ON ASYMPTOTIC CRITICAL VALUES OF A COMPLEX POLYNOMIAL

ZBIGNIEW JELONEK AND KRZYSZTOF KURDYKA

Abstract. Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial of degree d. We prove that the set of asymptotic critical values of f (i.e. values for which Malgrange's Condition fails) has at most $d^{n-1} - 1$ points. We give an asymptotically sharp bound for the number of bifurcation points of f. We give also an algorithm to compute this set.

1. Introduction.

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial. Over thirty years ago R. Thom proved that f is a $C^\infty-$fibration outside a finite set, the smallest such set is called *the bifurcation set of* f, we denote it by $B(f)$. In the natural way appears two fundamental questions: how to determine the set $B(f)$ and how to estimate the number of points of this set.

Let us recall that in general the set $B(f)$ is bigger than $K_0(f)$ - the set of critical values of f. It contains also the set $B_\infty(f)$ of bifurcations points at infinity. Briefly speaking the set $B_\infty(f)$ consists of points at which f is not a locally trivial fibration at infinity (i.e., outside a large ball). To control the set $B_\infty(f)$ one can use the set of *asymptotic critical values of* f

$$K_\infty(f) = \{y \in \mathbb{C} : \textit{ there is a sequence } x_l \to \infty \textit{ s.t. } f(x_l) \to y \textit{ and } \|x_l\|\|df(x_l)\| \to 0\}.$$

If $c \notin K_\infty(f)$, then it is usual to say that f satisfies *Malgrange's condition* at c. It is proved ([12], [13]), that $B_\infty(f) \subset K_\infty(f)$. Put $K(f) = K_0(f) \cup K_\infty(f)$. Thus we have that in general $B(f) \subset K(f)$, a simple proof of this fact is also given in the last section. Moreover, A. Parusiński ([12]) proved that $B(f) = K(f)$, if f has isolated singularities at infinity (in particular this holds for $n = 2$). The set of asymptotic critical values is always finite (we show this in the last section). The simplest case $n = 2$ was studied intensively (see e.g., [4], [5], [6], [11]) and it is rather well understood; the set $K_\infty(f)$ can be computed effectively and we have estimate $\#K_\infty(f) \leq \deg f - 1$. In this paper we consider the general case. We computed the set $K_0(f) \cup K_\infty(f)$ effectively and we estimated the number of asymptotic critical values. This problem was stated explicitly by P. Rabier [15] p. 689.

Usually bifurcation points of a polynomial were studied using stratification theory applied to the projective closure of the graph. Our approach is different. The main new argument in our paper is based on the idea that asymptotic critical values of a polynomial $f : \mathbb{C}^n \to \mathbb{C}$ can be detected by studying the set of non properness of the polynomial mapping $\Phi = (f, \frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}, h_{11}, h_{12}, ..., h_{nn})$, where $h_{ij} = x_i \frac{\partial f}{\partial x_j}$, $i, j = 1, \dots, n$. The set of non properness of a polynomial mapping was studied by the first author (see [7], [8]). In particular for mappings $\mathbb{C}^n \to \mathbb{C}^n$ there is a sharp estimate for its degree. This allows

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us to bound the number of points in $K(f)$, hence in $B(f)$. Note that for $n=1$ the set $\#K_\infty(f)$ is empty, hence $B(f)=K_0(f)$.

Actually we found a relation between $\deg f$, $\#K(f)$ and $\#K_\infty(f)$. Our results are following:

Theorem 1.1. *Let $f:\mathbb{C}^n\to\mathbb{C}$ be a polynomial of degree $d>0$. Let $a=\#K_\infty(f)$, $b=\#K(f)$. Then*
$$da+b\le d^n-1.$$

Corollary 1.1. *Let $f:\mathbb{C}^n\to\mathbb{C}$, $n\ge 2$, be a polynomial of degree $d>0$. Then*
$$\#K_\infty(f)\le (d^n-1)/(d+1)\le d^{n-1}-1.$$
In particular, $\#B_\infty(f)\le d^{n-1}-1$.

Proof. Indeed, we have $a\le b$, hence $(d+1)a\le d^n-1$ and $a\le (d^n-1)/(d+1)\le d^{n-1}-1$. □

In particular, for $n=2$ we recover a well-known fact [4], [11] that $\#B_\infty(f)\le d-1$.

Corollary 1.2. *Let $f:\mathbb{R}^n\to\mathbb{R}$, $n\ge 2$, be a polynomial of degree $d>0$. Then*
$$\#K_\infty(f)\le (d^n-1)/(d+1)\le d^{n-1}-1.$$
In particular, $\#B_\infty(f)\le d^{n-1}-1$.

Proof. Indeed, if $f_{\mathbb{C}}$ is a complexification of f, then $K_\infty(f)\subset K_\infty(f_{\mathbb{C}})$. □

So, in the real case this gives a better estimate than those given in [10]: $\#K_\infty(f)\le (2d+1)(4d-3)^n$.

Corollary 1.3. *Let $k=\mathbb{C}$ or $k=\mathbb{R}$. Let $f:k^n\to k$ be a polynomial of degree d. Let $B(f)$ denote the bifurcation set of the polynomial f. Then $\#B(f)\le\#K(f)<(d-1)^n+nd^{n-2}$.*

Proof. We have $B(f)\subset K_0(f)\cup K_\infty(f)=K(f)$. Put $b=\#K(f)$. Thus $b=a+e$, where $e\le\#K_0(f)$. We know that $\#K_0(f)\le(d-1)^n$ (see e.g. [1]), consequently $e\le(d-1)^n$. By Theorem 1.1 it follows that it is enough to estimate the maximum of the linear function $h(s,t)=s+t$ on the set $W=\{(s,t):(d+1)s+t\le d^n-1;\ t\le(d-1)^n\}$.

It is easy to check that the function h attains its maximum at the point $(s_0,t_0)=((d^n-1-(d-1)^n)/(d+1),(d-1)^n)$ and this maximum is equal to $m=(d-1)^n+(d^n-1-(d-1)^n)/(d+1)$. By the mean value theorem we have $d^n-(d-1)^n\le nd^{n-1}$ and consequently $b\le m<(d-1)^n+nd^{n-2}$. □

In section 2 we give a sequence of examples which shows that for fixed n and $d\to\infty$ the above estimate is asymptotically sharp. In the case of nonisolated critical values we have:

Corollary 1.4. *Let $f:\mathbb{C}^n\to\mathbb{C}$ be a polynomial of degree $d>0$. Assume that f has no isolated critical points. Then f has at most $d^{n-1}-1$ critical values.*

Proof. Indeed, since the polynomial f has no isolated critical points, we have that $K_0(f)\subset K_\infty(f)$, consequently $\#K_0(f)\le d^{n-1}-1$. □

In the third section we show:

Theorem 1.2. *Let $f:\mathbb{C}^n\to\mathbb{C}$ be a polynomial of degree $d>0$. Then the set $K(f)=K_0(f)\cup K_\infty(f)$ can be computed effectively.*

Here effectively means that we give an algorithm (based on Gröbner basis) which works actually on a computer. We tested it on some classical examples of polynomials. To our knowledge, up to now, there were no methods to compute asymptotic critical values (or bifurcation points) of a polynomial in more than 2 variables.

2. Estimates

Let us recall that a mapping $f : \mathbb{C}^n \to \mathbb{C}^m$ is *not proper* at a point $y \in \mathbb{C}^m$ if there is no neighborhood U of y such that $f^{-1}(\overline{U}))$ is compact. In other words, f is not proper at y if there is a sequence $x_l \to \infty$ such that $f(x_l) \to y$. Let S_f denote the set of points at which the mapping f is not proper. We have the following characterization of the set S_f (see [7], [8]):

Theorem 2.1. *Let $F = (F_1, ..., F_m) : \mathbb{C}^n \to \mathbb{C}^m$ be a generically-finite polynomial mapping. Then the set S_F is an algebraic subset of $\mathbb{C}^m$ and it is either empty or it has pure dimension $n-1$. Moreover, if $n = m$ then*

$$\deg\ S_F \leq \frac{(\prod_{i=1}^n \deg\ F_i) - \mu(F)}{\min_{1\leq i\leq n} \deg\ F_i},$$

where $\mu(F)$ denotes the geometric degree of F (i.e., it is a number of points in a generic fiber of F).

Recall that if $X \subset \mathbb{C}^n$ is an algebraic set of pure dimension r then by deg X (*degree* of X) we mean the number of points in the intersection of X with sufficiently general affine subspace of codimension r. In particular, if X is a hypersurface, then deg X is the degree of any generater of the ideal $I(X)$, which is simply the smallest degree of a (nonzero) polynomial vanishing on X.

In the proof of Theorem 1.1 we need following technical lemmas.

Lemma 2.1. *Let $B \subset A$ be algebraic subsets of $\mathbb{C}^{N+1}$, $\dim B < \dim A = n$. Let L be a line and M a linear subspace of $\mathbb{C}^{N+1}$, which contains L, $\dim M = n$. Assume that $L \not\subset B$, then there exists a linear projection $p : \mathbb{C}^{N+1} \to M$ such that p restricted to A is finite and $L \not\subset p(B)$. In particular p is proper on A.*

Proof. Take a point $a \in L \setminus B$. Let Λ be the Zariski closure of the cone $\bigcup \overline{ax},\ x \in B$. It is easy to see that $\dim \Lambda \leq n$. Let H_∞ be the hyperplane at infinity of $\mathbb{C} \times \mathbb{C}^N$. For any $Z \subset \mathbb{C} \times \mathbb{C}^N$ denote by $\tilde{Z}$ the projective closure of Z. Observe that

$$\dim H_\infty \cap (\tilde{\Lambda} \cup \tilde{\Gamma} \cap \tilde{M}) \leq n-1.$$

Thus, there is a projective subspace $Q \subset H_\infty$ of dimension $N-n$, which is disjoint with $(\tilde{\Lambda} \cup \tilde{A} \cap \tilde{M})$. Denote by $p_Q : \mathbb{P}^{N+1} \setminus Q \to \tilde{M}$ the linear projection determined by the subspace Q.

Now, let $p : \mathbb{C}^{N+1} \to M$ be the restriction of p_Q to $\mathbb{C}^{N+1}$. It is easily seen that p has desired properties, i.e., $p : A \to M$ is a finite mapping and $a \notin L \cap p(B)$. ☐

The following lemma follows from the Bezout theorem in the version of Vogel.

Lemma 2.2. *Let A be irreducible algebraic subvariety of $\mathbb{C}^{N+1}$ and let H be a linear subspace of $\mathbb{C}^{N+1}$. Assume that the set $H \cap A = \{a_1, ..., a_k\}$ is finite and that the germ $\mathbf{A}_{a_i}$ of the set A at the point a_i has n_i branches, $i = 1, ..., k$. Then* $\deg\ A \geq \sum_{i=1}^k n_i$.

To state our results in the full generality we need the following:

Definition 2.1. Let $\nabla f = (\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n})$. A critical value $c \in K_0(f)$ *is isolated*, if the set $\{\nabla f = 0\} \cap \{f = c\}$ is finite. The number $k_c = \#\{\nabla f = 0\} \cap \{f = c\}$ is *the multiplicity* of the critical value c.

Now we can prove Theorem 1.1. In fact we prove a slightly more general result:

Theorem 2.2. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial of degree $d > 0$. Assume that $K_0(f) = \{c_1, ..., c_k\}$ and that values $\{c_1, ..., c_l\}$ are isolated with multiplicities $k_1, ..., k_l$. Let $a = \#K_\infty(f)$, $b = \#K(f)$. Then*

$$da + b + \sum_{i=1}^{l}(k_i - 1) \leq d^n - 1.$$

Proof. Let us define a polynomial mapping $\Phi : \mathbb{C}^n \to \mathbb{C} \times \mathbb{C}^N$ by

$$\Phi = (f, \frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}, h_{11}, h_{12}, ..., h_{nn}),$$

where $h_{ij} = x_i \frac{\partial f}{\partial x_j}$, $i = 1, \dots, n$, $j = 1, \dots, n$. Consider the line $L := \mathbb{C} \times \{(0, ..., 0)\} \subset \mathbb{C} \times \mathbb{C}^N$. In further we identify this line with a copy of $\mathbb{C}$. Let us note that Φ is a birational mapping (onto its image), in particular it is generically finite. Indeed, denote $\Phi = (\phi_1, ..., \phi_{N+1})$. Since the polynomial f is not constant there exists j, such that $\frac{\partial f}{\partial x_j} \neq 0$. Thus the field $\mathbb{C}(\phi_1, ..., \phi_{N+1})$ contains all the functions

$$x_i = \frac{h_{ij}}{\partial f/\partial x_j}, \; i = 1, \dots, n,$$

and consequently the mapping Φ is birational (onto its image). By the definition of $K_\infty(f)$ and Φ we have

$$K_\infty(f) = L \cap S_\Phi,$$

where S_Φ denotes the set of point at which the mapping Φ is not proper. Recall that $K_\infty(f)$ is finite, hence also the set $L \cap S_\Phi$ is finite. Choose a linear space M of dimension n, which contains the line L. Denote $\Gamma = \Phi(\mathbb{C}^n)$, and by $\overline{\Gamma}$ its Zariski closure. Lemma 2.1 applied to $A = \overline{\Gamma}$ and $B = S_\Phi$ yields a projection $p : \mathbb{C}^{N+1} \to M$ which is finite on $\overline{\Gamma}$ and such that $L \not\subset p(S_\Phi)$.

Denote $X = p(S_\Phi)$. Then $K_\infty(f) \subset X$ and $L \not\subset X$. Since p is proper on $\overline{\Gamma}$, we obtain that $X = S_F$, where $F = p \circ \Phi$. Note that X is exactly the set of points at which the mapping F is not proper. Moreover, if a projection p is sufficiently general, then we have $F = (F_1, ..., F_n)$, where $\deg F_i = d$ for all $i = 1, ..., n$. Let us estimate the geometric degree $\mu(F)$ of F. Since the mapping Φ is birational we have $\mu(F) = \mu(p \circ \Phi) = \mu(p|_{\overline{\Gamma}}) = \deg \overline{\Gamma}$, where $p|_{\overline{\Gamma}}$ stands for restriction of p to $\overline{\Gamma}$. Hence it is enough to estimate the degree of $\overline{\Gamma}$. Let us consider a linear subspace $H = \mathbb{C}^{n+1} \times \{0, ..., 0\} \subset \mathbb{C}^{N+1}$ and take $A = \overline{\Gamma}$. Let us compute the set $H \cap A$. It is easy to see that $H \cap A = K(f) \cup \{\Phi(0)\}$. Indeed, let $y \in H \cap A$. We have two possibilities:

1) $y \in \overline{\Gamma} \setminus \Gamma$,

2) $y \in \Gamma$.

In the case 1) we have that $y \in K_\infty(f)$. In the case 2) we have $y = \Phi(x)$ for some $x \in \mathbb{C}^n$ and either $d_x f = 0$ and consequently $y \in K_0(f)$ or $d_x f \neq 0$ and then $x = 0$. Finally $y \in K(f) \cup \{\Phi(0)\}$ and $H \cap A \subset K(f) \cup \{\Phi(0)\}$. The converse inclusion is obvious.

By a linear change of coordinates we can always assume that $\Phi(0) \notin K(f)$. Moreover, if $c \in K_0(f)$ is an isolated critical value of multiplicity k_c, then the germ $\overline{\Gamma}_c$ has at least k_c

branches. Indeed, since the mapping $\Phi : \mathbb{C}^n \to \overline{\Gamma}$ is birational, we have that every critical point b_i, such that $c = \Phi(b_i)$ gives one branch. In fact, there are small neighborhoods U_i (in the strong topology) of points b_i, $i = 1, ..., k_c$, such that $\Phi(U_i)$ is a branch of $\overline{\Gamma}_c$. By Lemma 2.2 we have $\deg A \geq b + \sum_{i=1}^{l}(k_i - 1) + 1$, consequently $\mu(F) \geq b + \sum_{i=1}^{l}(k_i - 1) + 1$.

Now Theorem 2.1 yields that the degree of the variety $X \subset M$ is bounded by $(d^n - b - \sum_{i=1}^{l}(k_i - 1) - 1)/d$. So, the set $X \cap L$ has no more than $(d^n - b - \sum_{i=1}^{l}(k_i - 1) - 1)/d$ points. Finally we obtain that $\#K_\infty(f) \leq (d^n - b - \sum_{i=1}^{l}(k_i - 1) - 1)/d$ and that $da + b + \sum_{u=1}^{l}(k_i - 1) \leq d^n - 1$. This gives a proof of Theorem 2.2. $\square$

In the case, when f has only isolated critical points, we can improve our result.

Theorem 2.3. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial of degree $d > 0$. Assume that f has a finite number, say e critical points. If $a = \#K_\infty(f)$, then*

$$(d + 1)a + e \leq d^n - 1.$$

Proof. Since the mapping f has only isolated critical points, we have that the mapping Φ has finite fibers. We show that if $c \in K_0(f) \cap K_\infty(f)$ then the germ $\overline{\Gamma}_c$ has at least $k_c + 1$ branches. Indeed, since the mapping $\Phi : \mathbb{C}^n \to \overline{\Gamma}$ is birational, we have that every critical point b_i, such that $c = \Phi(b_i)$ gives one branch. In fact, there are small neighborhoods U_i (in the classical topology) of points b_i, $i = 1, ..., k_c$, such that $\Phi(U_i)$ is a branch of $\overline{\Gamma}_c$.

By our assumption there is a sequence $x_l \to \infty$ such that $\Phi(x_l) \to c$. It means that there is another branch at the point c, which comes from infinity. Thus the germ $\overline{\Gamma}_c$ has at least $k_c + 1$ branches.

Now Lemma 2.2 gives $\deg \overline{\Gamma} \geq a + \sum_{i=1}^{b} k_i + 1$ (as before we assume that $\Phi(0) \notin K(f)$). Thus $\mu(F) \geq a + \sum_{i=1}^{b} k_i + 1$ and Theorem 2.1 yields that the degree of the variety $X \subset M$ is bounded by $(d^n - a - \sum_{i=1}^{b} k_i - 1)/d$. So, the set $X \cap L$ has no more than $(d^n - a - \sum_{i=1}^{b} k_i - 1)/d$ points. Finally we obtain that $a \leq (d^n - a - \sum_{i=1}^{b} k_i - 1)/d$ and that $(d + 1)a + \sum_{i=1}^{b} k_i \leq d^n - 1$. This gives a proof of Theorem 2.3. $\square$

Corollary 2.1. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial of degree $d > 0$. Assume that f has only a finite number of critical points. Let $a = \#K_\infty(f)$ and $c = \#K_0(f)$. Then*

$$(d + 1)a + c \leq d^n - 1.$$

Example 2.1. We show that our estimates are nearly sharp. More precisely, we have:

For every $d > 0$ there are polynomials $g_n \in \mathbb{C}[x_1, ..., x_n]$; $n = 1, 2, ...,$ and $f_n \in \mathbb{C}[x_1, ..., x_n]$; $n = 2, 3, ...,$ of degree d, such that:

1) $\#B(g_n) = \#K(g_n) = (d - 1)^n$;

2) $\#B_\infty(f_n) = \#K_\infty(f) = (d - 1)^{n-1}$.

First we construct a polynomial g_n. Let us consider a polynomial of one variable $h(t) := t^d/d - t$. We have $h'(t) = t^{d-1} - 1$. The zeros of h' are precisely all roots of degree $d - 1$ of unity. Now consider a polynomial

$$g_n = \sum_{i=1}^{n} A_i h(x_i),$$

where numbers A_i are sufficiently general. It is easy to check that $\#K_0(g_n) = (d - 1)^n$ and that $K_0(f) = K(f) = B(f)$. Put $f_n(x_1, ..., x_n) := g_{n-1}(x_1, ..., x_{n-1})$. Of course, $K_0(g_{n-1}) = K_\infty(f_n) = B_\infty(f)$ and consequently $\#B_\infty(f_n) = (d - 1)^{n-1}$.

Remark 2.1. As we know we have $\#B(f) < (d-1)^n + nd^{n-2}$. Note that also $\#K_\infty(f) \leq (d^n-1)/(d+1) < (d-1)^{n-1} + nd^{n-2}$. Indeed, we have $\#K_\infty(f) \leq (d^n-1)/(d+1) = ((d-1)^n-1)/(d+1) + (d^n-(d-1)^n)/(d+1) < (d-1)^{n-1} + (nd^{n-1})/(d+1) < (d-1)^{n-1} + nd^{n-2}$. Thus we can remark that our estimates are sharp up to $O(d^{n-2})$.

3. Computations

In this section we use Gröbner basis to compute the set $K_0(f) \cup K_\infty(f)$ effectively. Let us recall the definition of Gröbner basis. Assume that in the set of monomials in $\mathbb{C}[x_1, ..., x_n]$ we have the ordering induced by the lexicographic ordering in $\mathbb{N}^n$, i.e., $a_\alpha x^\alpha > a_\beta x^\beta$, if $\alpha > \beta$ (in this paper we consider only this ordering). By $inP = a_d x^d$ we will denote the initial form of a polynomial $P = \sum a_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathbb{C}[x_1, \ldots, x_n]$, where $d = max\{\alpha = (\alpha_1, \ldots, \alpha_n);\ a_d \neq 0\}$. We have the following basic definition (see [14]):

Definition 3.1. *A finite subset $\mathcal{B} \subset I \subset \mathbb{C}[x_1, ..., x_n]$ of an ideal I is called a Gröbner basis of this ideal, if the set $\{inP;\ P \in \mathcal{B}\}$ generates the ideal generated by all initial forms of the ideal I.*

The Gröbner basis of the ideal I is a basis of this ideal, moreover it can be easily computed by arithmetical operations only. We have the following basic fact (see [14]):

Theorem 3.1. *Consider the ring $\mathbb{C}[x_1, ..., x_n; y_1, ..., y_m]$. Let $V \subset \mathbb{C}^n \times \mathbb{C}^m$ be an algebraic set and let $p : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^m$ denote the projection. Assume that $\mathcal{B}$ is a Gröbner basis of the ideal $I(V)$. Then $\mathcal{B} \cap \mathbb{C}[y_1, ..., y_m]$ is a Gröbner basis of the ideal $I(p(V))$ of polynomials vanishing on $p(V)$.*

Proof. Observe that $I(p(V)) = I(V) \cap \mathbb{C}[y_1, ..., y_m]$ and then to use [14], Proposition 4.3. □

Take again the mapping $\Phi = (\phi_1, ..., \phi_{N+1})$ as in the proof of Theorem 2.2. Recall that $\Gamma = \Phi(\mathbb{C}^n)$ and $\overline{\Gamma}$ stands for the Zariski closure of Γ. We have the following:

Lemma 3.1. $K_0(f) \cup K_\infty(f) = L \cap \overline{\Gamma}$.

Proof. It is easily seen that $K_0(f) \cup K_\infty(f) \subset L \cap \overline{\Gamma}$. Now assume that $y \in L \cap \overline{\Gamma}$. Then we have two possibilities: either $y \in \Gamma$ or $y \in \overline{\Gamma} \setminus \Gamma$. In the first case we easily obtain that $y \in K_0(f)$. Now we pass to the second case. Since the Zariski closure of Γ coincide with its closure in the strong topology, there is a sequence $x_l \to \infty$, such that $\Phi(x_n) \to y$. But this means that $\|x_l\|\|df(x_l)\| \to 0$, which is equivalent to $y \in K_\infty(f)$. □

Proof of Theorem 1.2. First we compute the ideal of the set $\overline{\Gamma}$. Let $V = graph(\Phi)$. The basis of the ideal $I(V)$ of the set V is given by polynomials $\{y_i - \phi_i(x)\}_{i=1,...,N+1}$. By Theorem 3.1 to compute a basis $\mathcal{B}$ of the ideal of $\overline{\Gamma}$ it is enough to compute a Gröbner basis $\mathcal{A}$ of the ideal $I(V)$ in $\mathbb{C}[x_1, ..., x_n; y_1, ..., y_{N+1}]$ and then to take $\mathcal{B} = \mathcal{A} \cap \mathbb{C}[y_1, ..., y_{N+1}]$. So Lemma 3.1 yields

$$K_0(f) \cup K_\infty(f) = \{y_1 \in \mathbb{C} : h(y_1, 0, ..., 0) = 0,\ for\ every\ h \in \mathcal{B}\}.$$

Example 3.1. Let us compute the set $K_0(f) \cup K_\infty(f)$ for polynomial $f(x_1, x_2, x_3) = x_1 + {x_1}^2 x_2 + {x_1}^4 x_2 x_3$ given in [12]. Using SINGULAR we obtained (in a few seconds) that the basis $\mathcal{B}$ after substituting $y_2 = ... = y_{13} = 0$ reduces to one polynomial in variable y_1, namely to y_1. Hence $K_0(f) \cup K_\infty(f) = \{0\}$. Since the zero fiber of f is reducible and a generic fiber is not, we have that $0 \in B(f)$. Hence finally $B(f) = K_\infty(f) = \{0\}$.

Remark 3.1. a) The set $K_0(f)$ alone can be computed effectively, too. Indeed, let us consider the ideal $I = (f - Z, \frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}) \subset \mathbb{C}[x_1, ..., x_n, Z]$. Let us compute a Gröbner basis $\mathcal{A}$ of the ideal I in $\mathbb{C}[x_1, ..., x_n, Z]$ and then take $\mathcal{B} = \mathcal{A} \cap \mathbb{C}[Z]$. Now $K_0(f) = \{z \in \mathbb{C} : h(z) = 0, \ for \ h \in \mathcal{B}\}$.

b) The set $K_\infty(f) = S_\Phi \cap L$ can be also computed effectively. To this aim it is enough to produce equations for S_Φ, this can be done using Gröbner basis techniques as well (see [16] and [7]).

Proposition 3.1. *Let σ be a subfield of $\mathbb{C}$ generated by all coefficients of a polynomial $f : \mathbb{C}^n \to \mathbb{C}$. Then all bifurcations points of the polynomial f are algebraic over the field σ. More precisely, if $\alpha \in B(f)$ then $(\sigma[\alpha] : \sigma) < d^n - d$.*

Proof. It follows from the basic properties of Gröbner basis that equations for $\overline{\Gamma}$ also have coefficients in σ. Now assume that $K(f) = K_0(f) \cup K_\infty(f) \neq \emptyset$. Thus all equations for the set $K(f)$ have coefficients in σ, too. Thus the GCD of these polynomials has coefficients in σ. Hence the reduced polynomial h which has as roots all points of $K_0(f) \cup K_\infty(f)$ belongs to $\sigma[Y]$. Now it is easy to see that (for $d > 1$) we have $b \leq d^n - d - 1$. Indeed, we can apply inequality $da + b \leq d^n - 1$. If $a = 0$, then $b = \# K_0(f) \leq (d-1)^n \leq d^n - d - 1$, otherwise $b \leq d^n - 1 - da \leq d^n - d - 1$. Thus deg $h \leq d^n - d - 1$ (note that $d > 1$, because $K(f) \neq \emptyset$). Finally: if $\alpha \in B(f)$, then $(\sigma[\alpha] : \sigma) < d^n - d$. $\square$

Remark 3.2. In fact $(\sigma[\alpha] : \sigma) \leq \max\{(d-1)^n, d^{n-1} - 1\}$. Indeed, by Remark 3.1 the sets $K_0(f)$ and $K_\infty(f)$ are also described by polynomials from $\sigma[X]$, consequently $h = h_1 h_2$, where deg $h_1 \leq (d-1)^n$, and deg $h_2 \leq d^{n-1} - 1$.

Corollary 3.1. *Let $f \in \mathbb{Q}[x_1, \ldots, x_n]$ and* $\deg f = d$*. If $y \in \mathbb{C}$ and a degree of y over $\mathbb{Q}$ is at least $d^n - d$, then y is not a bifurcation point of f.*

Corollary 3.2. *If $f \in \mathbb{Q}[x_1, \ldots, x_n]$ and $y, y' \in \mathbb{C}$ are transcendental over $\mathbb{Q}$, then $f^{-1}(y)$ and $f^{-1}(y')$ are diffeomorphic.*

4. Complementary Results

In order to make the paper self-contained we sketch now proofs of two facts: that $B(f) \subset K(f)$ and that the set $K(f)$ is finite if f is a polynomial.

Proposition 4.1. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial. Then $B(f) \subset K(f)$.*

Proof. Let $a \notin K(f)$. Without loss of a generality we can assume that $a = 0$. There is $\epsilon > 0$, such that $\|x\| \|\nabla f(x)\| > \epsilon$ for any $x \in \mathbb{C}^n$ satisfying $\|x\| \geq R$ and $|f(x)| < \eta$, where $R < +\infty$ is large enough, $\eta > 0$ is sufficiently small. Moreover, $\|\nabla_x f\| \neq 0$ for x; $|f(x)| < \eta$. Fix $\alpha \in \mathbb{C}$ with $|\alpha| < \eta$ and consider a vector field

$$V_\alpha(x) = \alpha \overline{\nabla f(x)} / \|\nabla f\|^2.$$

Let $x(y, \alpha, t)$ be a solution of the differential equation

$$x(t)' = V_\alpha(x), \text{ with } x(0) = y,$$

where $y \in f^{-1}(0)$. Let us note that $x(y, \alpha, t)$ is defined for $t \in [0, 1]$. Indeed, to see this it is enough to prove that the trajectory $x(y, \alpha, t), t \in [0, 1]$ does not escape to the infinity in a finite time. By our assumption we have $\|V_\alpha(x)\| < (|\alpha|/\epsilon)\|x\|$ for sufficiently big $\|x\|$. Let $r(t) = \|x(y, \alpha, t)\|^2$. We show that the function r is bounded. Assume that conversely $r(t) \to \infty$ as $t \to t_0 \in [0, 1]$. Take $t_1 < t_0$ sufficiently close to t_0. For $t_1 < t < t_0$ we have

$$r(t)' \leq 2\|x\| \|V_\alpha(x)\| < 2(|\eta|/\epsilon)\|x\|^2 = M r(t),$$

where $M = 2(|\eta|/\epsilon)$. It means that $\ln(r(t))' \leq M$ and consequently $r(t) \leq r(t_1)exp(M(t - t_1))$, but this implies that the set $r(t),\ t_1 \leq t \leq t_0$ is bounded, a contradiction.

It is easy to see that $f(x(y,\alpha,t)) = \alpha t$, i.e., the flow $x(y,\alpha,t),\ t \in [0,1]$ maps $f^{-1}(0)$ into $f^{-1}(\alpha)$. Now let $D_\eta = \{\alpha \in \mathbb{C} : |\alpha| < \eta\}$ and let $(y,\alpha) \in f^{-1}(0) \times D_\eta$ we put

$$h(y,\alpha) = x(y,\alpha,1)$$

It is easy to see that $h : f^{-1}(0) \times D_\eta \to f^{-1}(D_\eta)$ is a diffeomorphism. Thus $0 \notin B(f)$. $\square$

Now we sketch a proof (extracted from [10]) of the fact that the set $K_\infty(f)$ is finite if f is a polynomial. In the real case a simple proof (based on the existence of stratifications satisfying (w) condition of Verdier) is given in [9], however it can not be extended to the complex case in an obvious way.

Proposition 4.2. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial mapping. Then the set $K(f)$ is finite.*

Proof. Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial mapping. Of course, it is enough to prove that the set $K_\infty(f)$ is finite. Using Łojasiewicz's inequality at infinity we may find an integer N, depending on f, such that

(4.1)
$K_\infty(f) = \{y \in \mathbb{C} : \textit{there is a sequence}\ x_l \to \infty\ \textit{s.t.}\ f(x_l) \to y\ \textit{and}\ \|df(x_l)\| \leq \|x_l\|^{-(1+\frac{1}{N})}\}.$

We consider a family

$$\widetilde{\Sigma}_r = \{x \in \mathbb{C}^n : \|x\| \geq r,\ \|df(x)\| \leq \|x\|^{-(1+\frac{1}{N})}\},$$

where $r > 0$, and we put $\Delta_r = f(\widetilde{\Sigma}_r)$. Clearly, by (4.1)

$$K_\infty(f) = \bigcap_{r>0} \overline{\Delta}_r. \tag{4.2}$$

Let $\Sigma_r = \widetilde{\Sigma}_r \cap \{\|x\| = r\}$. For r large enough Σ_r has the same number (say k) of connected components. By a result of B. Teissier [17] (adapted to the semialgebraic case by Y. Yomdin [18], see also S. K. Donaldson [3]), there exists $M > 0$ such that any two points in a connected component of Σ_r can be joint, in Σ_r, by a piecewise smooth curve of length at most Mr. So, by the mean value theorem; if L_r is a connected component of Σ_r, then its image lies in a ball (in $\mathbb{C}$) of radius $Mr^{-\frac{1}{N}}$. By the semialgebraic triviality theorem of Hardt (see [1]), there exists a semialgebraic C^1 arc $\gamma : (a,\infty) \to \mathbb{C}^n$, such that

$$\gamma(r) \in \Sigma_r,\ \|\gamma(r)\| = r.$$

Since γ is semialgebraic, the angle between $\gamma(r)$ and $\gamma'(r)$ tends to 0 as $r \to \infty$. Hence $\|\gamma'(r)\| \to 1$ as $r \to \infty$. Thus we may assume that $\|\gamma'(r)\| \leq 2$. Let us compute now the length of $f \circ \gamma$

$$\int_r^{+\infty} |(f \circ \gamma)'(\zeta)| d\zeta \leq 2 \int_r^{+\infty} \zeta^{-(1+\frac{1}{N})} d\zeta = 2Nr^{-\frac{1}{N}}. \tag{4.3}$$

Let L_r be a connected component of Σ_r such that $\gamma(r) \in L_r$ and let $\widetilde{L}_r = \bigcup_{\zeta > r} L_\zeta$. Again by the semialgebraic triviality theorem $\widetilde{L}_r$ is a connected component of $\widetilde{\Sigma}_r$, for r large enough. By (4.3) and the estimate for diameter of $f(L_r)$ we deduce that $f(\widetilde{L}_r)$ lies in a ball of radius $(M + 2N)r^{-\frac{1}{N}}$. So the set

$$\bigcap_{r>0} \overline{f(\widetilde{L}_r)}$$

consists of only one point which is an asymptotic critical value, by (4.2). Hence $K_\infty(f)$ has at most k points, where k is the number of connected components of $\widetilde{\Sigma}_r$, for r large enough. $\square$

Remark 4.1. Note that k can be effectively estimated by the classical Thom-Milnor bounds for the number of connected components of a semialgebraic set (see [1]). But the estimate for $\#K_\infty(f)$ obtained in Corollary 1.1 is essentially better.

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(Z. Jelonek) Instytut Matematyczny, Polska Akademia Nauk, Św. Tomasza 30, 31-027 Kraków, Poland

E-mail address: najelone@cyf-kr.edu.pl

(K. Kurdyka) Laboratoire de Mathématiques, Université de Savoie, Campus scientifique, F-73376 Le Bourget du Lac cedex, France

E-mail address: kurdyka@univ-savoie.fr