Genericity of $\mathbf{a}_F$ and $\mathbf{w}_F$ Regularity Conditions and Equisingularity of Functions in a Family of Functions Definable in O-minimal Structures

Ta Le Loi - University of Dalat

Let $F : T \times X \to \mathbf{R}, (t, x) \mapsto F(t, x) = F_t(x)$, be a family of continuous functions. We consider the topological equivalence of F_t along T. ($F_t, F_{t'} : X \to \mathbf{R}$ are topologically equivalent iff there exist homeomorphisms $h : X \to X$ and $\lambda : \mathbf{R} \to \mathbf{R}$, such that $F_t \circ h = \lambda \circ F_{t'}$). In this note we will give the main ideas of the proof of the finiteness of topological classes of the families definable in o-minimal structures. The proof is based on Curve Selection with parameters and stratifications satisfying certain regularity conditions due to Whitney, Thom, and Verdier.

1. Motivation

Let $F : T \times K^n \to K, (t, x) = (t_1, \cdots, t_m, x_1, \cdots, x_n) \mapsto F(t, x) = F_t(x)$ be a family of analytic functions, where $K := \mathbf{R}$ or $\mathbf{C}$ and $T = K^m \times O \subset K^m \times K^n$.
The following regularity conditions are interesting and important in equisingularity theory.

Definition. Let us denote by T_pM the tangent space at the smooth point p of the analytic set M. Suppose A, B are linear subspaces at the origin in K^N, then the distance from A to B is defined by

$$d(A, B) = \sup \left\{ \frac{|\langle u, v\rangle|}{\|u\|\|v\|} : u \in B^{\perp} \setminus \{0\}, v \in A \setminus \{0\} \right\}$$

We are interested in the following conditions at $t \in T$:
($\mathbf{a}_F$) When p is a smooth point of $F^{-1}(F(p))$ and $p \to t$, we have

$$d(T_pF^{-1}(F(p)), T_tT) \to 0$$

($\mathbf{w}_F$) There exist $C > 0$ and a neigborhood U of t in $K^m \times K^n$ such that

$$d(T_pF^{-1}(F(p)), T_tT) \leq C\text{dist}(p, T),$$

when p is a smooth point of $F^{-1}(F(p))$, and $p \in U$.

Note 1. $T_pF^{-1}(F(p))^{\perp}$ is generated by the vector $\overline{\text{grad}F(p)}$ and hence

$$d(T_pF^{-1}(F(p)), T_tT) = \sup_{v \in T - \{0\}} \frac{\left| \langle \overline{\frac{\partial F}{\partial t}}(p), v \rangle \right|}{\|\text{grad}F(p)\| \|v\|}$$

So to examine the above conditions, we need to estimate $\left\| \frac{\partial F}{\partial t}(p) \right\| / \left\| \frac{\partial F}{\partial x}(p) \right\|$, when $p \to t$.

Note 2. For $K := \mathbf{C}$. Suppose $F(t, 0) = 0$ and $\frac{\partial F}{\partial x_i} \not\equiv 0$, for all $i \in \{1, \cdots, n\}$. Then F satisfies the condition ($\mathbf{w}_F$) at t if and only if there exist $K > 0$ and a neighborhood V of t such that

$$\left| \frac{\partial F}{\partial t_j}(p) \right| \leq K \|x(p)\| \left\| \frac{\partial F}{\partial x}(p) \right\|, \text{ for all } p \in V, \text{ and } j \in \{1, \cdots, m\}$$

In [Te], Teissier proved that there exists a nowhere dense complex subspace $B \subset T$ such that the condition (w_F) holds at $t \in T \setminus B$

Note 3. For $K := \mathbf{R}$, similar results are proved by Graffney (see [G]).

Note 4. If one of the above conditions holds at t, then by Thom Second Isotopy Lemma (see [Th], [Ma]), $F_{t'}$ is topologically equivalent to F_t, when t' is close to t.

2. O-minimal Structures.

In Real Algebraic and Analytic Geometry, the following classes of sets and their geometries are considered: the class of semialgebraic sets and the class of subanalyic sets (see the exposition [BM]).
These classes of sets have many nice properties. They are closed under Boolean operators and under proper projections. In particular, these classes have the finiteness property: each set in these classes has locally only one finite number of connected components and each of the components also belongs to the corresponding class.

In some problems, we have to treat sets or functions like x^{α} or e^{-1/x^2}, where $x > 0$ and α is an irrational number, which are not subanalytic at 0. Naturally, it requires an extension of classes mentioned above. The theory of *O-minimal structures* (see the definition below) is the most remarkable generalization (see the surveys [D] and [DM]). In recent years, many interesting o-minimal structrures have been established (see the examples below), and it is showed that o-minimal structures share many important results with those of semialgebraic geometry or subanalytic geometry. For example, sets in o-minimal structures admit Whitney stratification, and can be triangulated, and functions in the structures are piecewise trivial.

Definition
A *structure* on the real field $(\mathbf{R}, +, \cdot)$ is a sequence $\mathcal{D} = (\mathcal{D}_n)_{n\in\mathbf{N}}$ such that the following conditions are satisfied for all $n \in \mathbf{N}$:

(D1) $\mathcal{D}_n$ is a boolean algebra of subsets of $\mathbf{R}^n$.
(D2) If $A \in \mathcal{D}_n$, then $A \times \mathbf{R}$ and $\mathbf{R} \times A \in \mathcal{D}_{n+1}$.
(D3) If $A \in \mathcal{D}_{n+1}$, then $\pi(A) \in \mathcal{D}_n$, where $\pi : \mathbf{R}^{n+1} \to \mathbf{R}^n$ is the projection on the first n coordinates.
(D4) $\mathcal{D}_n$ contains $\{x \in \mathbf{R}^n : P(x) = 0\}$ for every polynomial $P \in \mathbf{R}[X_1, \cdots, X_n]$.

Structure $\mathcal{D}$ is called to be *o-minimal* if

(D5) Each set in $\mathcal{D}_1$ is a finite union of intervals and points.

A set belonging to $\mathcal{D}$ is called *definable* (in that structure). *Definable maps* in structure $\mathcal{D}$ are maps whose graphs are definable sets in $\mathcal{D}$.
$\mathcal{D}$ is *polynomially bounded* iff for every definable function $f : \mathbf{R} \to \mathbf{R}$, there exist $N \in \mathbf{N}$ and $M > 0$ such that $|f(t)| \leq t^N$, for $t > M$.

Examples. Given a collection of real-valued functions $\mathcal{F}$, the smallest structure on $(\mathbf{R}, +, \cdot)$ containing the graphs of all $f \in \mathcal{F}$ is denoted by $(\mathbf{R}, +, \cdot, \mathcal{F})$.
1. Let $\mathbf{R}_{\text{alg}}$ be the smallest structure on $(\mathbf{R}, \cdot, +)$. By Tarski-Seidenberg's Theorem, a subset $X \subset \mathbf{R}^n$ is definable in $\mathbf{R}_{\text{alg}}$ if and only if X is semialgebraic. Obviously, $\mathbf{R}_{\text{alg}}$ is o-minimal. This structure is polynomially bounded.
2. Let $\mathbf{R}_{\text{an}} = (\mathbf{R}, +, \cdot, \mathcal{A})$, where $\mathcal{A}$ is the class of all restricted analytic functions on $[-1, 1]^n$ $(n \in \mathbf{N})$. Definable sets in $\mathbf{R}_{\text{an}}$ are globally subanalytic sets, i.e $X \subset \mathbf{R}^n$ is definable in $\mathbf{R}_{\text{an}}$ if and only if X is subanalytic in the projective space $\mathbf{P}^n(\mathbf{R})$, where we identify $\mathbf{R}^n$ with an open set of $\mathbf{P}^n(\mathbf{R})$ via $(x_1, \cdots, x_n) \mapsto (1 : x_1 : \cdots : x_n)$. By Grabrielov's Theorem on the complement and a Lojasiewicz result on connected components of semianalytic sets, $\mathbf{R}_{\text{an}}$ is o-minimal. This structure is also polynomially bounded.
3. Let $\mathbf{R}_{\text{exp}} = (\mathbf{R}, +, \cdot, \exp)$. Wilkie [W1] proved that each definable sets in $\mathbf{R}_{\text{exp}}$ is

the image of the zeroset of a function in $\mathbf{R}[x_1, \cdots, x_N, e^{x_1}, \cdots, e^{x_N}]$, for some $N \in \mathbf{N}$ under a natural projection. By Khovanskii [Kh], these sets have only finitely connected components. So $\mathbf{R}_{\text{exp}}$ is an o-minimal structure. Note that $x^\alpha, e^{-1/x^2}$ $(x > 0$ and α is irrational) are definable functions in $\mathbf{R}_{\text{exp}}$ but not subanalytic at 0. This structure is not polynomially bounded.

4. More generally, let $f_1, \cdots, f_k : \mathbf{R}^n \to \mathbf{R}$ be a *Pfaffian chain* , i.e. $f_1, \cdots, f_k$ are smooth functions and there exist polynomials $P_{ij} \in \mathbf{R}[X_1, \cdots, X_n, Y_1, \cdots, Y_i]$ such that for all $x \in \mathbf{R}^n$

$$\frac{\partial f_i}{\partial x_j}(x) = P_{ij}(x, f_1(x), \cdots, f_i(x)) \qquad (i = 1, \cdots, k : j = 1, \cdots, n).$$

Let $\mathcal{P} = \mathcal{P}(f_1, \cdots, f_k)$ be the class of all functions of the form $f(x) = Q(x, f_1(x), \cdots, f_k(x))$, where $Q \in \mathbf{R}[X_1, \cdots, X_n, Y_1, \cdots, Y_k]$, (they are called *Pfaffian functions*). Then, by Khovanskii's Theory [Kh] and by a result of Wilkie [W2], $(\mathbf{R}, +, \cdot, \mathcal{P})$ is o-minimal.

3. Genericity of $\mathbf{a}_F$ and $\mathbf{w}_F$ conditions in o-minimal structures.

In this note we fix an o-minimal structure on $(\mathbf{R}, +, \cdot)$. "Definable" means definable in this structure. Let p be a positive integer.

Let $F : T \times \mathbf{R}^n \to \mathbf{R}, (t, x) = (t_1, \cdots, t_m, x_1, \cdots, x_n) \mapsto F(t, x) = F_t(x)$ be a family of C^p definable functions, where $T = T \times O \subset \mathbf{R}^m \times \mathbf{R}^n$.
Suppose $F(t, 0) = 0$ and $\dfrac{\partial F}{\partial x_i} \not\equiv 0$, for all $i \in \{1, \cdots, n\}$.

Proposition 1. *In the setup above. Let*

$$A_F = \{t \in T : \boldsymbol{a}_F \text{ holds at } t\}.$$

Then A_F is a definable set and $\dim(T \setminus A_F) < \dim T$.

Proposition 2. *In the setup above. Suppose that the structure is polynomially bounded. Let*

$$W_F = \{t \in T : \boldsymbol{w}_F \text{ holds at } t\}.$$

Then W_F is a definable set and $\dim(T \setminus W_F) < \dim T$.

Proof of Proposition 1. Note that the various logical operations on formulas correspond to operations on the sets these formulas define. So A_F is definable because it is defined

by the first-order formula (in the language of the structure).
We prove the second part of the proposition by contradiction. Suppose that $\dim(T \setminus A_F) = \dim T$. Then by Definable Choice [D, (1.2) Ch. 6] (similar to Curve Selection with parameters) and Note 1, there exist an open subset U of T , $c > 0, \epsilon > 0$, and a continuous definable map $x : U \times [0, \epsilon) \to \mathbf{R}^n$, such that

x is C^p on $U \times (0, \epsilon)$, and $x(t, 0) = 0$,

$F(t, x(t, s)) = s$, for all $t \in U$, $s \in (0, \epsilon)$, and for some $j \in \{1, \cdots, m\}$

$\left\|\frac{\partial F}{\partial t_j}(t, x(t, s))\right\| / \left\|\frac{\partial F}{\partial x}(t, x(t, s))\right\| \geq c$, for all $t \in U$, $s \in (0, \epsilon)$ $\qquad (*)$

On the other hand, differentiating with respect to t_j, we have

$$\frac{\partial F}{\partial t_j}(t, x(t, s)) + \langle \frac{\partial F}{\partial x}(t, x(t, s)), \frac{\partial x}{\partial t_j}(t, s)\rangle = 0$$

Hence $\left|\frac{\partial F}{\partial t_j}(t, x(t, s))\right| \leq \left\|\frac{\partial F}{\partial x}(t, x(t, s))\right\| \left\|\frac{\partial x}{\partial t_j}(t, s)\right\|$ $\qquad (**)$

By Uniform Bounds [DM, 4.18] (that can be seen as Lojasiewicz-type inequalities with parameters) , there exists a continuous definable function $\varphi : \mathbf{R} \to \mathbf{R}$, with $\varphi(0) = 0$, such that $\|x(t, s)\| \leq \varphi(s)$, for all $t \in U, s \in (0, \epsilon)$.
By Lemma 1.8 in [L1]: in o-minimal structures, if $\|M(t, s)\|$ is bounded on $U \times (0, \epsilon)$, then there exist a definable subset $B \subset U$ with $\dim B < \dim U$, and C^p definable functions $C, \tau : U \setminus B \to \mathbf{R}_+$ such that $\left\|\frac{\partial M}{\partial t_j}(t, s)\right\| \leq C(t)$, for all $t \in U \setminus B, s \in (0, \tau(t))$.
Applying the Lemma to $M(t, s) = \frac{x(t, s)}{\varphi(t)}$, we have $\left\|\frac{\partial x}{\partial t_j}(t, s)\right\| \leq C(t)\varphi(s)$. From this we get a contradiction between (**) and (*). $\diamond$

Proof of Proposition 2. W_F is definable by the same reasoning as for A_F.
The proof of the second part is similar to that of Proposition 1, but we have to keep track of the behaviour of the curve. Suppose, contrary to the assertion, that $\dim(T \setminus W_F) = \dim T$. Then by Definable Choice [D, 1.2 Ch. 6] and Note 1, there exist an open subset U of T , $\epsilon > 0$, and a continuous definable map $x : U \times [0, \epsilon) \to \mathbf{R}^n$, such that

x is C^p on $U \times (0, \epsilon)$, and $x(t, 0) = 0$,

$F(t, x(t, s)) = s$, for all $t \in U$, $s \in (0, \epsilon)$, and for some $j \in \{1, \cdots, m\}$

$\left\|\frac{\partial F}{\partial t_j}(t, x(t, s))\right\| / \|x(t, s)\| \left\|\frac{\partial F}{\partial x}(t, x(t, s))\right\| \to +\infty$, when $s \to 0^+$. $\qquad (*)$

Differentiating with respect to t_j, we have

$$\left|\frac{\partial F}{\partial t_j}(t,x(t,s))\right| / \|x(t,s)\| \left\|\frac{\partial F}{\partial x}(t,x(t,s))\right\| \leq \left\|\frac{\partial x}{\partial t_j}(t,x(t,s))\right\| / \|x(t,s)\|, \qquad (**)$$

for all $t \in U, s \in (0,\epsilon)$

By Puiseux-type expansion with parameters in polynomially bounded o-minimal structures [Mi, Prop. 5.2], there exist an open subset B of U, and $R > 0$, such that

$$x(t,s) = a(t)s^r + b(t,s)s^p,\ t \in B, s > 0 \text{ sufficiently small,}$$

where a, b is C^p, $a(t) \neq 0, r < p$ and $\lim_{s\to 0^+} b(t,s) = 0$, for all $t \in B$.

Applying Lemma 1.8 in [L2],we can suppose that $\left\|\frac{\partial b}{\partial t_j}(t,s)\right\|$ is bounded. Hence for all $t \in B$, we have

$$\left|\frac{\partial F}{\partial t_j}(t,x(t,s))\right| / \|x(t,s)\| \left\|\frac{\partial F}{\partial x}(t,x(t,s))\right\| \leq \left\|\frac{\partial a}{\partial t_j}(t)s^r + \frac{\partial b}{\partial t_j}(t,s)s^p\right\| / \|a(t)s^r + b(t,s)s^p\|$$

The right-hand side is bounded when s is sufficiently small. So (**) contradicts (*). $\diamond$

Note 5. If the structure is not polynomially bounded, the conclusion of Proposition 2 fails. A counter example (see [L1] for the details):

$$F(t,x) = x^t, t \in (0,+\infty), x \in [0,+\infty)$$

Note 6. From the propositions, by induction, one can stratify T by definable subsets

$$T = S^m \supset S^{m-1} \supset \cdots \supset S^0 \supset S^{-1} = \emptyset$$

such that for each $i \in \{0,\cdots,m\}$ $S^i \setminus S^{i-1}$ is a smooth manifold of dimension i or an empty set, and $\mathbf{a}_F$ or $\mathbf{w}_F$ holds along $S^i \setminus S^{i-1}$. Now we can come to the general situation.

4. Equisingularity of Functions in a Family of Definable Functions.

Fix an o-minimal structure on $(\mathbf{R},+,\cdot)$ and $p \geq 2$. By compactification, using the same argument as in the above propositions, induction, then applying Thom's Second Isotopy Lemma, we get the following theorem (for the details see [L2])

Theorem. *Let $X \subset \mathbf{R}^n, T \subset \mathbf{R}^m$ be definable sets. Let*

$$F : T \times X \to \mathbf{R}, (t,x) \mapsto F(t,x) = F_t(x)$$

be a continuous definable function. Then there exists a finite partition $T = \bigcup_{i=1}^{q} T_i$ into C^p definable manifolds, such that if t and t' are in the same connected component of T_i, $i = 1, \cdots, q$, F_t and $F_{t'}$ are topologically equivalent.

The theorem is an extension of [F], where Fukuda proved that the number of topological types of polynomial functions on $\mathbf{R}^n$ of degree $\leq d$ is finite.

References

[BM] E.Bierston & P.Milman, *Semianalytic and subanalytic sets,* Inst. Hautes EÙtudes Sci. Publ. Math. 67(1988), 5-42.

[D] L.van den Dries, *Tame Topology and o-minimal Structures,* LMS Lecture Note Series 248, Cambridge University Press (1998).

[DM] L.van den Dries and C.Miller, *Geometric Categories and o-minimal Structures,* Duke Math. J. 84, No 2(1996), 497-540.

[F] T.Fukuda, *Types topologiques de polynômes,* Inst. Hautes EÙtudes Sci. Publ. Math. 46(1976), 87-106.

[G] T.Graffney, *Multiplicities and equisingularity of ICIS germs,* Invent. Math., 123 (1996), 209-220.

[Kh] A.G.Khovanskii, *Fewnomials,* Trans. Math. Monographs AMS, Vol.88(1991).

[L1] T.L.Loi, *Verdier stratifications and Thom stratifications in o-minimal structures,* Illinois J. Math. 42, No 2 (1998), 347-356.

[L2] T.L.Loi, *Stratifications of families of functions definable in o-minimal structures,* Acta Math. Vietnam. Vol.27,2 (2002), 239-244.

[Ma] J.Mather, *Notes on topological stability,* University of Harvard (1970)

[Mi] C.Miller, *Expansion of the real field with power functions,* Ann. Pure Appl. Logic 68(1994), 79-94.

[Te] B.Teissier, *Cycles eùvanescents sections planes et condition de Whitney,* in Singulariteù à Cargèse, Asterisque, 7-8 (1973) .

[Th] R.Thom, *Ensembles et morphismes stratifieùs,* Bull. Amer. Math. Soc 75(1969), 240-289.

[W1] A.Wilkie, *Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function,* J. Amer. Math. Soc. 9(1996),1051-1094.

[W2] A.Wilkie, *A general theorem of the complement and some new o-minimal structures,* manuscript (1996).

Department of Mathematics

University of Dalat
Dalat - Vietnam
E-mail: taleloi@hotmail.com