SOME PHASE-THEORETICAL NOTES ON LOCALIZATION OF EIGENFUNCTIONS

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1. A Short Summary of the Present Status

The purpose of our rather formal approach is to elucidate the nature of localization phenomenon without relying, as far as possible, on approximation which often makes the nature obscure. It is practically inevitable to stick to simple, in particular one-dimensional models, to make the theory free from approximation.

It has been known that to discuss one-dimensional models, the method of phase is particularly powerful, though in principle it can handle multidimensional systems also. In this section a short summary of the present status of phase-theoretical treatment will be given.

Let us consider at first a system which can be described by a difference equation

$$-V u_{n-1} + \alpha_n u_n - V u_{n+1} = 0. \qquad (1.1)$$

Vibration of a mass-disordered linear chain, with nearest-neighbour harmonic interactions only, is an example of the systems of this type. In this case $V = k$ is spring constant of interactions, $\alpha_n = 2k - m_n \omega^2$, where m_n is mass of the nth atom, and u_n is its displacement. Anderson's model for an electron in disordered crystal is another example, in which V is overlap integral, $\alpha_n = E - \varepsilon_n$, where ε_n is energy at the nth site, and u_n is probability amplitude at the nth site.

Equation (1.1) can be written in a vector-matrix form

$$\begin{pmatrix} u_{n-1} \\ u_n \end{pmatrix} = \begin{pmatrix} \alpha_n/V & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} \qquad (1.2)$$

or

$$\psi_{n-1} = T_n \psi_n. \qquad (1.3)$$

We call $\psi_n \equiv (u_n, u_{n+1})$ and $T_n \equiv \begin{pmatrix} \alpha_n/V & -1 \\ 1 & 0 \end{pmatrix}$ state vector and transfer matrix respectively.

Upon introducing two linearly independent solutions u_n and v_n of (1.1), we may write

$$\begin{pmatrix} u_{n-1} & v_{n-1} \\ u_n & v_n \end{pmatrix} = \begin{pmatrix} \alpha_n/V & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_n & v_n \\ u_{n+1} & v_{n+1} \end{pmatrix} \qquad (1.4)$$

or

$$\mathbb{U}_{n-1} = \mathbb{T}_n \mathbb{U}_n . \tag{1.5}$$

Here we choose, as u_n and v_n, the solutions which satisfy respectively the boundary conditions

$$u_N = 1, \; u_{N+1} = 0 \quad \text{and} \quad v_N = 0, \; v_{N+1} = 1. \tag{1.6}$$

Then $\mathbb{U}_N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{I}$, so that we have

$$\mathbb{U}_{n-1} = \mathbb{T}_n \mathbb{T}_{n-1} \cdots \mathbb{T}_N . \tag{1.7}$$

If we introduce the "state ratio" $z_n \equiv u_{n-1}/u_n$, (1.1) can also be written

$$z_n = \frac{\alpha_n}{V} - \frac{1}{z_{n+1}} \tag{1.8}$$

The state ratio z_n is a real quantity so far as α_n is real. We can map it onto a unit circle on the complex plane by a Cayley transform $z_n' = \frac{z_n + i}{z_n - i}$, and define its phase by $z_n' = e^{i\delta_n}$. It is important that all the quantities $\mathbb{U}_n$, z_n (or z_n') and δ_n can be calculated successively by starting from their boundary value. It will be needless to say that we may start from the other end of the system $n = 0$, and proceed in the reverse direction.

In the phase theory we argue on the energy spectrum and the shape of wave functions upon the basis of the basic properties of δ_n. It is a monotonic increasing function of both n and E, and the increase of δ_0 by 2π with the increase of E means that there is one eigen-energy within this interval of E. In this way we can explain several characteristic features in the spectrum and the eigenfunctions of disordered systems, which can be described by transfer-matrix formalism (but not necessarily described by difference equation (1.1)).

In this lecture we confine our consideration on the localization of eigenfunctions.

Let $P(\delta)$ be the distribution of δ_n when we proceed in the direction of decreasing n. The phase δ_n', which is obtained when we proceed in the reverse direction, has a distribution function $P^+(\delta)$, which is entirely different from $P(\delta)$, for

almost all values of energy parameter E . Each value of E , at which an exception occurs, gives the eigenvalue of the system under consideration. The phase δ_n changes in general slowly with E, but in the vicinity of each exceptional value of E it changes rapidly for $n \lesseqgtr n_c$, n_c being a certain value of n depending on E, and sweeps an interval whose length is nearly 2π . Likewise, δ_n' changes with energy generally only slowly; but at each exceptional energy it changes rapidly for $n \gtrless n_c$. The eigenenergy is the value of E at which δ_n and δ_n' meet with each other during the rapid change of either δ_n or δ_n' or both.

Next consider the ratio of squared length of successive state vectors:

$$\frac{\|\psi_n\|^2}{\|\psi_{n-1}\|^2} = \frac{u_{n+1}^2 + u_n^2}{u_n^2 + u_{n-1}^2} = \frac{1}{z_{n+1}^2} \frac{1+z_{n+1}^2}{1+z_n^2} . \tag{1.9}$$

Taking the average of the logarithm, we have

$$\left\langle \ln \frac{\|\psi_n\|^2}{\|\psi_{n-1}\|^2} \right\rangle = \left\langle \ln \frac{1}{z_{n+1}^2} \right\rangle + \left\langle \ln \frac{1+z_{n+1}^2}{1+z_n^2} \right\rangle . \tag{1.10}$$

If the distribution of z_n is stationary, the last term will vanish, and we obtain, as a measure of average increase of the "intensity" of solution $\|\psi_n\|^2$, the quantity $\Gamma = \left\langle \ln \frac{1}{z_{n+1}^2} \right\rangle$. It has been argued that for most one-dimensional systems Γ is positive with probability 1 , irrespective of the direction of transfer. This means that the eigenfunction is exponentially localized around $n = n_c$, towards both directions. In this sense Γ is called "degree of localization".

The theoretical arguments sketched above have been given by Roberts, Makinson, Borland, Halperin, Hori and others [1) 2) 3) 4)]. Minami and Hori[5)] confirmed the argument also numerically. Matsuda and Ishii[6)] showed that for an isotopically disordered harmonic chain Γ is positive with probability 1. Recently Hirota and Ishii[7)] showed that the distribution function $P(\delta)$ and consequently Γ can be calculated analytically for Lloyd model in which ε_n obeys Cauchy distribution.

2. State-Ratio, Principal Minors of Secular Matrix and Self-Energy.

It is an important problem to find the quantity which measures the degree of localization in multidimensional systems also, and which reduces to L in one-dimensional case. Another important problem is to elucidate the relation between our concept of localization and that of Anderson[8]. These are difficult problems and only a small step towards this direction has just begun to be made. In the following some preliminary results are reported.

It was proved by Fujita[9] that the determinant of any square matrix $|A| = |\{a_{ij}\}|$ can be developed in terms of its principal minors:

$$|A| = a_{i_1 i_1} A^{i_1} + \sum_{m=2}^{N} {\sum_{(i_2 i_3 \cdots i_m)}}' (-1)^{m-1} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_m i_1} A^{i_1 i_2 i_3 \cdots i_m}, \quad (2.1)$$

where $A^{i_1 i_2 \cdots i_m}$ is the principal minor which is obtained by deleting from $|A|$ the i_1th, i_2th, $\cdots$, and i_mth rows and columns, and the prime means that the sum must be carried out only over all the sets of distinct indices $i_2, i_3, \cdots, i_m$. If we apply this expansion to the secular determinant of the Anderson model

$$S \equiv |E\mathbb{I} - \mathbb{H}| = \begin{vmatrix} \alpha_{-N} & -V_{-N,N+1} & \cdots & & -V_{-N,N} \\ -V_{-N+1,N} & \alpha_{N-1} & & & \\ & & \ddots & & \\ & & & & \alpha_N \end{vmatrix}, \quad (2.2)$$

we obtain

$$S = \alpha_{i_1} S^{i_1} - \sum_{m=2}^{N} {\sum_{(i_2, i_3, \cdots, i_m)}}' V_{i_1 i_2} V_{i_2 i_3} \cdots V_{i_m i_1} S^{i_1 i_2 \cdots i_m} \quad (2.3)$$

In the one-dimensional, diagonally disordered nearest-neighbour case, this becomes

$$S = \begin{vmatrix} \alpha_{-N} & -V & & & \\ -V & \alpha_{-N+1} & & & \\ & & \ddots & & \\ & & & \alpha_N & -V \\ & & & -V & \alpha_N \end{vmatrix} = \alpha_0 S^0 - V^2 S^{0,1} - V^2 S^{0,-1} \quad (2.4)$$

or

$$\frac{S}{S^0} = \alpha_0 - \frac{V^2}{\left(\frac{S^0}{S^{0,1}}\right)} - \frac{V^2}{\left(\frac{S^0}{S^{0,-1}}\right)}. \quad (2.5)$$

If we write

$$S_l \equiv \begin{vmatrix} \alpha_l & -V & & \\ -V & \alpha_{l+1} & \ddots & \\ & \ddots & & \alpha_N \end{vmatrix} \quad \text{and} \quad S_{-l} \equiv \begin{vmatrix} \alpha_{-N} & -V & & \\ -V & \alpha_{-N+1} & \ddots & \\ & \ddots & & \alpha_{-l} \end{vmatrix}, \tag{2.6}$$

then we have

$$S^0 = S_{-1} S_1, \quad S^{0,1} = S_{-1} S_2,$$

$$\frac{S^0}{S^{0,1}} = \frac{S_1}{S_2} = \alpha_1 - \frac{V^2}{\left(\frac{S_2}{S_3}\right)} = \alpha_1 - \cfrac{V^2}{\alpha_2 - \cdots - \cfrac{V^2}{\alpha_N}}. \tag{2.7}$$

On the other hand, we obtain from (1.8), by putting $t_n \equiv V z_n$,

$$t_n = \alpha_n - \frac{V^2}{t_{n+1}} = \alpha_n - \cfrac{V^2}{\alpha_{n+1} - \cdots}. \tag{2.8}$$

Thus it turns out that successive ratios of principal minors S_n / S_{n+1} just correspond to the state ratio multiplied by V : $V z_n = S_n / S_{n+1}$.

In more general cases, in which long-range interactions are present, the expansion (2.5) becomes much more complicated:

$$\frac{S}{S^0} = \alpha_0 - \sum_{m=1}^{N} {\sum_{(i_1, \cdots, i_m)}}' V_{0 i_1} V_{i_1 i_2} \cdots V_{i_m 0} \frac{S^{0 i_1 \cdots i_m}}{S^0}. \tag{2.9}$$

This may be rewritten

$$\alpha_0 - \frac{S}{S^0} = \sum_{m=1}^{N} {\sum_{(i_1 \cdots i_m)}}' V_{0 i_1} \frac{1}{\left(\frac{S^0}{S^{0, i_1}}\right)} V_{i_1 i_2} \frac{1}{\left(\frac{S^{0, i_1}}{S^{0, i_1, i_2}}\right)} V_{i_2 i_3} \cdots V_{i_{m-1} i_m} \frac{1}{\left(\frac{S^{0 i_1 \cdots i_{m-1}}}{S^{0 i_1 \cdots i_m}}\right)} V_{i_m 0} \tag{2.10}$$

which shows that the secular determinant S can be expressed in terms of all possible ratios $S^{0, i_1, \cdots i_{m-1}} / S^{0, i_1 \cdots i_m}$.

The formula (2.10) reminds us of the renormalized perturbation expansion of the self-energy considered by Anderson[8)]

$$\Delta_0(E) = {\sum_{n_1}}' V_{0 n_1} \frac{1}{\alpha_{n_1} - \Delta^0_{n_1}} V_{n_1 0} + {\sum_{\substack{n_1, n_2 \\ n_1 \neq 0 \\ n_2 \neq n_1, 0}}}' V_{0 n_1} \frac{1}{\alpha_{n_1} - \Delta^0_{n_1}} V_{n_1 n_2} \frac{1}{\alpha_{n_2} - \Delta^{0, n_2}_{n_2}} V_{n_2 0}$$

$$= \sum_{m=1}^{N} {\sum_{n_1 n_2 \cdots n_m}}' V_{0 n_1} \frac{1}{\alpha_{n_1} - \Delta^0_{n_1}} V_{n_1 n_2} \frac{1}{\alpha_{n_2} - \Delta^{0, n_1}_{n_2}} \cdots V_{n_m 0}. \tag{2.11}$$

Comparison of (2.10) and (2.11) gives identification

$$\Delta_n^{l_1 l_2 \cdots l_m} = \alpha_n - \frac{S^{l_1 l_2 \cdots l_m}}{S^{l_1 l_2 \cdots l_m ; n}} . \tag{2.12}$$

Thus it turned out that the self-energies are no other than the ratio of principal minors of the secular determinant.

Since the relation (2.12) can also be obtained directly from the very definition of Green's function, Fujita's argument provides us with a very simple alternative derivation of the renormalized perturbation expansion (2.11).

In the nearest-neighbour case, (2.11) becomes

$$\Delta_0(E) = \frac{V^2}{\alpha_1 - \Delta_1^0} + \frac{V^2}{\alpha_{-1} - \Delta_{-1}^0} \tag{2.13}$$

and

$$\Delta_0^1(E) \equiv \frac{V^2}{\alpha_2 - \Delta_2^{0,1}} = \cfrac{V^2}{\alpha_2 - \cfrac{V^2}{\alpha_3 - \cfrac{V^2}{\ddots - \cfrac{V^2}{\alpha_N}}}} . \tag{2.14}$$

If we write $\Delta_n^{0,1,\cdots n-1}(E) \equiv \Delta_n$ for $n \neq 0$, we have, for $n-1 \geq 1$,

$$\Delta_{n-1} = \frac{V^2}{\alpha_n - \Delta_n} = \cfrac{V^2}{\alpha_n - \cfrac{V^2}{\alpha_{n+1} - \cfrac{V^2}{\ddots - \cfrac{V^2}{\alpha_N}}}} \tag{2.15}$$

and

$$\Delta_n = \alpha_n - \frac{S^{n-1}}{S^n} = \alpha_n - \frac{S_n}{S_{n+1}} \tag{2.16}$$

The last formula gives the connection between the state ratio z_n and self-energy Δ_n

$$z_n = \frac{V}{\Delta_{n-1}} \tag{2.17}$$

Thus it turns out that our degree of localization is no other than the average of logarithm of squared self-energy, apart from the additive constant.

The relation (2.17) suggests that in multidimensional cases, or in the case in which long-range interactions are present, the degree of localization is given by some average over the quantities $\Delta_n^{l_1 l_2 \cdots l_m}$ or $S^{l_1 l_2 \cdots l_m} / S^{l_1 l_2 \cdots l_m ; n}$. But at present it is not

yet known whether this statement is correct or not. If it is correct, it is difficult to calculate the appropriate average because these quantities cannot be obtained successively by a recurrence formula, in contrast to the one-dimensional nearest-neighbour case.

3. A Possible Measure of Degree of Localization in the Case of Long-Range Interaction.

In this section we present a method by which we can avoid the difficulty mentioned in the end of the last section. As a simple example consider the Anderson model, in which only the nearest and next-nearest interactions are present. Schrödinger's equation of this system is

$$-V'u_{n-2} - Vu_{n-1} + \alpha_n u_n - Vu_{n+1} - V'u_{n+2} = 0 \qquad (3.1)$$

where $\alpha_n = E - \varepsilon_n$.

The secular matrix is

$$\mathfrak{S} \equiv \begin{pmatrix} \alpha_1 & -V & -V' & & \\ -V & \alpha_2 & -V & -V' & \\ -V' & -V & \alpha_3 & -V & -V' \\ & & & \ddots & \end{pmatrix} \qquad (3.2)$$

This can be partitioned as

$$\mathfrak{S} = \begin{pmatrix} A_1 & B_2 & & \\ B_2^T & A_2 & \ddots & \\ & & & \end{pmatrix} = \left(\begin{array}{cc|cc|c} \alpha_1 & -V & -V' & 0 & \\ -V & \alpha_2 & -V & -V' & \\ \hline -V' & -V & \alpha_3 & -V & \\ 0 & -V' & -V & \alpha_4 & \\ \hline & & & & \ddots \end{array}\right) \qquad (3.3)$$

By putting

$$u_j = \begin{pmatrix} u_{2j-1} \\ u_{2j} \end{pmatrix}, \qquad (3.4)$$

the secular equation can be written

$$\begin{pmatrix} A_1 & B_2 & & \\ B_2^T & A_2 & \ddots & \\ & & & \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \end{pmatrix} = 0, \qquad (3.5)$$

which is of course equivalent to

$$\begin{pmatrix} \alpha_1 & -V & -V' & & \\ -V & \alpha_2 & -V & -V' & \\ & & \ddots & & \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \end{pmatrix} = 0 \qquad (3.6)$$

The equation (3.1) may be written in a transfer-matrix form:

$$\mathcal{U}_{n-1} \equiv \begin{pmatrix} u_{n-1} \\ u_n \\ u_{n+1} \end{pmatrix} = \begin{pmatrix} -\frac{V}{V'} & \frac{\alpha_n}{V'} & -\frac{V}{V'} & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} u_{n-1} \\ u_n \\ u_{n+1} \\ u_{n+2} \end{pmatrix} = \mathcal{T}_n^{(0)} \mathcal{U}_n . \qquad (3.7)$$

If the system terminates at the $2N$th atom, we must put

$$\mathcal{U}_{2N} = \begin{pmatrix} u_{2N-1} \\ u_{2N} \\ 0 \\ 0 \end{pmatrix} . \qquad (3.8)$$

This is consistent with the form of the matrix $\mathcal{S}$. For the moment we shall assume that the system is semi-infinite in the direction of decreasing n .

Now we define the new transfer matrix by

$$\mathcal{T}_n \equiv \mathcal{T}_{2n-1}^{(0)} \mathcal{T}_{2n}^{(0)} . \qquad (3.9)$$

Then

$$\mathcal{T}_n = \begin{pmatrix} -\frac{V}{V'} & \frac{\alpha_{2n-1}}{V'} & -\frac{V}{V'} & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{V}{V'} & \frac{\alpha_{2n}}{V'} & -\frac{V}{V'} & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} .$$

$$= \begin{pmatrix} \frac{V^2}{V'^2} + \frac{\alpha_{2n-1}}{V'} & -\frac{V}{V'^2}\alpha_{2n} - \frac{V}{V'} & \frac{V^2}{V'^2} - 1 & \frac{V}{V'} \\ -\frac{V}{V'} & \frac{\alpha_{2n}}{V'} & -\frac{V}{V'} & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} . \qquad (3.10)$$

But since

$$\mathcal{B}_n^{T} = \begin{pmatrix} -V' & -V \\ 0 & -V' \end{pmatrix}$$

$$\mathcal{B}_n^{T^{-1}} = \frac{1}{V'^2} \begin{pmatrix} -V' & V \\ 0 & -V' \end{pmatrix}$$

$$-\mathcal{B}_n^{T^{-1}} \mathcal{A}_n = -\frac{1}{V'^2} \begin{pmatrix} -V' & V \\ 0 & -V' \end{pmatrix} \begin{pmatrix} \alpha_{2n-1} & -V \\ -V & \alpha_{2n} \end{pmatrix}$$

$$= -\frac{1}{V'^2}\begin{pmatrix} -V'\alpha_{2n-1} - V^2 & VV' + V\alpha_{2n} \\ VV' & -V'\alpha_{2n} \end{pmatrix}$$

$$-\mathrm{IB}_n^{T-1}\mathrm{IB}_n = -\frac{1}{V'^2}\begin{pmatrix} -V' & V \\ 0 & -V' \end{pmatrix}\begin{pmatrix} -V' & 0 \\ -V & -V' \end{pmatrix}$$

$$= -\frac{1}{V'^2}\begin{pmatrix} V'^2 - V^2 & -VV' \\ VV' & V'^2 \end{pmatrix}$$

$\mathbb{T}_n$ turns out to be

$$\mathbb{T}_n = \begin{pmatrix} -\mathrm{IB}_n^{T-1}\mathrm{IA}_n & -\mathrm{IB}_n^{T-1}\mathrm{IB}_n \\ \mathbb{I} & \mathbb{O} \end{pmatrix}. \tag{3.11}$$

If we use the new transfer matrix, the equation (3.7) can be written

$$\mathbb{X}_{n-1} = \mathbb{T}_n \mathbb{X}_n,$$

where

$$\mathbb{X}_n = \begin{pmatrix} \mathrm{u}_n \\ \mathrm{u}_{n+1} \end{pmatrix} = \begin{pmatrix} u_{2n-1} \\ u_{2n} \\ u_{2n+1} \\ u_{2n+2} \end{pmatrix} = \mathrm{x}_{2n}. \tag{3.12}$$

Now we consider two independent solutions u_n and v_n, which fulfill the boundary conditions

$$\mathrm{x}_{2N} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathrm{x}_{2N} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{3.13}$$

respectively. Then the matrix-vector

$$\mathbb{U}_n \equiv \begin{pmatrix} u_{n-1} & v_{n-1} \\ u_n & v_n \\ u_{n+1} & v_{n+1} \\ u_{n+2} & v_{n+2} \end{pmatrix} \equiv \begin{pmatrix} \mathbb{U}_n'' \\ \mathbb{U}_n^{21} \end{pmatrix} \tag{3.14}$$

also satisfies the equation

$$\mathbb{U}_{n-1} = \mathbb{T}_n \mathbb{U}_n, \tag{3.15}$$

with the boundary condition

$$\mathbb{U}_{2N} = \begin{pmatrix} \mathbb{I} \\ \mathbb{O} \end{pmatrix}. \tag{3.16}$$

Or if we consider other two solutions ω_n and χ_n which associate with the boundary conditions

$$\chi_{2N} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi_{2N} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{3.17}$$

then the matrix solution

$$\mathbb{U}_n = \begin{pmatrix} u_{n-1} & v_{n-1} & \omega_{n-1} & \chi_{n-1} \\ u_n & v_n & \omega_n & \chi_n \\ u_{n+1} & v_{n+1} & \omega_{n+1} & \chi_{n+1} \\ u_{n+2} & v_{n+2} & \omega_{n+2} & \chi_{n+2} \end{pmatrix} = \begin{pmatrix} \mathbb{U}_n^{11} & \mathbb{U}_n^{12} \\ \mathbb{U}_n^{21} & \mathbb{U}_n^{22} \end{pmatrix} \tag{3.18}$$

satisfies Eq.(3.15) with the boundary condition

$$\mathbb{U}_{2N} = \mathbb{I} \tag{3.19}$$

Next let us investigate the relation between the transfer matrices $\mathbb{T}_n$ and the secular matrix $\mathbb{S}$ or its determinant. Contrary to the case of the nearest-neighbour interaction, however, this becomes quite formidable unless we introduce a new concept of matrix-determinant of $\mathbb{S}$. This is defined in the same way as for the ordinary determinant by regarding each matrix entry in (3.37) as ordinary scalar matrix elements. We must of course be careful to keep the order of row-index in order to avoid the trouble which may be brought about by the non-commutability of the matrix entries. The matrix-determinant thus defined is not a scalar but a 2 x 2 matrix. Let us denote it by $\mathcal{S}$ and the ordinary determinant by $|S|$.

For the product of transfer matrices we have, for instance,

$$\mathbb{T}_{N-1}\mathbb{T}_N = \begin{pmatrix} -\mathbb{B}_{N-1}^{T^{-1}}\mathbb{A}_{N-1} & -\mathbb{B}_{N-1}^{T^{-1}}\mathbb{B}_{N-1} \\ \mathbb{I} & \mathbb{O} \end{pmatrix}\begin{pmatrix} -\mathbb{B}_N^{T^{-1}}\mathbb{A}_N & -\mathbb{B}_N^{T^{-1}}\mathbb{B}_N \\ \mathbb{I} & \mathbb{O} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbb{B}_{N-1}^{T^{-1}}\mathbb{A}_{N-1}\mathbb{B}_N^{T^{-1}}\mathbb{A}_N - \mathbb{B}_{N-1}^{T^{-1}}\mathbb{B}_{N-1} & \mathbb{B}_{N-1}^{T^{-1}}\mathbb{A}_{N-1}\mathbb{B}_N^{T^{-1}}\mathbb{B}_N \\ -\mathbb{B}_N^{T^{-1}}\mathbb{A}_N & -\mathbb{B}_N^{T^{-1}}\mathbb{B}_N \end{pmatrix}$$

(3.20)

From now on we shall specialize in our model and put

$$\mathbb{B}_n = -\mathbb{V} = \begin{pmatrix} V' & V \\ 0 & V' \end{pmatrix}$$

Then (3.20) becomes

$$\mathbb{T}_{N-1}\mathbb{T}_N = \begin{pmatrix} \mathbb{V}^{T^{-1}}\mathbb{A}_{N-1}\mathbb{V}^{T^{-1}}\mathbb{A}_N - \mathbb{V}^{T^{-1}}\mathbb{V} & -\mathbb{V}^{T^{-1}}\mathbb{A}_{N-1}\mathbb{V}^{T^{-1}}\mathbb{V} \\ \mathbb{V}^{T^{-1}}\mathbb{A}_N & -\mathbb{V}^{T^{-1}}\mathbb{V} \end{pmatrix} \tag{3.21}$$

Next we define the matrix

$$\mathbb{S}' = \begin{pmatrix} \mathbb{V}^{T^{-1}}\mathbb{A}_1 & -\mathbb{V}^{T^{-1}}\mathbb{V} & & & & & \\ -\mathbb{V}^{T^{-1}}\mathbb{V}^T & \mathbb{V}^{T^{-1}}\mathbb{A}_2 & & & & & \\ & & \ddots & & & & \\ & & & \mathbb{V}^{T^{-1}}\mathbb{A}_{k-1} & -\mathbb{V}^{T^{-1}}\mathbb{V} & & \\ & & & -\mathbb{V}^{T^{-1}}\mathbb{V}^T & -\mathbb{V}^{T^{-1}}\mathbb{V} & & \\ & & & & & \ddots & \\ & & & & & \mathbb{V}^{T^{-1}}\mathbb{A}_{N-1} & -\mathbb{V}^{T^{-1}}\mathbb{V} \\ & & & & & -\mathbb{V}^{T^{-1}}\mathbb{V}^T & \mathbb{V}^{T^{-1}}\mathbb{A}_N \end{pmatrix}$$

and denote by $\mathcal{S}'^{i_1 \cdots i_k}$ the matrix which is obtained by deleting the i_1th, i_2th,..., and i_kth rows and columns from $\mathcal{S}'$. Now

$$\mathcal{S}'^{1,2,\cdots,N-2} = \begin{pmatrix} W^{T^{-1}} A_{N-1} & -W^{T^{-1}} W \\ -W^{T^{-1}} W^T & W^{T^{-1}} A_N \end{pmatrix} \tag{3.23}$$

$$S'^{1,2,\cdots N-2} = W^{T^{-1}} A_{N-1} W^{T^{-1}} A_N - W^{T^{-1}} W W^{T^{-1}} W^T$$

$$= W^{T^{-1}} A_{N-1} W^{T^{-1}} A_N - W^{T^{-1}} W \tag{3.24}$$

In the same way we have

$$S'^{1,2,\cdots N-2,N} = W^{T^{-1}} A_{N-1}, \quad S'^{1,\cdots N-2,N-1} = W^{T^{-1}} A_N . \tag{3.25}$$

If we define

$$S'^{1,\cdots N-2,N-1,N} \equiv I, \tag{3.26}$$

then we find that (3.21) can be written

$$T_{N-1} T_N = \begin{pmatrix} S'^{1,2,\cdots N-2} & -S'^{1,\cdots N-2,N} W^{T^{-1}} W \\ S'^{1,\cdots N-2,N-1} & -S'^{1,\cdots N-2,N-1,N} W^{T^{-1}} W \end{pmatrix}. \tag{3.27}$$

In general we obtain the result

$$T_k T_{k+1} \cdots T_N = \begin{pmatrix} S'^{1,\cdots,k-1} & -S'^{1,\cdots k-1,N} W^{T^{-1}} W \\ S'^{1,\cdots k-1,k} & -S'^{1,\cdots k-1,k,N} W^{T^{-1}} W \end{pmatrix} \tag{3.28}$$

This can be proved by induction as follows. For k = 2 it has already been proved. If

$$\mathcal{T}_{k+1}\cdots\mathcal{T}_N=\begin{pmatrix} S^{1,\cdots,k} & -S^{1,\cdots k,N}W^{T^{-1}}W \\ S^{1,\cdots k,k+1} & -S^{1,\cdots k,k+1,N}W^{T^{-1}}W \end{pmatrix}, \tag{3.29}$$

then

$$\mathcal{T}_k\cdots\mathcal{T}_N=\mathcal{T}_k(\mathcal{T}_{k+1}\cdots\mathcal{T}_N)$$

$$=\begin{pmatrix} W^{T^{-1}}A_k & -W^{T^{-1}}W \\ \mathbb{I} & 0 \end{pmatrix}\begin{pmatrix} S^{1,\cdots,k} & -S^{1\cdots k,N}W^{T^{-1}}W \\ S^{1,\cdots k,k+1} & -S^{1,\cdots k,k+1,N}W^{T^{-1}}W \end{pmatrix}$$

$$=\begin{pmatrix} W^{T^{-1}}A_k S^{1,\cdots k}-W^{T^{-1}}WS^{1,\cdots k,k+1} & -W^{T^{-1}}A_k S^{1,\cdots k,N}W^{T^{-1}}W \\ & +W^{T^{-1}}WS^{1,\cdots k,k+1,N}W^{T^{-1}}W \\ S^{1\cdots k} & -S^{1,\cdots k,N}W^{T^{-1}}W \end{pmatrix} \tag{3.30}$$

But since

$$S^{1,\cdots,k-1}=W^{T^{-1}}A_k S^{1,\cdots k}-W^{T^{-1}}VV^{T^{-1}}V^TS^{1\cdots k+1}$$

$$S^{1,\cdots k-1,N}=W^{T^{-1}}A_k S^{1,\cdots k,N}-W^{T^{-1}}WW^{T^{-1}}V^TS^{1,\cdots k+1,N} \tag{3.31}$$

(3.30) just becomes (3.28). QED.

The theorem proved above gives a relation

$$U_m=\mathcal{T}_{m+1}\cdots \qquad \mathcal{T}_N U_N=\begin{pmatrix} S^{1,\cdots,m} \\ \\ S^{1,\cdots m+1} \end{pmatrix}.$$

(3.32)

Thus if we define the "norm" $\|U_n\|$ of the matrix-vector U_n by

$$\|U_n\|^2 = |U_n^{(11)}|^2 + |U_n^{(21)}|^2, \tag{3.33}$$

then we have

$$\frac{\|U_n\|^2}{\|U_{n-1}\|^2} = \frac{|S'^{1,\cdots,n}|^2 + |S'^{1,\cdots,n,n+1}|^2}{|S'^{1,\cdots,n-1}|^2 + |S'^{1,\cdots,n}|^2} \tag{3.34}$$

If we define

$$S'_{n+1} \equiv S'^{1,\cdots,n},$$

then this becomes

$$\frac{\|U_n\|^2}{\|U_{n-1}\|^2} = \frac{1}{\dfrac{|S'_{n+1}|^2}{|S'_{n+2}|^2}} \frac{1 + \dfrac{|S'_{n+1}|^2}{|S'_{n+2}|^2}}{1 + \dfrac{|S'_n|^2}{|S'_{n+1}|^2}} \tag{3.35}$$

Taking the average of logarithm, we get

$$\left\langle \ln \frac{\|U_n\|^2}{\|U_{n-1}\|^2} \right\rangle = \left\langle \ln \frac{|S'_{n+2}|^2}{|S'_{n+1}|^2} \right\rangle + \left\langle \ln \left\{ 1 + \frac{|S'_{n+1}|^2}{|S'_{n+2}|^2} \right\} \right\rangle + \left\langle \ln \left\{ 1 + \frac{|S'_n|^2}{|S'_{n+1}|^2} \right\} \right\rangle. \tag{3.36}$$

If it is assumed that the distribution of $|S'_{n+1}|^2/|S'_{n+2}|^2$ is stationary, the second and third terms of the r.h.s. cancel each other so that we have

$$\left\langle \ln \frac{\|U_n\|^2}{\|U_{n-1}\|^2} \right\rangle = \left\langle \ln |S'^{-1}_{n+1}||S'_{n+2}| \right\rangle. \tag{3.37}$$

If we define the matrices B_k and W by

$$V^{T-1} A_k \equiv B_k \quad \text{and} \quad V^{T-1} V = W, \tag{3.38}$$

then the first relation in (26) becomes

$$S'_k = B_k S'_{k+1} - W S'_{k+2} \tag{3.39}$$

or

$$S'_k S'^{-1}_{k+1} = B_k - W S'_{k+2} S'^{-1}_{k+1}$$

This is a recurrent formula for the matrix $S'_{k+2} S'^{-1}_{k+1}$, so that it can be calculated successively by starting from $S'_1 S'^{-1}_0 = B_N$.

The physical meaning of the norm $\| U_n \|$ is not so clear compared with the nearest-neighbour case. But this demerit may be saved at least to some extent by the following argument. In terms of the amplitudes of the solution it is

$$|U_n^{(1)}|^2 + |U_n^{(2)}|^2 = \begin{vmatrix} u_{n-1} & v_{n-1} \\ u_n & v_n \end{vmatrix}^2 + \begin{vmatrix} u_{n+1} & v_{n+1} \\ u_{n+2} & v_{n+2} \end{vmatrix}^2 . \tag{3.40}$$

In the nearest-neighbour case the determinant $\begin{vmatrix} u_{n-1} & v_{n-1} \\ u_n & v_n \end{vmatrix}$ is just Wronskian which is to be strictly conserved. But now Wronskian is given by the determinant (3.18) so that expression (3.40) is not necessarily conserved. If the system is periodic, however, this expression can at most oscillate. Therefore if it turns out that it increases or decreases without limit, then we can safely conclude that there occurs localication.

The above method may be applied not only to the cases in which interactions of longer range are present, but also to any miltidimensional systems.

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