8}$$

We now are in a position to replace reduced density matrices by distribution functions $\underline{n}_{\mathbf{p}}(\mathbf{r},t)$ in momentum and position space. We do this in two stages, examining first the so-called drift terms and then the collision integral itself.

2.1. Drift Terms

We define the Wigner transform distribution function[18] as

$$\underline{n}_{\mathbf{p}}(\mathbf{r},t)=\int d\mathbf{p}'\,e^{i\mathbf{p}'\cdot\mathbf{r}/\hbar}\left(\mathbf{p}+\frac{\mathbf{p}'}{2}\middle|\rho\middle|\mathbf{p}-\frac{\mathbf{p}'}{2}\right) \tag{2.9}$$

The underlining on $\underline{n}$ indicates that it is a 2 × 2 matrix in spin space. This quantity is usually interpreted as the distribution function giving the number of particles at momentum $\mathbf{p}$ at position $\mathbf{r}$ and time t. The normalization here is such that, in equilibrium, the diagonal elements of $\underline{n}$ are the standard spin-up and spin-down Fermi or Bose functions. The Wigner transform for an arbitrary two-body function $f(1,2)$ is

$$\begin{aligned}F_w(\mathbf{p}_1,\mathbf{r}_1,\mathbf{p}_2,\mathbf{r}_2)=&\int d\mathbf{p}_1'\,e^{i\mathbf{p}_1'\cdot\mathbf{r}_1/\hbar}\int d\mathbf{p}_2'\,e^{i\mathbf{p}_2'\cdot\mathbf{r}_2/\hbar}\\&\times\left(\mathbf{p}_1+\frac{\mathbf{p}_1'}{2},\mathbf{p}_2+\frac{\mathbf{p}_2'}{2}\middle|f(1,2)\middle|\mathbf{p}_1-\frac{\mathbf{p}_1'}{2},\mathbf{p}_2-\frac{\mathbf{p}_2'}{2}\right)\end{aligned} \tag{2.10}$$

We apply this formula to both sides of Eq. (2.8) and introduce matrix elements in spin space as well. Note that the trace over coordinate 2 is $1/h^3\int d\mathbf{r}_2\int d\mathbf{p}_2$. The result is

$$\begin{aligned}&\frac{\partial}{\partial t}(m_1|\underline{n}_{\mathbf{p}_1}(\mathbf{r}_1,t)|m_1')\\&\quad=\frac{1}{i\hbar h^3}\int d\mathbf{p}_1'\,e^{i\mathbf{p}_1'\cdot\mathbf{r}_1/\hbar}\int d\mathbf{p}_3\\&\qquad\times\sum_{m_3}\left(\mathbf{p}_1+\frac{\mathbf{p}_1'}{2};m_1\middle|H_1\middle|\mathbf{p}_3;m_3\right)\left(\mathbf{p}_3;m_3\middle|\rho_1\middle|\mathbf{p}_1-\frac{\mathbf{p}_1'}{2};m_1'\right)\\&\qquad+\frac{1}{i\hbar h^9}\int d\mathbf{r}_2\int d\mathbf{p}_2\int d\mathbf{p}_1'\int d\mathbf{p}_2'\int d\mathbf{p}_3\int d\mathbf{p}_4\,e^{i\mathbf{p}_1'\cdot\mathbf{r}_1/\hbar}\,e^{i\mathbf{p}_2'\cdot\mathbf{r}_2/\hbar}\end{aligned}$$

$$
\begin{aligned}
&\times \sum_{m_2,m_3,m_4} \left(\mathbf{p}_1+\frac{\mathbf{p}_1'}{2}, \mathbf{p}_2+\frac{\mathbf{p}_2'}{2}; m_1, m_2\middle| T_{12} \middle| \mathbf{p}_3, \mathbf{p}_4; m_3, m_4\right)\\
&\times\left[\left(\mathbf{p}_3; m_3\middle|\rho_1\middle|\mathbf{p}_1-\frac{\mathbf{p}_1'}{2}; m_1'\right)\left(\mathbf{p}_4; m_4\middle|\rho_2\middle|\mathbf{p}_2-\frac{\mathbf{p}_2'}{2}; m_2\right)\right.\\
&\left.+\eta\left(\mathbf{p}_3; m_3\middle|\rho_1\middle|\mathbf{p}_2-\frac{\mathbf{p}_2'}{2}; m_2\right)\left(\mathbf{p}_4; m_4\middle|\rho_2\middle|\mathbf{p}_1-\frac{\mathbf{p}_1'}{2}; m_1'\right)\right]\\
&+\text{h.c.}+(m_1|\underline{C}_2|m_1')
\end{aligned}
\tag{2.11}
$$

In this h.c. means Hermitian conjugate and C_2 represents the Wigner transformation of the terms in Eq. (2.8) having two factors of T. We will treat the C_2 terms separately below; they form a part of the collision integral. The other terms in Eq. (2.11) give the drift and mean-field terms and contribute to the collision integral.

The next step is to substitute the inverse transform of Eq. (2.9) into Eq. (2.11). This is easily seen to be

$$
(\mathbf{p}|\rho_1|\mathbf{p}') = \frac{1}{h^3}\int d\mathbf{r}\, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r}/\hbar}\, \underline{n}_{\frac{1}{2}(p+p')}(\mathbf{r}, t) \tag{2.12}
$$

We also assume that the T-matrix is a function of relative spatial variables only so that

$$
\begin{aligned}
&(\mathbf{p}_1, \mathbf{p}_2; m_1, m_2|T_{12}|\mathbf{p}_3, \mathbf{p}_4; m_3, m_4)\\
&\qquad = (\mathbf{p}_{12}; m_1, m_2|T_{12}|\mathbf{p}_{34}; m_3, m_4)\,\delta(\mathbf{P}_{12}-\mathbf{P}_{34})
\end{aligned}
\tag{2.13}
$$

where $\mathbf{p}_{12}=\frac{1}{2}(\mathbf{p}_1-\mathbf{p}_2)$ and $\mathbf{P}_{12}=(\mathbf{p}_1+\mathbf{p}_2)$ are the relative and center of mass momenta respectively. Furthermore, to achieve the standard collision integral form, such as that of the Uheling-Uhlenbeck integral,[24] which is a special case of our analysis, it is necessary to make use of the optical theorem. We write $T=\frac{1}{2}(T+T^\dagger)+\frac{1}{2}(T+T^\dagger)$ and $T^\dagger=\frac{1}{2}(T+T^\dagger)-\frac{1}{2}(T-T^\dagger)$ and then use the optical theorem in the form[9]

$$
T-T^\dagger = T^\dagger(R-R^\dagger)T \tag{2.14}
$$

Thus the terms in $T-T^\dagger$ transform into second order in T and are grouped with the C_2 terms to be considered below.

Further, it is convenient, in the exchange term in Eq. (2.11), to interchange dummy indices $\mathbf{p}_3$, $\mathbf{p}_4$ and m_3, m_4. Carrying out these operations leads to

$$
\begin{aligned}
&\frac{\partial}{\partial t}(m_1|\underline{n}_{\mathbf{p}_1}(\mathbf{r}_1, t)|m_1')\\
&\quad=\frac{1}{i\hbar h^3}\int d\mathbf{p}_1'\, e^{i\mathbf{p}_1'\cdot\mathbf{r}_1/\hbar}\int d\mathbf{p}_3\int d\mathbf{r}_3\, e^{-i(\mathbf{p}_3-\mathbf{p}_1+\mathbf{p}_1'/2)\cdot\mathbf{r}_3/\hbar}\\
&\quad\times\sum_{m_3}\left(\mathbf{p}_1+\frac{\mathbf{p}_1'}{2}; m_1\middle|H_1\middle|\mathbf{p}_3; m_3\right)(m_3|\underline{n}_{\frac{1}{2}(\mathbf{p}_3+\mathbf{p}_1'/2)}(\mathbf{r}_3)|m_1')\\
&\quad+\frac{1}{i\hbar h^9}\int d\mathbf{r}_2\int d\mathbf{p}_2\int d\mathbf{p}_1'\int d\mathbf{p}_2'\int d\mathbf{p}_3\int d\mathbf{p}_4\int d\mathbf{r}_3\int d\mathbf{r}_4\\
&\quad\times e^{i\mathbf{p}_1'\cdot\mathbf{r}_1/\hbar}\, e^{i\mathbf{p}_2'\cdot\mathbf{r}_2/\hbar}\, e^{-i(\mathbf{p}_3-\mathbf{p}_1+\mathbf{p}_1'/2)\cdot\mathbf{r}_3/\hbar}\\
&\quad\times e^{-i(\mathbf{p}_4-\mathbf{p}_2+\mathbf{p}_2'/2)\cdot\mathbf{r}_4/\hbar}\,\delta(\mathbf{P}_{12}+\mathbf{P}_{12}'/2-\mathbf{P}_{34})\\
&\quad\times\sum_{m_2,m_3,m_4}\left[\left(\mathbf{p}_{12}+\frac{\mathbf{p}_{12}'}{2}; m_1, m_2\middle|\tfrac{1}{2}(T_{12}+T_{12}^\dagger)\middle|\mathbf{p}_{34}; m_3, m_4\right)\right.\\
&\quad\times(m_3|\underline{n}_{\frac{1}{2}(\mathbf{p}_3+\mathbf{p}_1-\mathbf{p}_1'/2)}(\mathbf{r}_3)|m_1')(m_4|\underline{n}_{\frac{1}{2}(\mathbf{p}_4+\mathbf{p}_2-\mathbf{p}_2'/2)}(\mathbf{r}_4)|m_2)\\
&\quad+\eta\left(\mathbf{p}_{12}+\frac{\mathbf{p}_{12}'}{2}; m_1, m_2\middle|\tfrac{1}{2}(T_{12}+T_{12}^\dagger)\middle|-\mathbf{p}_{34}; m_3, m_4\right)\\
&\quad\left.\times(m_4|\underline{n}_{\frac{1}{2}(\mathbf{p}_4+\mathbf{p}_2-\mathbf{p}_2'/2)}(\mathbf{r}_4)|m_2)(m_3|\underline{n}_{\frac{1}{2}(\mathbf{p}_3+\mathbf{p}_1-\mathbf{p}_1'/2)}(\mathbf{r}_3)|m_1')\right]\\
&\quad+\text{h.c.}+(m_1|\underline{C}_2'|m_1')
\end{aligned}
\tag{2.15}
$$

in which $(m_1|\underline{C}_2'|m_1')$ now includes the terms in $T-T^\dagger$ as transformed by the optical theorem.

We want to identify a "quasiparticle" energy in Eq. (2.15). This quantity will be the Wigner transform of an effective single-particle Hamiltonian according to

$$
\underline{\varepsilon}_{\mathbf{p}}(\mathbf{r}) = \int d\mathbf{p}'\, e^{i\mathbf{p}'\cdot\mathbf{r}/\hbar}\left(\mathbf{p}+\frac{\mathbf{p}'}{2}\middle|H_1^{\text{eff}}\middle|\mathbf{p}-\frac{\mathbf{p}'}{2}\right) \tag{2.16}
$$

with inverse relation

$$
(\mathbf{p}|H_1^{\text{eff}}|\mathbf{p}') = \frac{1}{h^3}\int d\mathbf{r}\, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r}/\hbar}\,\underline{\varepsilon}_{\frac{1}{2}(\mathbf{p}+\mathbf{p}')}(\mathbf{r}, t) \tag{2.17}
$$

It is easy to identify H_1^{eff} by looking for terms that group appropriately with H_1, the first term of H_1^{eff}. Thus we find

$$\left(\mathbf{p}_1+\frac{\mathbf{p}_1'}{2};m_1\middle|H_1^{\text{eff}}\middle|\mathbf{p}_3;m_3\right)$$

$$=\left(\mathbf{p}_1+\frac{\mathbf{p}_2'}{2};m_1\middle|H_1\middle|\mathbf{p}_3;m_3\right)+\frac{1}{h^6}\int d\mathbf{r}_2\int d\mathbf{p}_2\int d\mathbf{p}_2'\int d\mathbf{p}_4\int d\mathbf{r}_4$$

$$\times e^{i\mathbf{p}_2'\cdot\mathbf{r}_2/\hbar}\,e^{-i(\mathbf{p}_4-\mathbf{p}_2+\mathbf{p}_2'/2)\cdot\mathbf{r}_4/\hbar}\,\delta(\mathbf{P}_{12}+\mathbf{P}_{12}'/2-\mathbf{P}_{34})$$

$$\times\sum_{m_2,m_4}\left[\left(\mathbf{p}_{12}+\frac{\mathbf{p}_{12}'}{2};m_1,m_2\middle|\tfrac{1}{2}(T_{12}+T_{12}^{\dagger})\middle|\mathbf{p}_{34};m_3,m_4\right)\right.$$

$$\times(m_4|\underline{n}_{\frac{1}{2}(\mathbf{p}_4+\mathbf{p}_2-\mathbf{p}_2'/2)}(\mathbf{r}_4)|m_2)$$

$$+\eta\left(\mathbf{p}_{12}+\frac{\mathbf{p}_{12}'}{2};m_1,m_2\middle|\tfrac{1}{2}(T_{12}+T_{12}^{\dagger})\middle|-\mathbf{p}_{34};m_4,m_3\right)$$

$$\left.\times(m_3|\underline{n}_{\frac{1}{2}(\mathbf{p}_4+\mathbf{p}_2-\mathbf{p}_2/2)}(\mathbf{r}_4)|m_2)\right] \tag{2.18}$$

We will return to a discussion of the character of the resulting $\underline{\varepsilon}_{\mathbf{p}}(\mathbf{r})$ in a moment. First we introduce this $\underline{\varepsilon}_{\mathbf{p}}(\mathbf{r})$ into the kinetic equation by use of Eq. (2.17). The resulting equation is

$$\frac{\partial}{\partial t}(m_1|\underline{n}_{\mathbf{p}_1}(\mathbf{r}_1,t)|m_1')$$

$$=\frac{1}{i\hbar h^6}\int d\mathbf{p}_1'\int d\mathbf{p}_3\int d\mathbf{r}_3\int d\mathbf{r}_4\,e^{i\mathbf{p}_1'\cdot\mathbf{r}_1/\hbar}$$

$$\times e^{-i(\mathbf{p}_3-\mathbf{p}_1+\mathbf{p}_1'/2)\cdot\mathbf{r}_3/\hbar}\,e^{-i(\mathbf{p}_1-\mathbf{p}_3+\mathbf{p}_1'/2)\cdot\mathbf{r}_4/\hbar}$$

$$\times\sum_{m_3}(m_1|\underline{\varepsilon}_{\frac{1}{2}(\mathbf{p}_1+\mathbf{p}_3+\mathbf{p}_1'/2)}(\mathbf{r}_4)|m_3)(m_3|\underline{n}_{\frac{1}{2}(\mathbf{p}_3+\mathbf{p}_1-\mathbf{p}_1'/2)}(\mathbf{r}_3)|m_1')$$

$$+\text{h.c.}+(m_1|\underline{C}_2'|m_1') \tag{2.19}$$

We assume that $\underline{\varepsilon}_{\mathbf{p}}(\mathbf{r})$ and $\underline{n}_{\mathbf{p}}(\mathbf{r})$ are slowly varying in space so that a local approximation which puts $\mathbf{r}_3$ and $\mathbf{r}_4$ equal to $\mathbf{r}_1$ is accurate. We will also keep lowest order nonlocal terms, which are second order in gradients coming from the Taylor series expansion of $\underline{\varepsilon}_{\mathbf{p}}(\mathbf{r}_4)$ and $\underline{n}_{\mathbf{p}}(\mathbf{r}_3)$. The operations are straightforward even if slightly messy. The gradient terms involve factors like $(\mathbf{r}_3-\mathbf{r}_1)\cdot\nabla_{\mathbf{r}_1}\underline{n}_{\mathbf{p}}(\mathbf{r}_1)$. The $(\mathbf{r}_3-\mathbf{r}_1)$ factor is changed into a momentum gradient by using

$$\int d\mathbf{r}_3(\mathbf{r}_3-\mathbf{r}_1)\,e^{i\mathbf{p}\cdot\mathbf{r}_3/\hbar}=-i\hbar\,e^{i\mathbf{p}\cdot\mathbf{r}_1/\hbar}\nabla_{\mathbf{p}}\int d\mathbf{r}_3\,e^{i\mathbf{p}\cdot(\mathbf{r}_3-\mathbf{r}_1)/\hbar} \tag{2.20}$$

The resulting derivatives of δ-functions are handled by integration by parts. Thus we end up with terms like the drift and mean-field terms in the Landau-Silin equation. We find

$$\frac{\partial}{\partial t}\underline{n}_{\mathbf{p}}(\mathbf{r},t)=\frac{1}{i\hbar}[\underline{\varepsilon}_{\mathbf{p}}(\mathbf{r}),\underline{n}_{\mathbf{p}}(\mathbf{r})]_-+\tfrac{1}{2}[\nabla_{\mathbf{r}}\underline{\varepsilon}_{\mathbf{p}}(\mathbf{r}),\nabla_{\mathbf{p}}\underline{n}_{\mathbf{p}}(\mathbf{r})]_+$$

$$-\tfrac{1}{2}[\nabla_{\mathbf{p}}\underline{\varepsilon}_{\mathbf{p}}(\mathbf{r}),\nabla_{\mathbf{r}}\underline{n}_{\mathbf{p}}(\mathbf{r})]_++\underline{C}_2' \tag{2.21}$$

The commutator gives rise to spin-rotation effects, including spin-waves, discussed in Refs. 3–5 and 14. The gradient terms are important in Landau's theory in giving rise to longitudinal wave phenomenon like zero sound.[15] They also renormalize certain equilibrium and transport properties.[15] For example, they lead to the factor $1/(1+F_0^a)$ in susceptibility and spin diffusion constants. For a dilute gas, these renormalizations take on the character of second virial corrections. Indeed one can show that they lead to the correct expression for the second virial correction to the pressure if one uses the correct formulation of dilute gas Landau paramaters in the interaction part of $\underline{\varepsilon}_{\mathbf{p}}(\mathbf{r})$.[25,26]

In order to compare our $\underline{\varepsilon}_{\mathbf{p}}(\mathbf{r})$ explicitly with that of the Landau-Silin equation we need to look more closely at the expression for $\underline{\varepsilon}_{\mathbf{p}}(\mathbf{r})$. We use Eq. (2.18) in (2.16) from which it follows that

$$(m_1|\underline{\varepsilon}_{\mathbf{p}_1}(\mathbf{r}_1)|m_1')$$

$$=(m_1|\underline{\varepsilon}_{\mathbf{p}_1}^0(\mathbf{r}_1)|m_1')+\frac{1}{h^6}\int d\mathbf{r}_2\int d\mathbf{r}_4\int d\mathbf{p}_2\int d\mathbf{p}_1'\int d\mathbf{p}_2'\int d\mathbf{p}_4$$

$$\times e^{i\mathbf{p}_1'\cdot\mathbf{r}_1/\hbar}\,e^{i\mathbf{p}_2'\cdot\mathbf{r}_2/\hbar}\,e^{-i(\mathbf{p}_4-\mathbf{p}_2+\mathbf{p}_2'/2)\cdot\mathbf{r}_4/\hbar}\,\delta(\mathbf{P}_{12}+\mathbf{P}_{12}'/2-\mathbf{p}_1+\mathbf{p}_1'/2-\mathbf{p}_4)$$

$$\times\sum_{m_2,m_4}\left[\left(\mathbf{p}_{12}+\frac{\mathbf{p}_{12}'}{2};m_1,m_2\middle|\tfrac{1}{2}(T_{12}+T_{12}^{\dagger})\middle|\tfrac{1}{2}(\mathbf{p}_1-\mathbf{p}_1'/2-\mathbf{p}_4);m_1',m_4\right)\right.$$

$$\times(m_4|\underline{n}_{\frac{1}{2}(\mathbf{p}_4+\mathbf{p}_2-\mathbf{p}_2'/2)}(\mathbf{r}_4)|m_2)+\eta\left(\mathbf{p}_{12}+\frac{\mathbf{p}_{12}'}{2};m_1,m_2\middle|\tfrac{1}{2}(T_{12}+T_{12}^{\dagger})\right.$$

$$\left.\times\middle|-\tfrac{1}{2}(\mathbf{p}_1-\mathbf{p}_1'/2-\mathbf{p}_4);m_4,m_1'\right)(m_4|\underline{n}_{\frac{1}{2}(\mathbf{p}_4+\mathbf{p}_2-\mathbf{p}_2'/2)}(\mathbf{r}_4)|m_2)\bigg] \tag{2.22}$$

where

$$\underline{\varepsilon}_{\mathbf{p}}^{0}(\mathbf{r})=\int d\mathbf{p}'\, e^{i\mathbf{p}'\cdot\mathbf{r}/\hbar}\left(\mathbf{p}+\frac{\mathbf{p}'}{2}\middle|H_1\middle|\mathbf{p}-\frac{\mathbf{p}'}{2}\right) \tag{2.23}$$

The expression for $\underline{\varepsilon}_{\mathbf{p}_1}(\mathbf{r}_1)$ is nonlocal. A local approximation is easily made by setting $\mathbf{r}_4=\mathbf{r}_1$. For the gradient terms in Eq. (2.21) this is all that is necessary in an approximation that is second order in gradients. However, for the spin-rotation commutator of Eq. (2.21) in which no gradients appear, it might seem necessary to go one more order in gradients in the expression for $\underline{\varepsilon}_{\mathbf{p}_1}(\mathbf{r}_1)$ to maintain a consistent approximation. However, we will show in Sec. 6 that such additional terms are expected to be small. Thus we give here only the local expression for $\underline{\varepsilon}_{\mathbf{p}_1}(\mathbf{r}_1)$.

To make our examination as easy as possible at this stage we also assume that the T-matrix is spin independent. Note that this is a major assumption, since the shielding factor S of Eq. (2.6) that appears in R of Eq. (2.5) modifies the Lippman-Schwinger equation for T, Eq. (2.4). Thus T is dependent on the distribution functions, the $\underline{n}_{\mathbf{p}}$, which appear in the intermediate states to ensure that scattering cannot occur into already filled states, etc. This, of course, makes T spin dependent. However, the phenomenological effective potential that we use later to describe ^{3}He-^{3}He interactions is spin *independent*. We make that assumption explicitly here. As discussed in Sec. 6, a more complete theory could possibly include a spin-dependent effective potential phenomenologically or via a fundamental computation. (Of course, because exchange effects are included automatically in the theory, the equivalent Landau parameters of the theory *are* spin dependent.)

With the local approximation and with a spin-independent T-matrix we find for the quasiparticle energy

$$\begin{aligned}\underline{\varepsilon}_{\mathbf{p}_1}(\mathbf{r}_1)=\underline{\varepsilon}_{\mathbf{p}_1}^{0}(\mathbf{r}_1)+\int d\mathbf{p}_2[&\mathrm{Re}(\mathbf{p}_{12}|T_{12}|\mathbf{p}_{12})\underline{I}\,\mathrm{tr}_s\,\underline{n}_{\mathbf{p}_2}(\mathbf{r}_1)\\ &+\eta\,\mathrm{Re}(\mathbf{p}_{12}|T_{12}|-\mathbf{p}_{12})\underline{n}_{\mathbf{p}_2}(\mathbf{r}_1)]\end{aligned} \tag{2.24}$$

where tr_s is a trace in spin space, Re means real part and $\underline{I}$ is the 2×2 unit matrix. The direct term of this expression has been found previously by Grossmann.[27] It was examined in detail by Rainwater and Snider[25] who showed that it does not agree fully with what one should get in the dilute limit of the Landau equation. These authors and Vetrovec and Carneiro[28] have given an expression for the Landau interaction in terms of phase shifts. Rainwater and Snider,[25] and Miyake *et al.*[26] showed that use of this interaction in the Landau theory yields the correct second virial coefficient for a dilute gas. The present expression for the Landau interaction yields only part of the second virial coefficient. Presumably this implies an error in the present theory, which we will estimate in Sec. 6. Recent work[21,23] has indicated that the error may reside in the decoupling approximation used to get from Eqs. (2.1) to (2.2).

2.2. Collision Integral

The collision integral is derived from the Wigner transform of the terms of Eq. (2.8) having two T factors. These arise from the second term on the right of Eq. (2.8) as well as the factors from the first term for which $T-T^\dagger$ was transformed into two T's by use of the optical theorem. We will use the full mechanism of the Wigner transformation although we are going to keep only the local terms in the resulting expression. This approach is more complicated than it need be to generate just the local terms—we could just treat a homogeneous system somewhat more easily—however, we want to set up the formalism that will allow a more general treatment of nonlocal terms at a later date.

The Wigner-transformed terms from Eq. (2.8) that produce second-order factors in the T-matrix are

$$\begin{aligned}&(m_1|\underline{C}_2'|m_1')\\
&=\frac{1}{i\hbar h^3}\int d\mathbf{r}_2\int d\mathbf{p}_2\int d\mathbf{p}_1'\int d\mathbf{p}_2'\int d\mathbf{p}_3\int d\mathbf{p}_4\, e^{i\mathbf{p}_1'\cdot\mathbf{r}_1/\hbar}\, e^{i\mathbf{p}_2'\cdot\mathbf{r}_2/\hbar}\\
&\quad\times\sum_{m_2,m_3,m_4}\left(\mathbf{p}_1+\frac{\mathbf{p}_1'}{2},\mathbf{p}_2+\frac{\mathbf{p}_2'}{2};m_1,m_2\middle|\tfrac{1}{2}(T_{12}-T_{12}^\dagger)\middle|\mathbf{p}_3,\mathbf{p}_4;m_3,m_4\right)\\
&\quad\times\left[\left(\mathbf{p}_3;m_3\middle|\rho_1\middle|\mathbf{p}_1-\frac{\mathbf{p}_1'}{2};m_1'\right)\left(\mathbf{p}_4;m_4\middle|\rho_2\middle|\mathbf{p}_2-\frac{\mathbf{p}_2'}{2};m_2\right)\right.\\
&\quad\left.+\eta\left(\mathbf{p}_3;m_3\middle|\rho_1\middle|\mathbf{p}_2-\frac{\mathbf{p}_2'}{2};m_2\right)\left(\mathbf{p}_4;m_4\middle|\rho_2\middle|\mathbf{p}_1-\frac{\mathbf{p}_1'}{2};m_1'\right)\right]\\
&\quad+\frac{1}{i\hbar h^3}\int d\mathbf{r}_2\int d\mathbf{p}_2\int d\mathbf{p}_1'\int d\mathbf{p}_2'\int d\mathbf{p}_3\int d\mathbf{p}_4\int d\mathbf{p}_a\\
&\quad\times\int d\mathbf{p}_b\int d\mathbf{p}_c\int d\mathbf{p}_d\, e^{i\mathbf{p}_1'\cdot\mathbf{r}_1/\hbar}\, e^{i\mathbf{p}_2'\cdot\mathbf{r}_2/\hbar}\\
&\quad\times\sum_{m_2,m_3,m_4}\ \sum_{m_a,m_b}\ \sum_{m_c,m_d}\left(\mathbf{p}_1+\frac{\mathbf{p}_1'}{2},\mathbf{p}_2+\frac{\mathbf{p}_2'}{2};m_1,m_2\middle|T_{12}\middle|\mathbf{p}_3,\mathbf{p}_4;m_3,m_4\right)\end{aligned}$$

$$\times (\mathbf{p}_3\,;m_3|\rho_1|\mathbf{p}_a\,;m_a)(\mathbf{p}_4\,;m_4|\rho_2|\mathbf{p}_b\,;m_b)$$
$$\times (\mathbf{p}_a,\mathbf{p}_b\,;m_a\,;m_b|T^{\dagger}_{12}|\mathbf{p}_c,\mathbf{p}_d\,;m_c,m_d)$$
$$\times \left(\mathbf{p}_c,\mathbf{p}_d\,;m_c,m_d\middle|R^{\dagger}_{12}(1+\eta P_{12})\middle|\mathbf{p}_1-\frac{\mathbf{p}_1'}{2},\mathbf{p}_2-\frac{\mathbf{p}_2'}{2}\,;m_1',m_2\right)+\text{h.c.} \tag{2.25}$$

As we discussed above, we use the optical theorem of Eq. (2.14) to convert the terms in $T-T^{\dagger}$ into second order terms. The resolvent operator, Eq. (2.5), is then used in the form

$$R=[P(E-H_{12}^0)^{-1}-i\pi\delta(E-H_{12}^0)](\tilde{\rho}(1)\tilde{\rho}(2)-\rho(1)\rho(2)) \tag{2.26}$$

in which P means principal part integral and $H_{12}^0=H_1+H_2$. In this expression for R the E refers to the energy of the intermediate state appearing to the right of the following T. In the corresponding expression for $R^{\dagger}(E)$ the E refers to the energy of the intermediate state to the left of the preceding $T^{\dagger}$. At this point it is convenient again to make the approximation that the T-matrix is spin independent to accommodate the form of the effective interaction that we will use in our numerical calculations. This simplifies our very complicated formulas somewhat. Note that Eq. (2.26) introduces two more factors of ρ. These allow the usual four factors of $\underline{n}_{\mathbf{p}}$ and $1+\eta\underline{n}_{\mathbf{p}}$ that one always finds in the collision integral for a degenerate system. We find the collision integral to become

$$(m_1|\underline{C}_2'|m_1')$$
$$=\frac{1}{i\hbar h^3}\int d\mathbf{r}_2\int d\mathbf{p}_2\int d\mathbf{p}_1'\int d\mathbf{p}_2'\int d\mathbf{p}_3\int d\mathbf{p}_4\int d\mathbf{p}_a$$
$$\times\int d\mathbf{p}_b\int d\mathbf{p}_c\int d\mathbf{p}_d\, e^{i\mathbf{p}_1'\cdot\mathbf{r}_1/\hbar}\,e^{i\mathbf{p}_2'\cdot\mathbf{r}_2/\hbar}$$
$$\times\sum_{m_2,m_a,m_b}\frac{1}{2}\left(\mathbf{p}_1+\frac{\mathbf{p}_1'}{2},\mathbf{p}_2+\frac{\mathbf{p}_2'}{2}\middle|T^{\dagger}_{12}\middle|\mathbf{p}_3,\mathbf{p}_4\right)$$
$$\times\left[\frac{P}{E_{cd}-E_{34}}-i\pi\delta(E_{cd}-E_{34})-\frac{P}{E_{12+}-E_{ab}}-i\pi\delta(E_{12+}-E_{ab})\right]$$
$$\times[(\mathbf{p}_3\,;m_1|\tilde{\rho}_1|\mathbf{p}_a\,;m_a)(\mathbf{p}_4\,;m_2|\tilde{\rho}_2|\mathbf{p}_b\,;m_b)$$
$$-(\mathbf{p}_3\,;m_1|\rho_1|\mathbf{p}_a\,;m_a)(\mathbf{p}_4\,;m_2|\rho_2|\mathbf{p}_b\,;m_b)](\mathbf{p}_a,\mathbf{p}_b|T_{12}|\mathbf{p}_c,\mathbf{p}_d)$$

$$\times\left[\left(\mathbf{p}_c\,;m_a\middle|\rho_1\middle|\mathbf{p}_1-\frac{\mathbf{p}_1'}{2}\,;m_1'\right)\left(\mathbf{p}_d\,;m_b\middle|\rho_2\middle|\mathbf{p}_2-\frac{\mathbf{p}_2'}{2}\,;m_2\right)\right.$$
$$\left.+\eta\left(\mathbf{p}_c\,;m_a\middle|\rho_1\middle|\mathbf{p}_2-\frac{\mathbf{p}_2'}{2}\,;m_1'\right)\left(\mathbf{p}_d\,;m_b\middle|\rho_2\middle|\mathbf{p}_1-\frac{\mathbf{p}_1'}{2}\,;m_2\right)\right]$$
$$+\frac{1}{i\hbar h^3}\int d\mathbf{r}_2\int d\mathbf{p}_2\int d\mathbf{p}_1'\int d\mathbf{p}_2'\int d\mathbf{p}_3\int d\mathbf{p}_4\int d\mathbf{p}_a$$
$$\times\int d\mathbf{p}_b\int d\mathbf{p}_c\int d\mathbf{p}_d\, e^{i\mathbf{p}_1'\cdot\mathbf{r}_1/\hbar}\,e^{i\mathbf{p}_2'\cdot\mathbf{r}_2/\hbar}$$
$$\times\sum_{m_2,m_a,m_b}\left(\mathbf{p}_1+\frac{\mathbf{p}_1'}{2},\mathbf{p}_2+\frac{\mathbf{p}_2'}{2}\middle|T_{12}\middle|\mathbf{p}_3,\mathbf{p}_4\right)(\mathbf{p}_3\,;m_1|\rho_1|\mathbf{p}_a\,;m_a)$$
$$\times(\mathbf{p}_4\,;m_2|\rho_2|\mathbf{p}_b\,;m_b)(\mathbf{p}_a,\mathbf{p}_b|T^{\dagger}_{12}|\mathbf{p}_c,\mathbf{p}_d)$$
$$\times\left\{\left[\left(\mathbf{p}_c\,;m_a\middle|\tilde{\rho}_3\middle|\mathbf{p}_1-\frac{\mathbf{p}_1'}{2}\,;m_1'\right)\left(\mathbf{p}_d\,;m_b\middle|\tilde{\rho}_4\middle|\mathbf{p}_2-\frac{\mathbf{p}_2'}{2}\,;m_2\right)\right.\right.$$
$$\left.-\left(\mathbf{p}_c\,;m_a\middle|\rho_3\middle|\mathbf{p}_1-\frac{\mathbf{p}_1'}{2}\,;m_1'\right)\left(\mathbf{p}_d\,;m_b\middle|\rho_4\middle|\mathbf{p}_2-\frac{\mathbf{p}_2'}{2}\,;m_2\right)\right]$$
$$+\eta\left[\left(\mathbf{p}_c\,;m_a\middle|\tilde{\rho}_3\middle|\mathbf{p}_2-\frac{\mathbf{p}_2'}{2}\,;m_2\right)\left(\mathbf{p}_d\,;m_b\middle|\tilde{\rho}_4\middle|\mathbf{p}_1-\frac{\mathbf{p}_1'}{2}\,;m_1'\right)\right.$$
$$\left.\left.-\left(\mathbf{p}_c\,;m_a\middle|\rho_3\middle|\mathbf{p}_2-\frac{\mathbf{p}_2'}{2}\,;m_2\right)\left(\mathbf{p}_d\,;m_b\middle|\rho_4\middle|\mathbf{p}_1-\frac{\mathbf{p}_1'}{2}\,;m_1'\right)\right]\right\}$$
$$\times\left[\frac{P}{E_{ab}-E_{12-}}+i\pi\delta(E_{ab}-E_{12-})\right]+\text{h.c.} \tag{2.27}$$

in which E_{12+} is the energy associated with the state

$$\left|\mathbf{p}_1+\frac{\mathbf{p}_1'}{2},\mathbf{p}_2+\frac{\mathbf{p}_2'}{2}\right)$$

etc. The first integral arises from the optical theorem terms and is therefore associated with forward scattering. The second integral is made up of the original two-T terms and represents lateral scattering.

We now introduce the inverse Wigner transform to change the ρ's into $\underline{n}_{\mathbf{p}}$'s. However, we immediately use the local approximation, i.e., in Eq. (2.12) we set the r argument of $\underline{n}_{\frac{1}{2}(\mathbf{p}+\mathbf{p}')}(\mathbf{r}, t)$ equal to $\mathbf{r}_1$, so that

$$(\mathbf{p}|\underline{\rho}_1|\mathbf{p}') = \delta(\mathbf{p}-\mathbf{p}')\underline{n}_p(\mathbf{r}_1, t) \tag{2.28}$$

If, as above, we assume that the T-matrix elements depend only on relative momenta, then the momentum integrals simplify considerably. Further we make use of standard symmetry properties[3] of T:

$$(\mathbf{p}|T|\mathbf{p}') = (-\mathbf{p}|T|-\mathbf{p}') = (-\mathbf{p}'|T|-\mathbf{p}) = (\mathbf{p}'|T|\mathbf{p}) \tag{2.29}$$

where the equalities follow by rotational invariance, time reversal invariance, or both simultaneously. We note as well that the spin matrix sums can be collapsed into 2 × 2 matrix products. All these operations are straightforward so we skip the details and immediately write the result that the collision integral, to be placed on the right side of Eq. (2.21) is

$$\begin{aligned}
\underline{C}_2' = {} & \frac{\pi}{\hbar}\int d\mathbf{p}_2 \int d\mathbf{p}_3 \int d\mathbf{p}_4\, \delta(E_{12}-E_{34})\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4) \\
& \times \{|(\mathbf{p}_{12}|T_{12}|\mathbf{p}_{34})|^2[[\underline{n}_{\mathbf{p}_3}, \tilde{\underline{n}}_{\mathbf{p}_1}]_+ \operatorname{tr}_s(\tilde{\underline{n}}_{\mathbf{p}_2}\underline{n}_{\mathbf{p}_4}) - [\tilde{\underline{n}}_{\mathbf{p}_3}, \underline{n}_{\mathbf{p}_1}]_+ \operatorname{tr}_s(\underline{n}_{\mathbf{p}_2}\tilde{\underline{n}}_{\mathbf{p}_4})] \\
& + \eta(\mathbf{p}_{12}|T_{12}|\mathbf{p}_{34})(\mathbf{p}_{34}|T_{12}^\dagger|-\mathbf{p}_{12})[[\underline{n}_{\mathbf{p}_3}\tilde{\underline{n}}_{\mathbf{p}_2}\underline{n}_{\mathbf{p}4}, \tilde{\underline{n}}_{\mathbf{p}_1}]_+ - [\underline{n}_{\mathbf{p}_3}\underline{n}_{\mathbf{p}_2}\underline{n}_{\mathbf{p}_4}, \underline{n}_{\mathbf{p}_1}]_+] \\
& + \eta(\mathbf{p}_{12}|T_{12}|\mathbf{p}_{34})^*(\mathbf{p}_{34}|T_{12}^\dagger|-\mathbf{p}_{12})^*[[\tilde{\underline{n}}_{\mathbf{p}_3}\underline{n}_{\mathbf{p}_2}\tilde{\underline{n}}_{\mathbf{p}4}, \underline{n}_{\mathbf{p}_1}]_+ - [\underline{n}_{\mathbf{p}_3}\underline{n}_{\mathbf{p}_2}\underline{n}_{\mathbf{p}_4}, \underline{n}_{\mathbf{p}_1}]_+]\} \\
& - \frac{1}{i\hbar}\int d\mathbf{p}_2 \int d\mathbf{p}_3 \int d\mathbf{p}_4\, \frac{P}{(E_{12}-E_{34})}\,\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4) \\
& \times \{|(\mathbf{p}_{12}|T_{12}|\mathbf{p}_{34})|^2[[\underline{n}_{\mathbf{p}_3}, \tilde{\underline{n}}_{\mathbf{p}_1}]_- \operatorname{tr}_s(\tilde{\underline{n}}_{\mathbf{p}_2}\underline{n}_{\mathbf{p}_4}) - [\underline{n}_{\mathbf{p}_3}, \underline{n}_{\mathbf{p}_1}]_- \operatorname{tr}_s(\underline{n}_{\mathbf{p}_2}\underline{n}_{\mathbf{p}_4})] \\
& + \eta(\mathbf{p}_{12}|T_{12}|\mathbf{p}_{34})(\mathbf{p}_{34}|T_{12}^\dagger|-\mathbf{p}_{12})[[\underline{n}_{\mathbf{p}_3}\tilde{\underline{n}}_{\mathbf{p}_2}\underline{n}_{\mathbf{p}4}, \tilde{\underline{n}}_{\mathbf{p}_1}]_- \\
& - [\tilde{\underline{n}}_{\mathbf{p}_1}\underline{n}_{\mathbf{p}_2}\underline{n}_{\mathbf{p}_4}, \underline{n}_{\mathbf{p}_1}]_-]\}
\end{aligned} \tag{2.30}$$

n which $\tilde{\underline{n}}_\mathbf{p} = 1 + \eta\underline{n}_\mathbf{p}$. All distribution functions are evaluated at $(\mathbf{r}_1, t)$. There are two general types of terms in (2.30): on-energy-shell and off-nergy-shell. The former are expected in a collision integral, and the latter arise from the forward scattering terms. The off-shell term involves commutators, and therefore adds a correction to the spin rotation term already found n the drift term. It is essentially the so-called I_2 term discussed previously in he literature[1,19] It is second order in T and third order in the distribution unctions and *vanishes in the Boltzmann limit* as speculated by Laloë.[19] No one has yet calculated its value but we will try to estimate its size in Sec. 6. It turns out that it is probably about a 10% correction to spin rotation.

In the Born approximation the on-shell terms reduce to those found previously by the present authors.[1] Furthermore, because we now include a T-matrix interaction, our results for $\underline{C}_2'$ reduce, in the Boltzmann limit, to precisely those previously found by Lhuillier and Laloë.[3] That reduction was limited to the Born approximation only in our previous work. It is easier to see this if one writes $\underline{I}_{\text{coll}} = \underline{C}_2'|_{\text{on shell}}$ in terms of real and imaginary parts of the T matrix elements as follows:

$$\begin{aligned}
\underline{I}_{\text{coll}} = {} & \frac{\pi}{\hbar}\int d\mathbf{p}_2 \int d\mathbf{p}_3 \int d\mathbf{p}_4\, \delta(E_{12}-E_{34})\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4) \\
& \times \{|(\mathbf{p}_{12}|T_{12}|\mathbf{p}_{34})|^2[[\underline{n}_{\mathbf{p}_3}, \tilde{\underline{n}}_{\mathbf{p}_1}]_+ \operatorname{tr}_s(\tilde{\underline{n}}_{\mathbf{p}_2}\underline{n}_{\mathbf{p}_4}) - [\tilde{\underline{n}}_{\mathbf{p}_3}, \underline{n}_{\mathbf{p}_1}]_+ \operatorname{tr}_s(\underline{n}_{\mathbf{p}_2}\tilde{\underline{n}}_{\mathbf{p}_4})] \\
& + \eta\,\mathrm{Re}[(\mathbf{p}_{12}|T_{12}|\mathbf{p}_{34})(\mathbf{p}_{34}|T_{12}^\dagger|-\mathbf{p}_{12})][[\underline{n}_{\mathbf{p}_3}\tilde{\underline{n}}_{\mathbf{p}_2}\underline{n}_{\mathbf{p}4}, \tilde{\underline{n}}_{\mathbf{p}_1}]_+ - [\tilde{\underline{n}}_{\mathbf{p}_3}\underline{n}_{\mathbf{p}_2}\underline{n}_{\mathbf{p}_4}, \underline{n}_{\mathbf{p}_1}]_+] \\
& + i\eta\,\mathrm{Im}[(\mathbf{p}_{12}|T_{12}|\mathbf{p}_{34})(\mathbf{p}_{34}|T_{12}^\dagger|-\mathbf{p}_{12})] \\
& \times [[\underline{n}_{\mathbf{p}_3}\tilde{\underline{n}}_{\mathbf{p}_2}\underline{n}_{\mathbf{p}4}, \tilde{\underline{n}}_{\mathbf{p}_1}]_+ + [\tilde{\underline{n}}_{\mathbf{p}_3}\underline{n}_{\mathbf{p}_2}\tilde{\underline{n}}_{\mathbf{p}_4}, \underline{n}_{\mathbf{p}_1}]_+ - 2[\underline{n}_{\mathbf{p}_3}\underline{n}_{\mathbf{p}_2}\underline{n}_{\mathbf{p}_4}, \underline{n}_{\mathbf{p}_1}]_+]\}
\end{aligned} \tag{2.31}$$

In the Boltzmann limit one sets all $\tilde{\underline{n}}$ to unity and the last four-n anticommutator to zero. The limit is straightforward for all the terms except the lateral spin rotation term. This is the last term in Eq. (2.31). It becomes

$$i\eta\,\mathrm{Im}[(\mathbf{p}_{12}|T_{12}|\mathbf{p}_{34})(\mathbf{p}_{34}|T_{12}^\dagger|-\mathbf{p}_{12})][2\underline{n}_{\mathbf{p}_3}\underline{n}_{\mathbf{p}_4} + [\underline{n}_{\mathbf{p}_2}, \underline{n}_{\mathbf{p}_1}]_+]$$

Interchange of 3 and 4 in one of the $\underline{n}_3\underline{n}_4$ terms produces $-\underline{n}_4\underline{n}_3$ because of the symmetry properties of the T matrix given in Eq. (2.29). A similar argument shows that the term in $\{\underline{n}_2, n_1\}_+$ disappears altogether giving the Boltzmann lateral spin rotation term of Ref. 3:

$$i\eta\,\mathrm{Im}[(\mathbf{p}_{12}|T_{12}|\mathbf{p}_{34})(\mathbf{p}_{34}|T_{12}^\dagger|-\mathbf{p}_{12})][\underline{n}_{\mathbf{p}_3}, \underline{n}_{\mathbf{p}_4}]_-$$

It is perhaps useful to summarize in one place the entire kinetic equation we will work with in the rest of this paper. It is

$$\begin{aligned}
& \frac{\partial}{\partial t}\underline{n}_\mathbf{p}(\mathbf{r}, t) - \frac{1}{i\hbar}[\underline{\varepsilon}_\mathbf{p}(\mathbf{r}), \underline{n}_\mathbf{p}(\mathbf{r})]_- - \tfrac{1}{2}[\nabla_\mathbf{r}\underline{\varepsilon}_\mathbf{p}(\mathbf{r}), \nabla_\mathbf{p}\underline{n}_\mathbf{p}(\mathbf{r})]_+ + \tfrac{1}{2}[\nabla_\mathbf{p}\underline{\varepsilon}_\mathbf{p}(\mathbf{r}), \nabla_\mathbf{r}\underline{n}_\mathbf{p}(\mathbf{r})]_+ \\
& \quad = \frac{\pi}{\hbar}\int d\mathbf{p}_2 \int d\mathbf{p}_3 \int d\mathbf{p}_4\, \delta(E_{12}-E_{34})\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4) \\
& \quad \times \{|(\mathbf{p}_{12}|T_{12}|\mathbf{p}_{34})|^2[[\underline{n}_{\mathbf{p}_3}, \tilde{\underline{n}}_{\mathbf{p}_1}]_+ \operatorname{tr}_s(\tilde{\underline{n}}_{\mathbf{p}_2}\underline{n}_{\mathbf{p}_4}) \\
& \quad - [\tilde{\underline{n}}_{\mathbf{p}_3}, \underline{n}_{\mathbf{p}_1}]_+ \operatorname{tr}_s(\underline{n}_{\mathbf{p}_2}\tilde{\underline{n}}_{\mathbf{p}_4})] \\
& \quad + \eta\,\mathrm{Re}[(\mathbf{p}_{12}|T_{12}|\mathbf{p}_{34})(\mathbf{p}_{34}|T_{12}^\dagger|-\mathbf{p}_{12})][[\underline{n}_{\mathbf{p}_3}\tilde{\underline{n}}_{\mathbf{p}_2}\underline{n}_{\mathbf{p}_4}, \tilde{\underline{n}}_{\mathbf{p}_1}]_+ - [\tilde{\underline{n}}_{\mathbf{p}_3}\underline{n}_{\mathbf{p}_2}\tilde{\underline{n}}_{\mathbf{p}_4}, \underline{n}_{\mathbf{p}_1}]_+] \\
& \quad + i\eta\,\mathrm{Im}[(\mathbf{p}_{12}|T_{12}|\mathbf{p}_{34})(\mathbf{p}_{34}|T_{12}^\dagger|-\mathbf{p}_{12})] \\
& \quad \times [[\underline{n}_{\mathbf{p}_3}\tilde{\underline{n}}_{\mathbf{p}_2}\underline{n}_{\mathbf{p}_4}, \tilde{\underline{n}}_{\mathbf{p}_1}]_+ + [\tilde{\underline{n}}_{\mathbf{p}_3}\underline{n}_{\mathbf{p}_2}\tilde{\underline{n}}_{\mathbf{p}_4}, \underline{n}_{\mathbf{p}_1}]_+ - 2[\underline{n}_{\mathbf{p}_3}\underline{n}_{\mathbf{p}_2}\underline{n}_{\mathbf{p}_4}, \underline{n}_{\mathbf{p}_1}]_+]\}
\end{aligned} \tag{2.32}$$

3. SPIN CURRENT

The method of analysis and solution of the kinetic equation given in Eq. (2.32) is similar to that summarized in Ref. 1. However, the analysis differs enough in its treatment that we must summarize it here. For one thing, our previous treatment used only an s-wave Born approximation; here we have a T-matrix which is a function of momentum. Furthermore, we treat mean-field terms more accurately here.

We follow the usual linearization procedure: a zero-order local-equilibrium distribution function $\underline{n}_{\mathbf{p}}^0$ causes the collision integral to vanish. The true solution

$$\underline{n}_{\mathbf{p}}=\underline{n}_{\mathbf{p}}^0+\delta\underline{n}_{\mathbf{p}} \tag{3.1}$$

is inserted into the kinetic equation. The collision integral is then linearized in $\delta\underline{n}_{\mathbf{p}}$.

We can write $\underline{n}_{\mathbf{p}}^0$

$$\underline{n}_{\mathbf{p}}^0=\tfrac{1}{2}(f_{\mathbf{p}}^0\underline{I}+\sigma_{\mathbf{p}}^0\hat{\mathbf{e}}\cdot\underline{\boldsymbol{\sigma}}) \tag{3.2}$$

where

$$f_{\mathbf{p}}^0=\sum_\sigma n_{\mathbf{p}\sigma}^0 \tag{3.3}$$

$$\sigma_{\mathbf{p}}^0=\sum_\sigma \sigma n_{\mathbf{p}\sigma}^0 \tag{3.4}$$

with $\hat{\mathbf{e}}$ the local direction of the magnetization (not necessarily along the external field), $\underline{\boldsymbol{\sigma}}$ the Pauli matrix vector, and

$$n_{\mathbf{p}\sigma}^0=[e^{\beta(\varepsilon_{\mathbf{p}\sigma}-\mu_\sigma)}-\eta]^{-1} \tag{3.5}$$

Here μ_0 is the chemical potential for spin species s that may depend on position and temperature. In our previous calculation we took $\varepsilon_{\mathbf{p}\sigma}$ to be just the kinetic energy and perhaps the Zeeman energy, because the energy conservation delta function that arises from our derivation is written in terms of only the eigenvalues of the single-particle Hamiltonian. However, it would seem more nearly in the spirit of the Landau theory if the single-particle energy in Eq. (3.5) included mean-field effects as given in Eq. (2.24). We can accommodate mean-field terms in the present energy delta function, without making an *ad hoc* modification of the theory, if we use an s-wave expression for them; such mean-field corrections cancel out in the difference $(\varepsilon_1+\varepsilon_2-\varepsilon_4-\varepsilon_4)$. Because these terms are small anyway this will be an adequate approximation for them. The relation between the T-matrix element and the effective potential used to approximate it is

$$(\mathbf{p}|T_{12}|\mathbf{p}')=\frac{1}{h^3}V(|\mathbf{p}-\mathbf{p}'|) \tag{3.6}$$

For an s-wave approximation we set $V(q)=V(0)$. The local equilibrium value of $n_{\mathbf{p}\sigma}^0$ must make the collision integral vanish and this will occur whether in it we use $\varepsilon_{\mathbf{p}\sigma}$ with or without a s-wave mean-field correction, so, on that basis as well, we can choose to include the latter. Thus for use in $n_{\mathbf{p}\sigma}^0$ we write the s-wave form of Eq. (2.24):

$$\varepsilon_{\mathbf{p}\sigma}=\varepsilon_{\mathbf{p}}^0+V(0)(n+\eta n_\sigma) \tag{3.7}$$

where $\varepsilon_{\mathbf{p}}^0$ is just the kinetic energy, so that $\partial\varepsilon_{\mathbf{p}}^0/\partial p_i=v_{\mathbf{p}i}$; n_σ is the density of spin species σ; and n is the total density. It is convenient to rewrite Eq. (3.5) in the form

$$n_{\mathbf{p}\sigma}^0=[e^{\beta(\varepsilon_{\mathbf{p}}^0-\tilde{\mu}_\sigma)}-\eta]^{-1} \tag{3.8}$$

with

$$\tilde{\mu}_\sigma=\mu_\sigma-V(0)(n+\eta n_s) \tag{3.9}$$

The mean-field terms have merely renormalized the chemical potentials. If the system were in equilibrium with an external field, $\varepsilon_{\mathbf{p}}^0$ would also include a Zeeman energy. However, here we envision a situation such as a spin-echo experiment in which the magnetization direction $\hat{\mathbf{e}}$ is not necessarily along the field $\mathbf{B}$. Nevertheless the momentum distribution remains the same—albeit out of equilibrium—and the effect of the field is considered to be included in the value of $\tilde{\mu}_\sigma$.

If we define part of the drift term of Eq. (2.21), not including the spin rotation term, as

$$\frac{D}{Dt}\underline{n}_{\mathbf{p}}(\mathbf{r},t)\equiv\frac{\partial}{\partial t}\underline{n}_{\mathbf{p}}(\mathbf{r},t)-\tfrac{1}{2}[\nabla_{\mathbf{r}}\underline{\varepsilon}_{\mathbf{p}}(\mathbf{r}),\nabla_{\mathbf{p}}\underline{n}_{\mathbf{p}}(\mathbf{r})]_+ +\tfrac{1}{2}[\nabla_{\mathbf{p}}\underline{\varepsilon}_{\mathbf{p}}(\mathbf{r}),\nabla_{\mathbf{r}}\underline{n}_{\mathbf{p}}(\mathbf{r})]_+ \tag{3.10}$$

then it becomes approximately

$$\begin{aligned}\frac{D}{Dt}\underline{n}_{\mathbf{p}}(\mathbf{r},t)=\frac{\partial\underline{n}_{\mathbf{p}}}{\partial t}-\frac{1}{2}\sum_{i,\sigma'}v_{\mathbf{p}i}\biggl\{&\frac{\partial n_{\mathbf{p}\sigma'}^0}{\partial\varepsilon_{\mathbf{p}}^0}\frac{\partial\mu_{\sigma'}}{\partial r_i}\underline{I}+\sigma'\frac{\partial n_{\mathbf{p}\sigma'}^0}{\partial\varepsilon_{\mathbf{p}}^0}\frac{\partial\mu_{\sigma'}}{\partial r_i}\hat{\mathbf{e}}\cdot\underline{\boldsymbol{\sigma}}\\&-\left(\sigma' n_{\mathbf{p}\sigma'}^0-\frac{V(0)}{2}\eta m\frac{\partial n_{\mathbf{p}\sigma'}^0}{\partial\varepsilon_{\mathbf{p}}^0}\right)\frac{\partial\hat{\mathbf{e}}}{\partial r_i}\cdot\underline{\boldsymbol{\sigma}}\biggr\}\end{aligned} \tag{3.11}$$

with $i=x,y,z$,

$$m=\sum_\sigma\sigma n_\sigma=\int\frac{d\mathbf{p}}{h^3}\sigma_{\mathbf{p}}^0 \tag{3.12}$$

the magnetization, and

$$n_\sigma = \int \frac{d\mathbf{p}}{h^3} n^0_{\mathbf{p}\sigma} \tag{3.13}$$

the density of spin species σ. In (3.11) we kept only the local equilibrium value $n^0_{\mathbf{p}\sigma}$ in all but the first term.

As in Ref. 1, we eliminate the chemical potential in favor of the magnetization m and density n by use of the Gibbs–Duhem relation

$$\sum_\sigma n_\sigma \nabla \mu_\sigma = 0 \tag{3.14}$$

and the relation

$$n_\sigma = \tfrac{1}{2}(\sigma m + n) \tag{3.15}$$

We find from differentiation of Eq. (3.13)

$$\nabla n_\sigma = G_\sigma \nabla \bar{\mu}_\sigma \tag{3.16}$$

where

$$G_\sigma = -\int \frac{d\mathbf{p}}{h^3} \frac{\partial n^0_{\mathbf{p}\sigma}}{\partial \varepsilon^0_{\mathbf{p}}} \tag{3.17}$$

n this equation gradients in the external field are neglected.

The use of Eqs. (3.9), (3.14), and (3.15) allows us to eliminate μ_σ from he relation, and after a bit of algebra we find

$$\nabla \mu_\sigma = \sigma t_\sigma \nabla m \tag{3.18}$$

here

$$t_\sigma = n_{-\sigma} \frac{\left\{\dfrac{1}{G_+ G_-} + V(0)(1+\eta)\left(\dfrac{1}{G_+} + \dfrac{1}{G_-}\right) + (2+\eta)\eta V(0)^2\right\}}{\left\{\left(\dfrac{n_+}{G_+} + \dfrac{n_-}{G_-}\right) + (2+\eta)V(0)n\right\}} \tag{3.19}$$

hen we substitute this back into the drift term Eq. (3.11) we find

$$\begin{aligned}\frac{D\underline{n}_{\mathbf{p}}}{Dt} = \frac{\partial \underline{n}_{\mathbf{p}}}{\partial t} + \frac{1}{2}\sum_{i\sigma'} v_{\mathbf{p}i}\Bigg\{&-\frac{\partial n^0_{\mathbf{p}\sigma'}}{\partial \varepsilon^0_{\mathbf{p}}}\sigma' t_{\sigma'}\frac{\partial m}{\partial r_i}\underline{I} - \frac{\partial n^0_{\mathbf{p}\sigma'}}{\partial \varepsilon^0_{\mathbf{p}}} t_{\sigma'} \frac{\partial m}{\partial r_i}\hat{\mathbf{e}}\cdot\underline{\sigma}\\ &+\left(\sigma' n^0_{\mathbf{p}\sigma'} - \frac{V(0)}{2}\eta m \frac{\partial n^0_{\mathbf{p}\sigma'}}{\partial \varepsilon^0_{\mathbf{p}}}\right)\frac{\partial \hat{\mathbf{e}}}{\partial r_i}\cdot\underline{\sigma}\Bigg\}\end{aligned} \tag{3.20}$$

The spin-rotation commutator can also be considered as part of the drift term. We have

$$\left.\frac{D\underline{n}_{\mathbf{p}}}{Dt}\right|_{\text{spin rot}} = \frac{i}{\hbar}[\underline{\varepsilon}_{\mathbf{p}}, \underline{n}_{\mathbf{p}}]_- \tag{3.21}$$

This commutator describes the precession about the external magnetic field as well as the mean-field exchange precession called identical-particle spin rotation. There is also a lateral scattering contribution to spin rotation as discussed at the end of Sec. 2. However, it depends on the imaginary part of the product of T-matrix elements as shown there. We are going to assume a *real* phenological effective potential for the T-matrix and so this lateral scattering term will not contribute. There is also the off-shell contribution in Eq. (2.30); we neglect this for now, but estimate its size in Sec. 6. Because of the relative magnitude of the spin rotation term of Eq. (3.21), we should *not* make an s-wave approximation to the mean-field contributions in $\underline{\varepsilon}_{\mathbf{p}}$. In fact it will turn out that the non-s-wave contributions are not very important, but that is an item that we want to test. From Eq. (2.24), Eq. (3.21) becomes, to first order in $\delta\underline{n}_{\mathbf{p}}$,

$$\begin{aligned}\left.\frac{D\underline{n}_{\mathbf{p}_1}}{Dt}\right|_{\text{spin rot}} = &-\frac{i\gamma}{2}[\underline{\sigma}\cdot\mathbf{B}, \underline{n}_{\mathbf{p}_1}]_- + \frac{i\eta}{\hbar}\int d\mathbf{p}_2\,\mathrm{Re}(\mathbf{p}_{12}|T_{12}|-\mathbf{p}_{12})\\ &\times\{[\underline{n}^0_{\mathbf{p}_2}, \delta\underline{n}_{\mathbf{p}_1}]_- + [\delta\underline{n}_{\mathbf{p}_2}, \underline{n}^0_{\mathbf{p}_1}]_-\}\end{aligned} \tag{3.22}$$

where the Zeeman term is explicitly included, with $\mathbf{B}$ the external field and γ the gyromagnetic ratio. If we write

$$\delta\underline{n}_{\mathbf{p}} = \tfrac{1}{2}(\delta f_{\mathbf{p}}\underline{I} + \delta\boldsymbol{\sigma}_{\mathbf{p}}\cdot\underline{\sigma}) \tag{3.23}$$

in analogy to Eq. (3.2), then we find

$$\begin{aligned}\left.\frac{D\underline{n}_{\mathbf{p}_1}}{Dt}\right|_{\text{spin rot}} = &\frac{\gamma}{2}[\mathbf{B}\times(\sigma^0_{\mathbf{p}_1}\hat{\mathbf{e}} + \delta\sigma_{\mathbf{p}_1})\cdot\underline{\sigma}] - \frac{\eta}{2\hbar}\int d\mathbf{p}_2\,\mathrm{Re}(\mathbf{p}_{12}|T_{12}|-\mathbf{p}_{12})\\ &\times\{\sigma^0_{\mathbf{p}_2}\hat{\mathbf{e}}\times\delta\boldsymbol{\sigma}_{\mathbf{p}_1} + \delta\boldsymbol{\sigma}_{\mathbf{p}_2}\times\sigma^0_{\mathbf{p}_1}\hat{\mathbf{e}}\}\cdot\underline{\sigma}\end{aligned} \tag{3.24}$$

From Eqs. (3.20) and (3.24) we can get an equation for the spin current by multiplying the kinetic equation by $\underline{\sigma} v_{\mathbf{p}j}/h^3$, integrating over $\mathbf{p}_1$, and taking a spin trace, because the current is given by

$$J_j(\mathbf{m}) = \int \frac{d\mathbf{p}}{h^3}\,\mathrm{tr}_s(v_{\mathbf{p}j}\,\underline{\boldsymbol{\sigma}}\,\delta\underline{n}_{\mathbf{p}}) = \int \frac{d\mathbf{p}}{h^3} v_{\mathbf{p}j}\,\delta\boldsymbol{\sigma}_{\mathbf{p}_1} \tag{3.25}$$

There are no mean-field corrections to this current in s-wave approximation. The resulting equation is easily found to be

$$\frac{\partial J_j(\mathbf{m})}{\partial t}+\alpha_\parallel \frac{\partial m}{\partial r_j}\hat{\mathbf{e}}+\alpha_\perp m\frac{\partial \hat{\mathbf{e}}}{\partial r_j}+\gamma \mathbf{B}\times J_j(\mathbf{m})$$
$$-\frac{\eta}{\hbar h^3}\int d\mathbf{p}_1\int d\mathbf{p}_2\,\mathrm{Re}(\mathbf{p}_{12}|T_{12}|-\mathbf{p}_{12})v_{\mathbf{p}_1 j}(\sigma^0_{\mathbf{p}_2}\hat{\mathbf{e}}\times\delta\boldsymbol{\sigma}_{\mathbf{p}_1}+\delta\boldsymbol{\sigma}_{\mathbf{p}_2}\times\sigma^0_{\mathbf{p}_1}\hat{\mathbf{e}})$$
$$=\int\frac{d\mathbf{p}}{h^3}\,\mathrm{tr}_s[v_{\mathbf{p}j}\underline{\boldsymbol{\sigma}}\underline{I}_{\mathrm{coll}}] \tag{3.26}$$

where

$$\alpha_\parallel=\sum_\sigma t_\sigma n_\sigma/m^* \tag{3.27}$$

and

$$\alpha_\perp=\frac{2}{3m^*m}\sum_{\sigma'}\sigma' k_{\sigma'}+\frac{\eta V(0)n}{2m^*} \tag{3.28}$$

where m^* is the hydrodynamic mass for an ^{3}He atom in liquid ^{4}He, and

$$k_\sigma=\int\frac{d\mathbf{p}}{h^3}\,\varepsilon^0_{\mathbf{p}}n^0_{\mathbf{p}\sigma} \tag{3.29}$$

We are going to consider a "quasi-steady-state" solution[5] of (3.26), which means that the only time dependence for $J_j(\mathbf{m})$ is that caused by the precession about the magnetic field. That is, $J_j(\mathbf{m})$ is assumed time independent in the frame rotating at the Larmor frequency. Henceforth, then, in our solution for $\delta\underline{n}_{\mathbf{p}}$ for inclusion in the current equation, we will neglect the term $\partial\underline{n}_{\mathbf{p}}/\partial t$ and the external field precession term. In order to solve Eq. (3.26) we must now give a variational form for $\delta\underline{n}_{\mathbf{p}}$.

To do this it is helpful to write Eq. (3.20) as a matrix notation in which the coordinate system for the Pauli matrices is $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, $\hat{\mathbf{e}}$, three unit vectors with $\hat{\mathbf{e}}$ along the direction of the magnetization as in Eq. (3.2) and $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ two unit vectors perpendicular to $\hat{\mathbf{e}}$. The drift term, excluding spin rotation, then becomes

$$\frac{D\underline{n}^0_{\mathbf{p}}}{Dt}=\sum_i v_{\mathbf{p}i}\begin{bmatrix}-\dfrac{\partial n^0_{\mathbf{p}+}}{\partial\varepsilon^0_{\mathbf{p}}}t_+\dfrac{\partial m}{\partial r_i} & \frac{1}{2}s_{\mathbf{p}}\phi_{i-}\\ \frac{1}{2}s_{\mathbf{p}}\phi_{i+} & \dfrac{\partial n^0_{\mathbf{p}-}}{\partial\varepsilon^0_{\mathbf{p}}}t_-\dfrac{\partial m}{\partial r_i}\end{bmatrix} \tag{3.30}$$

where

$$s_{\mathbf{p}}=\sigma^0_{\mathbf{p}}+\eta\frac{V(0)}{2}m\sum_\sigma\left(-\frac{\partial n^0_{\mathbf{p}\sigma}}{\partial\varepsilon^0_{\mathbf{p}}}\right) \tag{3.31}$$

and the ϕ_i are the components of the vector $\partial\hat{\mathbf{e}}/\partial r_i$, or

$$\frac{\partial\hat{\mathbf{e}}}{\partial r_i}=\phi_{iu}\hat{\mathbf{u}}+\phi_{iv}\hat{\mathbf{v}} \tag{3.32}$$

and

$$\phi_{i\xi}=\phi_{iu}+i\xi\phi_{iv} \tag{3.33}$$

where $\xi=\pm 1$. (Note that we now have *three* variables that can be ± 1: σ which represents the spin component along the direction $\hat{\mathbf{e}}$, η which is $+1$ for bosons and -1 for fermions, and now ξ, which gives the complex combinations of the u and v components of a vector perpendicular to $\hat{\mathbf{e}}$.)

In either form, Eq. (3.20) or (3.30) it is apparent that $D\underline{n}_{\mathbf{p}}/Dt$ contains terms that are transverse to the local magnetization direction $\hat{\mathbf{e}}$. A solution to the kinetic equation of the form of Eq. (3.1) requires[29] that $\delta\underline{n}_{\mathbf{p}}$ be orthogonal to the collisional invariants, 1, p, σ. A further condition for a solution to exist is that the drift term also be orthogonal to the collisional invariants. It is easy to verify that $D\underline{n}_{\mathbf{p}}/Dt$ is so orthogonal. A simple variational solution then, which insures the proper orthogonality, is a constant times the drift term itself. We assume therefore that

$$\delta\underline{n}_{\mathbf{p}}=\begin{bmatrix}\delta n_{\mathbf{p}+} & \frac{1}{2}\delta\sigma_{\mathbf{p}-}\\ \frac{1}{2}\delta\sigma_{\mathbf{p}+} & \delta n_{\mathbf{p}-}\end{bmatrix} \tag{3.34}$$

where

$$\delta n_{\mathbf{p}\sigma}=A_\parallel\sum_i v_{\mathbf{p}i}\frac{\partial n^0_{\mathbf{p}\sigma}}{\partial\varepsilon^0_{\mathbf{p}}}\sigma t_\sigma\frac{\partial m}{\partial r_i} \tag{3.35}$$

and

$$\delta\sigma_{\mathbf{p}\xi}=-A^\perp_\xi\sum_i v_{\mathbf{p}i}\sigma^0_{\mathbf{p}}\phi_{i\xi} \tag{3.36}$$

with the A's being variational constants. Note that the transverse distribution function correction $\delta\sigma_{\mathbf{p}\xi}$ depends on $\sigma^0_{\mathbf{p}}=n^0_{\mathbf{p}+}-n^0_{\mathbf{p}-}$, that is, all spins between the two Fermi sufaces are out of equilibrium. On the other hand, the longitudinal correction $\delta n_{\mathbf{p}\sigma}$ is, as usual, localized on the Fermi surface by the factor $\partial n^0_{\mathbf{p}\sigma}/\partial \varepsilon^0_{\mathbf{p}}$. In designing $\delta\sigma_{\mathbf{p}\xi}$ we have avoided including the small mean-field correction that occurs in the off-diagonal elements of $D\underline{n}_{\mathbf{p}}/Dt$. The resulting $\delta\underline{n}_{\mathbf{p}}$ remains orthogonal to the collisional invariants, but is easier to handle mathematically; appropriate adjustments will occur in the variational constants.

Because $\delta\underline{n}_{\mathbf{p}}$ has off-diagonal terms, so does $J_j(\mathbf{m})$, given in Eq. (3.25). That is, there are longitudinal and transverse components to $J_j(\mathbf{m})$, which means that they are parallel and perpendicular, respectively, to the local magnetization itself. We will see that there are two corresponding relaxation times given by the theory, one for longitudinal spin diffusion and another for transverse spin diffusion, and these two times *need not be equal*. The standard spin echo experiment detects the transverse spin current, but a recent experiment by Nunes *et al.*[11] has detected longitudinal spin diffusion. If we insert the trial forms for $\delta\underline{n}_{\mathbf{p}}$, we find simple expressions for the spin urrents. From Eqs. (3.25), (3.35), and (3.36) we find

$$J_j(\mathbf{m}) = J_j^{\parallel}(\mathbf{m}) + J_j^{\perp}(\mathbf{m}) \tag{3.37}$$

he longitudinal component is

$$\begin{aligned} J_j^{\parallel}(\mathbf{m}) &= \int \frac{d\mathbf{p}}{h^3} \sum_\sigma v_{\mathbf{p}j}\, \sigma\, \delta\underline{n}_{\mathbf{p}\sigma} \hat{\mathbf{e}} = \frac{\partial m}{\partial r_j} \sum_\sigma A_\parallel t_\sigma \int \frac{d\mathbf{p}}{h^3}\, v_{\mathbf{p}j}^2 \frac{\partial n^0_{\mathbf{p}\sigma}}{\partial \varepsilon^0_{\mathbf{p}}} \hat{\mathbf{e}} \\ &= -\frac{\partial m}{\partial r_j} \sum_\sigma A_\parallel \frac{t_\sigma n_\sigma}{m^*} \hat{\mathbf{e}} \end{aligned} \tag{3.38}$$

e note here that the particle current is

$$\begin{aligned} J_j^{f} &= \int \frac{d\mathbf{p}}{h^3} \sum_\sigma v_{\mathbf{p}j} \delta n_{\mathbf{p}\sigma} = \frac{\partial m}{\partial r_j} \sum_\sigma A_\parallel \sigma t_\sigma \int \frac{d\mathbf{p}}{h^3}\, v_{\mathbf{p}j}^2 \frac{\partial n^0_{\mathbf{p}\sigma}}{\partial \varepsilon^0_{\mathbf{p}}} \\ &= -\frac{\partial m}{\partial r_j} \sum_\sigma A_\parallel \frac{\sigma t_\sigma n_\sigma}{m^*} \end{aligned} \tag{3.39}$$

e require this to vanish in a diffusion experiment. The product $t_\sigma n_\sigma$ is lependent of σ as can be seen from Eq. (3.19) so that the particle current

automatically vanishes. Thus

$$J_j^{\parallel}(\mathbf{m}) = -D_\parallel \frac{\partial m}{\partial r_j} \hat{\mathbf{e}} \tag{3.40}$$

where

$$D_\parallel = \alpha_\parallel A_\parallel \tag{3.41}$$

We can easily relate $\alpha_\parallel$, from its definition in Eq. (3.27), to the average value of $v_{\mathbf{p}}^2$ and thus identify $A_\parallel$ as the longitudinal spin diffusion relaxation time. In the next section we will solve for $A_\parallel$.

The transverse spin current of Eq. (3.37) is

$$J_j^{\perp}(\mathbf{m}) = J_{ju}^{\perp}\hat{\mathbf{u}} + J_{jv}^{\perp}\hat{\mathbf{v}} \tag{3.42}$$

We can form the combinations

$$\begin{aligned} J_{j\xi}^{\perp} = J_{ju}^{\perp} + i\xi J_{jv}^{\perp} &= \int \frac{d\mathbf{p}}{h^3} v_{\mathbf{p}j}\, \delta\sigma_{\mathbf{p}\xi} = -A_\xi^{\perp} \phi_{j\xi} \int \frac{d\mathbf{p}}{h^3} v_{\mathbf{p}j}^2 \sigma^0_{\mathbf{p}} \\ &= -A_\xi^{\perp} \phi_{j\xi} \bar{\alpha}_\perp m \end{aligned} \tag{3.43}$$

where

$$\bar{\alpha}_\perp = \frac{2}{3m^* m} \sum_{\sigma'} \sigma' k_{\sigma'} \tag{3.44}$$

The solution for $A_\xi^{\perp}$ turns out to be complex so that we will need to wait for that solution before we try to identify a transverse diffusion constant and a relaxation time.

Finally we need to write the spin-rotation terms in matrix form. We can see from Eq. (3.24) that there are only off-diagonal, i.e., transverse, contributions to this quantity.

$$\begin{aligned} \left[\frac{D\underline{n}_{\mathbf{p}_1}}{Dt} \bigg|_{\text{spin rot}} \right]_\xi &= \frac{i\eta\xi}{2\hbar} \int d\mathbf{p}_2\, \mathrm{Re}(\mathbf{p}_{12}|T_{12}| - \mathbf{p}_{12}) \{\sigma^0_{\mathbf{p}_1} \delta\sigma_{\mathbf{p}_2\xi} - \sigma^0_{\mathbf{p}_2} \delta\sigma_{\mathbf{p}_1\xi}\} \\ &= \frac{i\eta\xi}{2\hbar} A_\xi^{\perp} \sum_i \phi_{i\xi} \int d\mathbf{p}_2\, \mathrm{Re}(\mathbf{p}_{12}|T_{12}| - \mathbf{p}_{12}) (v_{\mathbf{p}_1 i} - v_{\mathbf{p}_2 i}) \sigma^0_{\mathbf{p}_1} \sigma^0_{\mathbf{p}_2} \end{aligned} \tag{3.45}$$

We now move on to consider the linearization and calculation of the collision integral, and the solution of the kinetic equation for the variational parameters.

4. SOLUTION OF THE KINETIC EQUATION

Linearization of the collision integral of Eq. (2.1) leads us to

$$
\begin{aligned}
\underline{I}_{\text{coll}}=\frac{\pi}{\hbar}\int d\mathbf{p}_2\int d\mathbf{p}_3\int d\mathbf{p}_4\,\delta(E_{12}-E_{34})\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)\\
\times\{|(12|T|34)|^2[([\delta\underline{n}_3,\tilde{\underline{n}}_1^0]_++[\underline{n}_3^0,\delta\tilde{\underline{n}}_1]_+)\,\mathrm{tr}_s(\tilde{\underline{n}}_2^0\underline{n}_4^0)\\
+[\underline{n}_3^0,\tilde{\underline{n}}_1^0]_+\,\mathrm{tr}_s(\delta\tilde{\underline{n}}_2n_4^0+\tilde{\underline{n}}_2^0\delta\underline{n}_4)-(\tilde{\underline{n}}_i^0\leftrightarrow\underline{n}_i^0,\,\delta\tilde{\underline{n}}_i\leftrightarrow\delta\underline{n}_i)]\\
+\eta\,\mathrm{Re}[(12|T|34)(34|T^\dagger|21)][[\delta\underline{n}_3\tilde{\underline{n}}_2^0\underline{n}_4^0+\underline{n}_3^0\delta\tilde{\underline{n}}_2\underline{n}_4^0+\underline{n}_3^0\tilde{\underline{n}}_2^0\delta\underline{n}_4,\tilde{\underline{n}}_1^0]_+\\
+[\underline{n}_3^0\tilde{\underline{n}}_2^0\underline{n}_4^0,\delta\tilde{\underline{n}}_1]_+-(\tilde{\underline{n}}_i^0\leftrightarrow\underline{n}_i^0,\,\delta\tilde{\underline{n}}_i\leftrightarrow\delta\underline{n}_i)]\\
+i\eta\,\mathrm{Im}[(12|T|34)(34|T^\dagger|21)][[\delta\underline{n}_3\tilde{n}_2^0\underline{n}_4^0+\underline{n}_3^0\delta\tilde{\underline{n}}_2\underline{n}_4^0+\underline{n}_3^0\tilde{\underline{n}}_2^0\delta\underline{n}_4,\tilde{\underline{n}}_1^0]_+\\
+[\underline{n}_3^0\tilde{\underline{n}}_2^0\underline{n}_4^0,\delta\tilde{\underline{n}}_1]_++(\tilde{\underline{n}}_i^0\leftrightarrow\underline{n}_i^0,\,\delta\tilde{\underline{n}}_i\leftrightarrow\delta\underline{n}_i)\\
-2[\delta\underline{n}_3\underline{n}_2^0n_4^0+\underline{n}_3^0\delta\underline{n}_2\underline{n}_4^0+\underline{n}_3^0\underline{n}_2^0\delta\underline{n}_4,\underline{n}_1^0]_+-2[\underline{n}_3^0\underline{n}_2^0\underline{n}_4^0,\delta\underline{n}_1]_+]\}
\end{aligned}
\tag{4.1}
$$

in which ↔ means interchange. As indicated by Eq. (3.34), $\delta\underline{n}_{\mathbf{p}}$ has both diagonal and off-diagonal elements, while $\underline{n}_{\mathbf{p}}^0$ is completely diagonal. Thus a typical term in the linearized collision integral, Eq. (4.1), will have one nondiagonal distribution function factor and the others diagonal. It is then straightforward to carry out the necessary matrix multiplications to arrive at diagonal and off-diagonal elements of the collision integral.

4.1. Longitudinal Case

The diagonal elements turn out to be

$$
\begin{aligned}
I_{\text{coll}}|_\sigma^{\text{diag}}=\frac{2\pi}{\hbar}\int d\mathbf{p}_2\int d\mathbf{p}_3\int d\mathbf{p}_4\delta(E_{12}-E_{34})\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)\\
\times\Bigg\{|(12|T|34)|^2\Bigg[(\delta n_{3\sigma}\tilde{n}_{1\sigma}^0+n_{3\sigma}^0\delta\tilde{n}_{1\sigma})\sum_{\sigma'}\tilde{n}_{2\sigma'}^0n_{4\sigma'}^0+n_{3\sigma}^0\tilde{n}_{1\sigma}^0\\
\times\sum_{\sigma'}(\delta\tilde{n}_{2\sigma'}n_{4\sigma'}^0+\tilde{n}_{2\sigma'}\delta n_{4\sigma'})-(\tilde{n}_{i\sigma}^0\leftrightarrow n_{i\sigma}^0,\,\delta\tilde{n}_{i\sigma}\leftrightarrow\delta n_{i\sigma})\Bigg]\\
+\eta\,\mathrm{Re}[(12|T|34)(34|T^\dagger|21)][\delta n_{3\sigma}\tilde{n}_{2\sigma}^0n_{4\sigma}^0\tilde{n}_{1\sigma}^0+n_{3\sigma}^0\delta\tilde{n}_{2\sigma}n_{4\sigma}^0\tilde{n}_{1\sigma}^0\\
+n_{3\sigma}^0\tilde{n}_{2\sigma}^0\delta n_{4\sigma}\tilde{n}_{1\sigma}^0+n_{3\sigma}^0\tilde{n}_{2\sigma}^0n_{4\sigma}^0\delta\tilde{n}_{1\sigma}-(\tilde{n}_{i\sigma}^0\leftrightarrow n_{i\sigma}^0,\,\delta\tilde{n}_{i\sigma}\leftrightarrow\delta n_{i\sigma})]\\
+i\eta\,\mathrm{Im}[(12|T|34)(34|T^\dagger|21)][\delta n_{3\sigma}\tilde{n}_{2\sigma}^0n_{4\sigma}^0\tilde{n}_{1\sigma}^0+n_{3\sigma}^0\delta\tilde{n}_{2\sigma}n_{4\sigma}^0\tilde{n}_{1\sigma}^0\\
+n_{3\sigma}^0\tilde{n}_{2\sigma}^0\delta n_{4\sigma}\tilde{n}_{1\sigma}^0+n_{3\sigma}^0\tilde{n}_{2\sigma}^0n_{4\sigma}^0\delta\tilde{n}_{1\sigma}+(\tilde{n}_{i\sigma}^0\leftrightarrow n_{i\sigma}^0,\,\delta\tilde{n}_{i\sigma}\leftrightarrow\delta n_{i\sigma})
\end{aligned}
$$

$$
\begin{aligned}
-2(\delta n_{3\sigma}n_{2\sigma}^0n_{4\sigma}^0n_{1\sigma}^0+n_{3\sigma}^0\delta n_{2\sigma}n_{4\sigma}^0n_{1\sigma}^0+n_{3\sigma}^0n_{2\sigma}^0\delta n_{4\sigma}n_{1\sigma}^0\\
+n_{3\sigma}^0n_{2\sigma}^0n_{4\sigma}^0\delta n_{1\sigma})]\Bigg\}
\end{aligned}
\tag{4.2}
$$

It is easy to see that, since $\mathrm{Im}(12|T|34)(34|T^\dagger|21)$ is odd in the interchange of 3 and 4, and the remaining factors are even, that this last term vanishes even if T is not taken as real. In the remaining terms one extracts a factor $n_{3\sigma}^0n_{4\sigma'}^0\tilde{n}_{1\sigma}^0\tilde{n}_{2\sigma'}^0=\tilde{n}_{3\sigma}^0\tilde{n}_{4\sigma'}^0n_{1\sigma}^0n_{2\sigma'}$ (equality following by energy conservation) from each term and also uses

$$
\delta\tilde{n}_{p\sigma}=\delta(1+\eta n_{p\sigma})=\eta\delta n_{p\sigma}
\tag{4.3}
$$

$$
\frac{\delta n_{p\sigma}}{n_{p\sigma}^0}-\frac{\delta\tilde{n}_{p\sigma}}{\tilde{n}_{p\sigma}^0}=\frac{\delta n_{p\sigma}}{n_{p\sigma}^0\tilde{n}_{p\sigma}^0}
\tag{4.4}
$$

and

$$
\frac{\partial n_{p\sigma}^0}{\partial\varepsilon_{p\sigma}^0}=-\beta n_{p\sigma}^0\tilde{n}_{p\sigma}^0
\tag{4.5}
$$

Then we find

$$
\begin{aligned}
I_{\text{coll}}|_\sigma^{\text{diag}}=\frac{2\pi\beta}{\hbar}\int d\mathbf{p}_2\int d\mathbf{p}_3\int d\mathbf{p}_4\,\delta(E_{12}-E_{34})\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)\\
\times\Bigg\{(|(12|T|34)|^2+\eta\,\mathrm{Re}[(12|T|34)(34|T^\dagger|21)])n_{3\sigma}^0n_{4\sigma}^0\tilde{n}_{1\sigma}^0\tilde{n}_{2\sigma}^0\\
\times\left[\left(\frac{\partial n_{1\sigma}^0}{\partial\varepsilon_1^0}\right)^{-1}\delta n_{1\sigma}+\left(\frac{\partial n_{2\sigma}^0}{\partial\varepsilon_2^0}\right)^{-1}\delta n_{2\sigma}-\left(\frac{\partial n_{3\sigma}^0}{\partial\varepsilon_3^0}\right)^{-1}\delta n_{3\sigma}-\left(\frac{\partial n_{4\sigma}^0}{\partial\varepsilon_4^0}\right)^{-1}\delta n_{4\sigma}\right]\\
+|(12|T|34)|^2n_{3\sigma}^0n_{4-\sigma}^0\tilde{n}_{1\sigma}^0\tilde{n}_{2-\sigma}^0\\
\times\Bigg[\left(\frac{\partial n_{1\sigma}^0}{\partial\varepsilon_1^0}\right)^{-1}\delta n_{1\sigma}+\left(\frac{\partial n_{2-\sigma}^0}{\partial\varepsilon_2^0}\right)^{-1}\delta n_{2-\sigma}\\
-\left(\frac{\partial n_{3\sigma}^0}{\partial\varepsilon_3^0}\right)^{-1}\delta n_{3\sigma}-\left(\frac{\partial n_{4-\sigma}^0}{\partial\varepsilon_4^0}\right)^{-1}\delta n_{4-\sigma}\Bigg]\Bigg\}
\end{aligned}
\tag{4.6}
$$

We next substitute the form of Eq. (3.35) for $\delta n_{\mathbf{p}\sigma}$, multiply both sides of the diagonal elements of the kinetic equation by $v_{\mathbf{p}j}$ and integrate over $\mathbf{p}_1$. The left side of the equation contains the drift terms, given by the diagonal elements of Eq. (3.30); the right side the collision integral of (4.6). One can show that the term from Eq. (4.6) in $|(12|T|12)|^2 + \eta\,\mathrm{Re}(12|T|34)(34|T^\dagger|21)$ vanishes by momentum conservation and we find the result

$$\frac{n_\sigma}{m^*} = \frac{2\pi\beta}{\hbar h^3}\,\frac{A_\parallel(n_\sigma + n_-)}{n_{-\sigma}} \int d\mathbf{p}_1 \int d\mathbf{p}_2 \int d\mathbf{p}_3 \int d\mathbf{p}_4\, \delta(E_{12} - E_{34})$$

$$\times\, \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)|(12|T|34)|^2 n^0_{3\sigma} n^0_{4-\sigma} \tilde{n}^0_{1\sigma} \tilde{n}^0_{2-\sigma} v_{1j}(v_{1j} - v_{3j}) \tag{4.7}$$

We can solve this equation for the longitudinal relaxation time $\tau_\parallel = A_\parallel$. Clearly

$$\frac{1}{\tau_\parallel} = \frac{1}{A_\parallel} = \frac{2\pi\beta m^*}{\hbar h^3}\,\frac{(n_+ + n_-)}{n_+ n_-} \int d\mathbf{p}_1 \int d\mathbf{p}_2 \int d\mathbf{p}_3 \int d\mathbf{p}_4\, \delta(E_{12} - E_{34})$$

$$\times\, \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)|(12|T|34)|^2 n^0_{3\sigma} n^0_{4-\sigma} \tilde{n}^0_{1\sigma} \tilde{n}^0_{2-\sigma} v_{1j}(v_{1j} - v_{3j}) \tag{4.8}$$

This is essentially the same formula as we found in Refs. 30, but now the T-natrix element replaces the s-wave approximation $V(0)^2$. This result is used n Eq. (3.41) to find the longitudinal diffusion constant.

4.2. Transverse Case

To treat the transverse case we look at the off-diagonal terms in the rift terms and collision integral. The first group of transverse parts of the rift terms are given as the off-diagonal elements of Eq. (3.30). We multiply em by v_{1j}/h^3 and integrate over all $\mathbf{p}_1$. The integral is

$$\int \frac{d\mathbf{p}_1}{h^3}\, v_{1j}\, \frac{Dn^0_{\mathbf{p}1}}{Dt}\bigg|_\xi^{\text{offdiag}} = \frac{1}{2}\,\phi_{j\xi}\, a_\perp m \tag{4.9}$$

lso included in the off-diagonal drift terms is the spin-rotation term Eq. .45). Multiplying that by v_{1j}/h^3 and integrating over all $\mathbf{p}_1$ gives the integral call I_ξ^{spinrot} which is

$$I_\xi^{\text{spinrot}} = \frac{\pi i \eta}{h^4}\, \xi A_\xi^\perp \phi_{j\xi} \int d\mathbf{p}_1 \int d\mathbf{p}_2\, \mathrm{Re}(12|T|21)(n^0_{1+} - n^0_{1-})$$

$$\times\, (n^0_{2+} - n^0_{2-}) v_{1j}(v_{1j} - v_{2j}) \tag{4.10}$$

The off-diagonal elements of the collision integral of Eq. (4.1) are found be

$$I_{\text{coll}}^{\text{offdiag}} = \frac{\pi}{2\hbar} \int d\mathbf{p}_2 \int d\mathbf{p}_3 \int d\mathbf{p}_4\, \delta(E_{12} - E_{34})\delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)$$

$$\times \Big\{ |(12|T|34)|^2 \Big[\{(\tilde{n}^0_{1+} + \tilde{n}^0_{1-})\delta\sigma_{3\xi} + (n^0_{3+} + n^0_{3-})\delta\tilde{\sigma}_{1\xi}\}$$

$$\times \left(\sum_{\sigma'} \tilde{n}^0_{2\sigma'} n^0_{4\sigma'}\right) - (\tilde{n}^0_i \leftrightarrow n^0_i,\ \delta\tilde{\sigma}_i \leftrightarrow \delta\sigma_i) \Big]$$

$$+ \eta\,\mathrm{Re}[(12|T|34)(34|T^\dagger|21)]$$

$$\times \Big[(n^0_{1+} + n^0_{1-})(\delta\sigma_{3\xi} \tilde{n}^0_{2\xi} n^0_{4\xi} + n^0_{3-\xi}\delta\tilde{\sigma}_{2\xi} n^0_{4\xi} + n^0_{3-\xi}\tilde{n}^0_{2-\xi}\delta\sigma_{4\xi})$$

$$+ \left(\sum_{\sigma'} n^0_{3\sigma'} \tilde{n}^0_{2\sigma'} n^0_{4\sigma'}\right) \delta\tilde{\sigma}_{1\xi} - (\tilde{n}^0_i \leftrightarrow n^0_i,\ \delta\tilde{\sigma}_i \leftrightarrow \delta\sigma_i) \Big]\Big\} \tag{4.11}$$

in which $\frac{1}{2}\delta\tilde{\sigma}_\mathbf{p}$ is the off-diagonal part of $\delta\tilde{n}_\mathbf{p}$ or

$$\delta\tilde{\sigma}_\mathbf{p} = \eta\,\delta\sigma_\mathbf{p} \tag{4.12}$$

When we multiply this by v_{1i}/h^3 and integrate over $\mathbf{p}_1$, and then substitute the trial solution for $\delta\sigma_\mathbf{p}$ we are able to reduce the integral into a form amenable to computation. The calculation is straightforward but tedious and we will not give the details here but just quote the results. In the calculation we use the symmetry properties of the T-matrix elements (Eqs. (2.29)) and relations like

$$n^0_{3-} n^0_{4+} \tilde{n}^0_{1+} \tilde{n}^0_{2+} = e^{-\Delta} \tilde{n}^0_{3-} \tilde{n}^0_{4+} n^0_{1+} n_{2+} \tag{4.13}$$

where

$$\Delta = \beta(\tilde{\mu}_+ - \tilde{\mu}_-) \tag{4.14}$$

The calculation shows that the term in $\mathrm{Re}(12|T|34)(34|T^\dagger|21)$ vanishes. The net result is the integral

$$I_\xi^\perp = \frac{2\pi^2}{h^4}\, A_\xi^\perp \phi_{j\xi} \sinh(\Delta/2) \int d\mathbf{p}_1 \int d\mathbf{p}_2 \int d\mathbf{p}_3 \int d\mathbf{p}_4\, \delta(E_{12} - E_{34})$$

$$\times\, \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)|(12|T|34)|^2 (v_{1j} - v_{3j})^2$$

$$\times\, [e^{\Delta/2} n^0_{1-} n^0_{2-} + e^{-\Delta/2} n^0_{1+} n^0_{2+}]\tilde{n}^0_{3+}\tilde{n}^0_{4-} \tag{4.15}$$

Note the peculiar structure of this integral which has factors like $n^0_{1-} n^0_{2-} n^0_{3+} n^0_{4-}$ that seem to defy spin conservation in a collision. However, it is easy to show that the collision integral *does* conserve spin as it should, by integrating (4.11) over $\mathbf{p}_1$. The odd structure is due to the transverse character of the process in which spins in differing angular orientations collide as discussed in Sec. 1.

The entire result for the transverse elements of the kinetic equation now becomes

$$\alpha_\perp m = A^\perp_\xi (C_{\mathrm{coll}} + i\eta\xi C_{\mathrm{sr}}) \tag{4.16}$$

where we have cancelled out common factors and

$$\begin{aligned} C_{\mathrm{coll}} = \frac{4\pi^2}{h^4} \sinh(\Delta/2) \int d\mathbf{p}_1 \int d\mathbf{p}_2 \int d\mathbf{p}_3 \int d\mathbf{p}_4\, \delta(E_{12} - E_{34}) \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \\ \times |(12|T|34)|^2 (v_{1j} - v_{3j})^2 [e^{\Delta/2} n^0_{1-} n^0_{2-} + e^{-\Delta/2} n^0_{1+} n^0_{2+}] \bar{n}^0_{3+} \bar{n}^0_{4-} \end{aligned} \tag{4.17}$$

and

$$C_{\mathrm{sr}} = -\frac{2\pi}{h^4} \int d\mathbf{p}_1 \int d\mathbf{p}_2\, \mathrm{Re}(12|T|21)(n^0_{1+} - n^0_{1-})(n^0_{2+} - n^0_{2-}) v_{1j}(v_{1j} - v_{2j}) \tag{4.18}$$

The result of Eq. (4.16) proves immediately that

$$A^\perp_+ = (A^\perp_-)^* \tag{4.19}$$

The transverse spin current of Eq. (3.42) can be written as

$$J^\perp_j(\mathbf{m}) = \tfrac{1}{2}[(J^\perp_{j+} + J^\perp_{j-})\hat{\mathbf{u}} - i(J^\perp_{j+} - J^\perp_{j-})\hat{\mathbf{v}}] \tag{4.20}$$

Each of the $J^\perp_{j\xi}$ contains a factor $\phi_{j\xi}$ which we recall is related to the spatial derivative of the local magnetization direction $\hat{\mathbf{u}}$ according to Eqs. (3.32) and (3.33). It follows then that

$$\phi_{ju}\hat{\mathbf{v}} - \phi_{jv}\hat{\mathbf{u}} = \hat{\mathbf{e}} \times \frac{\partial \hat{\mathbf{e}}}{\partial r_j} \tag{4.21}$$

so that from Eq. (3.43)

$$\begin{aligned} J^\perp_j(\mathbf{m}) &= -\tfrac{1}{2}\bar{\alpha}_\perp m \left[(A^\perp_+ + A^\perp_-) \frac{\partial \hat{\mathbf{e}}}{\partial r_j} - i(A^\perp_+ - A^\perp_-)\hat{\mathbf{e}} \times \frac{\partial \hat{\mathbf{e}}}{\partial r_j} \right] \\ &= -\bar{\alpha}_\perp m \left[(\mathrm{Re}\, A^\perp_+) \frac{\partial \hat{\mathbf{e}}}{\partial r_j} + (\mathrm{Im}\, A^\perp_+)\hat{\mathbf{e}} \times \frac{\partial \hat{\mathbf{e}}}{\partial r_j} \right] \end{aligned} \tag{4.22}$$

Solving Eq. (4.16) for the real and imaginary parts of $A^\perp_+$, we finally find

$$J^\perp_j(\mathbf{m}) = -\frac{D_\perp}{1 + (\mu m/n)^2} \left[m \frac{\partial \hat{\mathbf{e}}}{\partial r_j} - \eta(\mu m^2/n)\hat{\mathbf{e}} \times \frac{\partial \hat{\mathbf{e}}}{\partial r_j} \right] \tag{4.23}$$

with

$$D_\perp = \frac{\alpha_\perp \bar{\alpha}_\perp m}{C_{\mathrm{coll}}} = \alpha_\perp \tau_\perp \tag{4.24}$$

where the transverse relaxation time is

$$\tau_\perp = \frac{\bar{\alpha}_\perp m}{C_{\mathrm{coll}}} \tag{4.25}$$

and the spin rotation parameter μ (not to be confused with the chemical potential) is given by

$$\mu m/n = \frac{C_{\mathrm{sr}}}{C_{\mathrm{coll}}} \tag{4.26}$$

Equation (4.23) is the Leggett equation, with the η showing that the equation applies to bosons as well as fermions. The transverse diffusion constant [Eq. (4.24)] is given by a different expression than the longitudinal diffusion constant [Eqs. (3.41) and (4.8)] showing that they can be different. One can show analytically that at high temperatures or low polarization the two different diffusion constants are the same up to some mean-field corrections. Notice that the diffusion constant contains a mean-field correction in $\alpha_\perp$ in analogy to that in Fermi liquid theory.

5. REDUCTION TO COMPUTABLE FORMS

We can summarize our results by writing the spin current in terms of a single equation:

$$J_j(\mathbf{m}) = -D_\parallel \frac{\partial m}{\partial r_j}\hat{\mathbf{e}} - \frac{D_\perp}{1 + (\mu m/n)^2} \left[m \frac{\partial \hat{\mathbf{e}}}{\partial r_j} - \eta(\mu m^2/n)\hat{\mathbf{e}} \times \frac{\partial \hat{\mathbf{e}}}{\partial r_j} \right] \tag{5.1}$$

where

$$D_\parallel = \alpha_\parallel \tau_\parallel \tag{5.2}$$

$$D_\perp = \alpha_\perp \tau_\perp \tag{5.3}$$

with $\alpha_\parallel$ and $\alpha_\perp$ given by Eqs. (3.27) and (3.28) respectively; $\tau_\parallel$ and $\tau_\perp$ given by Eqs. (4.8) and (4.25), and the spin rotation parameter is

$$\mu = \frac{nC_{sr}}{mC_{\text{coll}}} \tag{5.4}$$

where the integrals C_{coll} and C_{sr} are given by Eqs. (4.17) and (4.18).

In order to do computations we must reduce the integrals of Eqs. (4.8), (4.17), and (4.18) to a form suitable for numerical analysis. This is done by using the techniques developed in Ref. 30, in which we showed how such integrals can be reduced to two-fold form. The situation is slightly different here, however, because we are going to include the effect of the T-matrix momentum dependence. Nevertheless similar procedures continue to work well. The ability to reduce to two-fold form is independent of temperature, which allows us to calculate numerically results for *temperatures ranging all he way from completely degenerate to the Boltzmann statistics limit*. This ange is important because experiments on dilute solutions often give data n the intermediate temperature regime near the Fermi temperature. We resent one integral reduction here; the others are similar.

The T-matrix approximation we will consider is similar that made by ardeen *et al.*[12] in their treatment of dilute solutions. We write

$$(\mathbf{p}|T|\mathbf{p}') = \frac{1}{h^3} V(|\mathbf{p}-\mathbf{p}'|) \tag{5.5}$$

he effective potential that was fitted to data in Ref. 12 was a simple cosine unction. Later Ebner[13] used high temperature transport data to determine e phenomenological form

$$V(q) = |V_0|[\alpha_1 + \alpha_2 \exp[-C_2(q/2k_0)^2] + \alpha_3 \exp[-C_3(q/2k_0)^2]] \tag{5.6}$$

here $k_0 = 0.497\ \text{A}^{-1}\,\hbar$; $\alpha_1 = 10$; $\alpha_2 = -1.48$; $\alpha_3 = -9.60$; $C_2 = 4.0752$; $C_3 =$ 0564, and $V_0 = 1.298 \times 10^{-48}\ \text{J m}^3$. This potential is more suitable than thers in the literature because it was determined at high T and relatively gh concentration (1.3% and 5%) so that its validity extends to higher q alues than other forms. Thus we expect to be able to use it all the way into e Boltzmann limit where some of the data have been taken. We do not pect such a phenomenological potential to fit all the new data perfectly ell. Nevertheless, we will not attempt to adjust any potential parameters, d we find rather surprisingly that the theory does fit the new data essenlly within our estimate of the theoretical errors.

From Eqs. (4.17) and (4.25) we have

$$\frac{1}{\tau_\perp} = \frac{4\pi^2 \sinh(\Delta/2)}{\bar{a}_\perp m m^{*2} h^{10}} \int d\mathbf{p}_1 \int d\mathbf{p}_2 \int d\mathbf{p}_3 \int d\mathbf{p}_4\, \delta(E_{12}-E_{34})\,\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)$$
$$\times V(|\mathbf{p}_1-\mathbf{p}_3|)^2[(\mathbf{p}_1-\mathbf{p}_3)\cdot\hat{\mathbf{a}}]^2[e^{\Delta/2}n^0_{1-}n^0_{2-} + e^{-\Delta/2}n^0_{1+}n^0_{2+}]\tilde{n}^0_{3+}\tilde{n}^0_{4-} \tag{5.7}$$

in which $\hat{\mathbf{a}}$ is an arbitrary unit vector. We change variables in this integral to

$$\mathbf{p} = \mathbf{p}_1 - \mathbf{p}_3, \qquad \mathbf{P} = \tfrac{1}{2}(\mathbf{p}_1 + \mathbf{p}_3)$$
$$\mathbf{p}' = \mathbf{p}_4 - \mathbf{p}_2, \qquad \mathbf{P}' = \tfrac{1}{2}(\mathbf{p}_2 + \mathbf{p}_4)$$

The delta functions become $\delta(\mathbf{p}-\mathbf{p}')\delta(P_z - P'_z)m^*/p$ where we have taken $\mathbf{p}$ as the z-axis for the $\mathbf{P}$ and $\mathbf{P}'$ integrations. Carrying out the $\mathbf{p}'$ integration and the $\mathbf{P}$ and $\mathbf{P}'$ angular integrations gives us

$$\frac{1}{\tau_\perp} = \frac{16\pi^4 \sinh(\Delta/2)}{\bar{a}_\perp m m^* h^{10}} \int_0^\infty dp\, p \int_{-\infty}^\infty dP_z \int_0^\infty d\rho\, \rho \int_0^\infty d\rho'\, \rho' \int_0^\infty d\Omega_p$$
$$\times V(p)^2[\mathbf{p}\cdot\hat{\mathbf{a}}]^2[e^{\Delta/2}n^0_{1-}n^0_{2-} + e^{-\Delta/2}n^0_{1+}n^0_{2+}]\tilde{n}^0_{3+}\tilde{n}^0_{4-} \tag{5.8}$$

By writing the arguments of the distribution functions in terms of the variables

$$s^2 = \frac{\beta}{2m^*}\frac{p^2}{4}, \qquad t^2 = \frac{\beta}{2m^*}P_z^2 \tag{5.9}$$

$$y = \frac{\beta}{2m^*}\rho^2, \qquad y' = \frac{\beta}{2m^*}\rho'^2 \tag{5.10}$$

we can see that these arguments are all independent of the angular variables of $\mathbf{p}$, so that that integral may be done over the quantity $(\mathbf{p}\cdot\hat{\mathbf{a}})^2$. We find then

$$\frac{1}{\tau_\perp} = \frac{8(2\pi)^5(2m^*)^{9/2}\sinh(\Delta/2)}{3\bar{a}_\perp m m^* h^{10}\beta^{9/2}} \int_0^\infty ds\, s^3 V(s)^2 \int_{-\infty}^\infty dt \int_0^\infty dy \int_0^\infty dy'$$
$$\times \Bigg[e^{\Delta/2} \frac{1}{e^{y+(t+s)^2-\zeta_-}-\eta} \frac{-\eta}{e^{-y-(t-s)^2+\xi_+}-\eta} \frac{1}{e^{y'+(t-s)^2-\zeta_-}-\eta}$$
$$\times \frac{-\eta}{e^{-y'-(t+s)^2+\zeta_-}-\eta} + e^{-\Delta/2} \frac{1}{e^{y+(t+s)^2-\zeta_+}-\eta} \frac{-\eta}{e^{-y-(t-s)^2+\zeta_+}-\eta}$$
$$\times \frac{1}{e^{y'+(t-s)^2-\zeta_+}-\eta} \frac{-\eta}{e^{-y'-(t+s)^2+\zeta_-}-\eta}\Bigg] \tag{5.11}$$

where $\zeta_\sigma = \beta\bar{\mu}_\sigma$ and we have written the potential, which is a function of $|\mathbf{p}|$, as if it were a function of s. The integrals continue to apply to both fermions and bosons, however subsequent numerical work reported below will be for fermions only. The integrals over the y's can be done as well according to

$$\int_0^\infty dy \frac{1}{e^y a - \eta} \frac{1}{e^{-y}/b - \eta} = \frac{1}{1-(a/b)} \ln\left(\frac{1-\eta/a}{1-\eta/b}\right) \tag{5.12}$$

This gives us finally

$$\frac{1}{\tau_\perp} = \frac{2(2\pi)^5(2m^*)^{9/2}\sinh(\Delta/2)}{3\bar{\alpha}_\perp m m^* h^{10}\beta^{9/2}} \int_0^\infty ds\, s^3 V(s)^2 \int_{-\infty}^\infty$$
$$\times\, dt \frac{1}{\sinh(2ts+\Delta/2)\sinh(2ts)}$$
$$\times \ln\left[\frac{1-\eta\, e^{-(t+s)^2+\zeta_-}}{1-\eta\, e^{-(t-s)^2+\zeta_+}}\right] \ln\left[\frac{(1-\eta\, e^{-(t+s)^2+\zeta_-})(1-\eta\, e^{-(t+s)^2+\zeta_+})}{(1-\eta\, e^{-(t-s)^2+\zeta_-})(1-\eta\, e^{-(t-s)^2+\zeta_+})}\right] \tag{5.13}$$

This quantity is only a two-dimensional integral which is easily computed numerically. Note that we have not needed to make an s-wave approximation in order to carry out the calculations shown.

The other integrals we need in order to calculate the diffusion constants are reducible to simple forms in a similar manner, but we will not show the details here.

6. THEORY ERROR ANALYSIS

The theory we have presented has made several approximations and it is useful to analyze the size of the errors that we have made in those approximations. Further there are some terms that we have not yet been able to calculate and we should mention those and estimate their size, if possible. Many of the terms we consider here have been discussed more explicitly in the Boltzmann limit in Ref. 23.

Consider, for example Eq. (4.16), that is,

$$\alpha_\perp m = A_\xi^\perp (C_{\text{coll}} + i\eta\xi C_{\text{sr}}) \tag{6.1}$$

where from Eq. (3.28)

$$\alpha_\perp = \frac{2}{3m^* m} \sum_{\sigma'} \sigma' k_{\sigma'} + \frac{\eta V(0) n}{2m^*} \tag{6.2}$$

6.1. Mean-Field Drift Terms

The second term in $\alpha_\perp$ is a mean-field correction that is analogous to the factor $(1+F_0^a)$ in Fermi liquid theory. It arises from the terms $\nabla_r \varepsilon_\mathbf{p} \nabla_\mathbf{p} n_\mathbf{p}$ of the kinetic equation in Eq. (2.32). Such terms are really virial corrections to the transport coefficients, as well as to the equilibrium properties like pressure that can be calculated from the kinetic equation. We can estimate the importance of this correction by taking the ratio of this term to the first one in $\alpha_\perp$. From the definition of k_σ as an average energy we have that the mean-field term is of relative order $R_1 = V(0)n/\langle\varepsilon\rangle$ where $\langle\varepsilon\rangle$ is the average single particle kinetic energy. We can write $V(0)$ in terms of a scattering length, a, as $V(0) = \hbar^2 a/m^*\pi$ so that $R_1 = a\lambda^2/r_0^3$ where λ is the deBroglie wave length and r_0 is the mean particle spacing. In the Boltzmann limit $\lambda = \lambda_T \equiv h/(2\pi m^* kT)^{1/2}$ and R_1 is negligible. In the degenerate limit λ passes over to $\lambda = r_0$ as λ_T/r_0 grows to unity when the temperature is lowered. Thus the maximum value of this term is of order a/r_0 which can be related to the concentration, x, of the dilute solutions we consider. We have $a = -1\,A$, $r_0 = 3.5x^{-1/3}A$. The concentrations considered here range from 3.5×10^{-4} to 2×10^{-3} so that the largest value of $R_1 \approx 0.04$. This result agrees well with more precise calculations quoted in Sec. 7.

6.2. Nonlocal Lateral Scattering

In our derivation we have neglected gradient terms in the collision integral in the form $\hbar\nabla_r \ldots \nabla_\mathbf{p} \ldots$ (the gradient expansion is actually a series in $\hbar$). These terms are second order in the T-matrix and include a spatial gradient of a distribution function and a momentum gradient of either the distribution function or of the T-matrix element itself. Similar terms have been discussed in Ref. 23. Consider the real part of Eq. (6.1). The ratio of the collision integral to the drift term is $C_{\text{coll}}\tau_\perp/(\text{Drift}) = 1$ by the Boltzmann equation, where $\text{Drift} = \alpha_\perp m$. Correction terms having distribution function gradients in them will actually be additions to the mean-field drift terms $\nabla_r \varepsilon_\mathbf{p} \nabla_\mathbf{p} n_\mathbf{p}$, and should be compared to that term. We can get an idea of the order of such terms, which we will denote as C_{nl} by use of the relations

$$\nabla_\mathbf{p} n_{\mathbf{p}\sigma} = \frac{\partial n_{\mathbf{p}\sigma}}{\partial \varepsilon_\mathbf{p}} v_\mathbf{p} \qquad \text{and} \qquad \nabla_r n_{\mathbf{p}\sigma} = -\frac{\partial n_{\mathbf{p}\sigma}}{\partial \varepsilon_\mathbf{p}} \sigma t_\sigma \nabla_r m \tag{6.3}$$

We also have the identity

$$\frac{\partial n_{\mathbf{p}\sigma}}{\partial \varepsilon_\mathbf{p}} = \beta n_{\mathbf{p}\sigma}(1 - n_{\mathbf{p}\sigma}) \tag{6.4}$$

In the Boltzmann limit we set $1 - n_{\mathbf{p}\sigma}$ to 1 and the resulting integral then has the form $C_{\text{nl}} = (\hbar\beta/\tau)C_{\text{coll}}$, where τ is one of the diffusion relaxation times. Since $\tau = 1/(vna^2)$ in the Boltzmann limit, where v is an average velocity, we have $C_{\text{nl}}/(Drift) = hvna^2/kT = a^2\lambda_T/r_0^3$. Compared to the already small mean-field terms discussed in Sec. 6.1 above these are of relative order a/λ_T, which is small at low T. At high T they can become only as large as the negligible mean-field terms, and so are irrelevant. The terms involving a momentum derivative of the T-matrix are even smaller as we will see below in Sec. 6.4. For the degenerate case the analysis is much more difficult and we are unable to make a reliable estimate, but if we assume that λ_T goes over to r_0 then these corrections become of relative order (a/r_0) and continue to be small in that limit.

6.3. Three-Body Collision Terms

These should be smaller than the two-body terms we have considered by a factor of the small parameter found in the virial expansion. At high temperature in the Boltzmann limit this is $(a/r_0)^3$, a very small number. For low T, but still in the Boltzmann regime, one can show that the virial expansion is in $a\lambda_T^2/r_0^3$. Upon lowering the temperature to the degenerate regime one might expect λ_T to be replaced by r_0. For scattering processes on the Fermi surface the factor should be made smaller by factors of T/T_F and be unimportant. However, for transverse processes such a factor would not be present so that these terms might be as important as the mean-field terms discussed in Sec. 6.1. There is little hope of computing such terms. Moreover, whether a virial expansion for the transport coefficients has validity is an open question since the four-body term diverges.[31]

6.4. Spin Rotation Corrections

The term C_{sr} in Eq. (4.18) is the spin rotation integral. It is a forward scattering term, and so is first order in the T-matrix. The relative size of this is $\tau_\perp C_{sr}/Drift = \tau_\perp m V(0)/h = \mu$ where μ is the spin-rotation parameter in s-wave approximation [Eq. (5.4)]. For the Boltzmann limit this is of order λ_T/a, which is, of course, large at low T, since otherwise spin waves would not be observable in Boltzmann gases. For the degenerate limit the value is $(r_0/a)(T_F/T)^2$ until kT becomes less than the Zeeman energy where the temperature factor becomes a constant. Nonlocal corrections to spin rotation will involve forward scattering with a single T-matrix element and will have a spatial gradient of a distribution function and a momentum gradient of the T-matrix element. (A systematic analysis of nonlocal terms first order in T has been given by Meyerovich[32] for the Boltzmann limit.) We can evaluate the gradient of the T-matrix by using a quadratic approximation to the Ebner effective potential. The estimate is not difficult but we omit the details. We find the ratio of the nonlocal spin-rotation terms to the ordinary spin-rotation term in the Boltzmann limit is $C_{nlsr}/C_{sr} = (a^2/\lambda_T r_0)(x/0.05)^{2/3}$. The 0.05 arises because the scale of the Ebner potential is the Fermi momentum of the 5% solution. The ratio is quite negligible.

There is another correction to spin-rotation from lateral scattering. This exchange term depends on the imaginary part of a product of two T-matrix elements [see Eq. (2.31)]. We neglected this because our effective T-matrix is real. We know that, in the Boltzmann limit and at low T, this term is unimportant, but, for ^{3}He gas, becomes as important as the forward scattering term at higher T.[33] However, for the case of dilute solutions μ is likely to be so small when this high T regime is reached that experiments to observe it would be impossible anyway. At the low T end we have from Ref. 3, that the lateral scattering term is of order $(a/\lambda_T)^2$ when compared to the forward scattering part. For the degenerate case we expect that this becomes $(a/r_0)^2$. Thus it seems unimportant for all relevant T.

6.5. Variational Approximation

We have used the lowest order variational approximation. It is likely that there is nonnegligible error introduced by this approach. It would be best if we could calculate the higher-order terms, but this is extremely difficult for the large temperature range we consider here. Fortunately, there are calculations of the higher-order corrections in the Boltzmann limit[34] and, at least for longitudinal spin diffusion, at the degenerate limit.[35] In Ref. 34 it is shown that the lowest-order approximation of the spin-rotation parameter is in error by only a few percent over most of the range, but can be 20 or so percent right around 1 K for ^{3}He gas (near a minimum in μ). The spin diffusion constant is also only a few percent in error except right near a maximum at complete polarization where interference effects cause a 20% error. Comparison with exact calculations for the degenerate regime[35] shows that the variational approximation error at 10% at $M=0$, 8% at $M=0.6$, and 6% at complete polarization. Thus we estimate the average error in our present calculations at around 8% or less over most of the T range. However, we note that the longitudinal diffusion constant for the dilute solutions considered here also has a maximum at high T so we must be aware of the possibility that there is a greater error there. We will see that our calculations are consistent with errors of this magnitude when compared to experiment.

6.6. Ebner Effective Potential

The effective T-matrix $V(q)$ is phenomenological and involves many assumptions. $V(q)$ is non-retarded; it depends only on momentum transfer q and not on the center of mass momentum; it is independent of concentration and spin so that it is essentially a T-matrix based on free-particle propagators in intermediate states rather than interacting ones. The concentration dependence has been estimated by Fu and Pethick[36] who find corrections of order a/r_0 in the degenerate limit. Such a correction is of the same order as the mean-field corrections mentioned above that we have included. The problem with the Fu-Pethick terms in $V(q)$ is that they have been computed only for the degenerate, unpolarized limit, and at concentrations much larger than we consider here. We have tried to make a rough estimate of the effect of this correction by an s-wave approximation and find that it could be sizeable: For the largest concentration considered in the experiments we analyze, $x = 1.9 \times 10^{-3}$, we find that a transport property could be enhanced by as much as 15% even at the low concentrations that we consider! (At higher concentrations there seems to be considerable cancellation.) However, what is important is not the absolute enhancement, which might be incorporated in $V(0)$, but the relative change from one concentration to another. For the lowest concentration we consider, $x = 3.5 \times 10^{-4}$, the enhancement is estimated at 9%. Thus the relative change is only about 6%. In the Boltzmann limit it is easy to see that the Fu–Pethick correction will be of order $a\lambda_T^2/r_0^3$, which becomes increasingly smaller at higher temperature.

We noted in Sec. 2 that we expect the correct T-matrix to be spin dependent because of the screening factor in the resolvent operator R (see Eq. (2.26)). We might expect such effects to be included in a calculation for a polarized system that is the analog of the Fu–Pethick analysis. The contribution could possibly be of the same order as the Fu–Pethick correction.

The Ebner T-matrix is real, which means that lateral scattering contribuions to the spin rotation parameter are neglected. Fortunately we have seen above that this approximation makes a small error. Retardation corrections o the Ebner potential are of order $(v/c)^2$ where c is the velocity of sound n liquid ^{4}He. This is negligible.

6.7. The Virial Expansion Question

Some equilibrium properties can be calculated using the kinetic equaion. The momentum conservation law as derived from the kinetic equation ields an expression for a momentum tensor from which one gets the hydrotatic pressure. From the Boltzmann equation one finds the ideal gas law. This is surprising because one might have thought that the Boltzmann equaion considered two-body collisions completely. This question has been disussed often in the literature; it is the nonlocal terms such as the $\nabla_r \varepsilon_p \nabla_p n_p$ erms in the Landau equation and in Eq. (2.21) that give second virial oefficient corrections. We have included such terms but it turns out that this term, in the form we have it [our $\varepsilon_\mathbf{p}$ contains interaction terms depending on $\mathrm{Re}(\mathbf{p}|T|\mathbf{p})$ in Eq. (2.24)], gives only part of the second virial coefficient. This defect may be due to the inadequacy of the decoupling approximation that allows one to go from Eq. (2.1) to (2.2). A solution to this problem has been given recently by Laloë and coworkers[23] by introducing what is called the "free Wigner Transform." A similar analysis has also been given by Snider.[21] This result is still too recent to be included in our present theory. One way to estimate the discrepancy made here is to note that the correct Landau interaction for a dilute system, and one that gives the correct virial coefficient for the pressure, contains the factor $\sum_l (2l+1)\delta_l$ where δ_l is a phase shift in the partial wave expansion.[25,28] Our interaction term involving $\mathrm{Re}(12|T|12)$ is equivalent[25] to the factor $\sum_l (2l+1)\cos\delta_l \sin\delta_l$. The two results agree only for small δ_l. At low T, but still in the Boltzmann regime, the relative size of the correction is $(a/\lambda_T)^2$, which is small. This presumably goes over to $(a/r_0)^2$ in the degenerate case, also a small correction. At high T the difference is presumably as large as the term itself. However, the correction affects only mean-field terms, where it is completely negligible, and probably the spin-rotation term, where it is then the same order as the lateral spin rotation term discussed in Sec. 6.4. The distinction between δ_l and $\sin\delta_l \cos\delta_l$ interactions also appears in discussions of statistical and dynamical quasiparticles in Fermi liquid theory.[37] Also see Ref. 32 for a recent brief discussion of this question.

6.8. Off-Energy-Shell Corrections

We have seen above that there are off-shell corrections to spin rotation as given in Eq. (2.30). These are probably the I_2 terms discussed previously in the literature.[19] We have not yet tried to analyze these terms, but since, in comparison with the leading spin-rotation terms, they have an extra T-matrix element and an extra distribution function factor, we expect these terms to be of order $nV(0)/kT_F$ in the degenerate limit and $nV(0)/kT$ is the Boltzmann limit. These estimates translate to a/r_0 in the degenerate limit and $a\lambda_T/r_0^3$ in the Boltzmann limit, respectively, just like the mean-field terms of Sec. 6.1 or the Fu–Pethick corrections of Sec. 6.7. Thus they could be of order 10% corrections.

We see that the most important approximations that we ought to correct in future calculations are the low-order variational estimate, and the neglect of the Fu–Pethick corrections and off-shell spin-rotation terms. All of these should be straightforward to compute in the degenerate limit and possibly in the Boltzmann limit. Unfortunately most of the data exist in the intermediate region around T_F where the corrections are more difficult. We hope to return to this problem in the future.

7. COMPARISON WITH EXPERIMENT

We compute theoretical results by using the Ebner potential,[13] which was originally determined by fitting thermal conductivity, spin diffusion, and

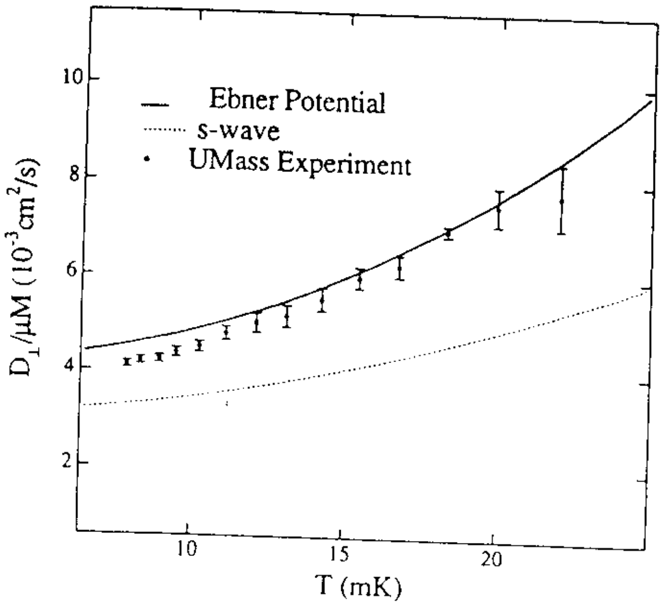

Fig. 4 Transverse diffusion constant $D_\perp$ divided by identical-particle spin-rotation parameter μ times polarization M. The experimental data is from the UMass spin-wave experiment (Ref. 10) with concentration $x=1.822\times 10^{-3}$ ($T_F=39.8$ mK) and $B\times 8.0$ T. The theoretical calculations result in the solid line when the Ebner potential $V(q)$ (Ref. 13) is used, and the dotted line for an s-wave approximation that uses only $V(0)$.

viscosity data at temperatures in the Boltzmann regime of temperatures and at the relatively high concentrations of 1.3% and 5%. The agreement in Ref. 13 was generally to within about 10-15%. We use this potential rather than others that exist in the literature because it remains valid at fairly high temperatures used for some of the data with which we compare our theory here. Also we find that the Ebner potential provides better agreement with the data than do other potentials. It has been stressed in the past that ideally one should have data and theory where the s-wave approximation is valid.[4] For an s-wave analysis we replace the Ebner potential with $V(0)$, the value at $q=0$. We find, however, that all experiments that we analyze extend over such a temperature range that the q-dependence of the potential is quite important for their interpretation. We have shown above how we may include this q-dependence in the theory. It is surprising that both the absolute magnitude and the q-dependence given by the Ebner potential give quite reasonable agreement between theory and experiment. We could take the point of view as in Ref. 10 that the experiments can be used to determine the potential more accurately. However, our analysis of Sec. 6 indicates that the error in the theoretical calculations is probably bigger than the discrepancy between theory and experiment so that we do not try to vary potential parameters here to improve the fit.

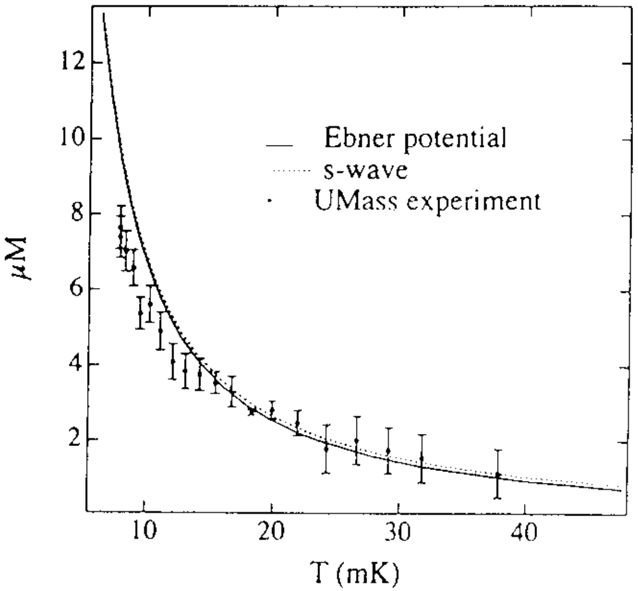

Fig. 5. μM versus temperature for the parameters $x=1.822\times 10^{-3}$, $B=8.0$ of the UMass experiment. The theory gives the solid line with the Ebner potential and the dotted line for the s-wave approximation.

The recent experiment of Candela *et al.*[10] involves spin-wave measurement of the quantities $D_\perp/\mu M$ and μM. The former quantity comes from the frequency of the spin-wave peaks, and the latter from their width. In Figs. 4 and 5 we plot these quantities from the present theory and from experiment versus temperature. For the conditions of the first UMass experiment, $x=1.82\times 10^{-3}$ ($T_F=39.8$ mK) and $B=8.0$ T, the s-wave results fall considerably below the $D_\perp/\mu M$ values, but use of the Ebner potential gives reasonable agreement. The s-wave value of μM differs little from its value with the potential. The maximum discrepancy with the $D_\perp/\mu M$ experiment is about 5% at lowest T and about 20% at low T for the μM data, although in the latter case the theory is almost always within the error bars of the experiment.

In Figs. 6 and 7, we show theory and experiment for a smaller concentration of $x=6.26\times 10^{-4}$. Here the Fermi temperature is $T_F=19.5$ mK. The agreement with the μM data is about the same as in the higher concentration (with a maximum disagreement of about 15% at the lowest T). The

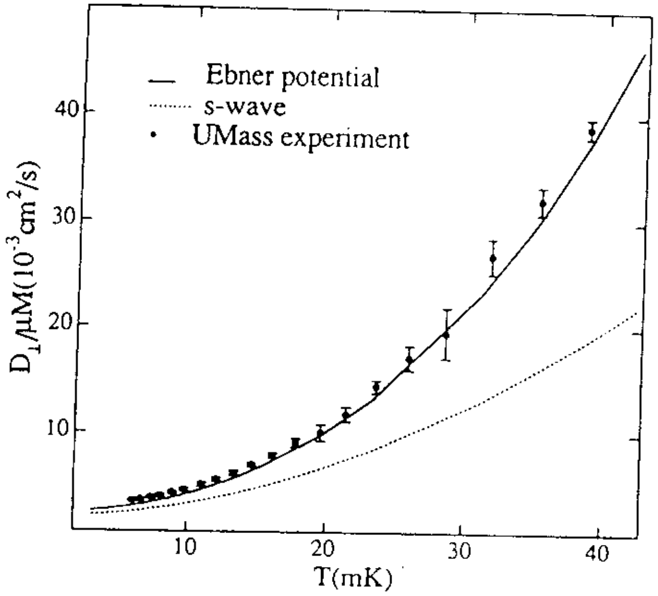

Fig. 6. $D_\perp/\mu M$ versus T for the parameters $x = 6.26 \times 10^{-4}$ ($T_F = 19.5$ mK) and $B = 8.0$ T of the UMass experiment. Notation is as in Fig. 4.

$D_\perp/\mu M$ theory is in reasonable agreement at the higher T but falls above the low T data by around 5%.

For the μM UMass data, the s-wave approximation's results are quite close to that of the full potential, while for the data involving $D_\perp$ it is clear that the full potential is needed.

In Fig. 8 we present the results of calculations of the longitudinal diffusion constant to compare with experiments performed by Nunes *et al.*[11] at Cornell. One experiment was at a concentration of $x = 3.5 \times 10^{-4}$ ($T_F = 13.3$ mK) and an external field of $B = 9.2$ T. We see from the figure that the theory with the Ebner potential does quite a good job with an error of only about 6% at the minimum in the curve. The s-wave approximation falls below the data over the whole range. The minimum in the curves comes in the crossover from degenerate Fermi behavior in which $D_\parallel$ is proportional to T^{-2} to Boltzmann behavior where it goes like $T^{1/2}$. The high T maximum occurs because the potential goes from attractive to repulsive at $q \sim 0.5$ A^{-1} and its integral over momentum tends to maximally cancel at a corresponding temperature. The position of the maximum in the theory is in good agreement with experiment. The second experiment was at a larger concentration of $x = 1.94 \times 10^{-3}$ ($T_F = 41.4$ mK). Here the calculations shown in Fig. 9 are not nearly in such good agreement with experiment. At the minimum the discrepancy is about 20%; at the maximum it is about 15%—descrepancies roughly in agreement with the maximum theory error estimates of Sec. 6. At present we have no idea why the lower concentration data are in so much better agreement with experiment than the high concentration data. The high concentration in the Cornell experiment is much the same as that in the high concentration case of the UMass experiment, and indeed we can multiply $D_\perp/\mu M$ by μM to extract $D_\perp$ from the data. Such data fall quite consistently on the curve of experimental $D_\parallel$ values in the range 10–20 mK. This happens to be right where the theoretical curve passes through the experimental Cornell data. Thus one could say that we were somewhat lucky in achieving such good agreement with the UMass data in Fig. 4. Note that the good agreement between the measurements of the transverse and longitudinal diffusion constants indicates that the temperature is not yet sufficiently low that these two quantities have diverged from one another as predicted by the theory. Actually the theory predicts that $D_\perp$ should be less than $D_\parallel$ by 3% at the lowest temperatures reached in the UMass experiment as shown in Fig. 3. This difference is clearly too small to be established from the experiment.

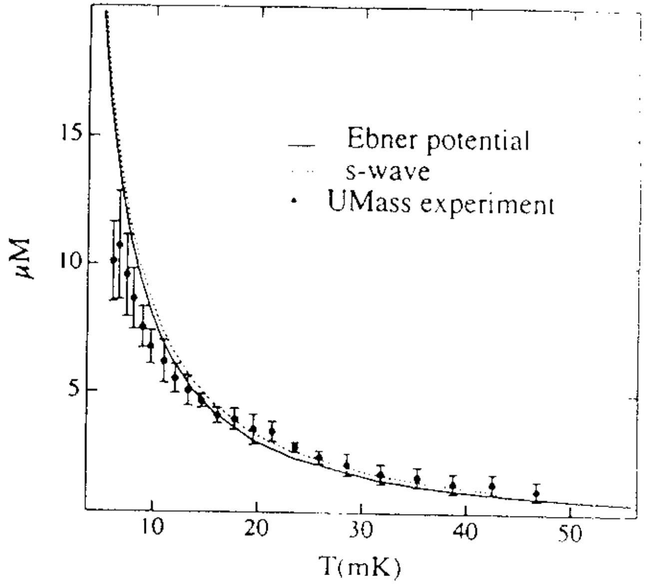

Fig. 7. μM versus T for the parameters $x = 6.26 \times 10^{-4}$, $B = 8.0$ T of the UMass experiment. Notation as in Fig. 5.

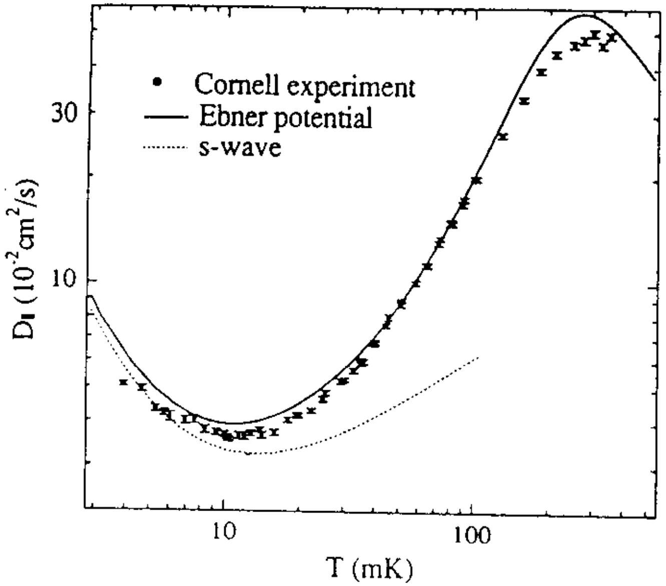

Fig. 8. Longitudinal spin diffusion coefficient $D_\parallel$ versus temperature for the parameters $x=3.5\times10^{-4}$ ($T_F=13.3$ mK) and $B=9.2$ T involved in the Cornell experiment (Ref. 11). The data are represented by solid circles. The theory curves are derived by use of the Ebner potential (solid line) and the s-wave approximation (dotted line).

We have discussed the size of the mean-field corrections briefly in Sec. 6. We examine here these corrections, which have been included in both longitudinal and transverse numerical calculations, quantitatively. These corrections are of two kinds: (a) a correction in the calculation of the polarization due to including the energy mean-field term of Eq. (3.9), in the Fermi functions $n_{p\sigma}$, and (b) one analogous to the $(1+F_0^a)$ factor in the diffusion constant of Fermi liquid theory. The former has been discussed previously.[38] Here it causes a change in the magnetization calculated for a given temperature and external field and a change in the effective chemical potentials that enter the calculations. For example, for the high concentration UMass parameters and at a temperature of 8 mK, the correction to M amounts to a decrease of 6%. However, the resulting change in $D_\perp$ is less than 0.5%. The effects of this correction are even smaller at higher temperatures. Corrections of the type (b) appears as correction terms to $\alpha_\perp$ and $\alpha_\parallel$ in Eqs. (3.28), and (3.27) with (3.19). In the high concentration UMass situation they result in an increase of D by about 7–8% at the lowest temperatures with smaller corrections at higher temperatures. Similar corrections occur in the high

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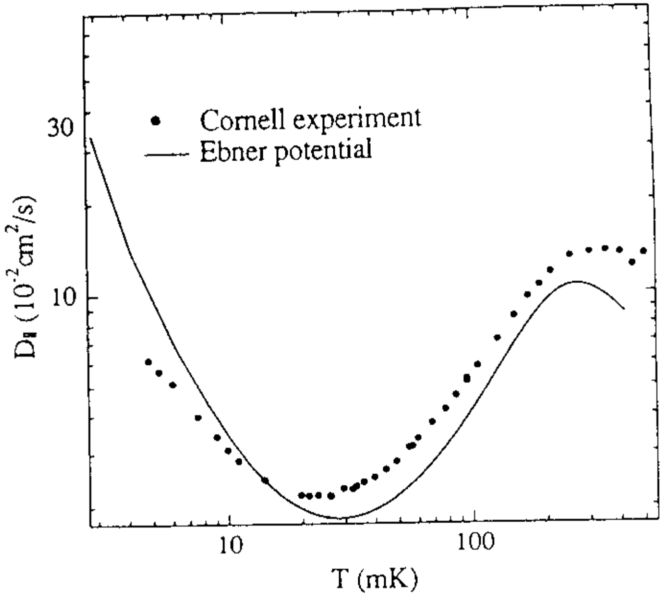

Fig. 9. $D_\parallel$ versus T for the parameters $x=1.94\times10^{-1}$ ($T_F=41.5$ mK) and $B=9.2$ T of the Cornell experiment. Notation is as in Fig. 8.

concentration longitudinal diffusion constant. We have used the s-wave approximation in all mean-field corrections.

In the low concentration Cornell data conditions the mean-field corrections are smaller because the concentration is smaller, but they are not completely negligible there either. At the lowest temperature of the data, the type (a) correction leads to a change in D of less than 0.5%, but the type (b) correction increases D by about 4% at the lowest temperatures.

Type (a) mean-field corrections also affect μM only a bit. The nonlocal corrections to μ could be classified as mean-field corrections (and therefore of type (b)), but as we have discussed above in Sec. 6 these should be small and have not yet been computed.

In a previous paper[39] we have presented theoretical results for the viscosity of polarized dilute solutions and compared them to vibrating wire experimental results. The details of the calculation were not given there and will not be given here either, because the theoretical treatment is the same as that described previously by the Nottingham group.[40] For completeness, however, we discuss the results here. Figure 10 shows the viscosity as a function of temperature and polarization. As found in Ref. 40, it is too difficult to compute the viscosity with the full q-dependence of the potential; thus we make a Taylor expansion of the Ebner potential and keep only

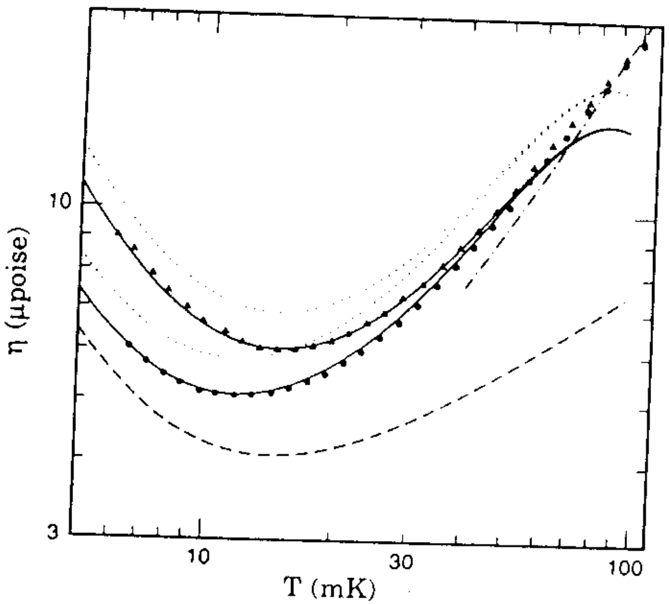

Fig. 10. Viscosity η versus temperature and magnetic field. The filled circles are UMass data for 1 T and the filled triangles for 8 T. The theory curves are as follows: Dotted lines are calculations for these two fields using a quadratic approximation to the Ebner potential $V(q)$. For the solid curves the Ebner potential parameter V_0 [Eq. (5.6)] was increased by 7%. The downturn in these curves is an artifact of the quadratic approximation and the dash-dotted lines show the calculation in the Boltzmann limit where the exact potential can be used. The dashed line is the s-wave approximation that uses the adjusted value of $V(0)$.

a quadratic approximation. This should be adequate for the low enough temperature. As one can see from the figure, the calculation with the unaltered Ebner potential is not in very good agreement with the absolute value of viscosity η. There may actually be some difficulty in accurately determining the absolute value experimentally. However, the polarization and temperature dependence of the theory are in excellent agreement with experiment and these relative values should be accurately determined in the experiment. To show this agreement we alter V_0, the overall coefficient of the potential, by 7% to shift the theory onto the data (curves shifted by about 14%). The curves agree up to the point where the quadratic approximation breaks down and the theory falls below the data. For temperatures higher than that we can then compute η by using the exact potential in the equations derivable from our theory for the Boltzmann limit. The dashed line shows this calculation which is again in good agreement with the experiment. The variational

error in the theory is about 7% in the degenerate system[11] at all relevant polarizations.

The calculations reported above show that we can provide a reasonable description, within known errors of the theory, of both longitudinal and transverse spin diffusion effects in dilute solutions of ^{3}He in liquid ^{4}He if we use an effective potential to describe the interactions. The theory, derived from a kinetic equation generalized to account for arbitrary spin polarization, has been improved over its previous Born and s-matrix approximations by including a T-matrix description of the interactions. This has the advantage of giving the proper Boltzmann limit and allowing the inclusion of a phenomenological effective interaction that was previously determined from transport experiments on more concentrated solutions at higher temperatures. Rather amazingly we achieve good agreement with experiment with the original unaltered Ebner potential. Future work ought to include improvements to the worst approximations discussed in Sec. 6, namely the variational approximation, the concentration dependence of the potential, and the off-energy-shell corrections.

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Magnetization Measurement of ^{3}He Film Adsorbed on Sintered Silver Powder

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The static magnetization of ^{3}He film adsorbed on sintered silver powder has been measured for coverages from 0.10 to 0.54 atoms/Å^2 in steps of 0.02 atoms/Å^2 and at a coverage of 4.5 atoms/Å^2. Below 0.20 atoms/Å^2 the magnetization well obeys the Curie law, while above that coverage a ferromagnetic tendency appears. The coverage dependence of the magnetization greatly differs from that for graphite. At 0.23 atoms/Å^2 the magnetization has a peak like that for graphite; however, the magnitude of the peak is not so large. Furthermore, a multipeak structure is found in the isotherm of the magnetization at 0.2 mK. It has two prominent peaks at 0.23 and 0.29 atoms/Å^2, and two other peaks are found at 0.35 and 0.41 atoms/Å^2. These appear periodically with an interval of 0.06 atoms/Å^2, which is nearly equal to the coverage of one liquid layer.

1. INTRODUCTION

There has been a great deal of interest recently in the ferromagnetism of adsorbed ^{3}He film. This ferromagnetism appears for temperatures only in mK region, and is thought to be caused by the spin exchange interaction between ^{3}He atoms.

The ferromagnetism was first observed by Ahonen *et al.*[1] in the experiment of liquid ^{3}He in contact with Mylar sheets as surface-induced ferromagnetism. They also demonstrated the ferromagnetism has its origin in the firs

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Nonlinear Leggett-Rice effect in spin polarised solid ^{3}He

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Spin diffusion involves the transport of the vector magnetisation, characterised by a diffusion *tensor*. A magnetic field destroys the system's isotropy so that while longitudinal diffusion is real, transverse diffusion becomes complex. The imaginary part leads to spin rotation since the spin current may no longer be parallel to the spin density. In spin polarised solid ^{3}He the molecular field of the exchange interaction leads to such an effect. We have evaluated the elements of the diffusion tensor to show this. A nonlinearity arises since the diffusion tensor so-calculated depends on the magnetisation. We show that this behaviour is described by the equation proposed by Leggett for phenomena in degenerate liquid ^{3}He. Consequently the nonlinear phenomena in liquid ^{3}He have their analogues in the spin polarised solid.

1. INTRODUCTION

It was predicted by Leggett that the interactions in a Fermi liquid would lead to significant effects on the spin dynamics [1]. He modelled the interactions by the molecular fields of Landau Fermi liquid theory. Since the molecular field is parallel to the magnetisation it can exert no torque on the magnetisation, and thus have no direct effect. However it will cause an extra precession of the spin current, which then influences the magnetisation through the equation of continuity. The consequence of this is the Leggett-Rice effect; the expected spin diffusion is modified and the rotating frame magnetisation obeys a more complicated, nonlinear equation:

$$\frac{\partial \boldsymbol{M}}{\partial t} = D_0 \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left[\frac{1}{1+\mu^2 M^2} \times \right.$$
$$\left. \times \left\{\frac{\partial \boldsymbol{M}}{\partial x_i} + \mu\left(\boldsymbol{M} \times \frac{\partial \boldsymbol{M}}{\partial x_i}\right) + \mu^2\left(\boldsymbol{M}.\frac{\partial \boldsymbol{M}}{\partial x_i}\right)\boldsymbol{M}\right\}\right] \qquad (1)$$

Here μ is the "spin rotation parameter", which governs the extent of the new behaviour.

2. NONLINEARITY

The nonlinearity is apparent from the dependence of Eq. (1) on the fractional magnetisation $\boldsymbol{M}$. This may be seen in a clear way by regarding this as a generalised diffusion phenomenon. The spin current is still linear in the gradient of the magnetisation (so the transport *is* diffusive) but the constant of proportionality, the diffusion tensor, depends on the magnitude and the orientation of the magnetisation. In other words Eq. (1) describes generalised diffusion, with a diffusion tensor D given by

$$\frac{D_0}{1+\mu^2 M^2}\begin{pmatrix} 1+\mu^2 M_x^2 & -\mu M_z + M_x M_y & \mu M_y + M_x M_z \\ \mu M_z + M_x M_y & 1+\mu^2 M_y^2 & -\mu M_x + M_y M_z \\ -\mu M_y + M_x M_z & \mu M_x + M_y M_z & 1+\mu^2 M_z^2 \end{pmatrix}$$

which takes a particularly simple form following an orthogonal transformation to a frame whose z axis points along the magnetisation vector. Then one finds

$$\mathsf{D} = \frac{D_0}{1+\mu^2 M^2}\begin{pmatrix} 1 & -\mu M & 0 \\ \mu M & 1 & 0 \\ 0 & 0 & 1+\mu^2 M^2 \end{pmatrix}. \qquad (2)$$

The full generality of this is demonstrated by Meyerovich [2]. The tensor is diagonal in the +, −, z representation; the transverse magnetisation has a complex diffusion coefficient $D_0/(1+i\mu M)$, while the longitudinal magnetisation has a real diffusion coefficient D_0. We see that Leggett-Rice phenomena are characterised by two quantities: the spin rotation parameter μ and the diffusion "constant" D_0. In the general case it turns out that D_0 may be different in the transverse the and longitudinal case; we denote these by $D_\perp^{(0)}$ and $D_\parallel^{(0)}$.

Recent research in Fermi liquids has shown that the Leggett-Rice effect leads to the existence of spin waves and anomalous spin diffusion as well as a host of exotic nonlinear phenomena. Furthermore the Leggett-Rice effect occurs even in Boltzmann gases due to quantum mechanical exchange interactions that cause what is often called "identical-particle spin rotation". In other words, these systems also obey Eq. (1).

3. SOLID ^{3}He

Exchange interactions can occur in solids, for example, in solid ^{3}He. The question arises, then, whether the spin dynamics of such solids can also be understood in terms of a Leggett-Rice effect. Does the magnetisation in solid ^{3}He obeys Eq. (1)? And if it does, then what are the parameters μ and D_0 for the system? These questions are answered in this paper.

For simplicity we consider Heisenberg pairwise exchange, with Hamiltonian

$$\mathcal{H}_x = -\sum_{i,j} J_{ij} I^i . I^j, \qquad (3)$$

where $J_{ij} = J$ for nearest neighbours. This (effective) Hamiltonian causes the evolution of the system's magnetisation, leading to the Leggett-Rice phenomena. At long wavelength the q Fourier component of the transverse magnetisation evolves according to e^{-Dq^2t} with $Dq^2 = D_\perp q^2 + i\omega_q$, where the effective diffusion constant (real part of D) is

$$D_\perp = \frac{D_\perp^{(0)}}{1 + (\mu M)^2}, \qquad (4)$$

with μ the spin rotation parameter and M the magnitude of the polarisation. The imaginary part of the diffusion gives rise to spin waves of frequency $-\omega_q$, where

$$\omega_q = \frac{D_\perp^{(0)} q^2 \mu M}{1 + (\mu M)^2}. \qquad (5)$$

The longitudinal case is treated similarly. We have evaluated the diffusion coefficients for this system by considering the time evolution of the magnetisation following the removal of a spatially sinusoudal magnetic field. This is essentially a linear response calculation, the response being linear in the *gradient* of the magnetisation. The magnetisation evolution is obtained as a moment-series expansion; this is an extension, to arbitrary polarisation, of our previous high-temperature results [2].

The computations of the moments involve extremely tedious spin traces and lattice sums with a much larger number of terms when the system is polarised than in the case of $M = 0$. We will present the complete results in a later publication. Spin traces have been computed in a mean-field approximation. The Figure shows the two diffusion constants. The most striking feature is that they behave so differently.

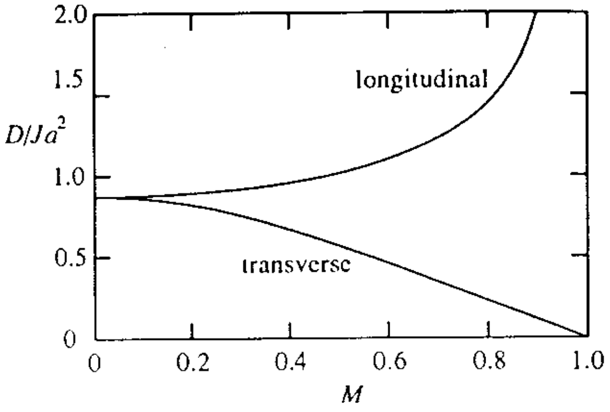

Figure 1. Longitudinal and transverse diffusion constants for isotropic exchange on an hcp lattice

$D_\perp$ is expected to vanish as $M \to 1$ because it represents the spin wave damping, which disappears at perfect polarisation. But we can understand it in a different way as well. With the aid of Eqs. (4) and (5) we can use our results to extract a spin-rotation parameter μ as a function of polarisation [4]. Let us do this analytically at high polarisation. There $D_\perp \propto (1 - M)$, as Fig. 1 clearly shows, so that $\mu M \propto M/(1 - M)$. Placing this form of μM back in Eq. (4), yields, for large M, an "ordinary diffusion constant" $D_\perp^{(0)} \propto (1 - M)^{-1}$, that is, divergent as $M \to 1$. This result is reasonable because of the long mean free path of spin waves in a highly polarised system. $D_\parallel$ on the other hand is considerably smaller because of the current precession about the magnetisation. A similar analysis in which we extract a $D_\perp^{(0)}$ and a μ can be carried out at any polarisation. At $M = 0$, we find $\mu = 2.3$. $D_\parallel$ is throughout qualitatively similar to $D_\perp^{(0)}$, as we might expect, but not equal to it, the former diverging as $(1 - M^2)^{-1/2}$.

Our overall conclusion is that, within the validity of the moment expansions/re-summations used, our results are quite consistent with the Leggett equation of motion for the magnetisation, Eq. (1), when allowing for different $D_\perp^{(0)}$ and $D_\parallel^{(0)}$

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The Leggett-Rice Effect in Solid ^{3}He and Other Magnetic Systems

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Abstract

We study spin diffusion in a polarized Heisenberg magnet in the paramagnetic regime. The longitudinal and transverse diffusion constants are given in terms of moments of correlation functions for arbitrary polarization. Polarization leads to a complex transverse diffusion constant, which is interpreted in terms of damped spin waves analogous to those in polarized Fermi liquids, and to a transverse diffusion constant that differs markedly from the longitudinal diffusion constant. The results are interpreted as a Leggett-Rice effect in the solid. We suggest NMR experiments in hcp solid ^{3}He for observing these phenomena.

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It was predicted by Leggett and Rice [1] that the interactions in a Fermi liquid would lead to molecular fields that would have significant effects on the spin dynamics. The magnetization obeys a complicated nonlinear equation known as the "Leggett equation". Research in Fermi liquids has shown that the Leggett-Rice (LR) effect involves the existence of spin waves and anomalous spin diffusion [1]- [7]. Furthermore the LR effect occurs even in Boltzmann gases due to quantum mechanical exchange interactions that cause what is often called "identical-particle spin rotation" [7].

Exchange interactions can occur in solids, for example, in solid ^{3}He. The question arises, then, whether the spin dynamics of such solids can also be understood in terms of an LR effect.

In the case where the magnetization differs only slightly from its equilibrium value, the Leggett equation reduces to a conventional diffusion equation, but with a *complex* polarization-dependent diffusion coefficient [1]. The imaginary part corresponds to spin wave propagation, while the real, diffusive, part may be interpreted as damping of these spin waves. The q Fourier component of the transverse magnetization varies according to $e^{i\omega_0 t}e^{-Dq^2 t}$ with $Dq^2 = D_\perp q^2 + i\omega_q$, where the effective diffusion constant is

$$D_\perp = \frac{D_\perp^{(0)}}{[1+\mu^2 M^2]}, \tag{1}$$

with μ the "spin rotation paramenter", M the magnitude of the polarization, and ω_0 the Larmor frequency. The spin wave frequency is $\omega_0 - \omega_q$, where

$$\omega_q = \frac{D_\perp^{(0)} q^2 \mu M}{[1+\mu^2 M^2]}. \tag{2}$$

The effective diffusion constant is diminished by the factor $1/(1+\mu^2 M^2)$ because the spin currents precess about the magnetization, which decreases the effective rate of spin transport. This feature is present in transverse diffusion in which the spin current has a component perpendicular to the magnetization, but is absent in longitudinal spin diffusion, whose diffusion coefficient we denote by $D_\|$.

We consider the solid as a polarized Heisenberg magnet in the paramagnetic state above any transition temperature. We expect that, at very high polarizations, the molecular field of the exchange interaction will result in spin waves whose frequency at $q = 0$ would be ω_0. As the polarization is lowered, the spin waves collide more strongly and become increasingly damped. Near zero polarization, as a consequence of exchange, the spin transport is described by pure spin diffusion. Thus it seems likely that in the solid at polarization less than unity, as in the fluid, the spin dynamics can be described in terms of a complex diffusion constant and an LR effect. A general justification of this requires the demonstration that the magnetization in a polarised Heisenberg magnet with exchange does indeed obey the nonlinear Leggett equation. In this Letter we address the special case of the behavior of the solid in the linear régime; work on the more general nonlinear problem is currently in progress. What our calculation shows is that, in the solid, $D_\perp$ is not equal to $D_\|$; and in fact $D_\| \to \infty$ as $M \to 1$ while $D_\perp \to 0$. Such a behavior is remarkable, independently of any interpretation in terms of the LR effect, and deserves experimental investigation. However, we will show that the behavior of the solid is compatible with Eqs. (1) and (2) so the LR effect provides a physical interpretation of the spin diffusion results.

For experimental purposes we have solid ^{3}He in mind because it is easily polarized in its paramagnetic phase by practical magnets. However, solid ^{3}He in its low-pressure bcc phase must be described in terms of multiparticle exchanges, including up to and possibly beyond 4-particle exchanges [8], [9]. We are at present unable to treat such a complicated exchange system. Here we study the simpler Heisenberg magnet, corresponding to the higher density hcp phase. The hcp phase is characterized mainly by triple exchange among particles on two different triangles in the lattice [8]. Fortunately, it is possible to write such an exchange Hamiltonian in terms of effective pair exchanges, with two different pair exchange integrals. We have then

$$H_x = -\sum_{i,j} J_{ij}\mathbf{I}^i\cdot\mathbf{I}^j, \tag{3}$$

where $J_{ij} = J$ for near neighbors within the basal plane, and $J_{ij} = J'$ for near neighbors out of the basal plane. Theory predicts that $J \approx J'$ [8], but that claim has not been tested by experiment. We note that our analysis is not confined to just hcp ^{3}He, but applies to any material described by a Heisenberg Hamiltonian.

The spin dynamics is studied by linear response theory. We imagine a small spatially oscillatory external gradient, switched on in the distant past so that the system has come to equilibrium under its influence. This field has an interaction Hamiltonian with the spin system given by

$$H_c = \sum_i [w_i^* I_+^i + w_i I_-^i], \tag{4}$$

in which $w_i = e^{i\mathbf{q}\cdot\mathbf{R}_i}$. One then turns off the sinusoidal field at $t = 0$ and watches the decay of the magnetization. The Fourier component of the magnetization is given by

$$M_+(\mathbf{q},t) = \sum_j w_j \mathrm{Tr}[\rho(t) I_+^j]. \tag{5}$$

We assume that w_i contains a small parameter so that we consider the linear response limit. The magnetization at time t is then given by

$$M_+(\mathbf{q},t) = e^{i\omega_0 t} G(t) M_+(\mathbf{q}, t=0), \tag{6}$$

where by following well-known procedures [10]–[12] we can show that

$$G(t) = \frac{1}{\chi_q}\int_{-\infty}^{\infty}\frac{d\omega}{\pi}e^{-i\omega t}\frac{\chi''(\mathbf{q},\omega)}{\omega-\omega_0} \tag{7}$$

with the absorptive part of the dynamic susceptibility $\chi''(\mathbf{q},\omega)$ given by

$$\chi''(\mathbf{q},\omega) = \sum_{i,j}\frac{w_i w_j^*}{2\hbar N}\int_{-\infty}^{\infty} dt\, e^{i\omega t}\mathrm{Tr}\left(\rho_0[I_-^i, \tilde{I}_+^j(t)]\right). \tag{8}$$

$G(t)$ describes the decay of the magnetization in the rotating frame. χ_q is the static suceptibility, which in our notation is $M_+(\mathbf{q}, t=0)/N$.

We assume that $M_+(\mathbf{q},t)$ satisfies a generalized diffusion equation analogous to the Leggett equation, i.e., with a complex diffusion constant. Therefore, we take the long-time behavior of $G(t)$ to be

$$G(t) \rightarrow e^{-(\gamma_q^\perp + i\omega_q)t}, \tag{9}$$

where $\gamma_q^\perp = D_\perp q^2$, with $D_\perp$ the transverse diffusion constant and $-\omega_q$ the spin-wave frequency in the rotating frame. The spin-wave frequency in the lab frame is then $f_q = \omega_0 - \omega_q$. We make no assumptions about the form of $\gamma_q^\perp$, that is, we do *not* assume that it has the form given by Eq. (1). The form will be determined by our computations.

Kubo and others [13], [14], [15] have derived a technique for relating the early time behavior of the correlation function $G(t)$ to the long-time behavior described by a diffusion constant. In the present case the result is

$$\gamma_q^\perp = -\mathrm{Re} \int_0^\infty G''(t')dt', \tag{10}$$

$$\omega_q = \tilde{M}_1 - \mathrm{Im} \int_0^\infty G''(t')dt', \tag{11}$$

where $\tilde{M}_n$ are the moments of $\chi''(\mathbf{q},\omega)/(\omega - \omega_0)$ or equally the expansion parameters in a time series for $G(t)$, and $G''(t)$ is the second derivative of $G(t)$.

To evaluate $\gamma_q^\perp$ we introduce $G''(t)$'s Fourier transform $\hat{G}''(\omega)$ into Eq. (10) to give

$$\gamma_q^\perp = -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \hat{G}''(\omega)\, \mathrm{Re}\int_0^\infty dt' e^{-i\omega t'} = -\frac{1}{2}\hat{G}''(0). \tag{12}$$

From Eq. (7) it is apparent that the Fourier transform is

$$\begin{aligned} \hat{G}''(\omega) &= -\frac{1}{\hbar^3 N \chi_q(\omega - \omega_0)} \sum_{i,j} w_i w_j^* \\ &\quad \times \int_{-\infty}^{\infty} dt \mathrm{Tr}\left(\rho_0[I_-^i, [H_x, [H_x, \tilde{I}_+^j(t)]]]\right). \end{aligned} \tag{13}$$

We can get rid of the outermost commutator, to make later calculations somewhat easier, by using the fluctuation-dissipation theorem [10], [12]. Together with Eq. (12) this gives

$$\begin{aligned} \gamma_q^\perp &= \frac{(e^{-\beta\hbar\omega_0} - 1)}{2\hbar^3 N \chi_q \omega_0} \sum_{i,j} w_i w_j^* \\ &\quad \times \int_{-\infty}^{\infty} dt \mathrm{Tr}\left(\rho_0 I_-^i [H_x, [H_x, \tilde{I}_+^j(t)]]\right). \end{aligned} \tag{14}$$

Our basic technique for reducing this result to a computable form is to express $\gamma_q^\perp$ in terms of moments [14], [16]- [20]. Consider the correlation function

$$\begin{aligned} k(t) &= \frac{\sum_{i,j} w_i w_j^* \mathrm{Tr}\left(\rho_0 I_-^i \tilde{I}_+^j(t)\right)}{\sum_{i,j} w_i w_j^* \mathrm{Tr}\left(\rho_0 I_-^i I_+^j\right)} \\ &= 1 - iM_1 t - \frac{1}{2}M_2 t^2 + \frac{i}{3!}M_3 t^3 + \cdots. \end{aligned} \tag{15}$$

This correlation function has been expanded in terms of moments M_n (not quite the same as the $\tilde{M}_n$ introduced above), a process that gives only the short-ranged behavior, which is all we will need here. Let the denominator of $k(t)$ be denoted as W_q. Then a quantity we need is

$$\sum_{i,j} w_i w_j^* \mathrm{Tr}\left(\rho_0 I_-^i [H_x, [H_x, \tilde{I}_+^j(t)]]\right)$$
$$= -\hbar^2 W_q \frac{d^2 k(t)}{dt^2} = \hbar^2 W_q M_2 s(t), \tag{16}$$

where

$$s(t) = 1 - i\frac{M_3}{M_2}t - \frac{1}{2}\frac{M_4}{M_2}t^2 + \ldots \tag{17}$$

We make a Gaussian approximation [14], [16]- [20]

$$s(t) = e^{-\alpha t^2/2} e^{-i\nu t} \tag{18}$$

with the coefficients determined, by Taylor expansion, to be $\alpha = \frac{M_4}{M_2} - \left(\frac{M_3}{M_2}\right)^2$ and $\nu = \frac{M_3}{M_2}$. Other mathematical forms are possible but the final result should not be too sensitive to the choice [20]. Once the form Eq. (16) and the ansatz Eq. (18) are substituted we can do the time integral in Eq. (14) explicitly leading to

$$\gamma_q^\perp = \left[\frac{W_q(e^{-\beta\hbar\omega_0} - 1)}{N\chi_q \hbar\omega_0}\right] \sqrt{\frac{\pi}{2\alpha}} M_2 e^{-\frac{M_3^2}{(2\alpha M_2^2)}}. \tag{19}$$

M_2 is already of order q^2 so that we can evaluate the quantity in square brackets at $q = 0$ to get $\gamma_q^\perp$ to leading order in q. By using the fluctuation-dissipation theorem on the static susceptibility, it is possible to show that, at $q = 0$, the quantity in square brackets is unity giving

$$\gamma_q^\perp = \sqrt{\frac{\pi}{2\alpha}} M_2 e^{-\frac{M_3^2}{(2\alpha M_2^2)}}. \tag{20}$$

A formula similar to this was derived in Ref. [14], but since the authors considered the unpolarized state, all the odd moments vanished so that the exponential factor was irrelevant. (In fact numerically it does not differ much from unity in our case.)

A similar analysis can be made for the longitudinal case. There one expresses the results in the moments, $\mathcal{M}_n$, given by expanding the correlation function $\mathcal{K}(t)$ according to

$$\mathcal{K}(t) = \frac{\sum_{i,j} w_i w_j \mathrm{Tr}\left(\rho_0 I_z^i \tilde{I}_z^j(t)\right)}{\sum_{i,j} w_i w_j \mathrm{Tr}\left(\rho_0 I_z^i I_z^j\right)}$$
$$= 1 - \frac{1}{2}\mathcal{M}_2 t^2 + \frac{1}{4!}\mathcal{M}_4 t^4 + \cdots. \tag{21}$$

There are, of course, no odd moments in the longitudinal case. The final result is

$$\gamma_q^{\parallel} = \sqrt{\frac{\pi}{2}} \frac{\mathcal{M}_2^{3/2}}{\mathcal{M}_4^{1/2}}. \tag{22}$$

This form is the same as in [14] and in the high temperature limit treated in Refs. [16], [17], and [20].

To evaluate the spin-wave frequency we make a random-phase approximation. We use only the first term in Eq. (11); we have estimated the second term and found it very small. The first moment is

$$\tilde{M}_1 = i\left[\frac{dG}{dt}\right]_{t=0} = \frac{-i}{\hbar^2 N\chi_q}\sum_{i,j} w_i w_j^* \times \int_{-\infty}^{0} d\tau e^{\epsilon\tau} e^{-i\omega_0 t}\mathrm{Tr}\left(\rho_0[I_-^i,[H_x,\tilde{I}_+^j(-\tau)]]\right). \tag{23}$$

We explicitly evaluate the commutator $[H_x, I_+^j]$ and then replace each occurrence of I_z^j by its average value $M/2$. The result is

$$\tilde{M}_1 = \frac{-iM}{\hbar^2 N\chi_q}(J_q - J_0)\sum_{i,j} w_i w_j^* \times \int_{-\infty}^{0} d\tau e^{\epsilon\tau} e^{-i\omega_0 t}\mathrm{Tr}\rho_0\left([I_-^i,\tilde{I}_+^j(-\tau)]\right), \tag{24}$$

where $J_q = \sum_R J(\mathbf{R})e^{-i\mathbf{q}\cdot\mathbf{R}}$. However one can show that the static susceptibility can be expressed in a formula that cancels out most of the expression in the numerator giving

$$\omega_q = \tilde{M}_1 = \frac{M}{\hbar}(J_0 - J_q). \tag{25}$$

The full spin wave frequency is then $f_q = \omega_0 - \omega_q$ a result originally found by Bogoliubov and Tjablikov [21]; corrections have been given by Callen [22] and others.

The computations of the moments involve extremely tedious spin traces and lattice sums with a much larger number of terms when the system is polarized than in the case of $M = 0$. For hcp ^{3}He, there is anisotropy introduced both by having $J \neq J'$ and by the non-Bravais lattice, however the latter effect is very small and, for simplicity in presentation, we neglect it here, and also consider only the case of the two exchange integrals equal. We will present the complete results in a later publication. Spin traces have been computed in a mean-field approximation. Figure 1 shows the two diffusion constants. We show D results for all polarizations although spin waves are expected to be well-defined only for large polarization. The most striking feature is that $D_\perp$ and $D_\parallel$ behave so differently.

$D_\perp$ is expected to vanish as $M \to 1$ because it represents the spin wave damping, which disappears at perfect polarization. But we can understand it in a different way as well. If Eqs. (1) and (2) apply here, we can use ω_q and $\gamma_q^\perp$ together to extract a spin-rotation parameter μ as a function of polarization [25]. Let us do this analytically at high polarization. There $D_\perp \propto (1 - M)$, as Fig. 1 clearly shows, so the ratio of ω_q to $\gamma_q^\perp$ gives $\mu M \propto M/(1 - M)$. Placing this form of μM back in Eq. (1), yields, for large M, an "ordinary diffusion constant" $D_\perp^{(0)} \propto 1/(1 - M)$, that is, divergent as $M \to 1$. This result is reasonable because of the long mean free path of spin waves in a highly polarized system. $D_\perp$ on the other hand is considerably smaller because of the current precession about the magnetization. A similar analysis in which we extract a $D_\perp^{(0)}$ and a μ can be carried out at any polarization. At

$M = 0$, we find $\mu = 2.3$. $D_{\perp}^{(0)}$ is throughout qualitatively similar to $D_{\|}$, as we might expect, but not equal to it [24], the latter diverging as $1/\sqrt{1 - M^2}$.

Our overall conclusion is that we have found a remarkable polarization dependence for $D_{\perp}^{(0)}$ and $D_{\|}$ and moreover that the linear behaviour of the solid is fully consistent with the Leggett equation [1] as represented by Eqs. (1) and (2). In a cw NMR experiment, a linear probe, the absorptive part of the dynamic susceptibility, is observed directly, which permits the observation of damped spin waves. In a spin-echo experiment the small tip-angle limit corresponds to the linear case, which gives the same information as the cw case discussed here.

For large tip angles, the LR effect involves nonlinear behavior because the effective diffusion constant $D_{\perp}$ depends on the polarization, which in turn depends on time. Leggett was able to solve the problem of the echo height as a function of time in the liquid exactly for all tip angles at low polarization. In the case of the solid we have established that the diffusion constant depends on the polarization, which will also lead to nonlinear behavior. As yet we have not established that the magnetization obeys the full nonlinear Leggett equation: we hope to report results on this and the behavior at arbitrary tip in a future publication. The resulting nonlinearities should lead to interesting phenomena [26].

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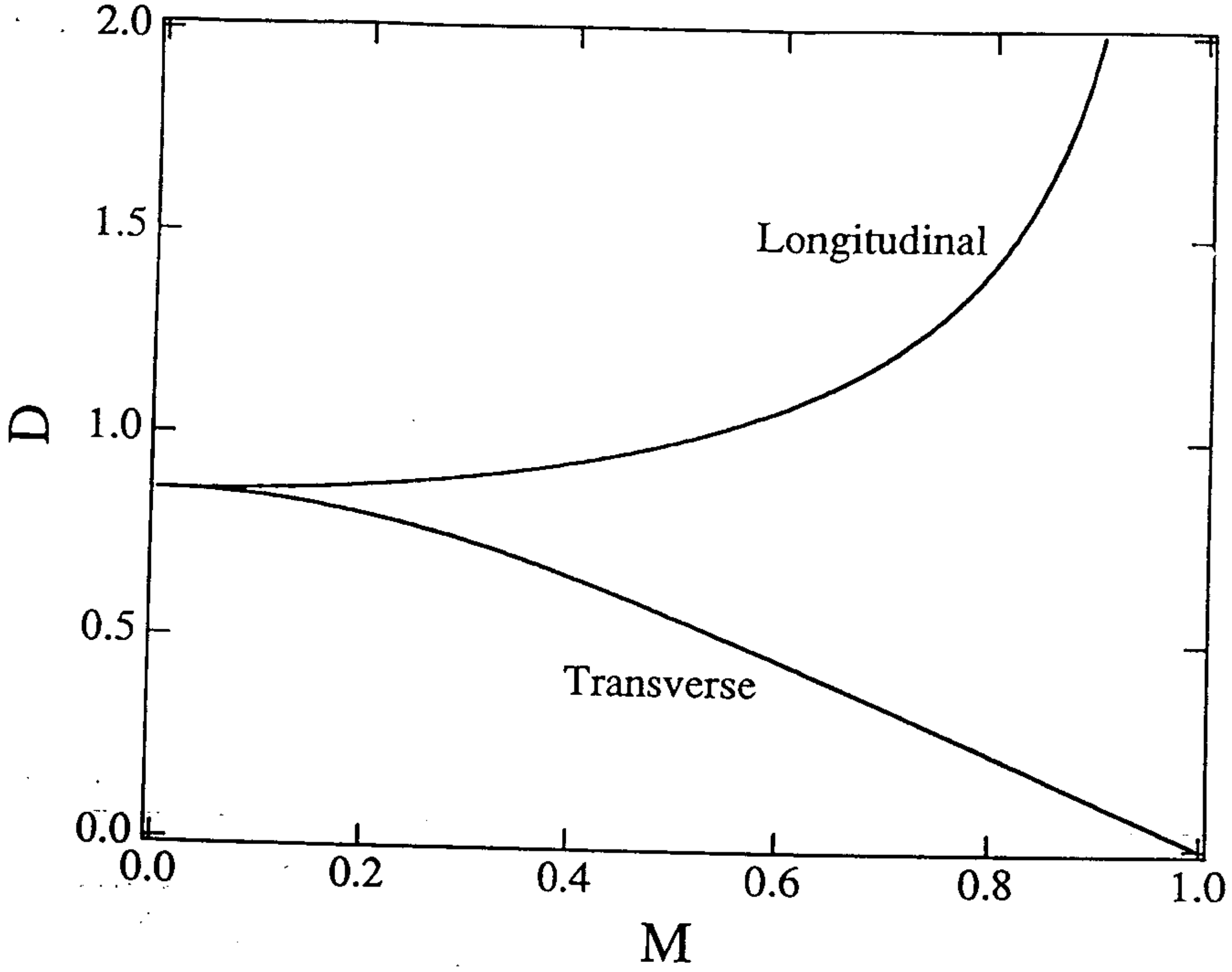
Figure Caption:

Fig. 1: The longitudinal and transverse diffusion constants as functions of polarization M for the case of isotropic exchange. The diffusion constants are given in units of $Ja^2/\hbar$, where a is a near-neighbor distance.


2.0
1.5
1.0
0.5
0.0
D
0.0
0.2
0.4
0.6
0.8
1.0
M
Longitudinal
Transverse

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Anisotropic Spin Diffusion and Multiple Spin Echoes in ^{3}He-^{4}He Solutions

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We propose a $\theta_1 - t_1 - \theta_2$ pulse-NMR experiment to detect the spin-diffusion anisotropy, $\Delta D = D_{\parallel} - D_{\perp}$, in degenerate spin-polarized ^{3}He-^{4}He mixtures, where $D_{\perp}$ and $D_{\parallel}$ are the transverse and longitudinal spin diffusion coefficients. In such an experiment the nonlinearity of the dynamics produces multiple spin echoes (MSE). At the ^{3}He concentration $x_3 \approx 4\%$ the spin-rotation parameter vanishes ($\mu M \rightarrow 0$), so that the nonlinearity of the equations of motion is entirely due to the anisotropy. In this situation, detection of MSE amounts to observation of ΔD. For slight anisotropy, i.e. $\Delta D/D_{\parallel} \lesssim 0.25$, we use a perturbation scheme similiar to that developed by Einzel et al. *(in that case, for small μM and small demagnetizing field) to calculate the second and third echo heights. For larger anisotropy we numerically calculate the echo heights. We find that for $\Delta D/D_{\parallel} = 0.5$ the heights are $\approx 2\%$ of the first echo, and should be detectable. The (θ_1, θ_2) tip-angle dependence of the ΔD echoes is different from that of the μM and demagnetization echoes, and furthermore, they occur at right angles to these echoes (in spin space). Thus, even when small spin-rotation and demagnetization effects are present, the $\theta_1 - t_1 - \theta_2$ experiment provides a sensitive means of detecting the anisotropy.*

1. INTRODUCTION

It is well established that spin diffusion in polarized Fermi fluids is anisotropic at low enough temperatures and high enough polarizations. In these systems, spin diffusion is parameterized by longitudinal and transverse diffusion coefficients, $D_{\parallel}$ and $D_{\perp}$, which characterize diffusion parallel and perpendicular, respectively, to the local magnetization density $\vec{M}$.[1] As demonstrated by detailed kinetic-theory calculations for dilute ^{3}He-^{4}He solutions, the anisotropy of spin diffusion reflects the underlying anisotropy of the diagonal and off-diagonal parts of the drift and collision terms of the kinetic equation for the spin-$\frac{1}{2}$ distribution matrix.[2-4] In the limit of zero polarization, spin diffusion is isotropic, and at degenerate temperatures both $D_\perp$ and $D_{\parallel}$ diverge as $1/T^2$. In the dilute limit,

$$D_\perp(T, P) = D_{\parallel}(T, P) = D_0(T) = \tfrac{1}{3}v_F^2\tau_T, \qquad P \to 0 \tag{1}$$

where $P \equiv (n_+ - n^-)/(n^+ + n^-)$ is the polarization, and $n^\pm$ are the occupation numbers for up and down spins. In Eq. (1), v_F is the Fermi velocity, $\tau_T = 3h^3C_D/(8\pi m_3^* a k_B^2 T^2)$ is the temperature-driven relaxation time, m_3^* is the quasi-particle effective mass, $a_T \simeq -1$ Å is the s-wave scattering length, and $C_D \equiv C(-1/3) \simeq 0.843$ is the Brokker-Sykes coefficient.[5]

At non-zero polarization the diffusion coefficients differ. $D_\perp$ decreases due to additional scattering between the spin-up and spin-down Fermi surfaces. For slight polarization, the dilute theory predicts that $D_\perp$ decreases according to[3-4]

$$D_\perp(T, P) = \frac{D_0(T)}{1 + [T_a(P)/T]^2}, \qquad P \ll 1 \tag{2}$$

where the anisotropy temperature at the equilibrium polarization is given by $T_a = (2C_D)^{1/2}(\pi\beta B_0/k_B)$, where B_0 is the polarizing external field. We can relate the polarization to the external field via $M_0 = \chi_0 B_0/\mu_0$ and $M = n_3\beta P$, where M_0 is the equilibrium magnetization, χ_0 is the Pauli susceptability, n_3 is the density of ^{3}He atoms, and β is the nuclear magnetic moment. For dense concentrations, $x_3 \gtrsim 1\%$, we take Eq. (2) as a semi-phenomenological form, where $D_0(T)$ and $T_a(P)$ can be measured in a small-tip $\theta - t_1 - 180$ spin-echo experiment.[6-7]

On the other hand, for small polarization, $D_{\parallel}$ increases slightly due to the larger average squared-velocity of the spin-up and spin-down Fermi surfaces;

$$D_{\parallel}(T, P) = D_0(T)[1 + \alpha P + O(P^2)] \tag{3}$$

where $\alpha \simeq 1$.[5]

The deviation of $D_\perp$ from T^{-2} dependence has been observed in pulsed-NMR spin-echo experiments of the $\theta - t_1 - 180$ type, although only in pure ^{3}He[6] and dense ^{3}He-^{4}He mixtures (2–4%).[7] The comparison between theory and experiment remains unsatisfactory since the kinetic theory applies only to dilute mixtures ($\lesssim 1\%$), and the proper Fermi liquid renormalization for dense systems is still unclear (although it seems to involve more interaction parameters).[3] Even so, the dense systems exhibit a $D_\perp(T, P)$ dependence which qualitatively confirms the basic phase-space arguments of the theory for dilute systems.

On the other hand, the polarization dependence of $D_{\parallel}$ has been mo difficult to observe, since it is rather small effect except at very hig polarizations ($>90\%$).[5] Nunes *et al.*[8] measured $D_{\parallel}$ in dilute mixtures b observing longitudinal spin diffusion between two reservoirs of up- an down-spins connected via a thin channel, but the polarization effects we smaller than the error bars.

The anisotropy in the spin-diffusion coefficients leads to nonline effects in the equation of motion of $\vec{M}$ which are in addition to the no linearities due to Leggett-Rice (LR) and demagnetization effects. O might say that the full kinetic equation is not tested until the nonline effects introduced by the anisotropy are observed, that is, until the *di ference* $\Delta D = D_{\parallel} - D_\perp$ is measured *directly*. To this end, we propose pulsed NMR multiple spin-echo (MSE) experiment of the $\theta_1 - t_1 -$ type at the concentration $x_3 \sim 3$–5% where the LR effect vanishes. We ref to this concentration as the critical concentration $x_{3(c)}$. Recently, $26° - t_1 - 180°$ spin-echo experiment was performed at the critical conce tration to measure the polarization dependence of $D_\perp$.[7] LR effects were n observed (there was no shift in the echo phase), which made a more preci determination of $D_\perp$ possible, since the LR effect introduces an addition anisotropy between the diffusion coefficients through $D_\perp \to D_\perp/(1 + \mu^2M$ (μM is the spin-rotation parameter) that is typically much larger than th polarization-induced anisotropy. The experiment was performed at th saturated vapor pressure in a magnetic field $B_0 = 8.8T$, in which case th critical concentration was found to be $x_{3(c)} = 3.8\%$. At this concentratio and field strength, the spin polarization is small $\approx 3\%$, and degenera temperatures are accessible ($T_F = 300$ mK) by dilution refrigeration. Th anisotropy temperature was determined to be $T_a = 15$ mK, so that at th lowest achieved temperatures, $T = 15$ mK, finite-polarization effec reduced $D_\perp$ by about 50%.

We propose that a $\theta_1 - t_1 - \theta_2$ experiment be performed with the sam parameters. It is well known that LR and demagnetization effects lead t the generation of MSE (higher harmonics) in a $\theta_1 - t_1 - \theta_2$ experiment.[9] In both cases, the local value of $\vec{M}$ affects the local precession frequency experienced by the diffusing spins. This spatially periodic variation in th precession frequency leads to refocusing of the spins at later times, resultin in multiple echoes. In contrast, the anisotropy of the diffusion coefficien affects the (spatial) motion of the spins, which in turn affects th accumulated phase of a diffusing spin via $\phi = \int \Omega[z(t)]\,dt$, where $z(t)$ is th trajectory of the spin. As with the LR and demagnetization effects, thi leads to a refocusing of the spins at later times. At the critical concentra tion, the nonlinearity of the equation of motion is entirely due to th anisotropy in the diffusion coefficients—and, possibly, the demagnetizatio field—although the latter effect should be small (see below). Detection of MSE at the critical concentration measures $\Delta D = D_{\parallel} - D_{\perp}$ directly.

In principle, this experiment also provides an indirect means for observing the polarization dependence of $D_{\parallel}$, although its effects in a $\theta_1 - t_1 - \theta_2$ experiment are expected to be small compared to those of $D_{\perp}$, and might not be discernable within experimental error.

In the following, we present a calculation of the MSE generated by anisotropy in the diffusion coefficients due to slight polarization. In Sec. II, we formulate the problem of calculating the echo heights, with the polarization dependence of the diffusion coefficients taken into account. In Sec. III, we carry out a perturbative calculation, similiar to that of Einzel *et al.*, which predicts second and third echoes whose heights are proportional to the anisotropy, ΔD. In Sec. IV, we extend the perturbation theory results to large anisotropy by calculating the echo heights numerically.

2. EQUATIONS OF MOTION

We consider a pulsed-NMR spin-echo experiment on a spin-polarized ^{3}He-^{4}He mixture.[10] If we neglect non-spin conserving processes (T_1 and T_2 are typically much longer than a spin-echo experiment) then the dynamics of the magnetization density is described by a continuity equation

$$\partial_t \vec{M}(\vec{x}, t) = -\sum_i \partial_i \vec{J}_i(\vec{x}, t) + \gamma \vec{M}(\vec{x}, t) \times [\vec{B}_{ext}(\vec{x}, t) + \vec{B}_d(\vec{x}, t)] \qquad (4)$$

where $\vec{J}_i$ is the ith component of the spin current, $\vec{B}_{ext}$ is the applied external field, and $\vec{B}_d$ is the local dipolar demagnetizion field. In the usual spin-echo experiment a large polarizing field B_0 is applied together with a linear field gradient, $\vec{B}_{ext} = (B_0 + Gz)\hat{z}$, so that the magnetization varies along the z-direction only. For an effectively unbounded sample with a large enough field gradient, the demagnetization field depends only on the local magnetization density, $\vec{B}_d = \mu_0(M_z\hat{z} - \vec{M}/3)$.[11] Additionally, spin-dependent interactions between the quasi-particles give rise to a molecular field, parallel to the local magnetization, about which the spin current precesses at a frequency Ω_{int} (the spin-rotation effect).[12]

For low-frequency, long-wavelength variations of $\vec{M}$ (the hydrodynamic limit), the spin current is given by[1,2]

$$\vec{J}_i = -D_{\parallel}\hat{e}\partial_i M - \frac{D_{\perp} M}{1 + (\mu M)^2}(\partial_i \hat{e} + \mu M \hat{e} \times \partial_i \hat{e}) \qquad (5)$$

where $\mu M \equiv \Omega_{int}\tau_{\perp}$ is the spin-rotation parameter, and $\tau_{\perp}$ is the relaxation time of the transverse spin current. We have written $\vec{M} = M\hat{e}$ in terms of the unit vector $\hat{e}$. The transport coefficients depend on z and t through their dependence on the polarization $P(z, t)$.

Substitution of Eq. (5) into the continuity equation results in a nonlinear equation of motion for $\vec{M}$. The nonlinearities are due not only to the demagnetization fields and LR effects, but also the anisotropic polarization dependence of the transport coefficients. Furthermore, anisotropic diffusion characterized by transverse and longitudinal coefficients is fundamentally nonlinear, even when $D_{\perp}$ and $D_{\parallel}$ are treated as constants.[13]

For high temperatures and low polarization the demagnetization and the Leggett-Rice effects are negligible, and diffusion is isotropic, $D_{\parallel} = D_{\perp}$, so that eqs. (4–5) reduce to the linear Torrey equation, which can be solved exactly for a uniform field gradient. In the case of a $\theta_1 - t_1 - \theta_2$ experiment the linear problem yields a single echo at time $t = \tau$ after the θ_2 pulse; no multiple spin echoes are expected. One can account for small nonlinearities by using first-order perturbation theory with the solutions of the linear problem as a basis set. Einzel *et al.* treat the case of very low polarization and degenerate temperatures, where $|\mu M| \ll 1$, demagnetization effects are small, and diffusion is essentially isotropic. Perturbation theory seems to describe the $\theta_1 - t_1 - \theta_2$ experimental data quite well, predicting a second echo at $t = 2t_1$ whose height is proportional to the LR and demagnetization terms.[9] At lower temperatures and higher polarizations, we can have large LR effects ($\mu M \gtrsim 1$) but essentially isotropic diffusion ($T \gg T_a$). This regime was studied by Bedford *et al.*, who were able to fit second and higher numerically calculated echoes to their experimental data.[10]

We are interested in a multiple spin-echo experiment at the ^{3}He critical concentration where the LR effect is absent, but the anisotropic polarization dependence of the spin diffusion coefficients is present. The LR effect is characterized by an internal precession frequency which, at the equilibrium, can be represented as $\Omega_{int}(P_0) = \lambda\Omega_0$, where λ is the molecular field parameter, and $\Omega_{int}(P_0)$ is the equilibrium polarization, $P_0 = P(B_0)$. In terms of the Landau interaction parameters, the parameter λ is given by $\lambda = (1 + F_0^a)^{-1} - (1 + F_1^a/3)^{-1}$. As a function of the ^{3}He concentration, λ changes sign at the critical concentration, $\lambda = 0$, i.e., $F_0^a = F_1^a/3$.[14] Bedell[15] observed that in this situation there is a singularity in the dispersion relation of the linearized kinetic equation, and suggested that in the neighborhood of $\lambda = 0$ the form of the relation switches from $\omega \sim k^2$ to $\omega \sim k$, so that Eq. (5) would be incorrect. However, Meyerovich and Musaelian[3] have recently shown that this only applies to short (microscopic) wavelengths; becaause of the zero-temperature attenuation of the spin current the dispersion relation retains the k^2 form, and the dynamics are purely diffusive in the $k \to 0$ (hydrodynamic) limit.

In the following, we assume that ^{3}He concentration is at or near the critical concentration, so LR effects can be neglected, i.e., $|\mu M_0| \ll 1$. We also assume that the relaxation rate of the transverse magnetization due to diffusion is large compared to the demagnetization frequency, $(D_0\gamma^2G^2)^{1/3} \gg \gamma\mu_0 M_0$ so that demagnetization effects are small. Both these conditions held, for example, in the experiment of Bedford *et al.* and Owers-Bradley *et al.* with ^{3}He-^{4}He mixtures.

To account for the spatial and temporal variation of the diffusion coefficients during the experiment we write,

$$\begin{aligned} D_\perp(T, M) &= \frac{D_0(T)}{1+\delta_\perp(M/M_0)^2} \\ D_\parallel(T, M) &= D_0(T)[1+\delta_\parallel(M/M_0)] \end{aligned} \tag{6}$$

where the parameters $\delta_{\perp,\parallel}$ depend only on the temperature and the equilibrium polarization, $\delta_\perp = [T_a(P_0)/T]^2$ and $\delta_\parallel = P_0$. At equilibrium, the anisotropy in the diffusion coefficients can be expressed as $\Delta D = D_\parallel - D_\perp = [\delta_\parallel + \delta_\perp/(1+\delta_\perp)]\, D_0$.

With these assumptions, the equations of motion for $\vec{M}(z, t)$ become

$$\begin{aligned} \partial_t \vec{M} &= -\partial_z \vec{J}_z + \gamma \vec{M} \times Gz\hat{z} \\ \vec{J}_z &= -D_0[1+\delta_\parallel(M/M_0)]\, \hat{e}\partial_z M - \frac{D_0}{1+\delta_\perp(M/M_0)^2} M\partial_z \hat{e} \end{aligned} \tag{7}$$

which are written for the Larmor frame. We put these equations into dimensionless form by writing $t = \tau t_1$, $z = \zeta/(\gamma G t_1)$,

$$\begin{aligned} &D^* = D_0\gamma^2G^2t_1^3,\ \vec{m} = \vec{M}/M_0, \text{ and } \vec{j} = \vec{J}_z\gamma G t_1^2/M_0; \\ &\partial_\tau \vec{m} = -\partial_\zeta \vec{j} + \vec{m} \times \zeta\hat{z} \\ &\vec{j} = -D^*(1+\delta_\parallel)\, \hat{e}\partial_\zeta m - \frac{D^*}{1+\delta_\perp m^2} m\partial_\zeta \hat{e} \end{aligned} \tag{8}$$

where t_1 is the interval between pulses in a $\theta_1 - t_1 - \theta_2$ experiment, and $2\pi/\gamma G t_1$ is the pitch length of the helix when the θ_2 pulse is applied.

It is convenient to express the transverse magnetization and spin current as $m^\pm = m_x \pm i m_y$, and $j^\pm = j_x \pm i j_y$. Further, we write $m^\pm \equiv e^{\mp i\zeta\tau} a^\pm$, $b = m_z$, and $j^\pm \equiv e^{\mp i\zeta\tau} c^\pm$, $d \equiv j_z$, where the functions $a^+(\zeta, \tau)$, $b(\zeta, \tau)$, $c^+(\zeta, \tau)$, $d(\zeta, \tau)$ have spatial period 2π, and b, d are real. Making these substitutions into Eq. (8), we get

$$\begin{aligned} \partial_\tau a^+ &= -\partial_\zeta c^+ + i\tau c^+ \\ \partial_\tau b &= -\partial_\zeta d \end{aligned} \tag{9}$$

and

$$\begin{aligned} c^+ &= -D^*\left(\delta_\parallel m + \frac{\delta_\perp m^2}{1+\delta_\perp m^2}\right) a^+ \partial_\zeta \ln m - \frac{D^*}{1+\delta_\perp m^2}(\partial_\zeta a^+ - i\tau a^+) \\ d &= -D^*\left(\delta_\parallel m + \frac{\delta_\perp m^2}{1+\delta_\perp m^2}\right) b\partial_\zeta \ln m - \frac{D^*}{1+\delta_\perp m^2}\partial_\zeta b \end{aligned} \tag{10}$$

where $m^2 = a^+a^- + b^2$.

In a $\theta_1 - t_1 - \theta_2$ experiment the initially uniform magnetization $\vec{M} = M_0\hat{z}$ is rotated about, say, the x-axis, by an angle θ_1, giving $m_y = \sin\theta_1$, $m_z = \cos\theta_1$, at time $t = -t_1$. Until the second pulse at $t = 0$ the magnetization evolves as a helix, with m_z uniform and constant, and the transverse magnetization described by $m^+ = iA(\tau)e^{-i\zeta\tau}$, where $A(\tau)$ satisfies the equation

$$\frac{d}{d\tau}\ln A = -\frac{D^*}{1+\delta_\perp(A^2+\cos^2\theta_1)} \tag{11}$$

This equation can be integrated from $\tau = -1$ to $\tau = 0$ to yield an implicit solution for $A(0)$,

$$(1+\delta_\perp\cos^2\theta_1)\ln[A(0)/\sin\theta_1] + \tfrac{1}{2}[A^2(0) - \sin^2\theta_1] = -D^*/3 \tag{12}$$

For $\delta_\perp = 0$ this reduces to the linear results, $A(0) = \sin\theta_1 \exp(-D^*/3)$.

At $t = 0$ a second pulse is applied that rotates the magnetization about the x-axis by a further angle θ_2, giving, for the initial conditions at $t = 0^+$,

$$\begin{aligned} a^+(\zeta, 0) &= i\cos\theta_1\sin\theta_2 - \frac{i}{2}A(0)[(1-\cos\theta_2)e^{i\zeta} + (1+\cos\theta_2)e^{-i\zeta}] \\ b(\zeta, 0) &= \cos\theta_1\cos\theta_2 - A(0)\sin\theta_2\cos\zeta \end{aligned} \tag{13}$$

To calculate the spin echoes in a $\theta_1 - t_1 - \theta_2$ experiment, one numerically integrates Eqs. (9–10) with initial conditions (12–13). The spatial integral of the transverse magnetization vanishes except at the times $t = nt_1$, where spin echoes occur whose heights are proportional to

$$E_n \equiv \frac{1}{2\pi}\int_0^{2\pi} e^{-in\zeta} a(\zeta, n)\, d\zeta \tag{14}$$

The size of the spin echoes depends on the parameters θ_1, θ_2, D^*, $\delta_\perp$, and $\delta_\parallel$ in a rather complicated way, so it is worthwhile to carry out a perturbative calculation for slight anisotropy, in which case the parametric dependence can be factored.

3. PERTURBATION THEORY RESULTS

In the following, we assume that the anisotropy is small, $\delta_{\perp,\parallel} \ll 1$, so that first-order perturbation theory applies. To first order in $\delta_\parallel$ we have $D_\perp = D_0(T)(1-\delta_\perp m^2) + O(\delta_\perp^2)$, so the anisotropy at equilibrium can be written as $\Delta D \simeq (\delta_\perp + \delta_\parallel) D_0$. Substituting this into Eqs. (8) we get the equation of motion for $\vec{m}$;

$$\partial_\tau m^+ - D^*\partial_{\zeta\zeta} m^+ + i\zeta m^+ = \delta_\perp D^*\partial_\zeta \left[\frac{1}{2} m^* \partial_\zeta(m^2) - m^2 \partial_\zeta m^+ \right] + \frac{1}{2}\delta_\parallel D^* \partial_\zeta \left[\frac{\partial_\zeta(m^2)}{(m^2)^{1/2}} m^+ \right] + O(\delta^2) \quad (15)$$

and

$$\partial_\tau m_z - D^*\partial_{\zeta\zeta} m_z = 0 + O(\delta) \quad (16)$$

where we have written $\delta \equiv \delta_\perp + \delta_\parallel$. In the equation for m_z we have dropped terms that are $O(\delta)$, since their effect on the spin echoes is $O(\delta^2)$.

We treat this problem by first solving the zero-order equation (the Torrey equation),

$$\partial_\tau \vec{m} = \vec{m} \times \zeta \hat{z} + D^* \partial_{\zeta\zeta} \vec{m} \quad (17)$$

and then using the zero-order solutions in the first-order (in δ) terms of Eq. (15).

Following Einzel *et al.*, we expand $\vec{m}$ in a Fourier series:

$$\begin{aligned} m^+ &= e^{-i\zeta\tau} a^+(\zeta,\tau) = e^{-i\zeta\tau} \sum_n a_n(\tau)\, e^{in\zeta} \\ m_z &= b(\zeta,\tau) = \sum_n b_n(\tau)\, e^{in\zeta} \end{aligned} \quad (18)$$

Substituting (18) into (17) and equating powers of $e^{i\zeta}$ and using the initial conditions, Eq. (13), we find the zero-order solutions, for $t > 0$,

$$\begin{aligned} a_{\pm 1}(\tau) &= \mp \frac{i}{2} \sin\theta_1 (1 \mp \cos\theta_2) \exp\left[-\frac{D^*}{3}((\tau \mp 1)^3 \pm 1 + 1) \right] \\ a_0(\tau) &= i \cos\theta_1 \sin\theta_2 \exp\left(-\frac{D^*}{3}\tau^3 \right) \\ b_{\pm 1}(\tau) &= -\frac{1}{2} \sin\theta_1 \sin\theta_2 \exp\left[-D^*\left(\tau + \frac{1}{3} \right) \right] \\ b_0 &= \cos\theta_1 \cos\theta_2 \end{aligned} \quad (19)$$

From (14) we see that the nth echo height is given by $E_n = a_n(n)$. Thus, from (19), the first echo is

$$E_1 = a_1(1) = -\frac{i}{2} \sin\theta_1 (1 - \cos\theta_2) \exp(-2D^*/3) \quad (20)$$

For small anisotropy the correction to (20) is $O(\delta)$.

To calculate the echo heights we substitute Eqs. (19) into the RHS of (15), and equate powers of $e^{i\zeta}$. Now, writing

$$m^2 = \left| \sum_{n=-1}^{1} a_n e^{in\zeta} \right|^2 + \left| \sum_{n=-1}^{1} b_n e^{in\zeta} \right|^2 \quad (21)$$

we see that the zero-order expression for m^2 is a finite series in powers of $e^{i\zeta}$. As an aside, we note that m^2 is not uniform (see Fig. [1]) for a $\theta_1 - t_1 - \theta_2$ experiment, even for the linear problem ($\delta = 0$). This point has not been fully appreciated in the literature, where the nonuniformity has been attributed to nonlinearities.[10] Now, only the $e^{i\zeta}$ and $e^{2i\zeta}$ terms in Eq. (21) give rise to multiple spin echoes. To order δ^2, one finds that a second and third echo are generated which are proportional to δ. Actually, the $\delta_\parallel$ term involves the Fourier expansion of $m = (m^2)^{1/2}$ which has higher harmonics, but for $n > 2$ these are typically small, resulting in very small

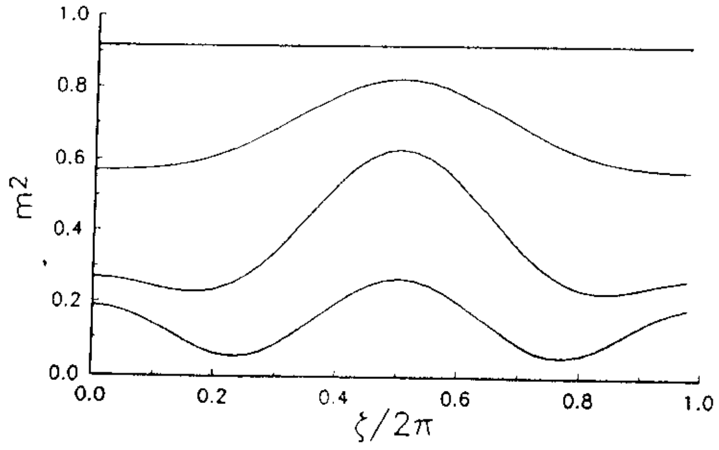


Fig. 1. The variation of $m^2(\zeta,\tau)$ for $\delta = 0$ over one spatial period for a $55 - t_1 - 90$ pulse sequence calculated from Eq. (13), with $D^* = 0.2$. In descending order the curves correspond to the times $T = 0, 1, 2, 3$.

higher echoes. Substituting (18), and (21) into (15), and equating powers of $e^{i\zeta}$, we obtain the differential equations for a_2 and a_3:

$$\frac{d}{d\tau}a_n + D^*(\tau-n)^2 a_n = -iD^*(\tau-n)\frac{1}{2\pi}\int_0^{2\pi} e^{-in\zeta}[\delta_{\parallel} f + \delta_{\perp} g]\, d\zeta + O(\delta^2) \tag{22}$$

with

$$\begin{aligned} f(\zeta,\tau) &= \frac{1}{2}\frac{\partial_\zeta(m^2)}{(m^2)^{1/2}}\, a^+ \\ g(\zeta,\tau) &= \frac{1}{2}\, a^+\partial_\tau(m^2) - m^2(\partial_\zeta a^+ - i\tau a^+) \end{aligned} \tag{23}$$

where we have integrated by parts and used the periodicity of a^+, b to eliminate the boundary term. Solving Eq. (22) with the initial conditions $a_2(0) = a_3(0) = 0$, we can write the echo heights as

$$E_n(D^*) = a_n(n) = \delta_{\parallel} F_n + \delta_{\perp} G_n \tag{24}$$

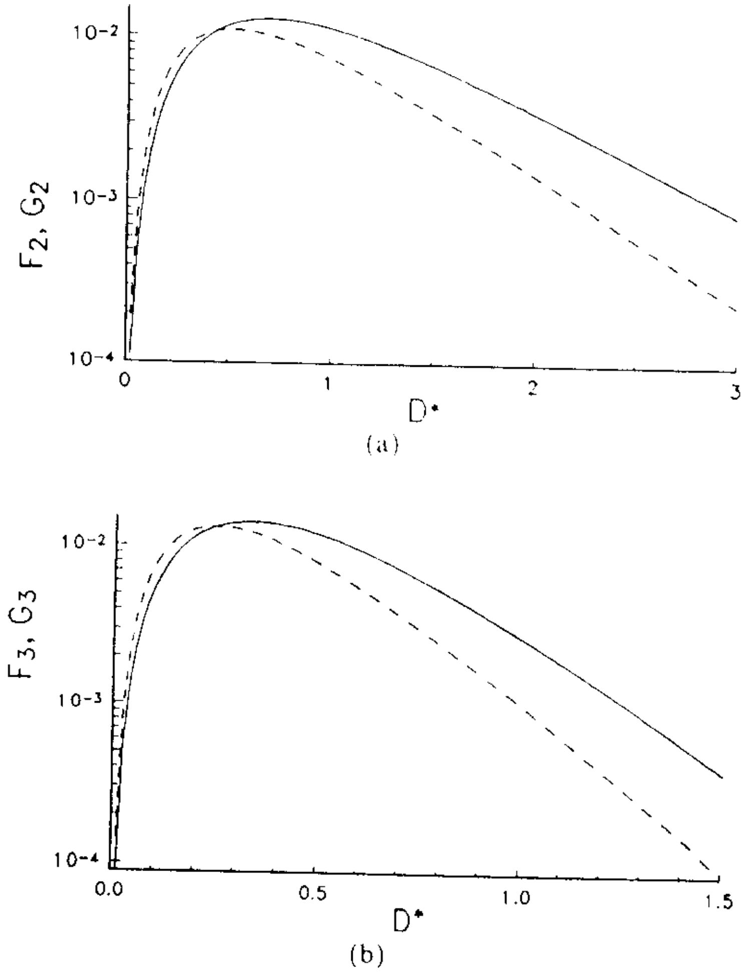
with

$$F_n(D^*) = -iD^*\int_0^n (\tau-n)\, e^{(1/3)D^*(\tau-n)^3}\left[\frac{1}{2\pi}\int_0^{2\pi} e^{-in\zeta} f(\zeta,\tau)\, d\zeta\right] d\tau \tag{25}$$

and likewise for $G_n(D^*)$, with $g(\zeta,\tau)$ in place of $f(\zeta,\tau)$. At this point we must resort to numerical calculation. However, by using (19) in Eqs. (23) we can factor out the (θ_1, θ_2)-dependence of G_n as

$$\begin{aligned} G_2 &\propto \cos\theta_1 \sin^2\theta_1\, h(\theta_2, D^*) \\ G_3 &\propto \sin^3\theta_1 \sin^3\theta_2 \end{aligned} \tag{26}$$

where the angular dependence of the function $h(\theta_2, D^*)$ increases monotonically from $h(0, D^*) = 0$ to $h(\pi/2, D^*)$. The angular dependence of F_n cannot be factored, but is numerically almost identical to that of G_n. In comparison to Einzel's results, the angular dependence of F_2, G_2, is markedly different from that of the second μM-echo,[9] $E_2[\mu M] \propto \sin^2\theta_1 \sin\theta_2(1-\cos\theta_2)$. (The demagnetization echoes have the same angular dependence as the μM-echoes). In particular, there is no second ΔD-echo for $\theta_1 = \pi/2$, while the second μM-echo is maximized at this angle (the second ΔD-echo is maximized when $\theta_1 = \cos^{-1}(1/\sqrt{3}) \simeq 55°$). In fact, for a $90 - t_1 - 90$ experiment, there are only odd-n echoes, even for arbitrarily large anisotropy. This is evident from Eqs. (9-10), since if a^+ consists of only odd powers of $e^{i\zeta}$, then $\partial_\tau a^+$ consists of only odd powers as well (m^2 consists of even powers). Furthermore, the ΔD echoes are imaginary (i.e., in the y-direction) while the μM and demagnetization echoes are real. (Recall that the x-axis is the rotational axis of the RF pulses.) The second ΔD echo occurs in the negative y-direction, while the third occurs in the positive. This feature persists for large anisotropy as



(a)

(b)

Fig. 2. The functions $F_n(D^*)$, $G_n(D^*)$. a) F_2 (solid line) and G_2 (dashed line) for a $55 - t_1 - 90$ experiment. b) F_3 (solid line) and G_3 (dashed line) for a $90 - t_1 - 90$ experiment.

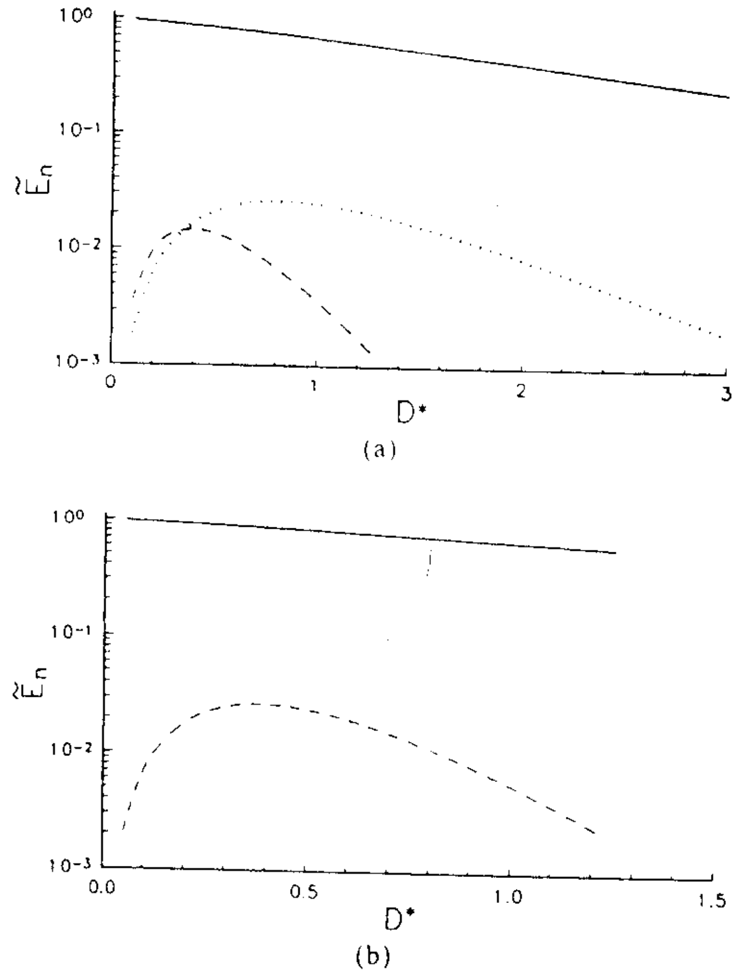


Fig. 3. a) Normalized heights of the first, second and third echoes, $\tilde{E}_n = E_n/[\sin\theta_1(1-\cos\theta_2)/2]$, for the pulse sequence $55 - t_1 - 90$ for $\delta_\perp = 1$, $\delta_\parallel = 0.03$. The solid line is the first echo height, the dotted line is the second, and the dashed line is the third. For this pulse sequence the second echo is maximized. b) The first and third echo heights for a $90 - t_1 - 90$ experiment with the same parameters as Fig. [3a]. There is no second echo for this pulse sequence.

well. Thus, the effects of anisotropy should be easily distinguishable from the effects of small LR and demagnetization effects.

As functions of D^*, the functions F_n, G_n are very similar to each other; F_2, G_2 and, hence the second echo, are maximum at $D^* \simeq 0.6$, while F_3, G_3, and the third echo are maximized at $D^* \simeq 0.3$. In Fig. [2] we show $F_n(D^*)$, $G_n(D^*)$ obtained by numerical integration of Eqs. (20–21) for the two pulse sequences, $90 - t_1 - 90$ and $55 - t_1 - 90$.

4. NUMERICAL RESULTS

In recent experiments, the smallest multiple spin echoes that are reported are typically ~0.5% – 1% of the first echo height.[9, 10] This corresponds to an anisotropy $\delta \sim 0.5$, i.e., $T \sim T_a$. Although this degree of anisotropy is obtainable experimentally, it is not quantitatively described by perturbation theory. In the following, we report the results of a numerical study of the multiple echoes due to large anisotropy ($\delta \gtrsim 0.5$). We restrict our attention to two cases: the second echo in a $55 - t_1 - 90$ experiment (where E_2 is maximized), and the third echo in a $90 - t_1 - 90$ experiment (where E_3 is maximized, and $E_2 = 0$).

To calculate the echo heights, we use a real-space finite-difference scheme to integrate Eqs. (9–10) with Eqs. (12–13) as the initial conditions. We use the experimental parameters of Bedford *et al.*, i.e., $P_0 = 3\%$ ($\delta_\parallel = 0.03$) and $T_a = 15$ mK. We also assume that LR and demagnetization effects are negligible, i.e., $\mu M_0 \to 0$, and $\gamma\mu_0 M_0/(D\gamma^2 G^2)^{1/3} \to 0$, which should be the case at the critical ³He concentration and at low enough temperatures and high enough field gradients.

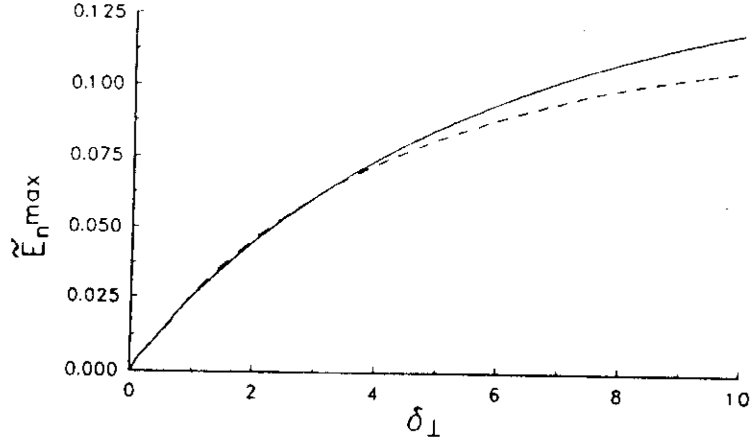


Fig. 4. Maximum normalized echo height for the second echo in a $55 - t_1 - 90$ experiment (solid line), and third echo in a $90 - t_1 - 90$ experiment (dashed line).

In Fig. [3] we show calculated spin echoes as functions of D^* for $\delta_\perp = 1$. The main effects of finite δ are not only an increase in the echo size but also a shift of the maximum echo height to larger D^*. To good approximation we find $D^*_{\max} \approx 0.25 + 0.125\delta_\perp$ for the third echo in a $90 - t_1 - 90$ experiment, and $D^*_{\max} \approx 0.55 + 0.25\delta_\perp$ for the second echo in a $55 - t_1 - 90$ experiment. The corresponding maximum (normalized) echo heights as functions of $\delta_\perp$ are shown in Fig. [4]. For the largest anisotropy shown, $\delta_\perp = 10$ ($T \approx 5$ mK), the second and third echoes should be $\approx 10\%$ of the first echo and should be easily detectable.

5. DISCUSSION

Of course, there will always be at least small LR and demagnetization effects present. We conclude by estimating the size of LR and demagnetization echoes in a 3.8% solution with $B_0 = 8.8T$ at a temperature $T = 15$ mK, as in the experiment of Owers-Bradley *et al.*[7] The height of the second demagnetization echo is given by $E_2^\beta = \beta f_\beta(D^*)$, where f_β has a maximum $f_\beta(1/8) \approx 0.18$, and $\beta \equiv \gamma\mu_0\chi_0 B_0/(D_0\gamma^2 G^2)^{1/3}$.[9] Using $D_0 T^2 \approx 70$ (cm^2 s^{-1} mK^2), $\chi_0 \approx 4 \times 10^{-9}$, and $G \approx 10$(Gauss/cm),[7] we find that $E_2^\beta \approx 6 \times 10^{-4}$ is small compared to the expected ΔD-echo heights, which are about 2%. Likewise, the second LR echo is given by $\alpha f_\alpha(D^*)$, where $f_\alpha(0.4) \approx 0.05$ is the maximum, and $\alpha \equiv \mu M_0$. Comparing the echo sizes, we find that LR effects are smaller than anisotropy effects for $\mu M_0 \lesssim 0.4$.

Finally, we remark that observation of δD echoes will be somewhat muddied by any polarization dependence of the molecular field parameter $\lambda(T, P)$. If the polarization dependence of λ is appreciable then we cannot simply set $\mu M = 0$ for a given spin-echo experiment. Meyerovich and Musaelin[3] parameterize this possible dependence as $\mu M = \lambda\{x - x_c(B_0 m, T)\}\, \Omega_0 m \tau_\perp(B_0, T)$ with $x_c(T, Hm) = x_c(T, 0) + k_H(2\beta Hm)/k_B T_F)$, where k_H is an unknown constant. Thus, if in the neighborhood of $x_{3(c)}$, $\lambda = (d\lambda/dx_3)|_{x_3 = x_{3(c)}}(x_3 - x_{3(c)})$, then we would expect μM to depend on m as

$$\mu M = [\lambda(T, 0) - \sigma m]\, \frac{\Omega_0 \tau_\perp(T, 0) m}{1 + \delta_\perp m^2} \tag{27}$$

with $\sigma = k_H(2\beta B_0/k_B T_F)(d\lambda/dx)|_{x_3 = x_{3(c)}}$. We expect the polarization dependence of λ to be a small effect. If it is detectable it would give rise to a second echo in a $90 - t_1 - 90$ experiment, in which case it could accounted for by putting Eq. (27) into Eq. (5) and keeping terms which are first order in μM. But before we speculate further, some experimental data is needed to see which direction the analysis should take.

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