The Existence of Morse Functions on Sets Definable in O-minimal Structures

Ta Lê Loi
Department of Mathematics
University of Dalat
Dalat - Vietnam

Abstract – In this note we present an application of the existence of Whitney stratification of sets definable in o-minimal structures to prove the existence of Morse functions on definable sets.

Morse theory is the study of the shape of a space via data given by Morse function defined on the space. For Morse theory of smooth manifolds we refer readers to the book by Milnor [M], for Morse theory of stratified spaces we refer to Goresky's and MacPherson's [GM]. [GM] proves the existence of Morse functions on closed Whitney stratified subanalytic sets. In this note we prove similar results for definable sets in o-minimal structures. Note that the spiral $\{(x,y) \in \mathbf{R}^2 : x = e^{-\varphi}\cos\varphi, y = e^{-\varphi}\sin\varphi, \varphi \geq 0\}$ or the oscillation $\{(x,y) \in \mathbf{R}^2 : y = x\sin\dfrac{1}{x}, x > 0\}$ has no Morse fucntions.

1. O-minimal structures. A *structure* on the real field $(\mathbf{R}, +, \cdot)$ is a sequence $\mathcal{D} = (\mathcal{D}_n)_{n\in\mathbf{N}}$ such that the following conditions are satisfied for all $n \in \mathbf{N}$:

- $\mathcal{D}_n$ is a Boolean algebra of subsets of $\mathbf{R}^n$.
- If $A \in \mathcal{D}_n$, then $A \times \mathbf{R}$ and $\mathbf{R} \times A \in \mathcal{D}_{n+1}$.
- If $A \in \mathcal{D}_{n+1}$, then $\pi(A) \in \mathcal{D}_n$, where $\pi : \mathbf{R}^{n+1} \to \mathbf{R}^n$ is the projection on the first n coordinates.
- $\mathcal{D}_n$ contains $\{x \in \mathbf{R}^n : P(x) = 0\}$, for every polynomial $P \in \mathbf{R}[X_1, \cdots, X_n]$.

Structure $\mathcal{D}$ is said to be *o-minimal* if

- Each set in $\mathcal{D}_1$ is a finite union of intervals and points.

A set belonging to $\mathcal{D}$ is said to be *definable* (in that structure). *Definable maps* in structure $\mathcal{D}$ are maps whose graphs are definable sets in $\mathcal{D}$.

The theory of o-minimal structures is a generalization of semialgebraic and subanalytic geometry. For the details we refer readers to surveys [D] and [DM].

In this note we fix an o-minimal structure on $(\mathbf{R}, +, \cdot)$. "Definable" means definable in this structure. Let p be a positive integer $p \geq 2$.

2. Stratifications. A *definable C^p Whitney stratification* of $X \subset \mathbf{R}^n$ is a partition $\mathcal{S}$ of X into finitely many subsets, called strata, such that:

- Each stratum is a C^p submanifold of $\mathbf{R}^n$ and also a definable set.
- For every $S \in \mathcal{S}$, $\overline{S} \setminus S$ is a union of some of the strata.
- For every $S, R \in \mathcal{S}$, if $S \subset \overline{R} \setminus R$, then (S, R) satisfies Whitney's conditions A and B (defined in [W]).

We say that stratification $\mathcal{S}$ is *compatible with* a class $\mathcal{A}$ of subsets of $\mathbf{R}^n$, if for each $S \in \mathcal{S}$ and $A \in \mathcal{A}$, $S \subset A$ or $S \cap A = \emptyset$.

The following theorem is proved in [L]

Theorem 1. *Given a finite collection $\mathcal{A}$ of definable sets in $\mathbf{R}^n$, there exists a definable C^p Whitney stratification of $\mathbf{R}^n$ compatible with $\mathcal{A}$.*

3. Tangents to definable sets. Let X be a definable subset of $\mathbf{R}^n$. Let $\mathcal{S}$ be a C^p Whitney stratification of X.
For $S \in \mathcal{S}$, the *conormal bundle* of S in $\mathbf{R}^n$ is defined by

$$T^*_S\mathbf{R}^n = \bigcup_{p \in S} \{\xi_p \in T^*_p\mathbf{R}^n : \xi_p|_{T_pS} = 0\}$$

Note that $T^*_S\mathbf{R}^n$ is a closed definable submanifold of $T^*\mathbf{R}^n$ and of dimension n.
A *generalized tangent space* Q at $p \in S$ is any plane of the form

$$Q = \lim_{x \to p} T_x R$$

where $R \in \mathcal{S}$ and $S \subset \overline{R}$.
The cotangent vector ξ_p is *degenerate* if there exists a generalized tangent space Q at p, $Q \neq T_pS$ such that $\xi_p|_Q = 0$.

Proposition 1. *The set of degenerate cotangent vectors which are conormal to S is a conical definable set of dimension $\leq n-1$.*

Proof. Let R be a stratum in $\mathcal{S}$ with $S \subset \overline{R} \setminus R$, and $\dim R = r$. Consider the mapping

$$g : R \to G_r(\mathbf{R}^n), \text{ defined by } g(x) = (x, T_xR)$$

The graph g of this mapping is a definable set of dimension r. So its closure $\overline{g}$ in $\mathbf{R}^n \times G_r(\mathbf{R}^n)$ is a definable set, and hence $\dim(\overline{g} \setminus g) \leq r - 1$.
Let

$$A_R = \{(\xi, p, Q) \in T^*_S\mathbf{R}^n \times R \times G_r(\mathbf{R}^n) : (p, Q) \in \overline{g} \setminus g, \xi_p|_Q = 0\}$$

Then A_R is definable. For each $(p,Q) \in \overline{g} \setminus g$ the fiber $A_R \cap T_p^*\mathbf{R}^n \times (p,Q)$ has dimension $\leq n - r$. Hence, $\dim A_r \leq \dim(\overline{g} \setminus g) + (n - r) = n - 1$.
Since there is a finite number of strata R in $\mathcal{S}$ such that $S \subset \overline{R} \setminus R$, the set of degenerate cotangent vectors which are conormal to S is of dimension $\leq n - 1$ □

4. Morse functions on definable sets. Let X be a definable subset of $\mathbf{R}^n$. Fix a definable C^p Whitney stratification $\mathcal{S}$ of X.
A *Morse function* $f : X \to \mathbf{R}$ is the restriction of a C^p function $\tilde{f} : \mathbf{R}^n \to \mathbf{R}$ such that

- For each $S \in \mathcal{S}$, the critical points of $f|_S$ are nondegenerate, i.e. if $\dim S \geq 1$, the Hessian of $f|_S$ at each critical point is nonsingular.
- For every critical $p \in S$ of $f|_S$, and for each generalized tangent space Q at p such that $Q \neq T_pS$, $d\tilde{f}(p)|_Q \neq 0$, i.e. $d\tilde{f}(p)$ is a nondegenerate cotangent vector.

Note that the definition depends on the stratification of X.

Let T be a definable C^p manifold. Let $F : T \times \mathbf{R}^n \to \mathbf{R}$, $F(t,x) = f_t(x)$ be a definable C^p function. Define $\Phi : T \times \mathbf{R}^n \to T^*\mathbf{R}^n$ by $\Phi(t,x) = d(f_t)(x)$. Consider the set of ' Morse parameters' $M(F,X) = \{t \in T : f_t|_X \text{ is a Morse function }\}$
Note that $M(F,X)$ is a definable set.

Theorem 2. *If Φ is a submersion, then* $\dim(T \setminus M(F,X)) < \dim T$

Proof. For each $S \in \mathcal{S}$, consider the following sets

$M_1 = M_1(S) = \{t \in T : f_t|_S \text{ has nondegenerate critical points}\}$, and
$M_2 = M_2(S) = \{t \in T : df_t(p) \text{ is a nondegenerate covector for each } p \in S\}$.

It is easy to check that M_1 and M_2 are definable sets. Since the collection $\mathcal{S}$ is finite, it is sufficient to prove that $\dim(T \setminus M_2 \cap M_2) < \dim T$.
Let

$$D = D(S) = \{\xi \in T_S^*\mathbf{R}^n : \exists p \in S, \xi_p \text{ is a degenerate cotangent vector }\}$$

Then D is a definable set. Let $\Phi_S : T \times S \to T^*S$, $\Phi_S(t,x) = (df_t|S)(x)$, and $\pi : T \times \mathbf{R}^n \to T$ be the natural projection. Since Φ is submersive, Φ_S is transverse to the zero section S of T^*S. So the set $V_1 = \Phi_S^{-1}(S)$ is a definable submanifold of $T \times S$. Furthermore, $t \in M_1$ if and only if t is not a critical value of $\pi|_{V_1}$. By Sard's theorem, $\dim(T \setminus M_1) < \dim T$.
On the other hands, Φ is transverse to each stratum of any Whitney stratification of D, and by Proposition 1, $\dim D \leq n - 1$, the set $V_2 = \Phi^{-1}(D)$ is a definable set of dimension $\leq \dim T - 1$. So $\dim(T \setminus M_2) = \dim \pi(V_2) \leq \dim T - 1$. The theorem follows. □

Corollary 1. *Consider the square of distance function*

$$F : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}, \quad F(t, x) = \|t - x\|^2$$

Let $M = \{t \in \mathbf{R}^n : F(t, .)$ *is a Morse function on* $X\}$*. Then* M *is a definable set and* $\dim(\mathbf{R}^n \setminus M) < n$

Corollary 2. *Let* $f : \mathbf{R}^n \to \mathbf{R}$ *be a definable* C^p *function. Consider the linear deformations of* f*:* $f + l$*, where* l *is a linear form of* $\mathbf{R}^n$*. Let*

$$M = \{l \in \mathbf{R}^{n*} : f + l \text{ is a Morse function on } X\}.$$

Then M *is a definable set and* $\dim(\mathbf{R}^{n*} \setminus M) < n$*.*

Remark. Using the same arguments as in the proof of Theorem 2, we have the following Propositon:

Proposition 2. *Let* $\mathcal{A}$ *be a finite collection of definable submanifold of* T^*S*. If* Φ *is submersive, then the set*

$$M(S, \mathcal{A}) = \{t \in T : df_t|_S \text{ is transverse to } \mathcal{A} \text{ in } T^*S\}$$

is a definable set and $\dim(T \setminus M(S, \mathcal{A})) < \dim T$*.*

To apply Morse theory to definable sets, one needs the following existence theorem:

Theorem 3. *Suppose that* X *is closed. Then there exists a definable* C^p *Morse function on* X *which is proper and has distinct critical values.*

Proof. By Corollary 1, there exists a definable C^p Morse function f on X which is proper. For each $S \in \mathcal{S}$, the set of critical points of $f|_S$ is finite, because it is definable and discrete. Let $x_1, \cdots, x_p$ be the critical points of $f|_S$, of all S in $\mathcal{S}$. Let $\varepsilon > 0$ such that the balls of radius ε $B(x_i, \varepsilon)$ and $B(x_j, \varepsilon)$ are disjoint when $i \neq j$. Let $\delta > 0$ such that for all $S \in \mathcal{S}$, $\|d(f|_S)(x)\| > \delta$, when x in $S \cap \cup_{i=1}^p \left(B(x_i, \varepsilon) \setminus B(x_i, \frac{\varepsilon}{2})\right)$. Choose a definable C^p function $\lambda : \mathbf{R}^n \to \mathbf{R}$, such that

λ is vanishing on $\mathbf{R}^n \setminus \cup_{i=1}^p B(x_i, \varepsilon)$,
$\|d\lambda(x)\| < \delta$, when $x \in \cup_{i=1}^p B(x_i, \varepsilon)$,
$\lambda|_{B(x_i, \frac{\varepsilon}{2})} = c_i$, with $c_1, \cdots, c_p$ satisfying: $f(x_i) + c_i \neq f(x_j) + c_j$, for $i \neq j$.

It is easy to check that $f + \lambda$ satisfies the required properties of the theorem. □

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e-mail: taleloi@hotmail.com