QUANTITATIVE GENERALIZED BERTINI-SARD THEOREM FOR SMOOTH AFFINE VARIETIES

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To the memory of our teacher and our friend Professor Stanisław Łojasiewicz.

Abstract. Let $X \subset \mathbb{C}^n$ be a smooth affine variety of dimension $n-r$ and let $f = (f_1, ..., f_m) : X \to \mathbb{C}^m$ be a polynomial dominant mapping. We prove that the set $K(f)$ of generalized critical values of f (which always contains the bifurcation set $B(f)$ of f) is a proper algebraic subset of $\mathbb{C}^m$. We give an explicit upper bound for the degree of a hypersurface containing $K(f)$. If $I(X)$ -the ideal of X is generated by polynomials of degree at most D and $\deg f_i \le d$, then $K(f)$ is contained in an algebraic hypersurface of degree at most $(d+(m-1)(d-1)+(D-1)r)^{n-r}D^r$. In particular if X is a hypersurface of degree D and $f : X \to \mathbb{C}$ is a polynomial of degree d, then f has at most $(d+D-1)^{n-1}D$ generalized critical values. This bound is asymptotically optimal for f linear. We give an algorithm to compute the set $K(f)$ effectively. Moreover, we obtain similar results in the real case.

1. Introduction.

There is a quite abundant literature about singularities at infinity of polynomials $f : \mathbb{C}^n \to \mathbb{C}$. This subject was initiated by R. Thom who proved, some 30 years ago, that there is a finite set $B \in \mathbb{C}$ such that f is a locally trivial fibration over the complement of B. The smallest such a set, denoted by $B(f)$, is called the set of *atypical values of* f. An effective (asymptotically sharp) bound for the number of points in $B(f)$ was given only recently by the authors [10]. In the paper we propose a study of a much more general situation.

Let X be a smooth affine variety over $k = \mathbb{R}$ or $\mathbb{C}$, of dimension $n-r$, and let $f : X \to k^m$ be a polynomial dominant mapping. In seventies Wallace [18], Varchenko [16] and Verdier [17] proved, that there exists a proper algebraic (or semi-algebraic in the real case) set $B \subset k^m$ such that

$$f : X \setminus f^{-1}(B) \to k^m \setminus B$$

is a locally trivial C^∞ fibration. We call the smallest such B *the bifurcation set of* f and we denote it by $B(f)$. In the natural way appears a question how to describe this set.

Since f may be nonproper, in general the set $B(f)$ is larger than $K_0(f)$ - the set of critical values of f. It contains also the set $B_\infty(f)$ of bifurcations points at infinity. Briefly speaking the set $B_\infty(f)$ consists of points at which f is not a locally trivial fibration at infinity (i.e., outside a compact set). The main difficulty is to understand the set $B_\infty(f)$. Usual way is to apply stratification theory to the projective closure of the graph of f.

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However by this method it is very difficult to give an explicit equation for $B_\infty(f)$ or even for a hypersurface which contains $B_\infty(f)$.

We follow another approach which in fact goes back to Ehresemann, Palais and Malgrange and was developed by P. Rabier [15]. To control the set $B_\infty(f)$ we use the set of *asymptotic critical values at infinity of* f [15]:

$$K_\infty(f) = \{y \in k^m :\ \exists_{x_l \in X,\ |x_l| \to \infty}\ s.t.\ f(x_l) \to y \ and \ |x_l|\nu(d_{x_l} f) \to 0\},$$

where ν stands for the distance of $d_{x_l} f$ to the space of degenerate linear maps on the tangent space to X at x_l. We explain in the next section different ways to compute ν.

We say that $K(f) = K_0(f) \cup K_\infty(f)$ is *the set of generalized critical values of* f. It follows from a general result of P. Rabier [15] (see also [9]) that $B_\infty(f) \subset K_\infty(f)$. Hence $B(f) \subset K(f)$ which means that f is a locally trivial fibration over the complement of $K(f)$.

In section 3 we prove that $K(f)$ is a semi-algebraic subset of k^m of measure 0. This generalizes the result from [12], where the case of polynomial mappings from $\mathbb{C}^n$ to $\mathbb{C}^m$ was studied.

In section 4 we prove that in the complex case $K(f)$ is actually a proper algebraic subset of $\mathbb{C}^m$. In fact we give an explicit description of $K(f)$ which allows us to estimate from above the degree of a hypersurface containing $K(f)$.

More precisely let $X \subset \mathbb{C}^n$ be a smooth affine variety of dimension $n - r$. Let $f = (f_1, ..., f_m) : X \to \mathbb{C}^m$ be a polynomial dominant mapping. Assume that $I(X)$ - the ideal of X is generated by polynomials of degree at most D and $\deg f_i \le d$.

Our main result Theorem 4.1 claims that $K(f)$ is contained in an algebraic hypersurface of degree at most $(d + (m-1)(d-1) + (D-1)r)^{n-r}(\deg X) \le (d + (m-1)(d-1) + (D-1)r)^{n-r}D^r$. In particular if X is a hypersurface of degree D and $f : X \to \mathbb{C}$ is a polynomial of degree d, then f has at most $(d + D - 1)^{n-1}D$ generalized critical values. Hence in this case we have that all fibers of f are smooth and diffeomorphic one to each other, with at most $(d+D-1)^{n-1}D$ exceptions. This generalizes (and slightly improves) the result from [7], where the case of polynomial mappings from $\mathbb{C}^n$ to $\mathbb{C}^m$ was studied.

In a particular interesting case of linear mapping (projections on line) this gives a bound D^n for the number of generalized critical values hence for the number of atypical values. We give an example which shows this bound is asymptotically optimal. Moreover, we obtain similar results in the real case.

In section 5 we give an algorithm to compute the set $K(f)$ effectively. All necessary results from linear algebra are given in section 2 . In particular we explain geometrically and analytically how to compute ν - the distance to singular operators, moreover we give several equivalent expressions for ν.

We based here on ideas from [7] and [12].

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2. Preliminaries.

Let $k = \mathbb{R}$ or $k = \mathbb{C}$. Let $X \cong k^n$, $Y \cong k^m$ be finite dimensional vector spaces (over k). We consider those space equipped with the canonical scalar (hermitian) products. Let us denote by $\mathcal{L}(X, Y)$ the set of linear mappings from X to Y and by $\Sigma = \Sigma(X, Y) \subset \mathcal{L}(X, Y)$

the set of non-surjective mappings. In this section we give several different expressions for a distance of an $A \in \mathcal{L}(X,Y)$ to the space Σ of singular operators. Let us recall the first following ([15]):

Definition 2.1. Let $A \in \mathcal{L}(X,Y)$. Set

$$\nu(A) = \inf_{||\phi||=1} ||A^*(\phi)||,$$

where $A^* : \mathcal{L}(Y^*, X^*)$ is adjoint operator and $\phi \in Y^*$.

Remark 2.1. *Recall (cf.* [15]*) that if* $A \in GL(X,Y)$*, then* $\nu(A) = ||A^{-1}||^{-1}$.

Moreover in [12] we have a following useful characterizations of $\nu(A)$:

Proposition 2.1. *Let* $A \in \mathcal{L}(X,Y)$*. Denote* $B_X(0,1) = \{x \in X;\ |x| \le 1\}$ *and* $B_Y(0,r) = \{y \in Y;\ |y| \le r\}$*. Then*

a) $\nu(A) = \sup\{r > 0 : B_Y(0,r) \subset A(B_X(0,1))\}$.

b) $\nu(A) = dist(A, \Sigma) = \inf_{B\in\Sigma} ||A - B||$.

Let $\alpha, \beta : \mathcal{L}(X,Y) \to \mathbb{R}_+$ be two non-negative functions. We shall say that α and β are *equivalent* (we write $\alpha \sim \beta$) if there are constants $c, d > 0$ such that

$$c\alpha(A) \le \beta(A) \le d\alpha(A)$$

for any $A \in \mathcal{L}(X,Y)$. We shall give below several functions equivalent to ν. Let $A = (A_1, ..., A_m) \in \mathcal{L}(X,Y)$ and let $\overline{A_i} = grad\ A_i$. Denote by $< (\overline{A_j})_{j\neq i} >$ the linear space generated by vectors $(\overline{A_j})$, $j \neq i$. Let

$$\kappa(A) = \min_{1\le i\le m} dist(\overline{A_i}, < (\overline{A_j})_{j\neq i} >),$$

be the *Kuo number of* A.

Proposition 2.2 ([12])**.** *The Kuo function* κ *is equivalent to* ν *of Rabier. More precisely*

$$\nu(A) \le \kappa(A) \le \sqrt{m}\nu(A).$$

Definition 2.2. Let $A \in \mathcal{L}(X,Y)$ and let $H \subset X$ be a linear subspace. We set

$$\nu(A,H) = \nu(A|_H),\ \ \kappa(A,H) = \kappa(A|_H),$$

where $A|_H$ denotes the restriction of A to H.

From Proposition 2.2 we get immediately:

Corollary 2.1. *We have* $\nu(A,H) \sim \kappa(A,H)$.

In fact we have also an explicit expression for $\kappa(A,H)$:

Proposition 2.3. *Let* $A = (A_1, ..., A_m) \in \mathcal{L}(X,Y)$ *and let* $H \subset X$ *be a linear subspace. Assume that* H *is given by a system of linear equations* $B_j = 0, j = 1, ..., r$*. Then*

$$\kappa(A,H) = \min_{1\le i\le m} dist(\overline{A_i}, < (\overline{A_j})_{j\neq i}; (\overline{B_j})_{j=1,\dots,r} >),$$

where $\overline{A_i} = grad\ A_i$ *and* $\overline{B_j} = grad\ B_j$.

Proof. The space $B =< \overline{B_j})_{j=1,\dots,r} >$ is the orthogonal supplement to H. Hence every vector $\overline{A_i}$ can be written as $a_i + b_i$, where $a_i \in H$, $b_i \in B$. Thus $dist(\overline{A_i}, < (\overline{A_j})_{j\neq i}; B) = dist(a_i, < (a_j)_{j\neq i} >)$ and since $grad(A_i|H) = a_i$ the proof is finished. □

Finally we introduce we function g' which will be useful in the explicit description of the set of generalized critical values:

Definition 2.3. Let $A \in \mathcal{L}(k^n, k^m)$, where $n \geq m + r$, and let $H \subset k^n$ be a linear subspace given by a system of independent linear equation $B_l = \sum b_{lk}x_k$, $l = 1, ..., r$. By abuse of notation we denote by A the matrix (in the canonical bases in k^n and k^m) of the mapping A. Let $C =$ be a $(m + r) \times n$ matrix given by rows $A_1, ..., A_m; B_1, ..., B_r$ (we identify $A_i = \sum a_{ik}x_k$ with the vector $(a_{j1}, ..., a_{jn})$, similarly for B_l). Let M_I, where $I = (i_1, ..., i_{m+r})$, denote a $((m+r) \times (m+r))$ minor of C given by columns indexed by I. Let $M_J(j)$ denote a $(m + r - 1) \times (m + r - 1)$ minor given by columns indexed by J and by deleting the jth row , where $1 \leq j \leq m$. Note that we delete only A_j rows ! We set

$$g'(A, H) = \max_I \{\min_{\{J \subset I,\ 1 \leq j \leq m\}} \frac{|M_I|}{|M_J(j)|}\},$$

(where we consider only numbers with $M_J(j) \neq 0$, if all numbers $M_J(j)$ are zero, we put $g'(A) = 0$).

In particular we get the following:

Proposition 2.4. *We have $g'(A, H) \sim \nu(A, H)$.*

We begin with

Lemma 2.1. *Let H be a linear subspace of k^n, $\dim H = p$, then there exists a coordinate linear subspace E, $\dim E = p$, such that*

$$\nu(\pi|_H) \geq \binom{n}{p}^{-1/2},$$

where $\pi|_H$ is the orthogonal projection on E restricted to H.

Proof. Recall that the canonical scalar (Hermitian) product on k^n induces a scalar (Hermitian) product on $\bigwedge^p k^n$, see eg. [4]. If $x = x_1 \wedge \cdots \wedge x_p$, and $y = y_1 \wedge \cdots \wedge y_p$, then we put

$$(x|y) = \det(x_i|y_j)_{i,j=1,\dots,n}.$$

Let $e_1, \dots, , e_n$ be the canonical basis of k^n and let $I = (i_1, \dots, i_p)$ be an multindex such that $1 \leq i_1 < \dots < i_p \leq n$. Denote $e_I = e_{i_1} \wedge \dots \wedge e_{i_p}$, then all e_I form an orthonormal basis of $\bigwedge^p k^n$. Let us choose some orthonormal basis $f_1, \dots, , f_p$ of H and put $f = f_1 \wedge \cdots \wedge f_p$. Clearly $f = \sum a_I e_I$, but $\|f\| = 1$, so

$$1 = \sum_I |a_I|^2.$$

We have $\binom{n}{p}$ positive summands, hence there is at least one I_0 such that $|a_{I_0}| \geq \binom{n}{p}^{-1/2}$.

We take as E the vector space generated by e_i, $i \in I_0$. Note that the Jacobian of the orthogonal projection $(\pi|_H)$ of H on E is exactly a_{I_0}. Let B be the unit ball in H, its image is an ellipsoid with semi-axes $0 < b_1 \leq b_2 \cdots \leq b_p \leq 1$. By the classical change of variables formula we see that the volume of $\pi(B)$ equals $|a_{I_0}|$ times volume of B. Hence

$$b_1 b_2 \dots b_p = |a_{I_0}|,$$

and consequently $b_1 \geq |a_{I_0}|$. It is an immediate consequence of Proposition 2.1 that $b_1 = \nu(\pi|_H)$. Hence the lemma follows. □

Suppose that $\nu(A, H) > 0$. Let us fix $I = (i_1, ..., i_{m+r})$ and assume that $M_J(j) \neq 0$ for all $\{J \subset I,\ 1 \leq j \leq m\}$. Note that by Lemma 2.1 such a I exists. Let E be the subspace generated by e_i, $i \in I$ and π the orthogonal projection on E. By Lemma 2.1 we may assume that $\nu(\pi|_H) \geq \delta$, where $\delta = \binom{n}{m+r}^{-1/2}$. Let $g = \nu(\pi|_H)^{-1}$, note that

$$\|g\|\nu(A|_H) \geq \nu((A|_H) \circ g) \geq \nu(A|_H)\ \nu(g)$$

and recall that $\nu(\pi|_H) = \|g\|^{-1}$. So it is enough to study $(A|_H) \circ g$ i.e the matrix obtained from A by deleting columns which are not in I. Thus we may assume that $n = m + r$. By Remark 2.1, $\nu(A, H) = \|h\|^{-1}$, where $h = (A|_{H \cap \ker^{\perp}})^{-1}$. Again by Lemma 2.1 we may consider the composition of h with the orthogonal projection on some coordinate m-plane, say generated by first m coordinates. Recall that the mapping $C : k^n \to k^{m+r}$ is invertible. By the formula for the inverse matrix (or directly by Cramer's rule) we can see that there are constants $\alpha, \beta > 0$ such that

$$\alpha\|h\|^{-1} \leq (\max_{\{J \subset I,\ 1 \leq j \leq m\}} \frac{|M_J(i)|}{|M_I|}) \leq \beta\|h\|^{-1}$$

which proves Proposition 2.4.

We end this section by giving another equivalent expression for ν (which will be useful in the proof of Theorem 4.1).

Definition 2.4. Let us take notation as in Definition 2.3. Put

$$q(A, H) = \frac{\max_I |M_I|}{\mathrm{max_{I,J\subset I,j}|M_J(j)|}},$$

(where we consider only numbers with $M_J(j) \neq 0$, if all numbers $M_J(j)$ are zero, we put $q(A) = 0$).

We have the following:

Proposition 2.5. *The function $q(A, H)$ is equivalent to $\nu(A, H)$.*

We leave the proof as an exercise (for details of the proof see [9], Corollary 2.2):

3. Rabier's Fibration Theorem and Sard Theorem for $K(f)$

Definition 3.1. Let $k = \mathbb{C}$ or $k = \mathbb{R}$ and let X be a smooth affine variety over k. Let $f : X \to k^m$ be a $k-$smooth mapping. Recall that we define the set of generalized critical values as:

$$K_\infty(f) = \{y \in k^m :\ \exists_{x_l \in X,\ |x_l| \to \infty}\ s.t.\ f(x_l) \to y \text{ and } |x_l|\nu df(x_l)) \to 0\}.$$

Remark 3.1. Note, that by virtue of results of section 2 we can replace the function ν by arbitrary function among κ, g, g'.

We have also an easy but important observation:

Remark 3.2. *Recall that we define by $K_0(f)$ the set of critical values of f. Let $k = \mathbb{C}$ or $k = \mathbb{R}$ and let X be a smooth affine variety over k. Let $f : X \to k^m$ be a $k-$smooth mapping. Then the set $K(f) = K_0(f) \cup K_\infty(f)$ is closed.*

Now we state the basic theorem which follows from the main result of Rabier [15]:

Theorem 3.1. *Let $k = \mathbb{C}$ or $k = \mathbb{R}$ and let X be a smooth affine variety over k of dimension $n - r \geq m$. Let $f : X \to k^m$ be a $k-$smooth mapping. Then*

$$f : X \setminus f^{-1}(K(f)) \to k^m \setminus K(f)$$

is a locally trivial fibration.

Recall that a value y of the map f is called *typical* if f is a C^∞ fibration over a neighborhood of y and *atypical* otherwise. Note that a typical value is not necessarily a value of f ! The set $B(f)$ is called the *bifurcation set* of f. Rabier's theorem [15] states that

$$B(f) \subset K(f) = K_0(f) \cup K_\infty(f).$$

A short, direct proof of the finite dimensional case of the Rabier Theorem (which is this what we actually need) is contained also in Jelonek note [9].

It is crucial to know that the set $K(f)$ is small, in particular that this set is nowhere dense. In [12] it was proven that for a polynomial mapping $f : k^n \to k^m$ the set $K(f)$ is semialgebraic of dimension less than m in the case $k = \mathbb{R}$, and algebraic of complex dimension less than m in the case $k = \mathbb{C}$. We give below a proof of this fact in the case of mapping $f : X \to k^m$, where X is smooth algebraic. We follow the idea from [12] simplifying it at some points.

Before we state the main result of this section we need some additional results. We need the fact, due to K. Kurdyka ([11]), that any semialgebraic set $A \subset \mathbb{R}^n$ is a finite union $A = \cup_i L^i$, where each L^i has the *Whitney property with constant* M: any two points $x, y \in L^i$ can be joined in L^i by a piecewise smooth arc of length $\leq M|x - y|$. What we need actually is a uniform version of the above decomposition, for families parameterized by finite dimensional spaces : if $B \subset \mathbb{R}^n \times \mathbb{R}^p$ and $t \in \mathbb{R}^p$, we write $B_t = \{x \in \mathbb{R}^n : (x, t) \in B\}$. Then from the method of [11], we obtain the following theorem

Theorem 3.2. *There exists $M = M(n) > 0$ such that any semialgebraic set $A \subset \mathbb{R}^n \times \mathbb{R}^p$ can be decomposed into a finite (and disjoint) union $A = \cup_{i\in I} L^i$, such that for each $t \in \mathbb{R}^p$, every set L^i_t has the Whitney property with constant M. So, in particular $A_t = \cup_{i\in I} L^i_t$ for each $t \in \mathbb{R}^p$. (Clearly, for a fixed $t \in \mathbb{R}^p$ some of L^i_t may be empty.)*

Further we shall use the following version of the curve selection lemma for semialgebraic sets (it can be easily obtained using a semialgebraic compactification of $\mathbb{R}^n$ and the classical curve selection lemma, see [2], [1]).

Lemma 3.1. *[Curve selection at infinity] Let $A \subset \mathbb{R}^n$ and let $\phi : A \to \mathbb{R}^q$ be a semialgebraic map. Assume that there exists a sequence $x_l \in A$ such that $|x_l| \to \infty$ and $\phi(x_l) \to y$, for some $y \in \mathbb{R}^q$. Then there exists a semialgebraic arc $\gamma : [\alpha, \beta) \to \mathbb{R}^n$ such that $\gamma(t) \in A$, $\lim_{t\to\beta} |\gamma(t)| = +\infty$ and $\lim_{t\to\beta} \phi(\gamma(t)) = y$.*

Now consider such a semialgebraic arc $\gamma : (\alpha, \beta) \to \mathbb{R}^n$. Since $|\gamma'(t)| > 0$ for t close to β, we may reparametrize γ in such a way that $\beta = +\infty$ and $|\gamma(r)| = r$. Under this assumption we have:

Lemma 3.2. $\lim_{r\to\infty} |\gamma'(r)| = 1$*; in particular, $\gamma'(r)$ is bounded for $r > 0$ large enough.*

Proof. Since γ is semialgebraic, $\lim_{r\to\infty} \dfrac{\gamma(r)}{|\gamma(r)|}$ and $\lim_{r\to\infty} \dfrac{\gamma'(r)}{|\gamma'(r)|}$ exist. Hence, it is easily seen that these limits are equal. In other words $\cos\alpha(r) \to 1$, as $r \to \infty$, where $\alpha(r)$ is the angle between $\dfrac{\gamma(r)}{|\gamma(r)|} = \dfrac{\gamma(r)}{|r|}$ and $\dfrac{\gamma'(r)}{|\gamma'(r)|}$. Differentiation of $|\gamma(r)|^2 = r^2$ yields $|\gamma'(r)| = \dfrac{1}{\cos\alpha(r)}$. This implies the lemma. □

In order to prove our Sard theorem, we also shall use the fact that for a fixed mapping f, the convergence of $\nu(df(x_l))$ in the definition of $K_\infty(f)$ is actually faster than $|x_l|^{-1}$.

To make this precise, for a differentiable function $f:\mathbb{R}^n\to\mathbb{R}^m$ and any $N\in\mathbb{N}^*$ we define

$$K^N_\infty(f)=\{y\in\mathbb{R}^k : \exists x_l\in\mathbb{R}^n, |x_l|\to\infty \text{ s.t. } f(x_l)\to y \text{ and } |x_l|^{1+\frac{1}{N}}\nu(df(x_l))\to 0\}.$$

We have

Lemma 3.3. *Let $f:X\to\mathbb{R}^m$ be a differentiable semialgebraic function. Then*

$$K_\infty(f)=\bigcup_{N=1}^{\infty}K^N_\infty(f).$$

Proof. Let $y\in K_\infty(f)$. By Lemma 3.1 there is a semialgebraic arc $\gamma:[\alpha,\beta)\to X$, such that $\lim_{t\to\beta}|\gamma(t)|=+\infty$, $\lim_{t\to\beta}|\gamma(t)|\nu(df(\gamma(t))=0$ and $\lim_{t\to\beta}\phi(\gamma(t))=y$. Let us consider semialgebraic functions $A(t)=|\gamma(t)|\nu(df(\gamma(t))$ and $B(t)=1/|\gamma(t)|$ defined on $[\alpha,\beta]$. We can assume that $zeros(B)=zeros(A)=\{\beta\}$, hence by the Łojasiewicz inequality (see [1], 2.3.11, p. 63), there is a constant $c>0$ and the integer $n>0$ such that $A\le cB^{1/n}$. Consequently, for $N>n$ we have $\lim_{t\to\beta}|\gamma(t)|^{1+1/N}\nu(df(\gamma(t))=0$. Thus $y\in K^N_\infty(f)$. □

The aim of this section is to prove the following:

Theorem 3.3. *Let $X\subset k^n$ be an affine variety of dimension $n-r$. Let $f:X\to k^m$ be a polynomial map. Then $K(f)$ is a closed semialgebraic set of a Lebesgue measure* 0. *In particular it is of dimension less than m.*

Proof. We can assume that $k=\mathbb{R}$. Clearly $K_\infty(f)$ is a semialgebraic subset of $\mathbb{R}^m$. By Lemma 3.3 and basic properties of the Lebesgue measure it is enough to prove that $K^N_\infty(f)$ is of measure 0 for any integer $N>0$.

Let $I(X)$ denote the ideal of functions vanishing on X, assume that $I(X)$ is generated by polynomials $b_1,\dots,b_w$. Since the variety X can be covered by at most $p=\binom{w}{r}$ Zariski-open subsets in which it is a complete intersection, we can assume that X is a complete intersection, that is $w=r$. So in particular, if $x\in X$, then $T_xX=\bigcap_{j=1}^r\{d_xb_j=0\}$. Let us write $f=(f_1,\dots,f_m)$ for components of f, where $f_i:\mathbb{R}^n\to\mathbb{R}$ is a polynomial.

By Proposition 2.2, we may replace the distance ν of Rabier by the distance $\kappa(d_xf,T_xX)$ of Kuo. For each $i\in\{1,\dots,m\}$, we define

$$D_i=\{x\in X:\kappa(df(x),T_X)=\operatorname{dist}(\nabla f_i(x),V_i(x))\}$$

where $V_i(x)$ is the vector space generated by $\nabla f_j(x)$, $j=1,\dots,m$ $j\ne i$ and by $\nabla b_j(x)$, $j=1,\dots,r$. Clearly each D_i is semialgebraic in $\mathbb{R}^n$ and $X=\bigcup_{i=1}^m D_i$, so

$$K_\infty(f)=\bigcup_{i=1}^m K_\infty(f_{|D_i})$$

where

$$K_\infty(f_{|D_i})=\{y\in\mathbb{R}^m:\exists x_l\in D_i, |x_l|^{1+1/N}\to\infty, f(x_l)\to y \text{ and } |x_l|\nu(df(x_l))\to 0\}.$$

We shall prove the following

Lemma 3.4. *$vol_m(K_\infty(f_{|D_i}))=0$ for each $i\in\{1,\dots,k\}$. In particular* $\dim K_\infty(f)<m$.

Proof. We will give the proof for $i=1$, we write $D=D_1$, $\overline{f}=(f_2,\dots,f_m)$.

Let us fix B an open ball in $\mathbb{R}^{m-1}$ and (α, β) an open bounded interval in $\mathbb{R}$. The lemma is clearly a consequence of

$$vol_m(K_\infty(f_{|D}) \cap (\alpha, \beta) \times B) = 0. \tag{3.1}$$

In order to prove equality 3.1, we construct a family of sets Δ_r such that

$$\overline{\Delta_r} \supset K_\infty(f_{|D}) \cap (\alpha, \beta) \times B \text{ and } vol_m(\overline{\Delta_r}) \to 0 \text{ as } r \to \infty.$$

We first define

$$\widetilde{\Sigma}_r = \{x \in D : |x| \geq r, f_1(x) \in (\alpha, \beta), \overline{f}(x) \in B \text{ and } |x|^{1+\frac{1}{N}}\kappa(df(x)) \leq 1\}$$

where $r > 0$, and put

$$\Delta_r = f(\widetilde{\Sigma}_r) \text{ and finally } \Delta = \bigcap_{r>0} \overline{\Delta}_r.$$

Every Δ_r is semialgebraic, hence we have $vol_m(\overline{\Delta}_r) = vol_m(\Delta_r)$ and consequently

$$vol_m(\Delta) = \lim_{r\to\infty} vol_m(\Delta_r)$$

since the family $(\Delta_r)_{r>0}$ is decreasing with respect to $r \to \infty$.

It is clear that

$$K_\infty(f_{|D}) \cap (\alpha, \beta) \times B \subset \Delta,$$

so it is enough to prove that $vol_m(\Delta) = 0$. First, using Fubini's theorem we write

$$vol_m(\Delta_r) = \int_B \mu_r(b)db$$

where db stands for the Lebesgue measure on $\mathbb{R}^{m-1}$, and

$$\mu_r(b) = vol_1(\{y_1 \in \mathbb{R} : (y_1, b) \in \Delta_r\}).$$

Clearly, each μ_r is measurable. Moreover, for fixed $b \in B$, the function $r \mapsto \mu_r(b) \geq 0$ is decreasing. Let

$$\mu(b) = \lim_{r\to\infty} \mu_r(b).$$

By Lebesgue's theorem on bounded convergence, we obtain

$$vol_m(\Delta) = \int_B \mu(b)db.$$

Now the final point in the proof of equality (3.1) is the fact that $m \equiv 0$, which follows from the next lemma. □

Lemma 3.5. *There exists a constant $c > 0$ such that, for r large enough*

$$\mu_r(b) \leq cr^{-\frac{1}{N}}.$$

Proof. To prove Lemma 3.5, we introduce the semialgebraic family

$$\Sigma_{r,b} = \widetilde{\Sigma}_r \cap \overline{f}^{-1}(b) \cap S(r),$$

where $b \in B$, $r > 0$, and next we write

$$\widetilde{\Sigma}_{r,b} = \widetilde{\Sigma}_r \cap \overline{f}^{-1}(b) = \bigcup_{s\geq r} \Sigma_{s,b}.$$

Note that

$$\mu_r(b) = vol_1(f_1(\widetilde{\Sigma}_{r,b})).$$

It follows from Theorem 3.2 that there exists a finite family $L^i \subset X \times \mathbb{R} \times \mathbb{R}^{m-1}, i \in I$ of semialgebraic sets such that

$$\Sigma_{r,b} = \bigcup_{i\in I} L^i_{r,b}.$$

Each $L^i_{r,b}$ has the Whitney property with constant M (some of $L^i_{r,b}$ may be empty).

Recall that the condition $|x|^{1+\frac{1}{N}}\kappa(df(x)) \le 1$ for $x \in \overline{f}^{-1}(b) = W_b$ means that

$$|\nabla f_{1|W_b}(x)| \le |x|^{-(1+\frac{1}{N})} \tag{3.2}$$

Hence, by the mean value theorem $f_1(L^i_{r,b})$ is a segment of length $d(r)$ where

$$d(r) \le 2Mr \sup_{L^i_{r,b}} |\nabla f_{1|W_b}| \le 2Mr^{-\frac{1}{N}}. \tag{3.3}$$

Fix $b \in B$, $i \in I$ and assume that $L^i_{r,b} \neq \emptyset$ for any r large enough. Applying the curve selection lemma at infinity, we obtain a semialgebraic arc $\gamma : [r, +\infty) \to \mathbb{R}^n$ such that $\gamma(\zeta) \in L^i_{\zeta,b}$. In particular, $\gamma(\zeta) \in \overline{f}^{-1}(b) = W_b$ and $|\gamma(\zeta)| = \zeta$.

By Lemma 3.2, we may suppose that $|\gamma'(\zeta)| \le 2$. So we can easily compute length of $f_1 \circ \gamma([r, +\infty))$; namely, by (3.2) we have

$$\int_r^{+\infty} |(f_1 \circ \gamma)'(\zeta)| d\zeta \le 2 \int_r^{+\infty} \zeta^{-(1+\frac{1}{N})} d\zeta = 2Nr^{-\frac{1}{N}}. \tag{3.4}$$

Thus, by (3.3) and (3.4), $f_1(\bigcup_{\zeta \ge r} L^i_{\zeta,b})$ is contained in a segment of length

$$(4M + 2N)r^{-\frac{1}{N}}.$$

Therefore $f_1(\widetilde{\Sigma}_{r,b})$ is contained in $\#I$ segments of this length. Put $c = (\#I)(4M + 2N)$; we have

$$\mu_r(b) \le cr^{-\frac{1}{N}}$$

and Lemma 3.5 follows. □

Finally from Lemma 3.4 and the usual semialgebraic Sard's theorem (see [1]) it follows that $\dim K(f) < m$. □

4. Estimation of the degree

In the proof of our next theorem we need following technical lemmas.

Lemma 4.1. *Let A be algebraic subsets of $\mathbb{C}^N$, $\dim A = n$. Let L, M be linear subspaces of $\mathbb{C}^N$, and $L \subset M$. Let $\dim M = n + 1$. Assume that $L \not\subset A$. Then there exists a linear projection $p : \mathbb{C}^N \to M$, such that p restricted to A is finite and $L \not\subset p(A)$.*

Proof. Take a point $a \in L \setminus A$. Let Λ be the Zariski closure of the cone $\bigcup \overline{ax},\ x \in A$. It is easy to see that $\dim \Lambda \le n + 1$. Let H_∞ be the hyperplane at infinity of $\mathbb{C} \times \mathbb{C}^N$. For any $Z \subset \mathbb{C} \times \mathbb{C}^N$ denote by $\tilde{Z}$ the projective closure of Z. Observe that

$$\dim H_\infty \cap (\tilde{\Lambda} \cup \tilde{\Gamma} \cap \tilde{M}) \le n.$$

Thus, there is a projective subspace $Q \subset H_\infty$ of dimension $N - n - 2$, which is disjoint with $(\tilde{\Lambda} \cup \tilde{A} \cap \tilde{M})$. Denote by $p_Q : \mathbb{P}^N \setminus Q \to \tilde{M}$ the linear projection determined by the subspace Q.

Now, let $p : A \to M$ be the restriction of p_Q to A. It is easily seen that p has desired properties, i.e., $p : A \to M$ is a finite mapping and $a \notin L \cap p(A)$. □

Lemma 4.2. *Let $F = (f_1, ..., f_n, f_{n+1}) : X - \to \mathbb{C}^{n+1}$ be a rational mapping. Assume that $\Gamma := cl(F(\mathbb{C}^n))$ is a hypersurface. Let $f_i = P_i/G$, where* $\deg P_i = d_i$ *and* $\deg G = d_0$. *Take $d = \max_{0 \le i \le n+1}\{d_i\}$. Then*

$$\deg \Gamma \le d^n \deg X/\mu(F),$$

where $\mu(F) = (\mathbb{C}(X) : \mathbb{C}(f_1, ..., f_{n+1}))$.

Proof. We can estimate the degree of Γ using the Bezout Theorem. Indeed, the number $(\deg \Gamma)\mu(F)$ is estimated by the number of solution of a generic system of equations:

$$\sum_{j=1}^{n+1} a_{ij} f_j(x) = c_i, \quad i = 1, ..., n, \ x \in X.$$

This system is equivalent to the system

$$\sum_{j=1}^{n+1} a_{ij} P_j(x) = c_i G(x), \quad i = 1, ..., n, \ x \in X,$$

and by the Bezout Theorem we have $\mu(F) \deg \Gamma \le d^n \deg X$. □

Theorem 4.1. *Let $X \subset \mathbb{C}^n$ be a smooth affine variety of dimension $n - r \ge m$. Assume that $I(X) = \{b_1, ..., b_w\}$, where deg $b_i \le D$. Let $f = (f_1, ..., f_m) : X \to \mathbb{C}^m$ be a polynomial dominant mapping and let deg $f_i \le d$. Then the set $K(f)$ is a proper algebraic subset of $\mathbb{C}^m$ and it is contained in a hypersurface of degree at most*

$$(d + (m-1)(d-1) + (D-1)r)^{n-r} \deg X \le (d + (m-1)(d-1) + (D-1)r)^{n-r} D^r.$$

Proof. Since X is smooth it means that there exists an open dense subset $U \subset X$, on which X is a complete intersection. As it will follows from the rest of the proof, we can assume, without loss of generality that $X = U$, i.e., that X is a complete intersection.

Let us recall notation of Definition 2.3. For $x \in \mathbb{C}$ let $A = d_x f$, and $B_l = d_x b_l$, $l = 1, \dots, r$. Let $A \in \mathcal{L}(k^n, k^m)$, where $n \ge m + r$, and let $T_x X = H \subset k^n$ be a linear subspace given by a system of independent linear equation $B_l = \sum b_{lk} x_k$, $l = 1, ..., r$. By abuse of notation we denote by A the matrix (in the canonical bases in k^n and k^m) of the mapping A. Let $C =$ be a $(m + r) \times n$ matrix given by rows $A_1, ..., A_m; B_1, ..., B_r$ (we identify $A_i = \sum a_{ik} x_k$ with the vector $(a_{j1}, ..., a_{jn})$, similarly for B_l).

For an index $I = (i_1, ..., i_{m+r}) \subset \{1, ..., n\}$ let $M_I(x)$ denote the $((m + r) \times (m + r))$ minor of C given by columns indexed by I. For integers $j \in I, 1 \le k \le m$ we denote by $M_{I(k,j)}(x)$ the $(m + r - 1) \times (m + r - 1)$ minor obtained by deleting jth column and kth row. Note that we delete only A_k, $1 \le k \le m$ rows !

Hence M_I and $M_{I(k,j)}$ are regular (restriction of polynomials) functions on X. We define now a family of rational functions on X:

$$W_{I(k,j)}(x) = M_I(x)/M_{I(k,j)}(x)$$

where for $M_{I(k,j)} \equiv 0$, we put $W_{I(k,j)} \equiv 0$. We write $b = (b_1, \dots, b_r)$ and $(f, b) : \mathbb{C}^n \to \mathbb{C}^m \times \mathbb{C}^r$, here we consider $f_1, \dots, f_m$, and $b_1, \dots, b_r$ as polynomials on $\mathbb{C}^n$.

Let $s = \binom{n}{m+r}$ and let $M_{I_1}, ..., M_{I_s}$ be all possible main minors of a matrix of $d_x(f, b)$. For every index I_l take a pair (k_l, j_l) which determine a $(m + r - 1) \times (m + r - 1)$ minor of M_{I_l}. We denote a sequence $(k_1, j_1), ..., (k_s, j_s)$ by $(k, j) \in \mathbb{N}^s \times \mathbb{N}^s$ and we consider a rational function:

$$\Phi_{(k,j)} = \Phi((k_1, j_1), ..., (k_s, j_s)) : X \to \mathbb{C}^m \times \mathbb{C}^N$$

where the first component of $\Phi_{(k,j)}$ is f and next components are $W_{I_i(k_i,j_i)}$, $i = 1, \dots, s$ and all products $x_l W_{I_i(k_i,j_i)}$, $i = 1, \dots, s$; $l = 1, \dots, m$.

We can assume that for some choice of l we have $W_{I_l(k_l,j_l)} \not\equiv 0$, and consequently dim $cl(\Phi_{(k,j)}(X)) = \dim X = n - r$. Here $cl(Y)$ stands for the closure of Y in the strong (or which is the same, in the Zariski topology). Let $\Gamma(k,j) = cl(\Phi_{(k,j)}(X))$.

Let us recall that $y \in K_\infty(f)$ if there exists a sequence $x \to \infty$ such that

$$f(x) \to y \textit{ and } |x| g'(x) \to 0,$$

were $g'(x) = g'(d_x f, T_x X)$ We have

Lemma 4.3.

$$K(f) = K_0(f) \cup K_\infty(f) = \mathbb{C}^m \cap \bigcup_{(k,j)} \Gamma(k,j),$$

where we identify $\mathbb{C}^m$ *with* $\mathbb{C}^m \times (0, ..., 0)$.

Proof. Indeed, if $y \in K_0(f)$ then there is a critical point $x_0 \in X$, such that $y = f(x_0)$. Since $d_{x_0} f$ is singular on $T_{x_0} X$, we have $g'(x_0) = 0$ consequently for any sequence $x \to x_0$, we have $g'(x) \to 0$. In particular for every index I_i, there are integers (k_i, j_i), such that $M_{I_i}/M_{I_r(k_i,j_i)}(x) \to 0$. This means that $y \in \Gamma(k,j) \cap \mathbb{C}^m$ where $(k,j) = ((k_1,j_1), ..., (k_s,j_s))$.

Similarly, if $y \in K_\infty(f)$, then there is a sequence $x \to \infty$, such that for every I_i there are integers (k_i, j_i), such that $|x| M_{I_i}/M_{I_r(k_i,j_i)}(x) \to 0$. This also gives $y \in \Gamma(k,j) \cap \mathbb{C}^m$ with $(k,j) = ((k_1,j_1), ..., (k_s,j_s))$.

Conversely, if $y \in \Gamma(k,j) \cap \mathbb{C}^m$, then we can choose a sequence $x \to a$, where $a \in \mathbb{C}^n$ or $a = \infty$, such that $M_{I_r}/M_{I_r(k_r,j_r)}(x) \to 0$ and $|x| M_{I_r}/M_{I_r(k_r,j_r)}(x_n) \to 0$. If $a \in \mathbb{C}^n$, this implies that all $M_I(a) = 0$, i.e. a is a critical point of f, hence $y = f(a) \in K_0(f)$. Otherwise we have that $|x||g'(x) \to 0$ and $f(x) \to y$, i.e. $y \in K_\infty(f)$. By Theorem 3.3 we have that $K(f) \neq \mathbb{C}^m$ hence $\mathbb{C}^m \cap \bigcup_{(k,j)} \Gamma(k,j) \neq \mathbb{C}^m$. □

In particular we have proved that the set $K(f)$ is algebraic. For an index $J = (i_1, ..., i_m) \subset \{1, ..., n\}$ and numbers $k \in I$ $j \in \{1, ..., m\}$ let $\alpha_{J(k,j)}$ denote a complex number. For every index I take

$$W_I(x) = M_I / (\sum_{J,k,j} \alpha_{J(k,j)} M_{J(k,j)}),$$

where $M_{J(k,j)}$ denotes the $(m-1) \times (m-1)$ minor which is created from M_J by deleting the jth row and kth column (recall that we delete only A_j rows!).

Now consider the rational mapping $\Phi : X \ni x - \to (f(x), W_{I_1}(x), x_1 W_{I_1}(x), ..., x_n W_{I_1}(x), ..., W_{I_s}(x), x_1 W_{I_s}(x), ..., x_n W_{I_s}(x)) \in \mathbb{C}^m \times \mathbb{C}^N$. Let $\Gamma = \overline{\Phi(X)}$.

Lemma 4.4. *For sufficiently general numbers* $\alpha_{J(k,j)}$ *we have*

$$K(f) = K_0(f) \cup K_\infty(f) = \Gamma \cap \mathbb{C}^m,$$

where we identify $\mathbb{C}^m$ *with* $\mathbb{C}^m \times (0, ..., 0)$.

Proof. Let us take a dense countable subset $E = \{y_1, y_2, y_3, ...\}$ of $K(f)$. By our previous consideration and by Proposition 2.5 for every y_k there is a sequence x_{kj}; $j = 1, 2, ...,$ where $x_{kj} \in X$ and $x_{kj} \to a_k$, where $a_k \in X$ or $a_k = \infty$, such that

a) $f(x_{kj}) \to y_k$,

b) if $|M_{J_{kj}(a_{kj},b_{kj})}(x_{kj})|$ denotes appropriate maximal minors, then for every index I we have: $|M_I(x_{kj})|/|M_{J_{kj}(a_{kj},b_{kj})}(x_{kj})| \to 0$ and $|x_{kj}||M_I(x_{kj})|/|M_{J_{kj}(a_{kj},b_{kj})}(x_{kj})| \to 0$ (the first limit is important for $a_k \in \mathbb{C}^n$ only).

Let us fix k. We assume that $a_k = \infty$, the other case can be done similarly. By the Dirichlet box principle we can assume that there exist J, a, b such that $|M_{J(a,b)}|(x_{kj})|$ are maximal among others minors of this type (if it is necessary we can pass to a subsequence of the sequence $\{x_{kj};\ j = 1, 2, ...\}$).

In particular we can assume that all ratios $|M_{J'(c,d)}(x_{kj})|/|M_{J(a,b)}(x_{kj})|$ are (defined and) bounded by 1. Thus (again after passing to a subsequence) we can assume that all this limits exists. Let $\gamma(k, J', c, d) = lim_{j\to\infty} M_{J'(c,d)}(x_{kj})/M_{J(a,b)}(x_{kj})$.

Since a countable family of hyperplanes can not fill the whole of affine space, we can find numbers $A_{J',c,d}$ such that for all $k = 1, 2, ...$ we have:

$$\sum_{J',c,d} A_{J',c,d}\gamma(k, J', c, d) = \epsilon_k \neq 0.$$

Now take $\alpha_{I(c,d)} = A_{I,c,d}$ as our general coefficients. Note that $E \subset \Gamma$. Indeed for big j we have:

$$|\sum \alpha_{I(c,d)} M_{I(c,d)}| \geq |M_{J(a,b)}(x_{kj})||\epsilon_k/2|$$

and consequently for every index I

$$|x_{kj}||M_I(x_{kj})|/(|\sum \alpha_{J'(k,j)} M_{J'(k,j)}(x_{kj})|) \leq |2/\epsilon_k||x_{kj}||M_I(x_{kj})|/|M_{J(a,b)}(x_{kj}| \to 0.$$

Thus we have $lim\ \Phi(x_{kj}) = y_k$, hence $E \subset \Gamma \cap \mathbb{C}^m$. Since $\Gamma \cap \mathbb{C}^m$ is a closed set, we obtain $K(f) \subset \Gamma \cap \mathbb{C}^m$.

Now take $y \in \Gamma \cap \mathbb{C}^m$. this means that there is a sequence $x_j \to a$, where $a \in C^n$ or $a = \infty$, such that $y = \lim f(x_j)$ and

$$|x_j||M_I(x_j)|/|\sum \alpha_{J'(k,j)} M_{J'(k,j)}(x_j)| \to 0.$$

As before we can assume that $a = \infty$ and that the minor $|M_J(a, b)(x_j)|$ is maximal (and non-zero) for every j and that all limits

$$\gamma(J', c, d) = lim_{j\to\infty} M_{J'(c,d)}(x_j)/M_{J(a,b)}(x_j)$$

exist. Let

$$\epsilon = \sum_{J',c,d} A_{J',c,d}\gamma(J', c, d).$$

If $\epsilon \neq 0$, then

$$|x_j||M_I(x_j)|/|\sum \alpha_{J'(k,j)} M_{J'(k,j)}(x_j)| \geq |1/(2\epsilon)||x_j||M_I(x_j)|/|M_{J(a,b)}(x_j)|$$

and since the first term tends to zero, we have also $|x_j||M_I(x_j)|/|M_{J(a,b)}(x_j)| \to 0$, what means that $y \in K(f)$. If $\epsilon = 0$ we can modify sequence x_j in this way that $\epsilon_j := \sum_{J',c,d} A_{J',c,d} M(J', c, d)/M(J, a, b) \neq 0$ (note that by the construction the function $\sum_{J',c,d} A_{J',c,d} M(J', c, d)(x) \not\equiv 0$). Of course $\lim \epsilon_j = \epsilon = 0$. We have

$$|\epsilon_j||x_j||M_I(x_j)|/|\sum \alpha_{J'(k,j)} M_{J'(k,j)}(x_j)| \geq |x_j||M_I(x_j)|/|M_{J(a,b)}(x_j)|$$

and again we have $|x_j||M_I(x_j)|/|M_{J(a,b)}(x_j)| \to 0$. Hence $y \in K(f)$ and $\Gamma \cap C^m \subset K(f)$. Finally we have $K(f) = \Gamma \cap \mathbb{C}^m$. □

Proof of the Theorem 4.1 Thus it is enough to estimate a degree of a hypersurface in which is contained the set $\Gamma \cap \mathbb{C}^m$. Let $M \subset \mathbb{C}^m \times \mathbb{C}^N$ be a linear subspace of dimension $n-r+1$, which contains the subspace $L = \mathbb{C}^m = \mathbb{C}^m \times (0, ..., 0)$. By Lemma 4.1 there is a projection $p : \Gamma \to M$, such that $p(\Gamma) \cap \mathbb{C}^m \neq \mathbb{C}^m$. Hence it is enough to estimate the degree of a hypersurface $p(\Gamma) \subset M$. To do this we use Lemma 4.2. In fact we have to estimate the degree of the image of the rational function $p \circ \Phi : X - \to M \cong \mathbb{C}^{n-r+1}$. In particular $p \circ \Phi = (P_1/Q, ..., P_{n-r+1}/Q)$, where

$$\deg P_i \leq d + ((m-1)(d-1) + r(D-1))$$

and

$$\deg Q \leq (m-1)(d-1) + r(D-1).$$

Finally by Lemma 4.2 we have

$$\deg p(\Gamma) \leq (d + (m-1)(d-1) + r(D-1))^{n-r} \mathrm{degX},$$

which proves the first estimate in the Theorem 4.1. To get the second one it is enough to apply the Bezout theorem. □

We deduce immediately an analogous statement in the real case:

Corollary 4.1. *Let $X \subset \mathbb{R}^n$ be a smooth affine variety of dimension $n - r \geq m$. Assume that $I(X) = \{b_1, ..., b_w\}$, where deg $b_i \leq D$. Let $f = (f_1, ..., f_m) : X \to \mathbb{R}^m$ be a polynomial dominant mapping and let deg $f_i \leq d$. The set $K(f)$ is a closed semi-algebraic subset of $\mathbb{R}^m$ and it is contained in a real hypersurface of degree at most $(d + (m-1)(d-1) + (D-1)r)^{n-r} D^r$.*

In a particularly interesting case where f is a polynomial function on hypersurface we have:

Corollary 4.2. *If X is a smooth hypersurface of degree D in $\mathbb{C}^n$ and $f : X \to \mathbb{C}$ is a polynomial of degree d, then the set $K(f)$ has at most $(d+D-1)^{n-1}D$ points. In particular f may have at most $(d + D - 1)^{n-1}D$ atypical fibers.*

The following example shows that the estimate in Corollary 4.2 is asymptotically sharp.

Example 4.1. Let $X \subset \mathbb{P}^n$ be a smooth projective hypersurface of degree D. It is well-known that the degree of the dual hypersurface to X is $D(D-1)^{n-1}$ - see e.g., [3]. This means in particular that there is a projective subspace $W \subset \mathbb{P}^n$ of codimension 2, such that there is exactly $D(D-1)^{n-1}$ tangent hyperplanes to X which contain W. Let us take an affine system of coordinates in $\mathbb{P}^n$ in this way that W is contained in the hyperplane at infinity, and all these tangent spaces which contain W are different from the hyperplane at infinity. Let f be a homogeneous linear function which describe a subspace W. Then f considered as polynomial on the affine part of X has at least $D(D-1)^{n-1}$ generalized critical values, because it has at least $D(D-1)^{n-1}$ singular fibers. This means that our estimation is nearly sharp.

Clearly Corollary 4.1 yields in the real case:

Corollary 4.3. *If X is a smooth hypersurface of degree D in $\mathbb{R}^n$ and $f : X \to \mathbb{R}$ is a polynomial of degree d, then the set $K(f)$ has at most $(d + D - 1)^{n-1}D$ points. In particular f has at most $(d + D - 1)^{n-1}D$ non-generic fibers and consequently there is at most $(d + D - 1)^{n-1}D + 1$ types of generic fibers of f.*

Let us note that from the proof of Theorem 4.1 follows, that to give an estimation of degree of $K(f)$ it is not necesarry to know the system of generators of $I(X)$ (what in general is difficult). It is enough if we will find any polynomials $\{h_1, ..., h_r\} \subset I(X)$ ($r =$codim X) such that $Jac(h_1, ..., h_r)$ does not vanish identically on X and then we can put $D = \max \{ \deg h_i\}$. Below we show that in this way we can always take $D = \deg X$.

Theorem 4.2. *Let $X \subset \mathbb{C}^n$ be a smooth affine variety of dimension $n - r \geq m$. Let $f = (f_1, ..., f_m) : X \to \mathbb{C}^m$ be a polynomial dominant mapping and let deg $f_i \leq d$. Assume that* $\deg X = D$. *Then the set $K(f)$ is a proper algebraic subset of $\mathbb{C}^m$ and it is contained in a hypersurface of degree at most*

$$(d + (m - 1)(d - 1) + (D - 1)r)^{n-r}D.$$

Proof. It is enough to construct polynomials $h_1, ..., h_r$ of degree $D = \deg X$, which vanish on X and for which $Jac(h_1, ..., h_r)$ does not vanish identically on X.

Let us take a point $x \in X$. Let S be the closure of the union of all secants xy, where $y \in X$ is another point of X. It is easy to see that dim $S \leq n - r + 1$ and the projective closure $\overline{S}$ of S contains the projective closure of X.

Now on the hyperplane at infinity H_∞ let us choose a system of homogeneous coordinates $x_1, ..., x_{n-r}, x_{n-r+1}, ..., x_n$ in this way that for every $j > n - r$ we have $\{x_1 = 0, ..., x_{n-r} = 0, x_j = 0\} \cap \overline{S} = \emptyset$. Of course every sufficiently general system of coordinates has this property. The coordinate system on H_∞ we can extend in an obvious way to a coordinate system on the whole of $\mathbb{P}^n$ (by adding new variable x_0).

Now for every $j > n - r$ let us consider the projection $\pi_j : X \ni x \to (x_1, ..., x_{n-r}, x_j) \in \mathbb{C}^{n-r+1}$. By the construction the mapping π_j is proper and birational (the last property follows from the fact that $(\pi_j)^{-1}(\pi_j(x)) = \{x\}$ and that π_j is smooth at x). The image $X_j := \pi_j(X)$ is a hypersurface in $\mathbb{C}^{n-r+1}$. Let h_s be a reduced equation of X_{n-r+s}. Then h_s vanishes on X and $\frac{\partial h_s}{\partial x_{n-r+s}}$ does not vanish identically on X. Now it is easy to check that polynomials $h_1, ..., h_r$ (of degree $D = \deg X$) vanish on X and $Jac(h_1, ..., h_r)$ does not vanish identically on X. □

Corollary 4.4. *Let X be a smooth algebraic variety of dimension k and degree D in $\mathbb{C}^n$ and let $f : X \to \mathbb{C}$ be a polynomial of degree d. Then the set $K(f)$ has at most $(d + (D - 1)(n - k))^k D$ points. In particular f may have at most $(d + (D - 1)(n - k))^k D$ atypical fibers.*

Corollary 4.5. *Let X be a smooth algebraic variety of dimension k and degree D in $\mathbb{R}^n$ and let $f : X \to \mathbb{R}$ be a polynomial of degree d. Then the set $K(f)$ has at most $(d + (D - 1)(n - k))^k D$ points. In particular f may have at most $(d + (D - 1)(n - k))^k D$ atypical fibers and consequently there is at most $(d+(D-1)(n-k))^k D+1$ types of generic fibers of f.*

We can also apply our results to rational functions $f : \mathbb{C}^n \to \mathbb{C}^m$. For simplicity we formulate the result for a rational function $f : \mathbb{C}^n \to \mathbb{C}$ (we live to formulate all obvious definition to the reader). The general case can be done similarly.

Corollary 4.6. *If $f = P/Q : \mathbb{C}^n \to \mathbb{C}$ is a rational function of degree d (i.e.,* max $\deg P, \deg Q = d$*), then the set $K(f)$ has at most $(2d + 1)^n(d + 1)$ points. In particular f may have at most $(2d + 1)^n(d + 1)$ atypical fibers.*

Proof. Let us consider the hypersurface $X = \{(x, z) \in \mathbb{C}^n \times \mathbb{C} : zQ(x) = 1\}$. Thus the function f can be considered as polynomial function $F = zP(x)$ on the hypersurface X.

Since this hypersurface and the function F have degrees at most $d+1$, we have that the set $K(f)$ has at most $(2d+1)^n(d+1)$ points. □

Now let us consider the special case $n = m$. Let us recall that a mapping $f : X \to k^n$ is *not proper* at a point $y \in k^m$ if there is no neighborhood U of y such that $f^{-1}(\overline{U}))$ is compact. In other words, f is not proper at y if there is a sequence $x_l \to \infty$ such that $f(x_l) \to y$. Let S_f denote the set of points at which the mapping f is not proper. It is well-known (see [5], [6]) that in the case $(k = \mathbb{C})$ the set S_f is either an empty set or it is a hypersurface. We can repeat word by word the proof of Proposition 4.1 in [7] to obtain:

Proposition 4.1. *Let $f = (f_1, ..., f_n) : X \to \mathbb{C}^n$ be a dominant polynomial mapping. Then the set $K(f)$ of generalized critical values is either the empty set (and then f is an automorphism) or it is a hypersurface. Moreover, $K(f) = K_0(f) \cup S_f = B(f)$. In particular the bifurcation set $B(f)$ is either the empty set (and then f is an automorphism) or it is a hypersurface.*

In the real case $K(f)$ need not be a hypersurface, in particular $K_\infty(f)$ may be of codimension at least 2 (see [8]). However arguing as in the proof of Proposition 3.1 in [12] we can easily prove the following.

Proposition 4.2. *Assume that X is smooth and algebraic of dimension n and let $f = (f_1, ..., f_n) : X \to \mathbb{R}^n$ be a polynomial mapping (or more generally C^1 with semialgebraic graph). Then $K_\infty(f) = S_f$ and consequently $B(f) = K_0(f) \cup S_f$.*

5. Computations

In this section we use Gröbner basis to compute the set $K(f)$ effectively. Let us recall the definition of Gröbner basis. Assume that in the set of monomials in $\mathbb{C}[x_1, ..., x_n]$ we have the ordering induced by the lexicographic ordering in $\mathbb{N}^n$, i.e., $a_\alpha x^\alpha > a_\beta x^\beta$, if $\alpha > \beta$ (in this paper we consider only this ordering). By $inP = a_d x^d$ we will denote the initial form of a polynomial $P = \sum a_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathbb{C}[x_1, \ldots, x_n]$, where $d = max\{\alpha = (\alpha_1, \ldots, \alpha_n);\ a_d \neq 0\}$. We have the following basic definition (see [14]):

Definition 5.1. *A finite subset $\mathcal{B} \subset I \subset \mathbb{C}[x_1, ..., x_n]$ of an ideal I is called a Gröbner basis of this ideal, if the set $\{inP;\ P \in \mathcal{B}\}$ generates the ideal generated by all initial forms of the ideal I.*

The Gröbner basis of the ideal I is a basis of this ideal, moreover it can be easily computed by arithmetical operations only. We have the following basic fact (see [14]):

Theorem 5.1. *Consider the ring $\mathbb{C}[x_1, ..., x_n; y_1, ..., y_m]$. Let $V \subset \mathbb{C}^n \times \mathbb{C}^m$ be an algebraic set and let $p : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^m$ denote the projection. Assume that $\mathcal{B}$ is a Gröbner basis of the ideal $I(V)$. Then $\mathcal{B} \cap \mathbb{C}[y_1, ..., y_m]$ is a Gröbner basis of the ideal $I(p(V)) = I(cl(p(V)))$.*

Proof. Observe that $I(p(V)) = I(V) \cap \mathbb{C}[y_1, ..., y_m]$ and then to use [14], Proposition 4.3. □

Theorem 5.2. *Let X be a smooth affine variety of dimension $n - r$ and let $I(X) = \{b_1, \ldots, b_w\}$. Let $f = (f_1, ..., f_m) : X \to \mathbb{C}^m$ be a polynomial dominant mapping. Then the set $K(f)$ can be computed effectively.*

Proof. Let $I(X) = \{b_1, ..., b_w\}$. Let $b_1, ..., b_r \in I(X)$ be polynomials such that rank $\{grad\ b_1, ..., grad\ b_r\} = r$ on some non-empty open subset of X. Let us consider the rational mapping:

$$\Phi((k_1,j_1),...,(k_s,j_s)) : X \ni x - \to (f(x), W_{I_1(k_1,j_1)}(x), x_1 W_{I_1(k_1,j_1)}(x), ..., x_n W_{I_1(k_1,j_1)}(x),$$

$$..., W_{I_s(k_s,j_s)}(x), x_1 W_{I_s(k_s,j_s)}(x), ..., x_n W_{I_s(k_s,j_s)}(x)) \in \mathbb{C}^m \times \mathbb{C}^N,$$

which are constructed exactly as in the proof of Theorem 4.1. Recall that

$$\Gamma((k_1,j_1),...,(k_s,j_s)) = cl(\Phi((k_1,j_1),...,(k_s,j_s))(\mathbb{C}^n)).$$

We know also that

$$K(f) = (\bigcup_{((k_1,j_1),...,(k_s,j_s))} \Gamma((k_1,j_1),...,(k_s,j_s))\ ) \cap L,$$

where $L = \mathbb{C}^m \times (0,...,0)$. First we compute the ideal of the set $\Gamma((k_1,j_1),...,(k_s,j_s))$.

To do this we can restrict the mpping $\Phi(k,j)$ to any open dense subset U on which this mapping is regular. In particular we can choose the set $U = X \setminus \bigcup_{r=1}^s \{M_{I_r(k_r,j_r)} = 0\}$. The set U can be identified with the set

$$V((k_1,j_1),...,(k_s,j_s)) := \{(x,z_1,...,z_s) \in \mathbb{C}^n \times \mathbb{C}^s : b_j = 0, j = 1,...,w; M_{I_r(k_r,j_r)} z_r = 1;\ \ r = 1,...,s\}.$$

Now we can consider a morphism

$$\Psi((k_1,j_1),...,(k_s,j_s)) : V((k_1,j_1),...,(k_s,j_s)) \ni (x,z) \to (f(x), z_1 M_{I_1(k_1,j_1)}(x), x_1 z_1 M_{I_1(k_1,j_1)}(x),$$

$$..., x_n z_1 M_{I_1(k_1,j_1)}(x), ..., z_s M_{I_s(k_s,j_s)}(x), x_1 z_s M_{I_s(k_s,j_s)}(x), ..., x_n z_s M_{I_s(k_s,j_s)}(x)) \in \mathbb{C}^m \times \mathbb{C}^N.$$

Denote $\Psi((k_1,j_1),...,(k_s,j_s)) := (\psi_1(x,z),...,\psi_{m+N}(x,z))$. It is easy to see that

$$\Gamma((k_1,j_1),...,(k_s,j_s)) = cl(\Psi((k_1,j_1),...,(k_s,j_s))(V((k_1,j_1),...,(k_s,j_s)))).$$

Let $G((k_1,j_1),...,(k_s,j_s)) = graph(\Psi((k_1,j_1),...,(k_s,j_s)))$. The basis of the ideal I of the set $G((k_1,j_1),...,(k_s,j_s))$ in the ring $\mathbb{C}[x,z,y]$ is given by polynomials

$$\{z_r M_{I_r(k_r,j_r)}(x) - 1\}_{r=1,...,s} \cup \ \{y_i - \psi_i(x,z)\}_{i=1,...,m+N}.$$

Thus by Theorem 5.1 to compute a basis $\mathcal{B}((k_1,j_1),...,(k_s,j_s))$ of the ideal of the set $cl(\Gamma((k_1,j_1),...,(k_s,j_s))$, it is enough to compute a Gröbner basis $\mathcal{A}((k_1,j_1),...,(k_s,j_s))$ of the ideal I in $\mathbb{C}[x_1,...,x_n,z_1,...,z_s;y_1,...,y_{m+N}]$ and then to take

$$\mathcal{B}((k_1,j_1),...,(k_s,j_s)) = \mathcal{A}((k_1,j_1),...,(k_s,j_s)) \cap \mathbb{C}[y_1,...,y_{m+N}].$$

Consequently, $K(f) = \bigcup_{((k_1,j_1),...,(k_s,j_s))} \{y \in \mathbb{C}^m :\ \ h(y,0,...,0) = 0\ for\ every\ h \in \mathcal{B}((k_1,j_1),...,(k_s,j_s))\}$. □

Corollary 5.1. *Let X be a smooth affine variety and let $I(X) = \{b_1,\ldots,b_w\}$. Let $f = (f_1,...,f_m) : X \to \mathbb{C}^m$ be a dominant polynomial mapping. Let σ be a subfield of $\mathbb{C}$ generated by all coefficients of polynomials f_i and b_j. Then there exists a finite family $\{g_1,...,g_s\}$ of polynomials from $\sigma[y_1,...,y_m]$, such that*

$$K(f) = \{y \in \mathbb{C}^m : g_i(y) = 0,\ i = 1,...,s\}.$$

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