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"PHYSICS OF CONDENSED MATTER AT PLANETARY PRESSURES"  
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DESCRIPTION OF THE LIQUID STATE  
(Lectures I & II)

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I.C.T.P. RESEARCH WORKSHOP

PHYSICS OF CONDENSED MATTER AT PLANETARY PRESSURES

DESCRIPTION OF THE LIQUID STATE

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INTRODUCTION

The liquid state of matter is of extreme importance in all aspects of the Universe. In the present context we are concerned with liquids within the special context of planets and particularly of planetary interiors. This means particularly under extreme conditions of pressure and temperature.

The theory of the behaviour of liquids under forces of various kinds is now a very extensive subject. The liquid can be regarded either as an entirely macroscopic entity without an atomic sub-structure (this is the point of view of classical hydrodynamics) or it can be treated as a collection of atoms in continual interaction. We shall consider both aspects in these lectures. There is no possibility of covering more than a small fraction of the work in four lectures and only some parts of the total work can be touched on. In being selective I have chosen those parts of the subject that I have been directly involved with over the years and which have some direct relevance to the description of planetary behaviour. In particular, I have stressed the background to the formulae often used in the description of planetary conditions to make clear the limitations that are there although not always realised. Planetary material is more complicated than the simple material usually involved in the theory.

Planetary material is a most complicated thing to describe but we must start somewhere. It is obvious that we should start where the laboratory work is reliable even though this is not immediately relevant to the material of interest. The important thing is an understanding of liquids in these cases; only with understanding are we likely to be able to progress to more complicated material. With the laboratory to guide us in the first instance we can then use the planets themselves as giant laboratories for the investigation of the properties of matter under conditions more extreme than we can achieve in the laboratory at the present time. It is in this spirit that I present these lectures.

The background literature to the theoretical study of the liquid state is vast and, to provide some common ground for the lectures and

for discussion later, I have prepared the accompanying general comments and commentary. This is in no way intended to be complete and is not to be regarded as a text book coverage of the material of the lectures or to be a logically ordered and complete account of the field. Rather the notes are of the nature of an "aid memoir" to provide some basic ideas to stop the lectures becoming too formal. The lectures will extend the work and will not be a simple re-run of this material. There are a list of references and students are advised to spend some time before the lectures studying the notes and the books. The accompanying notes will inevitably contain errors of various kinds and I would be grateful if students will draw these to my attention as they are found.

## A FEW SELECTED REFERENCES

### LECTURE 1

Books of Physical Tables.

### LECTURE 2

There is an enormous literature here, but of particular general usefulness are

L.D. Landau and E.M. Lifshitz, 1959 Fluid Mechanics, being Vol. 6 of Course of Theoretical Physics, Pergamon Press Ltd.

J.S. Trefil, 1975, Introduction to the Physics of Fluids and Solids, Pergamon Press Inc.

H.N.V. Temperley and D.H. Trevena, 1978, Liquids and their Properties, Ellis Horwood (John Wiley).

H.J. Kreuzer, 1981, Nonequilibrium Thermodynamics and its Statistical Foundations, Oxford University Press.

S. Chandrasekhar, 1961, Hydrodynamic and Hydromagnetic Stability, Clarendon, Oxford.

D.J. Tritton, 1977, Physical Fluid Dynamics, Van Nostrand Reinhold.

### LECTURES 3 and 4

The earlier books are still relevant because many of the original problems still remain to be solved. The following books provide a general review of various aspects of the subject.

P.A. Egelstaff, 1967, An Introduction to the Liquid State, Academic Press.

G.H.A. Cole, 1967, The Statistical Theory of Classical Dense Simple Fluids, Pergamon Press.

Editor: C. Croxton, 1978, Progress in Liquid Physics, John Wiley.

E.M. Lifshitz and L.P. Pitaevskii, 1981, Physical Kinetics, being Vol. 10 of Course of Theoretical Physics, Pergamon Press.

I. Prigogine, 1962, Non-equilibrium Statistical Mechanics, Interscience, New York.

I. Z. Fisher, 1962, Statistic Theory of Liquids, (translation with commentary) Chicago Univ. Press.

I. Gyarmati, 1970, *Non-equilibrium Thermodynamics*, Springer-Verlag, Berlin.

D.N. Zubarev, 1974, *Non-equilibrium Statistical Mechanics*, Consultants Bureau, New York.

S.A. Rice and P. Gray, 1965, *The Statistical Mechanics of Simple Liquids*, Interscience.

G.H. Wannier, 1966, *Statistical Physics*, John Wiley.

## LECTURE 1

### PHYSICAL DESCRIPTION OF LIQUID BEHAVIOUR

#### AND THE RELATION TO A PLANETARY BODY

We are familiar, in everyday life on Earth under gravity, with material forming liquids, solids and gases. So strong is this experience that we accept that these are indeed the three states of matter. The distinction between the states is sharp in gross terms but becomes blurred if the observed properties are assessed critically, and over an extended period of time. Nevertheless, the concept of the three idealised states of matter is useful as a clear backcloth against which we can assess the properties of real matter.

#### 1.1. The Three States of Matter

There are two general conditions of real matter which allow the three idealised states to be specified:

- i. Whether or not the material is condensed, with a free surface;
- ii. Whether or not the material can withstand shear stresses.

These conditions allow for further ramifications.

##### 1.1.1. Condensed and Uncondensed Matter.

Condensed matter has a characteristic volume for given physical conditions and so shows a particular density (mass per unit volume). For a solid the volume can be in contact with a vacuum but a liquid must always be in contact with its vapour. This means a solid can exist without a closed container but this is not so for a liquid (although the volume may be very large and the containment may not involve a simple "lid" - for instance, gravity may contain an

## THE THREE STATES OF MATTER

atmosphere). The vapour associated with a liquid is an example of uncondensed (gaseous) matter. A gas will fill the entire volume available to it and its density will depend on the magnitude of the volume it occupies. There is no characteristic density for a gas, although the density is (usually) low by comparison with condensed matter (which lies within the general range  $10^3 - 10^4 \text{ kg m}^{-3}$ ).

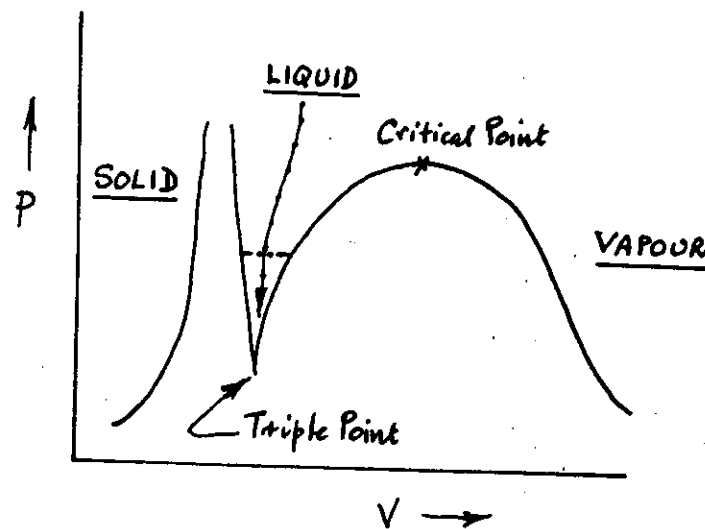
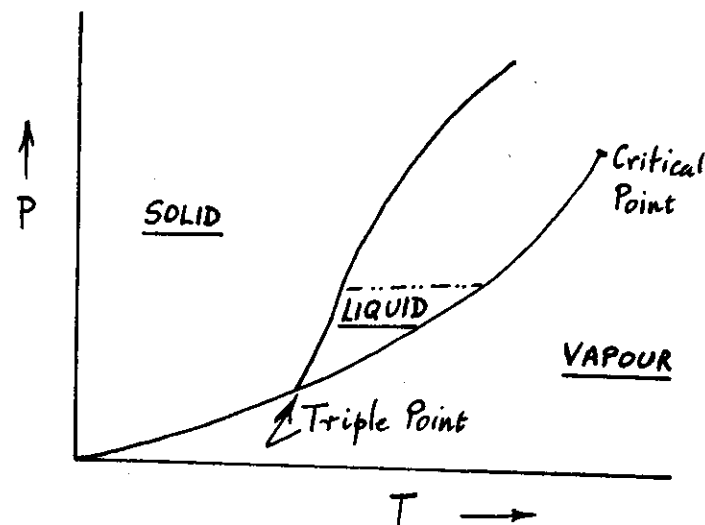
The range of densities is mirrored in a range of compressibilities. The compressibility of condensed matter is low, and that for a solid is rather lower than for the liquid. The compressibility of a gas is high. Compressibility is important in the science of planetary interiors because the compression within a planet is a controlling factor for conditions inside. The inverse of compressibility is the bulk modulus of the material and this quantity plays a prominent role in the description of planetary interiors.

### 1.1.2. Effect of Shear Stresses.

Condensed matter can itself have one of two forms depending on its response to a shear force. If the matter has strength to withstand shear forces it is a solid; otherwise it is a liquid. A liquid, in consequence, flows under the action of forces whereas a solid does not.

### 1.1.3. Fluidity.

The two conditions listed above as characterising the three states of matter can be re-interpreted in another way. All matter can be split into two groupings depending upon whether it can withstand shear forces or not. That material which can is said to be solid. That which cannot shows flow under the action of shear stresses and is said to be fluid. The fluid which shows a free surface and has a fixed density (for given physical conditions) and is condensed is called a



liquid; otherwise the fluid is called a gas.

It is seen that, in a sense, a liquid viewed this way falls mid-way between a solid and a gas in having fluidity with no resistance to shear stresses (like a gas), while having a free surface and a particular substantial density for given physical conditions (like a solid). This mid-way position makes liquid properties of extreme importance in a wide range of physical circumstances; it has also been used as a method of achieving a theoretical understanding of this state of matter.

Of course the real world is not as simple as this as we shall see. A familiar intermediate case is pitch; if this is struck impulsively it will break like a brittle solid while if it is left for a period of days under the action of a shear force it will flow like a viscous fluid. There are many other examples.

### 1.2. The Phase Diagram

A liquid volume can exist generally only if the surface is in contact with vapour (gas) of the same material at sufficient pressure, for a given temperature. There is a critical pressure below which the liquid will evaporate. At very low pressures, therefore, matter is either solid or gaseous. At high temperatures the liquid phase will exist only if the external pressure is sufficiently high. It follows that the liquid phase exists only within a specific range of pressure and temperature, the precise range in a particular case being a characteristic of the material. This general behaviour is conveniently represented by a construct called a phase diagram. The two phase diagrams summarise the relations between pressure  $P$ , temperature  $T$  and volume  $V$  specifying the equation of state of the system.

One of the phase diagrams is a plot of pressure against

temperature (at constant volume) and has the form shown in Figure 1.1. The lines represent the dividing conditions between associated two phases and there is one point (called the triple point) where the three phases of matter can coexist in equilibrium. There is a second point, called the critical point, where liquid properties are lost, the material becoming a very dense gas. There is, beyond the critical point, no distinction between the liquid and gaseous phases.

The second phase diagram is a plot of pressure against volume (at constant temperature) and is shown in Figure 1.2. Again the critical and triple points are indicated. The liquid region is shown, occupying the restricted regions above the triple point. There is not a sharp region of demarcation for the liquid at the upper ends.

### 1.3. Types of Liquid: I

It has become conventional to recognise six general categories of liquid. We shall find later that the arrangement is based primarily on the type of interaction between the constituent atoms and molecules; this will influence the chemical behaviour of the material and it is the physical characteristics that will allow the divisions to be recognised in the present context. The categories are as follows:

- i. Simple liquids. These have spherical molecules (and in molecular terms interact with the van der Waals force). Examples are Argon and Methane.
- ii. Diatomic liquids. These have diatomic molecules. They are similar to the simple liquids but the effects of the electrical quadrupoles must be accounted for. Examples are molecular hydrogen and molecular nitrogen.
- iii. Metals. These form a separate group (being characterised at the atomic level by long ranged Coulomb interaction forces). Examples are

Sodium, Mercury, Iron and so on.

iv. Polar liquids. These show the effects of electric dipole moments and are composed of asymmetric molecules. Examples are Hydrogen Bromide, Hydrogen Chloride, and so on.

v. Associated liquids form a separate category with unusual properties. Water fits into this division and is very difficult to describe from the theoretical point of view.

vi. Complicated liquids composed of large molecules. Internal modes of molecular motion are crucial in the description of these liquids.

This is only a rough division but we will find it useful in the arguments to follow. Generally speaking the macroscopic description of the liquids has made progress in describing the behaviour of members in all six classes, but the microscopic description has made significant progress only for liquids falling within the first three categories.

#### 1.4. Dissipative Processes

The presence of an external action (a force) on a material element of matter causes a corresponding reaction and the effect normally leads to the dissipation of energy into heat (ordered in to disordered molecular motion). Thus, the presence of an electromotive force  $V$  across an electrically conducting material at rest leads to the flow of electric current  $j$ . For simple materials (in which conditions are homogeneous and isotropic) containing small potential gradients the electric current produced is proportional to the electromotive force and this is expressed by Ohm's law. The constant of proportionality (being the electric current per unit potential difference) is the electrical resistance  $R = j/V$  of the material. As a second example, the non-uniform distribution of a solute concentration

$n$  in a solvent leads locally to a current  $j$  of solute proportional to the local concentration gradient ( $\text{grad } n$ ) and directed in such a way as to lead to a uniform distribution of solute throughout the volume. This is a diffusion process and for low concentration gradients the current is proportional to the concentration gradient (Fick's law). The current flow per unit concentration gradient is the (constant) diffusion coefficient  $D$  of the solution and  $D = -j/\text{grad } n$ , the negative sign showing the current to flow down the concentration gradient.

For a fluid in incompressible flow there is a force  $F_1/A$  per unit area in the direction  $i$  perpendicular to the flow and proportional to the velocity gradient  $dv/dx_i$  in that direction (Stokes' law of fluid friction). The constant of proportionality is the shear viscosity  $\eta$  of the fluid; then  $F_1/A = \eta dv/dx_i$ . The effect is to eliminate differences of velocity in the fluid and so reduce the differential motion to one of uniformity.

These various and different processes all describe irreversible processes in the material in which an initial condition is lost as time proceeds and which can only be maintained by the expenditure of energy of some particular type, different in each case. Dissipative processes are a universal feature of the real world. In particular, fluid viscosity is an integral part of the flow of a fluid and this is particularly the case for planetary conditions. The peculiar conditions inside a planetary body usually make the viscosities extremely high.

#### 1.5. The Macro- and Micro- Worlds

The world of our immediate perception can be described in one of two complimentary ways. First, there is the immediate impression of

the macroscopic world in which the description is based on an empirical statement (for instance, Fick's law of diffusion, or Stokes' law of fluid friction) together with laws of motion for the material. The description is complete within itself and the various coefficients of the processes (diffusion coefficient or shear viscosity) are treated as empirical parameters describing the behaviour of the material. This forms the macroscopic description of the system. This approach has provided the great theoretical descriptions of physics of which fluid mechanics is an example.

The macroscopic appearance of matter is superficial because the observed properties of matter are, in fact, an average of the collective properties of the constituent atoms and molecules of which the material is composed. The macroscopic properties of matter can be related to the atomic properties through some statistical averaging process and this forms the microscopic description. More especially the empirical constants (such as diffusion or shear viscosity) are related directly to atomic effects. In particular, these parameters (which must be assigned empirically in the macroscopic theories) are now accessible to calculation using the microscopic description. Once the atomic properties are assigned the macroscopic behaviour is, at least in principle, defined.

Whether it is appropriate to use the macroscopic or microscopic description in any particular case is a matter of choice. Generally speaking, the macroscopic approach is the most direct for calculations if the empirical constants are known; this may not be so for complicated materials. We shall be concerned with both approaches in what follows.

### 1.6. The Three States of Matter in Microscopic Terms

The constituent atoms in a gas are in free motion, moving independently between collisions with other atoms. The time for a collision is infinitesimal in comparison with that between collisions. Most of the volume of the gas is empty space accounting for the very high compressibility and very low density. The total energy  $E$  of each atom (or molecule) is part kinetic  $E_k$  (of motion) and part potential  $E_v$  (of interaction). The collective mean kinetic energy of random atomic motion is recognised macroscopically as temperature, the speed being greater the higher the temperature. The gas can, then, be specified by the condition  $E_k \gg E_v$ . As the temperature is reduced the energy of interaction during collisions becomes more significant and at a sufficiently low temperature (the condensation/boiling temperature) the atoms become bound and held together. The material has now condensed. Because the atoms are in close contact the compressibility is small. Each atom is able to execute considerable (periodic or bound) motion about a fixed position and vibrates. In terms of the energy  $E_k \sim E_v$ , the two mean energies being of comparable magnitude. This is the liquid state, the kinetic energy providing the fluidity and the potential energy ensuring low compressibility and high density. As the temperature is reduced further the kinetic energy falls and we achieve the condition  $E_v \gg E_k$ . The kinetic energy now has little effect and we have reached the solid state of matter. The temperature at which this condition applies is the melting/freezing temperature. The three idealised states of matter can, in this way, be represented in a simple way in atomic terms. It is realised that a controlling influence on these conditions is the strength of the

interaction force between the atoms and molecules. In what follows we are concerned particularly with the condition that the mean kinetic energy of each atom is of comparable magnitude to the mean potential energy of interaction between any representative pair of atoms.

#### 1.7. Types of Liquid: II

The duality of description of the different types of liquid recognised in Section 1.5 is well illustrated by the specification of liquid types in terms of the properties of the constituent atoms.

The matter is condensed because the individual atoms attract each other sufficiently strongly to hold the accumulation together. There is a limit to the degree of condensation, however, because the atoms themselves resist compression beyond a given amount. In atomic terms, the atoms will repel each other at small separation distances (when their electron clouds are forced close together) but attract each other at larger separation distances. The nature of this attraction/repulsion depends on the structure of the atoms involved. The forces between the atoms are ultimately electrical, and involve a lack of geometrical symmetry of the atom resulting in the formation of an electric multipole.

##### 1.7.1. Interatomic Binding.

The interaction between atoms is described by taking account of the coulomb interactions between the complete set of nuclei and electrons present. The precise details then depend on the number of atoms in the aggregate although saturation effects make the result of the calculation depend almost entirely on the interaction between a small number of atoms, for instance as few as two. The approach involves the solution of the Schrödinger equation for the system and presents many mathematical difficulties. Indeed, definitive

interaction expressions are still only available for the simpler atomic aggregations. It is sufficient for our present purposes to appeal to semi-empirical expressions for the interactions.

The Periodic Table of the elements, in which chemically similar elements are collected together in ordered groupings, reflects at the macroscopic level the arrangement, at the atomic level, of the extranuclear electrons. The inert elements (Helium, Argon, Neon, etc) are associated with closed electron shells which represent truly stable electronic configurations. The attractive forces associated with a closed electron shell are weak and the interaction with neighbouring atoms is correspondingly small. The other elements, which are chemically active, are characterised by partially filled shells and the chemical interactions arise primarily from the tendency of partially filled shells to become closed. An isolated atom cannot achieve this and remain isolated because it would cease to be electrically neutral. A collection of atoms can, however, effectively contain closed shells by "sharing" electrons with neighbours in a particular way depending on the number of electrons necessary to close the shell. The different types of interatomic forces is a reflection of the variants of this sharing procedure.

We can recognise three types of interaction as the ideal although it must be stressed that each is really a special form of the general interaction. The first involves closed shells themselves and is summarised as the van der Waals force; the second is the ionic force between atoms differing by one electron from the closed shell structure; and the third is the valence force where two or more electrons must be gained or lost to achieve the closed shell structure. Metals are a particular form where electrons are shared by



all the atoms of the aggregate simultaneously.

#### 1.7.2. Electronegative and Electropositive Elements.

Elements in group I of the Periodic Table have one s-electron (for instance lithium, sodium or potassium) and would achieve a closed shell if this electron were lost. This would leave the atom electrically positive overall; elements which would achieve closed shell structures in this way are said to be electropositive. Elements in group 7, on the other hand, would achieve a closed shell configuration if an electron were gained; since they would then become negatively charged they are said to be electronegative. Fluorine, chlorine and bromine are examples of these elements. The affinity of two materials to interact and form a molecule can be expressed as an example of the appropriate electronegativity of the atom. The electronegativity of the elements has been set onto a scale by Pauling with the range 0 to 4. On his scale fluorine is given the value 4, oxygen 3.5, nitrogen 3.0, chlorine 3.0, carbon 2.5 and sulphur 2.5. All other elements have lower values. The cosmic abundance of the elements makes the high values for oxygen, carbon and nitrogen particularly significant. The electronegativities of several of the more important planetary elements are collected in Table 1.1.

Element	E	Element	E
oxygen	3.5	iron	1.8
carbon	2.5	gold	2.4
nitrogen	3.0	sulphur	2.5
		iodine	2.5
		bromine	2.8
		chlorine	3.0
		fluorine	4.0

Table 1.1. The electronegativities of several elements of planetary interest.

#### 1.7.3. Ionic Bonding.

An electronegative and electropositive element placed together can each achieve a closed shell structure by transference of an electron, the interaction between them then being due to the force between a positive and a negative ion. The resulting molecule is said to be ionic and sodium chloride is the typical ionic material. That the binding is very strong is shown by the melting and boiling points of sodium chloride which are 1074K and 1686K respectively.

From the geophysical point of view oxygen (group 6) is strongly electronegative and forms ionic compounds with elements of substantially lower electronegativity ( $< 1.6$ ). This includes sodium (0.9), calcium (1.0), magnesium (1.2), aluminium (1.5) and zinc (1.7) together with the rare earth and alkaline earth materials. This is the basis for the silicates common in the terrestrial planets.

#### 1.7.4. Valence Bonding.

This arises when two electronegative or two electropositive atoms are in close proximity. The binding is stronger the smaller the electron deviation from the closed shell arrangement. The outer shell electrons of each atom become associated with the other in such a way that the effective arrangement around each is a closed shell although the overall number must be different. The binding exists only if the electron shells of each atom overlap with the other and it is this continued close proximity which constitutes the binding force. The binding energy is the excess of the sum of the separate energies over that for the combined system. The sharing of the valence electrons in this way leads to the binding being called valence binding. The range of such binding can be wide and the strength covers a wide range and depends on the number of electrons that must be lost to achieve the

closed shell configuration. For sodium, where only one electron need be lost, the binding is quite weak and this is reflected in the low melting point for sodium of 370K (roughly room temperature). This is typical of electropositive elements. Electronegative elements involve more energy and as an example the temperature for molecular dissociation for fluorine is 1300K. Diatomic molecules also form for chlorine, oxygen and nitrogen in the same way.

The covalent bond is particularly effective between atoms with closely similar values of the electronegativity. For the terrestrial planets sulphur ( $E = 2.5$ ) is a central element from an abundance point of view and this affinity provides a wide range of sulphides, particularly with the third transition metals.

It will be realised that the distinction between pure ionic and pure valence bonding is not necessarily strong.

#### 1.7.5. The Metal Bond.

The covalent bond takes on a special form in metals. Here the atoms fit together particularly closely and the sharing of valence electrons is particularly complete. Indeed, the ions form a lattice structure with long range ordering and the electrons are shared by effectively all the atoms in the sample. The extreme mobility of the electrons provides the electric and thermal properties characteristic of metals. By their nature metals tend to remain pure but here again there is a gradual variation of features between the pure covalent and pure metal bond.

At ordinary pressures the lightest elements do not form metal structures because the thermal motions of the atoms are too great. Since a metallic form is determined primarily by the ordering of the atoms, strong compression can be expected to counteract the thermal

motion. In particular, at pressures of the order  $2 \times 10^9 \text{ N m}^{-2}$  graphite converts to the diamond structure and at about  $2 \times 10^{11} \text{ N m}^{-2}$  hydrogen atoms becomes sufficiently compressed to form a metallic solid. This is only slightly lower than the pressure at the centre of the Earth (about  $4 \times 10^{11} \text{ N m}^{-2}$ ) and at the centre of Saturn (about  $6 \times 10^{11} \text{ N m}^{-2}$ ). Correspondingly, diamond becomes metallic at a pressure of about  $2 \times 10^{13} \text{ N m}^{-2}$  and helium becomes metallic at a pressure of about  $1 \times 10^{14} \text{ N m}^{-2}$ . This is closely similar to the pressure at the centre of Jupiter. The metallic state is important in planetary problems because of the high pressures met within the larger planetary bodies.

#### 1.7.6. Semiconductor structure.

We will do no more here than recognise the existence of semiconductor structures which involve the ionic or covalent bondings but have special circumstances associated with them. Semiconduction could well have important implications in planetary science.

#### 1.7.7. van der Waals forces.

In this simplest case of a closed electronic shell the atom is completely spherical and examples are Argon or Helium. The interaction force between any two atoms can be regarded as arising from the formation of transient dipoles but other multipoles are also involved and the analogy must not be taken too far. The attractive force at large distances is found from quantum mechanical arguments to vanish like  $r^{-7}$  with the separation distance  $r$  and there is no angular dependence. The repulsive force at small separation distances is open to some doubt but many atoms are well represented by a dependence of the form  $r^{-13}$  or some such power. The force can be represented by an interaction potential  $\psi$  of the form

$$\psi(r) = a_1 r^{-12} - a_2 r^{-6} \quad (1)$$

first empirically proposed by Lennard-Jones and Devonshire. This (12-6) potential has played an important part in the development of the microscopic theory.

It might be expected that more complicated atoms can be represented in a similar way and it is often convenient to use the more general form

$$\psi(r) = a_1 r^{-m} - a_2 r^{-n} \quad (2)$$

proposed by Mie. It becomes the (12-6) form by setting  $m = 12$  and  $n = 6$ ; it is often convenient to leave  $m$  as a selectable parameter and takes the form

$$\psi(r) = a_1 r^{-m} - a_2 r^{-6} \quad (3)$$

setting  $n = 6$  has some theoretical support. The atomic size  $r_0$  is that value of  $r$  for which the force vanishes; this gives

$$r_0 = [na_2/ma_1]^{1/(n-m)} \quad (4)$$

The energy  $E_0$  corresponding to this size is found by inserting (4) into (3). Alternatively, the parameters  $a_1, a_2$  can be deduced if  $r_0, E_0$  are known from experiment.

#### 1.7.8. More complicated expressions.

The van der Waals potential applies to spherical atoms and molecules but most atoms and molecules are not spherically symmetric. The asymmetry can be accounted for in one of two general ways. One is to put the appropriate angular dependence into the formulae and average the resulting expression over the angles to produce a mean symmetrical field. The other is to assume a mean field as a general background on which specific angular dependences are added. For instance, many molecules have a dipole form of interaction and this specific angular dependence can be added with an appropriate dipole moment deduced empirically.

#### 1.7.9. Pair and triplet forces.

The simpler atoms interact with pair interaction forces so the total potential energy of interaction for a collection of  $N$  atoms is written as the sum of the interaction between the constituent pairs of atoms according to

$$\Psi(1,2,3,\dots,N) = \sum_{i < j}^N \psi(i,j) \quad (5)$$

where  $\psi(i,j)$  is the interaction potential between the representative pair of atoms  $i$  and  $j$ . This superposition of pair forces is most convenient for calculations but it is sometimes necessary to suppose a more complicated form. Thus triplet forces may also be appropriate so that

$$\Psi(1,2,3,\dots,N) = \sum_{i < j}^N \psi(i,j) + \sum_{i < j < k}^N \varphi(i,j,k) \quad (6)$$

where  $\varphi(i,j,k)$  is the potential energy between the representative triplet of atoms  $i, j$  and  $k$ . The problem of specifying  $\psi(i,j)$  and  $\varphi(i,j,k)$  has many difficulties and the triplet contribution is not often included.

#### 1.8. The State of Planetary Material

The conditions within a planetary body are dictated particularly by the pressure. The equilibrium of the body does not involve thermal energy so the material density is a function of the pressure alone. This means that the conditions are controlled by the bulk modulus of the material, which is the dependence of the density on the pressure.

##### 1.8.1. Energy of atoms.

The equilibrium form of an atom results from the balance between the electrostatic pressure of attraction between the constituent

charges and the electron degeneracy pressure. For a hydrogen atom this involves the precise balance between two equal and opposite pressures of magnitude about  $3 \times 10^{12} \text{ N m}^{-2}$  while the pressures are higher for heavier atoms (because the electric forces are stronger).

The energy associated with an atom is measured in terms of the Rydberg  $R_y = 13.6 \text{ e.v.} = 2.178 \times 10^{-18} \text{ J}$  and this is the ionisation energy for a hydrogen atom. The energy for more complicated atoms is correspondingly larger, but the Rydberg remains the appropriate unit for measurement. The interaction energy is smaller than this by about two orders of magnitude although the details vary from one element to another. This means, as a very broad rule, that the interaction energy is of order  $10^{-20} \text{ J}$  or about  $10^{-1} \text{ e.v.}$  We shall use this magnitude in many of the arguments to follow.

The interaction energy can be linked to an equivalent temperature  $T_e$  according to

$$E = k_B T_e \quad (7)$$

where  $k_B (= 1.38 \times 10^{-23} \text{ J K}^{-1})$  is the Boltzmann constant. As to orders of magnitude the interaction energy  $E = 10^{-20} \text{ J}$  corresponds to a temperature  $T_e \sim 10^3 \text{ K}$  as the temperature where the interaction energy becomes small in comparison with the kinetic energy. This is a very imprecise estimate in any particular case but will be of interest later as an indication of the melting temperature of a typical material.

#### 1.8.2. Effect of the pressure.

The application of pressure on a collection of atoms has a very profound effect if the pressure, and so the associated energy, is high. There are two magnitudes of importance. One is an energy comparable to that of interatomic attraction; the other is that

comparable to the energy of the atom itself. The first external energy will make the interparticle forces inoperative; the second will cause the atoms to become disrupted and so ionised. The pressure associated with ionisation will be about  $3 \times 10^{12} \text{ N m}^{-2}$  (incidentally corresponding to an equivalent temperature of about  $10^5 \text{ K}$ ) while the pressure associated with the interatomic forces is about  $10^{10} \text{ N m}^{-2}$ .

In the planetary context pressure will increase with depth from the surface and the pressure  $P(R)$  at the distance  $R$  from the centre (the planet being assumed spherical) will be given, as to orders of magnitude, by  $P(R) \sim \bar{\rho}(R)g(R)R$  where  $\bar{\rho}(R)$  is the mean density of the material within the sphere of radius  $R$  and  $g(R)$  is the acceleration of gravity at the radius  $R$ . For  $\bar{\rho}(R) = 3 \times 10^3 \text{ kg m}^{-3}$  and  $g(R) \sim 10 \text{ m s}^{-2}$  the pressure is then calculated from  $P(R) \sim 3 \times 10^4 R \text{ nm}^{-2}$ . A pressure appropriate to ionisation is obtained at a depth of about  $10^8 \text{ m}$  while for overcoming the interparticle forces the depth will need to be about  $10^6 \text{ m}$ . These estimates are very rough but portray two important features for planetary interiors:

- i. There will be a maximum size for a planetary body if this is defined as a body where pressure ionisation does not occur. Detailed calculations show this maximum mass for a hydrogen body is only slightly higher than the mass of Jupiter. Apparently, Jupiter is close to the maximum mass and size for a planetary body of that composition;
- ii. There is a level where interatomic forces are inoperative, the atoms being uncoupled in the direction of the gravitational field. For equilibrium conditions the pressure is homogeneous and so the atoms are uncoupled in all directions. This will mean that the atoms are free to change their relative arrangements and so behave macroscopically as a fluid. The condition of fluidity is surprising

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and will apply for lesser pressures if the material is regarded as a fluid with high viscosity. Then the necessary pressure can be reduced by perhaps an order of magnitude so the depth becomes about 100km. The material will then show flow and the figure will be that of symmetry overall (spherical for a non-rotating body, ellipsoidal for a rotating body). The precise depth will also depend on the temperature of the material because the viscosity is strongly dependent on the temperature. For a body like the Earth the temperature increases rapidly with the depth and 33km is usually taken as the depth where the Adams-Williamson method can begin, that is where the material is behaving as a fluid.

This can be used to give a rather imprecise lower limit for the size of a planetary body if it is regarded as a body showing a figure of symmetry. This will be of the order of 100km.

## 1.8.3. Hydrostatic equilibrium.

The equilibrium form of a liquid under planetary circumstances will be a condition of hydrostatic equilibrium. This surprising conclusion is in fact well known; the well established method of Adams and Williamson (proposed some 50 years ago now) for exploring the interior of the Earth by seismic waves is based on the condition of hydrostatic equilibrium for the material.

The equilibrium of the planetary body is expressed as the balance between the force of gravity at the depth R holding the material together and the gradient of the hydrostatic pressure  $dP(R)/dR$  resisting compression. This gives the expression

$$dP(R)/dR = - \rho(R)g(R) \quad (8)$$

the negative sign showing the pressure to increase with depth.

This expression is rearranged to involve the bulk modulus  $K(R)$

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by using the definition

$$K(R) = \rho(R)dP(R)/d\rho(R) \quad (9)$$

appropriate to the depth R. Consequently (8) is converted into the equation for the density gradient

$$d\rho(R)/dR = -\rho^2(R)g(R)/K. \quad (10)$$

By relating g to the mass and radius of the body and K to seismic wave speeds (10) reduces to the usual Adams-Williamson iterative expression for calculating the internal distribution of density and pressure.

## 1.8.4. Empirical Equation of State.

The conditions of equilibrium within the body are determined by the bulk modulus K and not the temperature. The magnitudes for the bulk modulus cover a very wide range with  $K(\text{solid}) > K(\text{liquid}) \gg K(\text{gas})$ . This is seen from Table 1.2.

Material	Bulk Modulus K (N m <sup>-2</sup> )	Compressibility $k = 1/K$ (m <sup>2</sup> N <sup>-1</sup> )
silicate-type	10 <sup>11</sup>	10 <sup>-11</sup>
water-ice	10 <sup>10</sup>	10 <sup>-10</sup>
liquid	10 <sup>9</sup>	10 <sup>-9</sup>
gas	1/P	P

TABLE 1.2. The bulk modulus and compressibility for silicate-type (amorphous) materials, water-ice, liquids and gases as to general orders of magnitude. P is the gas pressure.

An approximate equation of state is obtained by expanding K in a power series in the pressure P

$$K(P) = K_0 + BP + CP^2 + \dots \quad (11)$$

and it is often an acceptable approximation to terminate the series at the first term. In (11), B, C and so on, are parameters characteristic of the material.

For the linear relation  $K = K_0 + BP$  and using (9) we integrate the resulting expression to obtain P as function of the density  $\rho$ . This is

$$P = K_0/B[(\rho/\rho_0)^B - 1], \quad (12)$$

an equation of state first proposed by Mannaghan. This is equivalent to the condition

$$K/\rho^B = K_0/\rho_0^B. \quad (13)$$

The inclusion of further terms in the expansion (11) will lead to more refined formulae but the accuracy of the calculation is hardly increased in most cases. One reason for this is that we have not got accurate information about either the material forming the planet or the physical nature of the material.

For silicate materials it is found empirically that  $B = 3.5$  is a useful approximation; for iron  $B = 3.3$ ; for ice  $B = 3$ ; and for condensed hydrogen/helium mixtures  $B = 1.9 - 2.1$  has been used in various investigations.

#### 1.8.5. Planetary shape.

The behaviour of planetary material as a liquid, albeit a strange one and of high viscosity, is compatible with the observed figures of the planets. It is a remarkable fact that all the planets, together with the larger satellites, have shapes reminiscent of a rotating liquid. The major planets (Jupiter, Saturn, Uranus and Neptune) are fluid anyway but their satellites are not and the terrestrial planets are definitely solid on the short time scale. Our previous arguments allow this apparent paradox to be resolved. We take

the Earth as a typical member of the terrestrial planets.

The pressure below about 30km, accounting also for the temperature, is such that conditions are those of hydrostatic equilibrium. The figure is then that of a fluid body in rotation. The time to achieve such a figure, starting from some arbitrary shape, is something we shall have to consider in a later lecture. There are, then, two layers; one (called the crust) accounting for the upper regions which is effectively solid from the present point of view, and the other (involving the largest volume of the material) which is fluid. There could be a surface distribution of liquid such as the oceans of the Earth but this is not important for the overall structure. The symmetry figure of the planet arises from the flow properties of the interior. This is masked from the outside by the crustal region which will mirror the internal shape to some extent but not precisely. The overall figure viewed from outside is then generally one of liquid symmetry but not exactly so. The degree to which it shows this symmetry from a distance will depend on the rheological structure of the crust and particularly its temperature. The creep of solid material will be crucial in this connection.

## LECTURE 2

### THE DESCRIPTION OF A MACROSCOPIC LIQUID

The description of the behaviour of a liquid under the action of forces of various kinds is one of the earliest and most successful of the theoretical constructs of classical physics. The theory makes no reference to the atomic structure but specifies the liquid by a set of empirical macroscopic parameters such as viscosity, thermal conductivity, and so on. The liquid is supposed to obey the laws of Newtonian mechanics modified to apply to a fluid continuum. This means that the conservation laws of mass, momentum (including linear and angular) and energy are used as an expression of the behaviour of a liquid. The particular application depends upon the precise properties of the liquid. The greatest development of the theory has involved simple liquids of which water is the example. Extensions of the theory to include more complicated liquids have been slow to develop. We begin our analysis with the case of a simple liquid. We shall use classical physics throughout.

#### 2.1. Fluid Point

The fluid is a continuum (except for special regions such as occur in shock waves) and special interpretation is needed to apply Newton's laws of motion in this case. The special interpretation centres on the definition of the fluid point.

Suppose a hypothetical surface is drawn in the liquid enclosing a chosen point. The mass of liquid  $dM$  isolated in this way in this hypothetical small volume  $dV$  has a mean density  $\rho = dM/dV$ . The liquid will have all the properties of the bulk liquid and will be

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indistinguishable from it. We now cause the volume  $dV$  to decrease down to a point (say a pin's head) which is literally a point in macroscopic terms even though microscopically it is still a large volume. The point volume will still contain liquid of the same density as before and having all the properties of the bulk liquid. The hypothetical point constructed in this way is called a fluid point and plays the same role for a fluid that the point mass plays in particle mechanics. The continuum liquid is, in this way, regarded as a collection of an indefinitely large number of interacting fluid points.

Each fluid point is not isolated from the remainder but interacts with forces that are recognised by way of the pressure and the various dissipative coefficients describing the liquid.

#### 2.2. Local Thermodynamic Equilibrium

The recognition of the elastic and thermal properties of the liquid require, for their definition, a state of thermodynamic equilibrium. The liquid may be in motion and such equilibrium is obviously not applicable. Nevertheless quantities such as the pressure and temperature still have meaning for a flowing liquid and can still be defined in the usual way provided the definition is made appropriately. The crucial thing is to recognise that equilibrium is known to exist only if it can be demonstrated experimentally and experiments always involve errors.

Choose some point in the liquid and enclose it with a volume selected so that each liquid property of interest has the same equilibrium value within the errors of experiment of the apparatus used. All the liquid properties within this small volume will satisfy the requirements of thermodynamic equilibrium within a prescribed

accuracy. The same process can be applied to a neighbouring point and the volume established where thermodynamic equilibrium applies. Of course, the equilibrium of the second point will not be the same as that of the first point although the difference will be small if the associated volumes are small. The process can be repeated to other points until the whole liquid volume is included. The result is a description of the liquid using thermodynamic parameters which are properly defined everywhere but which vary from one region to another within the liquid. By choosing the points indefinitely close together the variation of the parameters are virtually continuous throughout the liquid.

The concept of local thermodynamic equilibrium is central to the full development of the subject. It is well satisfied in practical cases. As an extreme case consider the Sun. Its central temperature is about  $10^6$  K while the surface temperature is about  $5 \times 10^3$  K. The radius is about  $7 \times 10^8$  m. There is then a temperature gradient across the Sun's radius which, were it linear, would be about  $10^{-3}$  K per metre. This is an extremely small gradient. Surprising as it may seem, it is a good approximation to this accuracy of measurement to suppose the Sun to be in thermodynamic equilibrium locally in the neighbourhood of every point within its volume.

### 2.3. Expressions of Balance

Having established the fluid point and the condition of local thermodynamic equilibrium we can begin the study of the conservation of liquid properties. This is achieved by considering the balance of quantities within a liquid volume.

Consider a volume  $dV$  within the liquid about a fixed point in space (relative to some coordinate axes fixed outside - this is

sometimes called the Euler description). Let us be concerned with the way some physical scalar quantity  $Q$  (which may be mass, energy, or a component of momentum, or any other liquid property) varies within the liquid.

The quantity of  $Q$  within  $dV$  can change during the time interval  $dt$  for one of two reasons and no more! Either  $Q$  can pass across the boundary (increasing  $Q$  if inwards or decreasing  $Q$  if outwards) or  $Q$  can be produced (positive - or annihilated, negative) within the volume. The passage of  $Q$  across the boundary will result from a current  $C$  of  $Q$  over the whole closed surface (positive inwards, negative outwards) while the positive production (negative annihilation) within  $dV$  is written  $P$ . Then we have the expression for the balance

$$dQ/dt = [C + P]. \quad (1)$$

There is no other possibility.

The statement (1) is a truism and can be rearranged by introducing the volume element  $dV$  and surface element  $dS$  explicitly.

#### 2.3.1. The density of $Q$ .

The volume element  $dV$  is introduced explicitly by writing  $Q$  in terms of the quantity  $q$  per unit volume according to

$$Q = \int_V q dV. \quad (2)$$

Then

$$\begin{aligned} dQ/dt &= (d/dt) \int_V q dV \\ &= \int_V dQ/dt \end{aligned} \quad (3)$$

because the volume element  $dV$  is assumed fixed in magnitude and position.



## EQUATION OF BALANCE

### 2.3.2. The current through the boundary.

There is a flux of current  $d\mathbf{J}$  of  $Q$  (per unit time per unit surface area) and for the entire surface we write, for the time interval  $dt$

$$C = - dt \oint \mathbf{J} \cdot d\mathbf{S},$$

the negative sign accounting for the situation that an outward (positive) flow of  $Q$  is associated with a decrease of  $Q$  in the volume. According to the divergence theorem this is written in terms of the volume integral

$$C = - dt \int_V \text{div} \mathbf{J} dV. \quad (4)$$

### 2.3.3. The production term.

Finally we introduce the production  $\Phi$  (or annihilation) per unit volume per unit time so that

$$P = \int_V \Phi dV. \quad (5)$$

### 2.3.4. Equation of balance.

The expression for the balance is obtained by inserting (3), (4) and (5) into (1) to give

$$dt dV [dq/dt + \text{div} \mathbf{J} - \Phi] = 0. \quad (6)$$

This relation is to apply to any arbitrary volume and can be true only if the integrand is zero

$$dq/dt + \text{div} \mathbf{J} - \Phi = 0. \quad (7)$$

This is the equation of balance. It is applied to particular situations by assigning the three quantities  $q$ ,  $\mathbf{J}$  and  $\Phi$ .

There is often convenience in using component notation for the various components and operators so that we write

$$\text{div} = \sum_{j=1}^3 \partial/\partial x_j$$

## CONTINUITY EQUATION

### 2.4. Mass Balance and Conservation: Continuity Equation

The balance equation (7) is applied to continuum mass flow by setting  $Q = \rho$ , the local density of the liquid. If further mass is conserved  $\Phi = 0$ . The current flow is the flow of mass so for the  $i$ -th component  $J_i = \rho v_i$  where  $v_i$  is the  $i$ -th component of the velocity of the fluid point. Then (7) takes the form, in component notation

$$\partial \rho / \partial t + \partial(\rho v_j) / \partial x_j = 0. \quad (8)$$

This expression can be rewritten by making the expansion

$$\partial(\rho v_i) \partial x_j = \rho \partial v_j / \partial x_j + v_j \partial \rho / \partial x_j.$$

Then (8) becomes

$$\partial \rho / \partial t + v_j \partial \rho / \partial x_j + \rho \partial v_j / \partial x_j = 0.$$

We introduce the concept of the derivative moving with the fluid by defining the operator

$$D/Dt = \partial/\partial t + v_j \partial/\partial x_j. \quad (9)$$

Then (8) takes the final form

$$D \rho / Dt + \rho \partial v_j / \partial x_j = 0. \quad (10)$$

This is the equation of continuity.

For an incompressible liquid  $D \rho / Dt = 0$ . Consequently

$$\partial v_j / \partial x_j = 0 \quad (11)$$

that is, the divergence is zero. This condition can apply to a real fluid flow even though the fluid be compressible provided conditions are such that the effects of compression are negligibly small. In practice for ordinary fluids this usually amounts to the restriction that the fluid speed does not exceed about 1/3 the local speed of sound.

The general expression (10) must be modified if there is a source of liquid by including the term  $\Phi$  neglected so far. Such an extension would be necessary if one component of a multicomponent

## VELOCITY POTENTIAL

system were involved (for instance ions and electrons that may combine or dissociate during some process, or a series of chemical reactions).

In general, if there are  $n$  components labelled  $c$  with densities

$\rho_a$  the set of equations

$$\partial \rho_a / \partial t + \text{div}(\rho_a \underline{v}_a) = \sum_{r=1}^r w_{ar} J_r$$

must be satisfied. Here  $w_{ar} J_r$  is the rate of production of component  $a$  with the  $r$ -th of  $r$  chemical reactions,  $J_r$  is the reaction rate for the reaction representing the total mass transformation per unit volume per unit time, and  $w_{ar}$  is proportional to the stoichiometric coefficient with which component  $a$  appears in the chemical equations for the reaction  $a$ .

The total density is defined by

$$\rho = \sum_a \rho_a$$

and the centre of mass velocity  $\underline{v}$  by

$$\rho \underline{v} = \sum_a \rho_a \underline{v}_a$$

The diffusion current is then

$$j_a^{\text{diff}} = \rho_a (\underline{v}_a - \underline{v})$$

and the continuity equation is

$$D \rho_a / Dt + \rho_a \text{div} \underline{v} + \text{div} j_a^{\text{diff}} = \sum_{r=1}^r w_{ar} J_r$$

with

$$D \rho / Dt + \rho \text{div} \underline{v} = 0.$$

These are complicated equations to deal with particularly for the chemical reactions likely to be of planetary interest.

### 2.5. Velocity Potential

It is a general result of the vector calculus that  $\text{div grad } A = 0$  if  $A$  is any scalar quantity. It follows from (11) that the velocity vector  $\underline{v}$  for incompressible flow can be written as the gradient of a scalar quantity  $\varphi$ , called the velocity potential. Then

## MOMENTUM BALANCE

$$v_i = - \partial \varphi / \partial x_i. \quad (12)$$

The introduction of the velocity potential for incompressible flows provides an important simplification because the flow can be described by an equation involving a scalar rather than the vector velocity itself. The problem of specifying boundary conditions will be considered later.

### 2.6. Momentum Balance

The linear momentum of a fluid point is  $\rho \underline{v}$  and each component  $\rho v_i$  (with  $i = x, y, z$  in Cartesian coordinates) is a scalar quantity. We make the identification  $Q = \rho v_i$ . The momentum current is a vector flow for each component and we denote the  $i$ -th component by  $J_{ij}$ . The current is therefore represented by a second rank tensor and is associated with six component parts.

The current is itself the sum of two separate contributions of very different character. One, written  $J_{ij}^{\text{con}}$ , describes the flow of momentum convected with the liquid as it flows. Then

$$J_{ij}^{\text{con}} = \rho v_i v_j \quad (13)$$

for the  $j$ -th component of the  $i$ -th component of momentum.

The second component,  $J_{ij}^s$ , describes a conduction of momentum due to the interaction between contiguous liquid points. Its form depends both on the physical nature of the liquid and on the details of the motion. The origin of this contribution is in the interaction between the atoms and, because the interatomic forces are short ranged, is essentially a local transfer of momentum at each point in the liquid. The explicit form of the conductive transfer will be considered further later.

The total momentum flux is then written as the sum

$$J_{ij} = \rho v_i v_j + J_{ij}^s. \quad (14)$$

$J_{ij}^S$  is the stress tensor.

The production contribution will not now be zero because contributions will arise from the action of external forces of various kinds. Because the time rate of change of momentum is to be equated to the force acting the function  $\Phi$  is to be equated directly to the sum of the (suppose)  $n$  forces  $\rho F_i = \sum_{k=1}^n \rho_k F_{ki}$  acting on the liquid volume where  $\rho_k$  is the density appropriate to the action of the  $k$ -th force. The linear momentum is not then generally conserved within a volume and the behaviour of the liquid is linked to outside influences. Linear momentum is, of course, conserved for the total liquid-forces system treated as a single entity.

Inserting (13) and (14) into (11) gives the equation for local momentum balance in the form

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j + J_{ij}^S) = \sum_{k=1}^n \rho_k F_{ki}. \quad (15)$$

This expression applied to a fixed stationary volume but is rearranged into a form moving with the liquid by introducing the total derivative (9). Then (15) becomes instead

$$\rho \frac{Dv_i}{Dt} = \sum_{k=1}^n \rho_k F_{ki} - \sum_{j=1}^3 \frac{\partial J_{ij}^S}{\partial x_j}. \quad (16)$$

The stress tensor  $J_{ij}^S$  takes a simple form when the liquid is at rest; then it involves only the hydrostatic pressure  $P$ . The flux is more complicated when the liquid is in motion and, provided the motion is not too strong, the stress tensor is written as the sum of the two contributions, one applying to the liquid at rest and the other,  $J_{ij}^V$ , to the liquid in motion and is associated with viscous dissipation:

$$J_{ij} = P \delta_{ij} + J_{ij}^V. \quad (17)$$

Here  $\delta_{ij}$  is the Krönicker delta.

Then (16) becomes successively

$$\rho \frac{Dv_i}{Dt} = \sum_{k=1}^n \rho_k F_{ki} - \frac{\partial P}{\partial x_i} - \sum_{j=1}^3 \frac{\partial J_{ij}^V}{\partial x_j}$$

or

$$\rho \frac{Dv_i}{Dt} = \sum_{k=1}^n \rho_k F_{ki} - \frac{\partial P}{\partial x_i} - \sum_{j=1}^3 \frac{\partial J_{ij}^V}{\partial x_j}. \quad (18)$$

This is the general form of the momentum equation. Application of this equation to the description of a particular physical liquid flow is made by giving explicit expressions for  $J_{ij}^V$  and  $F_{ki}$ . This will be considered later.

## 2.7. The Euler Equation: Inviscid Flows

There are cases of liquid flow where the viscous stress makes a negligible contribution to the momentum balance so that  $J_{ij}^V$  can be disregarded. This means the effects of dissipative processes (and particularly of the viscosity) are irrelevant, the liquid behaving as if no dissipation were involved. Such a liquid is said to be ideal because it is not met in real liquids except approximately.

In such cases (18) is simplified: in vector form the equation is then

$$\rho \frac{Dv}{Dt} = -\text{grad } P + \rho F. \quad (19)$$

This equation is called the Euler equation of motion.

## 2.8. The Viscous Stress

The form to be assigned to the stress tensor  $J_{ij}^V$  depends on the physical nature of the liquid and on the definition of stress and

strain in the material.

The action of an external stress is to cause a strain in the material; the strain is defined as the displacement  $\xi_i$  of the equilibrium position of an arbitrary point in the material from what it would have been without the stress. Conditions are never homogeneous and it is the gradient of the strain  $\partial \xi_i / \partial x_j$  at any point that determined the conditions of strain. This is the quantity which characterises the stress in an elastic solid but in a liquid, which cannot withstand stress, it is the time rate of change of strain  $\partial / \partial t (\partial \xi_i / \partial x_j) = \partial v_i / \partial x_j$  that is the significant quantity. Because the liquid cannot withstand shear it will not maintain a gradient of shear within itself so we can write the statement of proportionality

$$J_{ij}^s \propto \partial v_i / \partial x_j \quad (20)$$

where there is a constant of proportionality to be added. It is convenient to rewrite the expression for the strain in a form which separates explicitly the change of volume (dilatational effect), the change of shape (non-deviatorial effects) and any effects of rotation. For this purpose we write the identity

$$\begin{aligned} \frac{\partial v_i}{\partial x_j} = & \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \sum_{l=1}^3 \frac{\partial v_l}{\partial x_l} \delta_{ij} \right) \\ & + \frac{1}{3} \sum_{l=1}^3 \frac{\partial v_l}{\partial x_l} \delta_{ij} + \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right). \end{aligned} \quad (21)$$

The first term describes the change of shape and includes all non-diagonal symmetric terms (this is the deviatoric contribution); the second term describes the dilatational (change of volume) effects; while the third term involves the anti-symmetric terms important in the cases of rotation as we shall see in a moment.

For the simpler liquids the three contributions arise separately from the corresponding stresses. Thus a pure dilatational stress will produce only a change of volume and so on. It is natural to assume for such a liquid that the total stress is the sum of the three components dilatation, shear and rotation respectively

$$J_{ij}^v = J_{ij}^{(1)} + J_{ij}^{(2)} + J_{ij}^{(3)} \quad (22)$$

so, in conjunction with (20), we suppose further that the separate strains arise only from the corresponding stresses. From (20), (21) and (22) we make the association for such a liquid

$$\begin{aligned} J_{ij}^{(1)} &= - (a_1/2) \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \sum_{l=1}^3 \frac{\partial v_l}{\partial x_l} \delta_{ij} \right) \\ J_{ij}^{(2)} &= - (a_2/3) \sum_{l=1}^3 \frac{\partial v_l}{\partial x_l} \delta_{ij} \\ J_{ij}^{(3)} &= - (a_3/2) \left\{ \left( \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right) - 2w \right\}. \end{aligned} \quad (23)$$

The negative sign in each expression accounts for the loss of momentum to the liquid volume due to the action of the dissipative effects. The three positive constants  $a_i$  ( $i = 1, 2, 3$ ) are characteristic of the liquid and determine the degree of dissipation in the liquid for given flow conditions. They are related to the coefficients of bulk (dilatational) viscosity  $\zeta$ , shear viscosity  $\eta$  and rotational viscosity  $\eta_{rot}$  according to

$$\zeta = \frac{a_2}{3}, \quad \eta = \frac{a_1}{2}, \quad \eta_{rot} = \frac{a_3}{2}. \quad (24)$$

These various formulae express the rheology of the liquid in macroscopic terms without reference to the atomic sub-structure.

If the liquid is in rotation with mean angular velocity  $w$  the

atomic sub-structure can be relevant if the molecules are non-symmetric because the rotation can cause a torque on them which dissipates energy. It is this effect which is described by the rotational viscosity  $\eta_{rot}$ . Concern then is with the difference between the rotational characteristics of the liquid and the mean angular rotation, that is with  $[\text{curl } \underline{v} - \underline{\omega}]$ .

The stress tensor can then be written in the final form

$$J_{ij}^v = -\eta \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \sum_{l=1}^3 \frac{\partial v_l}{\partial x_l} \delta_{ij} \right) - \frac{2}{3} \sum_{l=1}^3 \frac{\partial v_l}{\partial x_l} \delta_{ij} - \eta_{rot} \left\{ \left( \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right) - 2\underline{\omega} \right\}. \quad (25)$$

The solid body rotational contribution  $\underline{\omega}$  is included for completeness. This is the expression for the stress that is to be inserted into (18) to provide the final equation of motion of the liquid.

This approach to viscous behaviour was first realised by George Stokes. It has the important feature that  $J_{ij}^v$  is related directly to the velocity of the liquid (together with three empirical constants) and its use converts (18) into an equation for the liquid velocity as the only unknown. More complicated expressions for the viscous dissipation could be a more accurate expression of the precise conditions in a given liquid under particular conditions but the resulting expression would probably be virtually unusable from the theoretical point of view.

## 2.9. The Navier-Stokes Equation

Inserting (25) into (18) provides the equation of motion

$$\rho \frac{Dv_i}{Dt} = \sum_{k=1}^n \rho_k F_{ki} - \frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j} \left\{ \eta \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \sum_{l=1}^3 \frac{\partial v_l}{\partial x_l} \delta_{ij} \right) \right\} + \frac{\partial}{\partial x_j} \left\{ \eta_{rot} \sum_{l=1}^3 \frac{\partial v_l}{\partial x_l} \delta_{ij} \right\} \quad (26)$$

in the absence of rotation. The expression simplifies greatly if the coefficients of viscosity are constant throughout the liquid. In that case (26) becomes, in vector notation

$$\rho \frac{D\underline{v}}{Dt} = \rho \underline{F} - \text{grad } P + \eta \nabla^2 \underline{v} + \left( \frac{\eta}{3} + \eta_{rot} \right) \text{grad div } \underline{v}. \quad (27)$$

There is a further important simplification for incompressible flows for then  $\text{div } \underline{v} = 0$ . Then (27) takes the final form

$$\rho \frac{D\underline{v}}{Dt} = \rho \underline{F} - \text{grad } P + \eta \nabla^2 \underline{v}. \quad (28)$$

This is what is usually meant by the Navier-Stokes equation.

The coordinates used for (28) in a particular case will depend on the geometry of the system involved. Transformation of coordinates must be made with care and it is safest to replace the nabla operator in (28) according to the formula

$$\text{curl curl } \underline{v} = \text{grad div } \underline{v} - \nabla^2 \underline{v}.$$

The quantity  $\underline{\omega} = \text{curl } \underline{v}$  is called the vorticity. It is introduced into the theory by the term involving the viscosity in the Navier-Stokes equation.

It is remarkable that the motion of the liquid including dissipation is described by the velocity alone and that the particular liquid is characterised by the single parameter  $\eta$ . When this

## IMPORTANT FORCES

parameter is zero (no dissipation) the Navier-Stokes equation (28) reduces to the Euler equation (19).

### 2.10. Boundary Conditions

The various equations are to be solved subject to boundary conditions. The universal condition for a real liquid (and macroscopic gas as well for that matter) is that of no slip at an impenetrable boundary. This is expressed by requiring the vector velocity  $\underline{v}$  to vanish at the boundary. There are two component statements.

The fact that the boundary is impenetrable means that the normal component of the velocity  $v_n$  must vanish at the boundary. This condition will also apply to the Euler equation (19). The second condition applies in the presence of viscous forces. It is an empirical fact, for any liquid and for any surface material, that the liquid is at rest on the surface so there is no liquid flow parallel to the surface. If  $v_t$  is the component of the liquid velocity transverse to the surface the full conditions are

$$v_n = 0 \quad v_t = 0 \quad (\text{at the surface}) \quad (29)$$

The first condition alone applies for an ideal liquid flow.

These conditions again are remarkable in their generality and imply special flows near the boundary. These were recognised by Prandtl as forming a boundary layer for the liquid.

### 2.11. Important Forces

There are two forces acting bodily on a liquid in motion which are particularly important from the point of view of planetary science. One is the effect of gravity,  $\underline{F}_g$  while the other is the Lorentz force,  $\underline{F}_L$ .

If  $\underline{g}$  is the local acceleration of gravity

$$\underline{F}_g = \rho \underline{g}$$

## HYDROMAGNETICS

though we have assumed implicitly so far that conditions are isothermal any temperature variations within the liquid will affect the density and this force contribution is of great importance in thermal flows as we shall see later.

The Lorentz force is operative if the liquid is a conductor of electricity and is permeated by an externally maintained magnetic field as it flows. We will consider this case in the next Section; for the present we need notice only that the force contribution in (19), (27) or (28) will generally have the form

$$\underline{F} = \rho \underline{g} + \rho \underline{F}_L \quad (30)$$

### 12. Hydromagnetics

Suppose the liquid to conduct electricity and to have an electrical conductivity  $\sigma$ . If it is permeated by a magnetic field  $\underline{B}$  then the motion of the liquid relative to the field constitutes the motion of a conductor through the field. Suppose  $\underline{J}$  to be the electric current formed by the moving liquid. The force acting on an element of the liquid is the Lorentz force

$$\underline{F}_L = [\underline{J} \times \underline{B}] \quad (31)$$

In a linear conduction process the electric current is related to the electric and magnetic fields, respectively  $\underline{E}$  and  $\underline{B}$ , according to Ohm's law

$$\underline{J} = \sigma \{ \underline{E} + [\underline{J} \times \underline{B}] \} \quad (32)$$

so  $\underline{E}$ ,  $\underline{B}$  and  $\underline{J}$  are related through Maxwell's electromagnetic equations

$$\begin{aligned} \text{curl } \underline{B} &= \underline{J} & \text{curl } \underline{E} &= -\frac{\partial \underline{B}}{\partial t} \\ \text{div } \underline{E} &= 0 & \text{div } \underline{B} &= 0. \end{aligned} \quad (33)$$

The displacement current has been neglected since there are supposed to be no free charges. Inserting (32) in (33) and eliminating  $\underline{E}$  we

obtain the equation for the magnetic field

$$\frac{\partial \mathbf{B}}{\partial t} = \text{curl} [\mathbf{v} \times \mathbf{B}] + \frac{1}{\sigma \mu} \nabla^2 \mathbf{B}. \quad (34)$$

This equation can be solved once  $\mathbf{v}$  is specified; inserting (31) into (30) and the result into (27) gives the equation of motion for a simple liquid. This equation now involves  $\mathbf{B}$  as well as  $\mathbf{v}$  so the new (27) is coupled with (34) to give both  $\mathbf{v}$  and  $\mathbf{B}$ . The mutual nature of the flow field and the magnetic field is clear for this case.

The expression (34) is the basis of the theory of the dynamo action for the production of planetary magnetic fields. The first term on the right hand side describes the convection of field by the fluid while the second term describes the diffusion of the field. If the convection concentrates magnetic field faster than the diffusion allows it to disperse the field will be amplified, otherwise it will decay. The problem then is the production of suitable fluid flow patterns to cause the field to be maintained or amplified.

### 2.13. Equation for the Vorticity

The Navier-Stokes equation (28) can be rearranged to show the vorticity  $\underline{\omega} = \text{curl } \underline{v}$  explicitly. Using the vector relation

$$\underline{v} \cdot \text{grad } \underline{v} = \text{grad}(v^2/2) - [\underline{v} \times \text{curl } \underline{v}]$$

and taking the curl operation of each term in (28) gives

$$\frac{\partial \underline{\omega}}{\partial t} = \text{curl} [\underline{v} \times \underline{\omega}] + \eta \nabla^2 \underline{\omega} \quad (35)$$

where we have assumed the body forces to be conservative (and so derivable from a potential energy function).

Comparing (34) and (35) it is seen that the equations for the vorticity and the magnetic field are of the same form.

### 2.14. The Effects of Rotation

New forces are introduced by rotation for the observer rotating with the body. For a liquid rotating with constant angular speed  $\underline{\Omega}$  the total inertia acceleration of a liquid element moving with velocity  $\underline{v}$  distance  $\underline{r}$  from the rotation axis is modified to become

$$(\underline{Dv}/\underline{Dt})^s = \underline{Dv}/\underline{Dt} + 2[\underline{\Omega} \times \underline{v}] - \text{grad}([\underline{\Omega} \times \underline{r}]^2)$$

where the superscript  $s$  refers to an inertia frame of reference taken to be stationary. Then (28) is modified to become in the rotating frame of reference

$$\rho \frac{\underline{Dv}}{\underline{Dt}} = \rho \underline{F} - \text{grad } P + 2\rho [\underline{v} \times \underline{\Omega}] + \eta \nabla^2 \underline{v} \quad (36)$$

where the modified pressure  $P_1$  is

$$P_1 = P - \rho [\underline{\Omega} \times \underline{r}]^2.$$

The modified pressure behaves in all ways as a normal pressure and we need not distinguish between  $P$  and  $P_1$  provided we realise that the effects of rotation would need to be accounted for in any practical application of the formula.

### 2.15. Two Theorems

There are two results, closely analogous, that follow from (36) without body forces and with the Lorentz force included.

#### 2.15.1. Taylor-Proudman Theorem.

Consider steady ( $\partial \underline{v}/\partial t = 0$ ), slow ( $\underline{v} \cdot \text{grad } \underline{v} = 0$ ) inviscid ( $\eta = 0$ ) flow of a liquid in the absence of forces ( $\underline{F} = 0$ ). Then (36) reduces to the form

$$[\underline{v} \times \underline{\Omega}] = 0 \quad \text{i.e.} \quad \Omega_j \partial v_i / \partial x_j = 0.$$

The velocity vector is therefore perpendicular to the direction of the

rotation axis and this condition does not change. The flow is steady and two-dimensional with symmetry about the rotation axis. This result is called the Taylor-Proudman theorem. It has been well demonstrated in laboratory experiments.

#### 2.15.2. The Magnetic Analogue.

Suppose a uniform magnetic field  $B_1$  permeates the liquid at rest. If the system is now perturbed into a steady state the liquid will have the small velocity  $v_i$  and the magnetic field will be augmented by the small field  $h_i$ . Then (28) with (31) and (34) becomes

$$B_j \partial v_i / \partial x_j = 0.$$

This shows that motions cannot vary in the direction of the field  $B$ . All slow motions in the presence of a uniform magnetic field are necessarily two-dimensional. Magnetic effects and rotation effects often have similar effects on liquid flow.

#### 2.16. Conservation of Energy

The energy  $e$  per unit mass associated with a liquid region has two components. One is the kinetic energy  $e_k$  of motion and the other is the thermodynamic internal energy  $u$  per unit mass so that  $e = e_k + u$ . We make the identification  $Q = \rho e = \rho v^2/2 + \rho u$ . For an ideal liquid (with no dissipation) the conservation of energy is expressed as

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho v^2 + \rho u \right) = - \sum_j \frac{\partial}{\partial x_j} \left[ \rho v_j \left( \frac{1}{2} v^2 + h \right) \right] \quad (37)$$

where  $h = u + P/\rho$  is the enthalpy or total heat. For a real liquid, with dissipation, thermal conductivity  $\lambda$  and the heating effects of

viscosity must also be taken into account. It follows from (18) that

$$\begin{aligned} \frac{\partial}{\partial t} (\rho e) = & \sum_j \frac{\partial}{\partial x_j} \left( \rho F_j \right) - \sum_{i,j} \rho \delta_{ij} \frac{\partial v_i}{\partial x_j} \\ & - \sum_{i,j} J_{ij}^v \frac{\partial v_i}{\partial x_j} - \rho e \sum_j \frac{\partial v_j}{\partial x_j} \end{aligned} \quad (38)$$

where a contribution to the heating arises from the action of work done by the external forces (first term on the right hand side), the pressure (second term), the viscous forces (the third term) and dilatation (the last term).

There is an additional term now, however, in that heat can be conducted across the boundary to the local volume. If  $q$  is the heat flux (heat passing per unit area per unit time) we use Fourier's law

$$q = - \lambda \text{ grad } T \quad (39)$$

the negative sign accounting for the heat flow from hotter to colder regions.

We remember that we assume conditions of local thermodynamic equilibrium within the liquid so the expression for the internal energy can also be rearranged. From thermodynamics we can write

$$du = T ds + (P/\rho^2) d\rho \quad dh = T ds + dP/\rho$$

where  $T$  is the temperature and  $s$  is the entropy per unit mass (entropy density). Collecting these various contributions together there results the equation for the entropy

$$\rho T Ds/Dt = J_{1j}^v \partial v_i / \partial x_j + \text{div}(\lambda \text{ grad } T). \quad (40)$$

In the absence of dissipative effects this reduces to the statement of entropy conservation  $Ds/Dt = 0$ .

The equation (40) for the entropy is transformed into an equation for the temperature by using the thermodynamic relation



$$Ds/Dt = c_p DT/Dt,$$

where  $c_p$  is the specific heat capacity at constant pressure per unit mass of the liquid. Then (40) is transformed to the form

$$\rho c_p DT/Dt = \text{div}(\lambda \text{grad} T) + J_{ij}^Y \partial v_i / \partial x_j \quad (41)$$

This is the equation for the temperature distribution within the liquid. It is associated with the dimensionless parameter

$$\alpha = \lambda / \rho c_p \quad (42)$$

called the thermometric conductivity.

#### 2.17. A Bouyancy Approximation of Boussinesq

The equations (27) and (28) involve the density only through the inertia and force terms. The gravitational force is an important special case because it will introduce convection into a liquid which contains a temperature gradient along the gravitational field direction. The effect of temperature variation in the liquid are important in connection with gravity but offer a small perturbation effect to the inertia force. It is a useful and obvious approximation, therefore, to account for temperature variations only in connection with the force term. The density assigned to the calculation of inertia is chosen as the mean for the region of interest. This approximation of accounting for temperature change only in connection with the gravity force was first introduced by Boussinesq and the approximation is usually named after him.

According to this approximation we write for the gravity force  $F_G$  the expansion

$$F_G = \rho_0 [1 - \beta \Delta T]$$

where  $\rho_0$  is the mean density which will appear also in the expression for the inertia force.  $\beta$  is the thermal expansion coefficient of the liquid and  $\Delta T$  is the difference in temperature

across the region of interest in the direction of  $\underline{g}$ . It is this term in the equation of motion that introduced the possibility of thermal convection into the discussion.

#### 2.18. Dimensionless Numbers

The equations derived so far have two essential deficiencies for planetary studies. First they have an extremely complicated mathematical form and analytic solutions are very hard to find. This can, of course, be alleviated to some extent by the use of numerical methods. Second they describe simple material which is hardly relevant in detail to planetary material. None the less the theory we have outlined does allow us to understand some simple liquid situations but more than this it allows us to recognise certain relationships between dimensionless groupings of variables (that is numbers) for simple cases. These relationships can be expected to be relevant to more complicated materials as well and in this way the simple theory will have provided an invaluable tool for the description of complicated systems.

There is also a very good mathematical reason for introducing dimensionless numbers into the arguments even if analytic solutions are sought. Interest is always with the most general solutions of equations and these are best obtained by solving the equations in non-dimensional form. Boundary conditions must also be expressed this way.

##### 2.18.1. The Comparison of the Strengths of Forces.

The various forces and energies appearing in the equations derived so far can be expressed in order of magnitude terms for a liquid with characteristic speed  $U$  in a region of characteristic size  $L$ . From (36) for steady conditions we isolate the following forces and orders of magnitude.

inertia force	$F_I$	$\rho(\underline{v} \cdot \text{grad}) \underline{v}$	$\sim \rho U^2/L$
bouyancy force	$F_B$	$\rho \underline{E} \Delta T$	$\sim \rho \beta \Delta T$
gravity force	$F_G$	$\rho \underline{E}$	$\sim \rho \beta$
viscous force	$F_V$	$\eta \nabla^2 \underline{v}$	$\sim \eta U/L^2$
rotation force	$F_R$	$\rho [\underline{v} \times \underline{\Omega}]$	$\sim \rho \Omega U$
magnetic force	$F_M$	$[\text{curl} \underline{B} \times \underline{B}]$	$\sim B^2/L$

## 2.18.2. Energies.

convection	$E_c$	$\rho(\underline{v} \cdot \text{grad } T)$	$\sim \rho U T/L$
conduction	$E_{\text{con}}$	$\lambda \nabla^2 T$	$\sim \lambda T/L^2$
viscous	$E_v$	$\eta (\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})^2$	$\sim \eta (U/L)^2$
and finally for the mass			
inertia mass	$M_I$	$\rho v_i / \rho t$	$\sim \rho / t$
dilatation	$M_d$	$\text{div } \underline{v}$	$\sim U/L$

## 2.18.3. Numbers.

Comparing the effects of one influence with another (or taking the ratio of two terms of the same dimensions) provides a number expressing the relative effectiveness of the two contributions in influencing the equation involved. These numbers characterise the effects of the various forces and influences on the liquid. It is natural to choose the inertia characteristics as the standard since these represent the condition of the liquid in the absence of other forces. We therefore form the following ratios, each traditionally named as indicated:

$$\text{Reynolds number } Re = F_I/F_V = \rho L U / \eta$$

$$\text{Froude number } Fr = F_I/F_G = U / (gL)^{1/2}$$

$$\text{Grashof number } G = (Re)^2 F_I/F_B$$

$$= \rho \beta \Delta T L^3 / \eta^2$$

$$\text{Rossby number } Ro = F_I/F_R = U / \Omega L$$

So

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$$\text{Alfven number } Al = F_I/F_M = \rho U^2 / B^2.$$

From the energy contributions we have

$$\text{Peclet number } Pe = E_c/E_{\text{con}} = UL/\lambda$$

$$\text{Prandtl number } Pr = \eta / \alpha$$

$$\text{Rayleigh number } Ra = \rho \beta \Delta T L^3 / \eta \alpha$$

while from the mass expression we find

$$\text{Strouhal number } St = L/U \tau.$$

Further relationships can be deduced; for instance  $Pe = Re \times Pr$ . The dimensional approach to liquid flow problems is often invaluable, particularly in complicated situations and its usefulness in planetary science cannot be overstressed.

There is also a boundary condition to yield a further number. Heat will be conducted to the surface from (or to) the inside. This will have entered through the surface and the heat flux  $q$  can be represented by  $q = h \Delta T$  where  $h$  is a heat transfer coefficient and  $\Delta T$  the temperature difference between the surface and outside. Then we can write

$$q = (h L / \lambda) \lambda \Delta T / L$$

which has the form of the Fourier law but with  $Nu = h L / \lambda$  as a number determining the extent of the heat flow over that of pure conduction in the liquid.  $Nu$  is called the Nusselt number. These various numbers can be supplemented by others applying to particular circumstances.

## 2.19. Some Applications to Planetary Science

As an application of some of the ideas developed so far we consider some cases of importance in planetary problems.

## 2.19.1. Viscosity limited convection.

The effect of a temperature difference  $\Delta T$  across a region of thickness  $L$  is to cause convection in the material. The motion will

need to overcome viscous dissipation forces. Consider the motion in which the buoyancy force  $F_b$  is balanced by an opposing viscous force  $F_v$ . From the previous formulae the condition for motion is

$$\rho \Delta T \epsilon_s > \eta U/L^2.$$

This implies the condition

$$\rho \epsilon_s \Delta T L^2 / \eta U > 1$$

or that

$$Ra > Pe.$$

Since  $Pe > 0$  we see that there is a specific value of  $Ra > 0$  for which motion is possible.

The characteristic speed  $U$  of the motion is

$$U \sim \rho \epsilon_s L^2 \Delta T / \eta,$$

but this cannot be calculated directly because essentially none of the physical parameters are known. Some indication of the order of magnitude of the speed for the terrestrial interior follows from geophysical observation. The Atlantic Ocean has opened up to the extent of 3000 miles ( $\sim 5000$  km) in some  $200 \times 10^6$  years. This implies a mean speed of material of  $U \sim 10^{-9} \text{ m s}^{-1}$ . If this reflects a motion underneath the crust the lower motion may have this speed or more - say  $10^{-8} \text{ m s}^{-1}$ . For a lower layer of thickness say  $10^6 \text{ m}$ ,  $\Delta T \sim 10^3 \text{ K}$ ,  $\rho \sim 10^{-5}$  and  $g \sim 10$  we find for the mean kinematic viscosity  $\eta \sim 10^{20} \text{ m}^2 \text{ s}^{-2}$ . This is comparable to other independent estimates. For the high speed the estimate is lower by a factor 10.

This example is typical of many; there is no precise numerical magnitude as a result but an indication of an order of magnitude of a variable whose magnitude is probably the least well known of the group involved.

## 2.19.2. Relaxation Time.

Suppose the material of the last section takes time  $t$  to traverse the distance  $L$ . Then

$$t \sim \eta / g L \rho \Delta T.$$

For the magnitudes used there we find  $t \sim 10^8$  years. This is enormously long in comparison with the typical periods of seismic waves for terrestrial type material ( $\sim$  seconds) or body oscillations ( $\sim$  hours or days) that the material is certainly solid from the seismic point of view even though it shows flow characteristics over a longer time period. In this way the simultaneous liquid and solid behaviours of the material can be understood within the compass of different time scales.

## 2.19.3. Pressure - viscosity balance.

Towards the centre of a large planet the pressure is high and the balance can then be considered of the motion due to pressure countered by the dissipative action of viscosity. As to orders of magnitude

$$P/L \sim \eta U/L^2$$

which provides a relaxation time  $t$  of order

$$t \sim \eta / P.$$

For the values of the variables used above this gives  $t \sim 10^2$  years. Such conditions may be associated with the core of the Earth and it might be significant that this time is of the same order as that for the secular variation of the Earth's magnetic field.

## 2.19.4. Inertia and buoyancy.

The action of buoyancy is to change the static condition of the material and cause convection. As the motion starts the speed is low and viscous forces will not be of sufficient magnitude to be

important. In that case there is a balance between the inertia and bouyancy forces.

The condition of balance is

$$\rho \beta \Delta T > \eta U^2/L.$$

This leads to the condition

$$G > (Re)^2$$

or in terms of the Rayleigh number

$$Ra > (Re)^2(Pr).$$

The Rayleigh number must exceed a certain value for this motion to be stable. The precise value of the critical Rayleigh number for motion has been calculated by several workers as an example of the variational approach to fluid flow problems. The present elementary approach shows, however, that there is a critical Rayleigh number and this is consistent with the concept of the adiabatic temperature gradient within the material due to compression. No motion is possible unless the temperature gradient exceeds the adiabatic value.