On some subring of formal power series*

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Abstract

For each $n \in \mathbb{N}$, we denote by $\mathbb{C}[[z_1, \ldots, z_n]]$ the ring of formal power series with complex coefficients and $\mathbb{C}\{z_1, \ldots, z_n\}$ the ring of convergent power series. For each $n \in \mathbb{N}$, we consider $\mathcal{A}_n \subset \mathbb{C}[[z_1, \ldots, z_n]]$ a subalgebra containing $\mathbb{C}\{z_1, \ldots, z_n\}$. In the first part we give some conditions under which $\mathcal{A}_n$ possesses good properties as module over $\mathbb{C}\{z_1, \ldots, z_n\}$. In the end we give condition under which each $\mathcal{A}_n$ is noetherian algebra and we give some examples.

Introduction

Let $\mathbb{C}[[z_1, \ldots, z_n]]$ be the ring of formal series; we give some conditions under which a subring $\mathcal{A}_n$ of $\mathbb{C}[[z_1, \ldots, z_n]]$ is noetherian. As an application we prove that the subring of $\mathbb{C}[[z_1, \ldots, z_n]]$ defined by the growth of the coefficients is a noetherian ring. This result has been proved by J. Chaumat and A.M. Chollet [2]. Ours method is more simple and general and can be applied in others contexts.

1 background .

We denote by $\mathbb{C}[[z_1, \ldots, z_n]]$ the ring of formal powers series and by $\mathcal{O}_n$ the ring of germs of holomorphic functions at the origin in $\mathbb{C}^n$. Finally $\mathbb{C}[z_1, \ldots, z_n]$ is the ring of polynomial with complex coefficients.

If $\lambda = (\lambda_1, \ldots, \lambda_p) \in \mathbb{C}^p$, then $P(z_n, \lambda) = z_n^p + \sum_{i=1}^{p} \lambda_i z_n^{p-i}$ is called a generic polynomial in z_n of degree p. Let $\sigma = (\sigma_1, \ldots, \sigma_p) : \mathbb{C}^p \to \mathbb{C}^p$ be the polynomial map such that each σ_i is (modulo a sign) the i^{eme} symmetric function and:

$$P(z_n, \sigma(\xi)) = (z_n - \xi_1) \ldots (z_n - \xi_p),$$

where $\xi = (\xi_1, \ldots, \xi_p)$.

Let $A : \mathbb{C}^n \times \mathbb{C}^p \to \mathbb{C}^n \times \mathbb{C}^p$ be the mapping defined by $A(z, \xi) = (z, \sigma(\xi))$. The rank of A, $rk(A)$, is the rank of the jacobian matrix of A considered as a matrix over the field $[\mathcal{O}_{n+p}]$ (quotient field of the ring $\mathcal{O}_{n+p}$). We see that $rk(A) = n + p$; hence the induced mapping:

$$A^* : \mathbb{C}[[z, \xi]] \to \mathbb{C}[[z, \xi]],$$

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defined by $A^*(\psi(z,\xi)) = \psi(z,\sigma(\xi))$ is injective.

Definition 1 *A weak formal Weierstrass system (w.f.w.s.) is the data, for each $n \in \mathbb{N}$, of a ring $\mathcal{A}_n$, such that:*

1) *$\forall n$, $\mathcal{O}_n \subset \mathcal{A}_n \subset \mathbb{C}[[z_1,\dots,z_n]]$.*

2) *If $\varphi : (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$ is a mapping with components in $\mathcal{O}_n$ and if $f \in \mathcal{A}_p$, then $f \circ \varphi \in \mathcal{A}_n$.*

3) *For each $n \in \mathbb{N}$, the ring $\mathcal{A}_n$ is closed under division by coordinates, this means that, if $f \in \mathcal{A}_n$ and $f = (z_i - \alpha)g$, where $g \in \mathbb{C}[[z]]$ and $\alpha \in \mathbb{C}$; then $g \in \mathcal{A}_n$.*

4) *Let $\sigma : (\mathbb{C}^p,0) \to (\mathbb{C}^n,0)$ be a holomorphic mapping. Suppose that $rk(\sigma) = n$. Let $f \in \mathbb{C}[[z_1,\dots,z_n]]$ such that $f \circ \sigma \in \mathcal{A}_p$; then $f \in \mathcal{A}_n$.*

Exemple 1 *The system $\mathcal{O} = (\mathcal{O}_n)_n$ is a weak formal Weierstrass system. The property 4) of the last definition is the main result of [5].*

The aim of this paper is to construct a w.f.w.s. $\mathcal{A} = (\mathcal{A}_n)_n$, such that $\mathcal{A}_n \subset \mathbb{C}[[z]]$ and $\mathcal{A}_n$ contains strictly $\mathcal{O}_n$. Before to that, let us give some properties of a w.f.w.s.

2 Generic division theorem

Théorème 1 *Let $\mathcal{A} = (\mathcal{A}_n)_n$ be a w.f.w.s. and let $P(z_n,\lambda)$ be a generic polynomial in z_n of degree p. If $f \in \mathcal{A}_n$, then there exist unique element $q(\lambda,z) \in \mathcal{A}_{n+p}$ and $r_j \in \mathcal{A}_{n-1+p}$, $1 \le j \le p$, such that*

$$f = P.q + \sum_{j=1}^{p} r_j z_n^{p-j}.$$

Proof
Let $f \in \mathcal{A}_n$; then by 2), $f_1 = f \circ A \in \mathcal{A}_{n+p}$; besides, if $z' = (z_1,\dots,z_{n-1})$:

$$f_1(z',z_n,\xi) - f_1(z',\xi_1,\xi) = (z_n - \xi_1) f_2(z,\xi)$$

and $f_1(z',\xi_1,\xi)$, f_2 are in $\mathcal{A}_{n+p}$ by 2) and 4). By repeating the processes, we get at the end:

$$f_1(z,\xi) = g(z,\xi)(z_n - \xi_1)\dots(z_n - \xi_p) + \sum_{i=1}^{p} g_i(z',\xi) z_n^{p-i},$$

with $g \in \mathcal{A}_{n+p}$ and $g_i \in \mathcal{A}_{n-1+p}$, $1 \le i \le p$. We see that g and g_i, $1 \le i \le p$ are symmetric with respect $(\xi_1,\dots,\xi_p)$. By the formal Newton's theorem, there exist $q \in \mathbb{C}[[z_1,\dots,z_n,\xi_1,\dots,\xi_p]]$, $r_i \in \mathbb{C}[[z_1,\dots,z_{n-1},\xi_1,\dots,\xi_p]]$, $1 \le i \le p$, such that:

$$g = q \circ A \text{ and } r_i = g_i \circ A, \ 1 \le i \le p.$$

By assumption 4); $q \in \mathcal{A}_{n+p}$ and $r_i \in \mathcal{A}_{n-1+p},\ 1 \leq i \leq p$. Since A^* is injective, we have proved the theorem.

Let $h \in \mathcal{O}_n - \{0\}$; after making a linear change of coordinates on $(z_1, \ldots, z_n)$, we can suppose that h is regular in z_n of order p, i.e. $h(0,0) = \frac{\partial h}{\partial z_n}(0,0) = \ldots = \frac{\partial^{p-1} h}{\partial z_n^{p-1}}(0,0) = 0$ while $\frac{\partial^p h}{\partial z_n^p}(0,0) \neq 0$. By the Weierstrass division theorem [8]; $h = Q.P$ where $Q \in \mathcal{O}_n$ is unit and $P(z', z_n) = z_n^p + \sum_{j=1}^p a_j(z') z_n^{p-j},\ a_j \in \mathcal{O}_{n-1}$ and $a_j(0) = 0,\ 1 \leq j \leq p$.

Corollaire 1 *Let $h \in \mathcal{O}_n - \{0\}$ be regular in z_n of order p. If $f \in \mathcal{A}_n$, there are unique $\tilde{q} \in \mathcal{A}_n$ and $\tilde{r_j} \in \mathcal{A}_{n-1},\ 1 \leq j \leq p$, such that:*

$$f = h\,\tilde{q} + \sum_{j=1}^{p} \tilde{r_j}(z') z_n^{p-j}.$$

Proof.
We can suppose $f = QP$, where $Q \in \mathcal{O}_n$ is unit and $P(z', z_n) = z_n^p + \sum_{j=1}^p a_j(z') z_n^{p-j},\ a_j \in \mathcal{O}_{n-1},\ a_j(0) = 0,\ 1 \leq j \leq p$. By theorem 1, we make division of f by a generic polynomial in z_n of degree p. If we replace $a(z') := (a_1(z'), \ldots, a_p(z'))$ in $\lambda = (\lambda_1, \ldots, \lambda_p)$ and we put $\tilde{q} = Q^{-1} q(a(z'), z),\ \tilde{r_j}(z') = r_j(z', a(z')),\ 1 \leq j \leq p$; the corollary follows.

3 algebraic properties

As consequence of the last corollary, we deduce some flatness properties. For each $n \in \mathbb{N}$, we put $M_n = \frac{\mathbb{C}[[z]]}{\mathcal{A}_n}$.

Proposition 1 *For each $n \in \mathbb{N}$, M_n is a flat $\mathcal{O}_n$-module.*

Proof.
By [7], we have to show that if $I \subset \mathcal{O}_n$ is an ideal, then $Tor_1^{\mathcal{O}_n}(\frac{\mathcal{O}_n}{I}, M_n) = 0$.
From the exact sequence:

$$0 \to \mathcal{A}_n \to \mathbb{C}[[z]] \to M_n \to 0$$

we deduce the longer exact sequence of "Tor".

$$\ldots \to Tor_1^{\mathcal{O}_n}(\frac{\mathcal{O}_n}{I}, \mathbb{C}[[z]]) \to Tor_1^{\mathcal{O}_n}(\frac{\mathcal{O}_n}{I}, M_n) \to \frac{\mathcal{A}_n}{I\mathcal{A}_n} \to \frac{\mathbb{C}[[z]]}{I\mathbb{C}[[z]]} \to M_n \otimes_{\mathcal{O}_n} \frac{\mathcal{O}_n}{I} \to 0.$$

Since $\mathbb{C}[[z]]$ is a flat $\mathcal{O}_n - module$; $Tor_1^{\mathcal{O}_n}(\frac{\mathcal{O}_n}{I}, \mathbb{C}[[z]]) = 0$; hence, we have the exact sequence:

$$0 \to Tor_1^{\mathcal{O}_n}(\frac{\mathcal{O}_n}{I}, M_n) \to \frac{\mathcal{A}_n}{I\mathcal{A}_n} \to \frac{\mathbb{C}[[z]]}{I\mathbb{C}[[z]]}.$$

We see then M_n is a flat $\mathcal{O}_n$-module if and only if, the last arrow is injective i.e $I\mathbb{C}[[z]] \cap \mathcal{A}_n = I\mathcal{A}_n$.
We will prove this by induction on n. Suppose $n = 1$; then the ideal I is principal and the result holds by property 3) of definition 1. We suppose $n > 1$ and the result holds for $n-1$. Let

$(h_1, \ldots, h_q)$ be a system of generators of I. We can suppose h_1 is a distinguished polynomial in z_n of degree p. By dividing $(h_2, \ldots, h_q)$ with h_1, we can suppose that $h_2, \ldots, h_q$ are polynomials in z_n of degree $< p$. Let $f = \sum_{i=1}^{q} h_i f_i,\ \ f_i \in \mathbb{C}[[z]],\ 1 \le i \le q$, and $f \in \mathcal{A}_n$. After making a Weierstrass division of $f_2, \ldots, f_q$ and f by h_1, we can suppose that $f, f_2, \ldots, f_q$ are polynomials in z_n of degree $< p$ (f has its coefficients in $\mathcal{A}_{n-1}$). We deduce then that f_1 is also a polynomial of degree $< p-1$. Let us identify the subset of $\mathcal{O}_n$ of elements in $\mathcal{O}_{n-1}[z_n]$ of degree less or equal to $2p-1$ with $(\mathcal{O}_{n-1})^{2p-1}$. In this identification we can replace the ideal I by a submodule $N \subset (\mathcal{O}_{n-1})^{2p-1}$. In order to prove the assumption for n, we have to show:

$$N\mathbb{C}[[z']] \cap \mathcal{A}_{n-1}^{2p-1} = N\mathcal{A}_{n-1}, \quad z' = (z_1, \ldots, z_{n-1}),$$

but this equality follows by the inductive hypothesis, hence the proposition.

In the following section we will give an example of w.f.w.s. that we are interested.

4 Formal series of class M

4.1 notations and definitions

Fix a sequence $(m_n)_{n=0}^{\infty}$ with $m_n = e^{\mu(n)}$, where μ is a nonnegative, increasing, convex function defined in $\{t \in \mathbb{R} \ / \ t \ge 0 \ \}$, $\mu(0) = 0$, $\frac{\mu(t)}{t} \to \infty$ as $t \to \infty$. Since μ is convex; for each $t \ge 0$ the derivative of μ at the right of t exists, say $\mu'_d(t)$, and the function $t \to \mu'_d(t)$ is increasing. We suppose that there exists $a > 0$ such that $\mu'_d(t) \le at$. We put $M_n = n! m_n$. The sequence $M = (M_n)_{n \in \mathbb{N}}$ will be called the class M.

4.2 formal series of class M

If $\varphi \in \mathbb{C}[[z_1, \ldots, z_n]]$, $\varphi = \sum_{\omega \in \mathbb{N}^n} \varphi_\omega z_1^{\omega_1} \ldots z_n^{\omega_n}$, and $C > 0$; we put:

$$\|\varphi\|_{M,C} = Sup_m \, Sup_{|\omega|=m} \frac{|\ \varphi_\omega\ |}{C^m M_m} \in [0, \infty].$$

Definition 2 *A formal power series $\varphi \in \mathbb{C}[[z_1, \ldots, z_n]]$ is said to be in the class M, if there exists a constant $C > 0$ such that $\|\varphi\|_{M,C} < \infty$.*

In the following, $\mathbb{C}[[z]]_{M,n}$, $z = (z_1, \ldots, z_n)$, denotes the collection of all $\varphi \in \mathbb{C}[[z]]$ in the class M.

Remarque 1 *If $a, b \in \mathbb{R}$, let $\tilde{\mu}(t) = \mu(t) + at + b$ and put $\tilde{m}_n = e^{\tilde{\mu}(t)}$. We can then define an other class $\tilde{M}$. We can easily see that $\mathbb{C}[[z]]_{M,n} = \mathbb{C}[[z]]_{\tilde{M},n}$; hence the class does not change when μ is replaced by $\tilde{\mu}$.*

Since the function $t \to \mu(t)$ is convex, we can prove :

$$M_j M_{n-j} \leq M_n, \text{ for } 0 \leq j \leq n.$$

Using this inequality, it is easy to show that $\mathbb{C}[[z]]_{M,n}$ is a ring for each $n \in \mathbb{N}$.

Since $M_p \geq p^p$; $\mathbb{C}[[z]]_{M,n}$ contains $\mathcal{O}_n, \ \forall n \in \mathbb{N}$.

4.3 One dimensional characterization of formal series of class M

If $\varphi \in \mathbb{C}[[z]]$; we put $H_j(\varphi) \in \mathbb{C}[z_1, \dots, z_n]$ the homogeneous polynomial of degree j in the expansion of φ; we have then $\varphi = \sum_{j=0}^{\infty} H_j(\varphi)(z)$. Let Ω be an open set of the sphere $S^{n-1} \subset \mathbb{C}^n$; if $\xi \in \Omega$; $\varphi_{|\xi}$ denotes the formal series $\sum_{j=1}^{\infty} H_j(\varphi)(\xi) t^j \in \mathbb{C}[[t]]$ called the restriction of φ to the line $t \to \xi t$.
For each $\xi \in \Omega$ and $m \in \mathbb{N}$; we put:

$$\theta_m(\xi) = \frac{\mid H_m(\varphi)(\xi) \mid}{M(m)},$$

and

$$\theta(\xi) = Sup_m \theta_m(\xi).$$

Proposition 2 *Let Ω be a nonempty open subset of S^{n-1} and $\varphi \in \mathbb{C}[[z]]$; we assume that for each $\xi \in \Omega$, there is a constant $C_\xi > 0$ such that $\theta(\xi) \leq C_\xi$; then $\varphi \in \mathbb{C}[[z]]_M$.*

<u>Proof</u>
The function θ is lower semicontinuous; by Baire's theorem, there exists $\Omega_1 \subset \Omega$ an open subset, $\Omega_1 \neq \emptyset$, and a constant $C_1 > 0$, such that:

$$\forall \xi \in \Omega_1, \ \theta(\xi) \leq C_1.$$

By a result in [6]; there is a constant C_{2,Ω_1}, such that:

$$Sup_{|\xi|=1} \mid H_m(\varphi)(\xi) \mid \leq C_{2,\Omega_1}^m Sup_{\xi \in \Omega_1} \mid H_m(\varphi)(\xi) \mid, \ \forall m \in \mathbb{N}.$$

In view of Bernstein inequality; there exists a constant C_3, such that:

$$C_3^m \mid \varphi_\omega \mid \leq Sup_{|\xi|=1} \mid H_m(\varphi)(\xi) \mid, \ \forall m \in \mathbb{N},$$

where $\omega \in \mathbb{N}^n, \ \mid \omega \mid = m$.
We put $\rho = \frac{C_{2,\Omega_1}}{C_3}$; we have then $\|\varphi\|_{M,\rho} < \infty$, hence the result.

Lemme 1 *Let $\varphi = \sum_{j=0}^{\infty} \varphi_j t^j \in \mathbb{C}[[t]]$ and $\mu \in \mathbb{N}$. We put $h = \varphi^\mu$; then $h = \sum_{\nu=0}^{\infty} h_\nu t^\nu$, where:*

$$h_\nu = \sum \frac{\mu!}{k_1! \dots k_l!} \varphi_{\mu_1}^{k_1} \dots \varphi_{\mu_l}^{k_l},$$

and $\mu = k_1 + \dots + k_l$. The sum is taken over all sets $\{\mu_1, \dots, \mu_l\}$ of distinct elements in $\mathbb{N}^l$ and $(k_1, \dots, k_l) \in (\mathbb{N} - \{0\})^l$, $l = 1, 2, 3, \dots$ such that $k_1\mu_1 + \dots + k_l\mu_l = \nu$.

Proof.
The lemma is an easy consequence of the following:
if $a_1, \ldots, a_q \in I\!R$ and $\nu \in I\!N$; then

$$(a_1 + \ldots + a_q)^\nu = \sum \frac{\nu!}{k_1! \ldots k_q!} a_1^{k_1} \ldots a_q^{k_q},$$

the sum is taken over all $(k_1, \ldots, k_q) \in I\!N^q$ such that $k_1 + \ldots + k_q = \nu$.

Remarque 2 *1) Let $\varphi = (\varphi_1, \ldots, \varphi_p) \in \mathbb{C}[[t]]^p$, $\varphi_j = \sum_{\nu=0}^{\infty} \varphi_{j,\nu} t^\nu$, $1 \leq j \leq p$; and for each $\nu \in I\!N$, we put $\varphi_\nu = (\varphi_{1,\nu}, \ldots, \varphi_{p,\nu}) \in \mathbb{C}^p$. Let $\alpha = (\alpha_1, \ldots, \alpha_p) \in I\!N^p$ and put $h = \varphi^\alpha = \varphi_1^{\alpha_1} \ldots \varphi_p^{\alpha_p}$. As in the previous lemma, we have $h = \sum_\nu h_\nu t^\nu$, where:*

$$h_\nu = \sum \frac{\alpha!}{k_1! \ldots k_l!} \varphi_{\mu_1}^{k_1} \ldots \varphi_{\mu_l}^{k_l},$$

$\alpha = k_1 + \ldots + k_l$ and the sum is taken over all sets $\{\mu_1, \ldots, \mu_l\}$ of distinct elements in $I\!N$ and $(k_1, \ldots, k_l) \in (I\!N^p - \{0\})^l$, $l = 1, 2, 3, \ldots$ such that $\mu_1 \mid k_1 \mid + \ldots + \mu_l \mid k_l \mid= \nu$.

2) Since the sequence $\frac{M_p}{p!} = e^{\mu(p)}$ is logarithmically convex we have, by [3], :

$$\frac{M_s}{s!} (\frac{M_{\mu_1}}{\mu_1!})^{t_1} \ldots (\frac{M_{\mu_l}}{\mu_l!})^{t_l} \leq (\frac{M_1}{1!})^s \frac{M_\nu}{\nu!},$$

for every $t_1, \ldots, t_l \in I\!N$ and $\mu_1, \ldots, \mu_l \in I\!N$, such that $\mu_1 t_1 + \ldots \mu_l t_l = \nu$ and $k_1 + \ldots + k_l = s$.
If $k_1, \ldots, k_l \in I\!N^p$, $\mu_1, \ldots, \mu_l \in I\!N$ such that $\mu_1 \mid k_1 \mid + \ldots \mu_l \mid k_l \mid= \nu$ and $k_1 + \ldots + k_l = \alpha$; then

$$\frac{M_{|\alpha|}}{\mid \alpha \mid!} (\frac{M_{\mu_1}}{\mu_1!})^{|k_1|} \ldots (\frac{M_{\mu_l}}{\mu_l!})^{|k_l|} \leq (\frac{M_1}{1!})^{|\alpha|} \frac{M_\nu}{\nu!}$$

.

3) Let $a > 0$, put $g(z_1, \ldots, z_p) = \prod_{j=1}^p \frac{1}{(1 - a z_j)}$, $\psi_1(t) = \ldots = \psi_p(t) = \sum_{\nu > 0} t^\nu$ and $\psi(t) = (\psi_1(t), \ldots, \psi_p(t))$. The function $g(\psi(t))$ is holomorphic in a neighborhood of the origin. We have $g(\psi(t)) = \sum_\nu q_\nu t^\nu$, where, by 1),

$$q_\nu = \sum \frac{\alpha!}{k_1! \ldots k_l!} a^{|\alpha|},$$

the sum is taken as in 1).

Proposition 3 *Let $f \in \mathbb{C}[[z]]_{M,p}$ and suppose that $\varphi_j = \sum_{\nu=0}^{\infty} \varphi_{j,\nu} t^\nu$ in$\mathbb{C}[[t]]_{M,1}$, $\varphi_{j,0} = 0$, $1 \leq j \leq p$; then $f(\varphi_1, \ldots, \varphi_p) \in \mathbb{C}[[t]]_{M,1}$.*

Proof

Put $f = \sum_{\mu \in \mathbb{N}^p} f_\mu z_1^{\mu_1} \dots z_p^{\mu_p}$; then $f(\varphi_1, \dots, \varphi_p) = \sum_{\nu=1}^{\infty} h_\nu t^\nu$, where :

$$h_\nu = \sum \frac{\alpha!}{k_1! \dots k_l!} \varphi_{\mu_1}^{k_1} \dots \varphi_{\mu_l}^{k_l},$$

and the sum as in 1) of remark 2.

There are constants $c, \rho > 0$ such that:

$$| f_\alpha | \leq \rho c^{|\alpha|} M_{|\alpha|}, \quad \forall \alpha \in \mathbb{N}^p,$$

$$| \varphi_{j,\nu} | \leq \rho c^\nu M_\nu, \quad \forall \nu \in \mathbb{N}, \ \forall j = 1, \dots, p.$$

We have then:

$$| h_\nu | \leq \rho c^\nu \sum \frac{\alpha!}{k_1! \dots k_l!} (c\rho)^{|\alpha|} (\mu_1!)^{|k_1|} \dots (\mu_l!)^{|k_l|} (| \alpha |!) \frac{M_{|\alpha|}}{| \alpha |!} (\frac{M_{\mu_1}}{\mu_1!})^{|k_1|} \dots (\frac{M_{\mu_l}}{\mu_l!})^{|k_l|}.$$

By 2) of remark 2 and using the trivial inequality $\frac{(\mu_1!)^{|k_1|} \dots (\mu_l!)^{|k_l|} (|\alpha|!)}{\nu!} \leq 1$, we have:

$$| h_\nu | \leq \rho c^\nu M_\nu \sum \frac{\alpha!}{k_1! \dots k_l!} (c M_1 \rho)^{|\alpha|}.$$

By 3) of remark 2, if we put

$$q_\nu = \sum \frac{\alpha!}{k_1! \dots k_l!} (c M_1 \rho)^{|\alpha|},$$

there are $c_1 > 0$, and $\rho_1 > 0$, such that,$| q_\nu | \leq c_1^\nu \rho_1$; hence, $| h_\nu | \leq \rho \rho_1 (c c_1)^\nu M_\nu$, which proves the lemma.

Proposition 4 *Let* $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ *be a holomorphic mapping and let* $f \in \mathbb{C}[[y_1, \dots, y_p]]_{M,p}$*; then* $\psi := f(\varphi) \in \mathbb{C}[[z_1, \dots, z_n]]_{M,n}$.

Proof

By lemma 1, we have to prove that the restriction of ψ to the line $t \to \xi t$ is in $\mathbb{C}[[t]]_{M,1}$, for each $\xi \in S^{n-1}$. Let $\xi \in S^{n-1}$, since $\varphi_{|\xi} = (\varphi_{1|\xi}, \dots, \varphi_{n|\xi}) \in \mathbb{C}[[z]]_{M,1}$; by proposition 3, we have the result.

Proposition 5 *Let* $f \in \mathbb{C}[[z]]_{M,n}$ *and* $\alpha \in \mathbb{C}$*; suppose that* $f = (z_i - \alpha) g(z)$*, where* $g \in \mathbb{C}[[z]]$*; then* $g \in \mathbb{C}[[z]]_{M,n}$.

proof

Put $f = \sum_{\omega \in \mathbb{N}^n} f_\omega z^\omega$ and $g = \sum_{\mu \in \mathbb{N}^n} g_\mu z^\mu$. Suppose $\alpha = 0$. For each $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{N}^n$ such that $g_\mu \neq 0$; we have $g_\mu = f_{\omega_\mu}$, where $\omega_\mu = (\mu_1 \dots, \mu_{i-1}, \mu_i + 1, \mu_{i+1}, \dots, \mu_n)$; hence $| g_\mu | \leq C \rho^{|\mu|+1} M_{|\mu|+1}$. By (*) of paragraph 6, there exists a constant $c_1 > 0$ such that $M_{|\mu|+1} \leq c_1^{|\mu|+1} M_{|\mu|}$; hence the result.

Now if $\alpha \neq 0$, then the function $\frac{1}{z_i - \alpha}$ is holomorphic in a neighborhood of the origin in $\mathbb{C}$, hence $\frac{1}{z_i - \alpha} \in \mathcal{O}_1 \subset \mathcal{O}_n$, and the result follows, since $\mathbb{C}[[z]]_{M,n}$ is a ring containing $\mathcal{O}_n$.

Proposition 6 *$\mathbb{C}[[t]]_{M,1}$ is a local algebra; its maximal ideal is*

$$\underline{m}_{M,1} = \{f \in \mathbb{C}[[z]]_{M,1};\ \ f(0) = 0\}.$$

Proof

Let $\varphi \in \mathbb{C}[[t]]_{M,1}$ with a non-zero constant term φ_0. Considering $\varphi_0^{-1}\varphi$ instead of φ, we can suppose $\varphi_0 = 1$. We put $\varphi = 1 - \psi$; $\psi \in \mathbb{C}[[t]]_{M,1}$ with constant term equal to zero. Let $h(\tau) = \sum_{n=0}^{\infty} \tau^n \in \mathbb{C}\{\tau\}$ (the ring of convergent power series). Since $\mathbb{C}\{\tau\} \subset \mathbb{C}[[t]]_{M,1}$; by proposition 2, we have $h(\psi) \in \mathbb{C}[[t]]_{M,1}$. We see that $\varphi h(\psi) = 1$, so φ has an inverse in $\mathbb{C}[[t]]_{M,1}$; hence the proposition. The following corollary was announced in [2] paragraph 2. The authors use a result of E.M. Dynkin [4] for functions. But for a general class we have not a version of Borel extension theorem for the ring $\mathbb{C}[[z]]_{M,n}$ (for example if the class is quasianalytic, that is the series $\sum_n \frac{M_n}{m_{n+1}}$ is not convergent). We give here,in the following corollary, a direct proof of this fact.

Corollaire 2 *$\mathbb{C}[[z]]_{M,n}$ is a local algebra; its maximal ideal is*

$$\underline{m}_{M,n} = \{f \in \mathbb{C}[[z]]_{M,n};\ \ f(0) = 0\}.$$

Proof

Let $\varphi \in \mathbb{C}[[z]]_{M,n}$ with a non-zero constant term; then φ admits an inverse $\psi \in \mathbb{C}[[z]]$. We will show that $\psi \in C[[z]]_{M,n}$. Let $\xi \in S^{n-1}$, then the restriction of ψ to the line $t \to \xi t$, $\psi_{|\xi}$, is the inverse of $\varphi_{|\xi} \in \mathbb{C}[[t]]_{M,1}$; by the last proposition, $\psi_{|\xi} \in \mathbb{C}[[t]]_{M,1}$, for all $\xi \in S^{n-1}$; hence the corollary, by proposition 2.

By proposition 5, the maximal ideal, $\underline{m}_{M,n}$, is generated by $(z_1, \ldots, z_n)$.

The proof of the following theorem is the same as the proof given in [5] for convergent power series instead of formal series of class M. For completeness we will outline the proof in the last section. We put $y = (y_1, \ldots, y_p)$.

Théorème 2 *Let $\varphi : (\mathbb{C}^p, 0) \to (\mathbb{C}^n, 0)$ be a holomorphic mapping with $rk(\varphi) = n$. For each $f \in \mathbb{C}[[z]]$ such that $f(\varphi) \in \mathbb{C}[[y]]_{M,p}$, we have $f \in \mathbb{C}[[z]]_{M,n}$.*

On the whole, we have proved that, if M is a class as in 4.1; the system $(\mathbb{C}[[z]]_{M,n})_n$ satisfies the properties 1), 2), 3) and 4) of definition 1, hence $(\mathbb{C}[[z]]_{M,n})_n$ is a w.f.w.s.

5 Formal Weierstrass system

Definition 3 *Let $\mathcal{B} = (\mathcal{B}_n)_n$ be a w.f.w.s.; we said that $\mathcal{B}$ is a formal Weierstrass system (f.w.s.), if the following conditions are satisfied:*

1) $\forall n \in \mathbb{N}$, $\mathcal{B}_n$ is a local ring with maximal ideal denoted by $\underline{m}_n$ generated by $(z_1, \ldots, z_n)$.

2) Put $y = (y_1, \ldots, y_p)$ and let $f_1(z, y), \ldots, f_p(z, y) \in \mathcal{B}_{n+p}$ such that $f_1(0, 0) = \ldots = f_p(0, 0) = 0$ and the jacobien $\frac{D(f_1,\ldots,f_p)}{D(y_1,\ldots,y_p)}(0, 0) \neq 0$; then there are $\varphi_1, \ldots, \varphi_p \in \mathcal{B}_n$, $\varphi_1(0) = \ldots = \varphi_p(0) = 0$ such that $f(z_1, \varphi_1(z)) = \ldots = f_p(z, \varphi(z)) = 0$.

Proposition 7 *Let $\mathcal{B} = (\mathcal{B}_n)_n$ be a formal Weierstrass system; then, for each $n \in I\!N$, $\mathcal{B}_n$ is a local regular ring of dimension n. Besides, the Weierstrass division theorem holds in the system $\mathcal{B} = (\mathcal{B}_n)_n$, $n \in I\!N$.*

We can deduct then the following:

Corollaire 3 *Let $\mathcal{B} = (\mathcal{B}_n)_n$ be a formal Weierstrass system; then $\frac{\mathbb{C}[[z_1,\ldots,z_n]]}{\mathcal{B}_n}$ and $\mathbb{C}[[z_1, \ldots, z_n]]$ are flat modules over $\mathcal{O}_n$, $\forall n \in I\!N$.*

As in the analytic case, the Artin's theorem [1] is also true in this situation, more precisely: Let $f_1, \ldots, f_q \in \mathcal{B}_{n+p}$ and consider the system of implicit equations:

$$f_1(z, y) = \ldots = f_q(z, y) = 0,$$

with $f_1(0, 0) = \ldots = f_q(0, 0) = 0$.
Let $\psi(z) = (\psi_1(z), \ldots, \psi_p(z)) \in (\mathbb{C}[[z]])^p$, $\psi(0) = 0$, be a formal solution of this system. Then for each $\nu \in I\!N$, there exists a solution $\psi_\nu \in (\mathcal{B}_n)^p$, $\psi_\nu(0) = 0$, such that $\psi - \psi_\nu \in \underline{m}_n^\nu (\mathbb{C}[[z]])^p$.

Proof of proposition 7

Let $f \in \mathcal{B}_n$, we suppose that f is regular of order p with respect z_n. By theorem 1 we can make division of f by the generic polynomial $P(z_n, \lambda) = z_n^p + \sum_{j=1}^p \lambda_j z_n^{p-j}$:

$$f = qP + \sum_{j=1}^{p} r_j(z', \lambda) z_n^{p-j},$$

where $q \in \mathcal{B}_{n+p}$; $r_j \in \mathcal{B}_{n-1+p}$, $1 \le j \le p$. Since f is regular of order p with respect z_n, we can easily see that:

$$q(0, 0) = 0; \quad r_j(0, 0) = 0, \;\; 1 \le j \le p,$$

and

$$\frac{D(r_1, \ldots, r_p)}{D(\lambda_1, \ldots, \lambda_p)}(0, 0) \neq 0.$$

By condition 2) in definition 4, there are $\psi_1(z'), \ldots, \psi_p(z') \in \mathcal{B}_{n-1}$, $\psi_j(0) = 0$, $1 \le j \le p$, such that:

$$f = q(z, \psi(z'))P(z_n, \psi(z')),$$

where $\psi(z') = (\psi_1(z'), \ldots, \psi_p(z'))$. We see then that f is equivalent, in $\mathcal{B}_n$, to the distinguished polynomial : $z_n^p + \sum_{j=1}^p \psi_j(z') z_n^{p-j} \in \mathcal{B}_{n-1}[z_n]$.
Let $h \in \mathcal{B}_n$; we can make division of h by the generic polynomial $P(z_n, \lambda)$, and hence by $P(z_n, \psi(z'))$, so by f, after the substitution $\lambda \to \psi(z')$:

$$h = fQ + \sum_{j=1}^{p} h_j(z') z_n^{p-j},$$

with $Q \in \mathcal{B}_n$; $h_j \in \mathcal{B}n-1$, ,$1 \leq j \leq p$, and this decomposition is unique.
Since the Weierstrass theorem is true in $\mathcal{B}_n$, $\forall n$; we deduce that $\mathcal{B}_n$ is a noetherian ring for all $n \in I\!N$. We have the inclusions:

$$\mathcal{O}_n \subset \mathcal{B}_n \subset \mathbb{C}[[z_1, \ldots, z_n]],$$

which implies $\hat{\mathcal{B}}_n = \mathbb{C}[[z_1, \ldots, z_n]]$; $\hat{\mathcal{B}}_n$ is the completion of $\mathcal{B}_n$ with respect the topology defined by the maximal ideal $\underline{m}_n$. Since the completion of the local noetherian ring $\mathcal{B}_n$ is the ring of formal series $C[[z_1, \ldots, z_n]]$, we deduce that $\mathcal{B}_n$ is a regular ring of dimension n; and the proposition is proved.

Théorème 3 *Let M be a class as in 4.1, the weak formal Weierstrass system $(\mathbb{C}[[z]]_{M,n})_n$ is a formal Weierstrass system.*

Proof
By corollary 2, condition 1) of definition is satisfied. By remark 1, we can suppose $M_1 = 1$. Since μ is convex, we have, for all $n \in I\!N$, $n\mu(n-1) \leq (n-1)\mu(n)$. Applying this repeatedly, we get:

$$(p-1)\mu(q) \leq C(q-1)\mu(p-1), \quad \forall p \geq q \geq 2$$

where C is a constant.
Hence the class satisfies the following:

$$(\frac{M_q}{q!})^{\frac{1}{q-1}} \leq C(\frac{M_p}{p!})^{\frac{1}{p-1}}, \;\; 2 \leq q \leq p.$$

By a result of [3], the implicit function theorem holds in the ring $\mathbb{C}[[z]]_{M,n}$; hence the theorem.

6 Outline of the proof of theorem 2

In the following, all the considered morphisms between rings of formal series are induced by holomorphic functions. We keep the notations of theorem 2 and put $y = (y_1, \ldots, y_p)$, $z = (z_1, \ldots, z_n)$; let $\varphi^* : \mathbb{C}[[z]] \to \mathbb{C}[[y]]$ be the homomorphism induced by φ. Recall that $M_n = n!e^{\mu(n)}$ where μ is as in 4.1. Let $\tilde{M} = (\tilde{M}_n)_n$ where $\tilde{M}_n = n^n e^{\mu(n)}$. We can easily see that $\mathbb{C}[[z]]_{\tilde{M},n} = \mathbb{C}[[z]]_{M,n}$, so we suppose that $M_n = n^n e^{\mu(n)}$. We put, for $t \geq 1$, $m(t) = t\log t + \mu(t) - \mu(1)$ and $m(t) = 0$ for $0 \leq t \leq 1$. We see that m is convex and there exists $b > 0$ such that $m'_d(t) \leq bt + 1$, $\forall t \geq 0$.
For all $p \in I\!N$ and $j \in I\!N$, we have:

$$m(p+j) - m(j) \leq j(b(p+j)+1).$$

We put $A_j = e^{jb}$ and $\rho_j = e^j$; then we have:

$$(*) \qquad M_{p+j} \leq \rho_j A_j^{(p+j)} M_j.$$

Definition 4 *We say that φ is strongly M-injective, if for each $f \in \mathbb{C}[[z]]$ such that $\varphi(f) \in \mathbb{C}[[y]]_{M,p}$; then $f \in \mathbb{C}[[z]]_{M,n}$.*

Exemples 1 *1) Let $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a mapping such that $\varphi(z_1, \dots, z_n) = (z_1, \dots, z_{i-1}, z_i z_j, z_{i+1} \dots$ then φ is strongly M-injective.*
Indeed, if $f = \sum_\omega f_\omega z^\omega \in \mathbb{C}[[z]]$ such that $\varphi^*(f) \in \mathbb{C}[[y]]_{M,p}$; then, for each $\omega \in \mathbb{N}^n$, $| f_\omega | \leq c\rho^{|\omega|+\omega_j} M_{|\omega|+\omega_j}$. So the result by inequality (*). ,

2) Suppose that $\varphi(z_1, \dots, z_n) = (z_1, \dots, z_{i-1}, z_i^q, z_{i+1} \dots, z_n)$, $q \in \mathbb{N}$. We can easily see that φ is strongly M-injective.

3) Put $w = (w_1, \dots, w_s)$; if $\varphi^* : \mathbb{C}[[z]] \to \mathbb{C}[[y]]$ and $\psi^* : \mathbb{C}[[y]] \to \mathbb{C}[[w]]$ are homomorphisms such that φ^* and ψ^* are strongly M-injective; then $\psi^* \circ \varphi^*$ is strongly M-injective.

Proof of the theorem 2

The proof uses an algorithm, introduced in [5], which consists of modifying φ^* by a finite number of steps. Each step preserves the rank and it is strongly M-injective.

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