# The theorem of the complement for a quasi subanalytic set *

Abdelhafed ELKHADIRI†
Faculty of Sciences
Department of Mathematics.
B.P 133. Kenitra, 14000, Morocco.‡
E.mail: kabdelhafed@hotmail.com

*Dedicated to Professor Jean Claude Tougeron*

**Abstract**

Let $X \subset (\mathbb{R}^n, 0)$ be a germ of a set at the origin. We suppose $X$ is described by a subalgebra, $C_n(M)$, of the algebra of germs of $C^\infty$ functions at the origin (see 2.1). This algebra is quasianalytic. We show that the germ $X$ has almost all the properties of germs of semianalytic sets. In the end we study the projection of such germs and prove a version of Gabrielov's theorem.

## Introduction

The aim of this paper is to study germs, at the origin in $\mathbb{R}^n$, of some sets defined as finite union of sets of the form:

$$\{x \ / \ \varphi_0(x) = 0, \varphi_1(x) > 0, \ldots, \varphi_q(x) > 0 \ \},$$

where $\varphi_0, \ldots, \varphi_q$, are elements of a subalgebra, say $C_n(M)$, of the algebra of $C^\infty$ germs at the origin. We will call those germs: quasi semianalytic germs and their projections quasi subanalytic. We suppose that our algebra contains the germs of real analytic functions at the origin and it is quasianalytic, that is, if $f \in C_n(M)$ such that its Taylor's series at the origin, say $T_0 f$, is zero, then the germ $f$ is null. It 's well know, [3], that the Weierstrass division theorem does not hold in $C_n(M)$, and we don't know if this algebra is noetherian or not; so we can't completely follow the methods used in the classical case, i.e when $C_n(M)$ is the algebra of analytic germs, to study quasi semi analytic germs and their projections.

By using an elementary blowings up of $\mathbb{R}^n$ with smooth center, we can prove that, after a finite number of blowings up, we can transform any $f \in C_n(M)$, modulo a product by an

*Mathematics Subject Clasification . Primary 32Bxx, 14Pxx, Secondary 26E10
†Key words: quasianalytic functions, subanalytic and semianalytic sets, Gabriolov's theorem.
‡Recherche menée dans le cadre du projet PARS MI 33.

invertible element in $C_n(M)$, to a monomial (proposition 7). This implies that $C_n(M)$ is topologically noetherian that is, every decreasing sequence of germs is stationary. This property is enough for us to extend some well know properties of semianalytic germs ( stratification, locally finite number of connected components,...) to the quasi semianalytic germs. We prove also that the closure and each connected component of a quasi semianalytic germ is quasi semianalytic. Tarski-Seidenberg theorem is not true in this class of germs, so in section 8 we study the quasi subanalytic germs. The main results are theorem 7 which gives a uniform bound of the number of connected components of the fibers of a projection restricted to a bounded quasi subanalytic set and lemma 7 which shows that the dimension of quasi semianalytic germ is well behaved. At the end we prove the complement theorem for quasi subanalytic germs.

There is a priprint of J.P. Rolin, P. Speissegger and A.J. Wilkie entitled "Quasianalytic Denjoy-Carleman class and o-minimality" where is proved also the complement theorem. Our approach is different:
The reader can see that the normalization algorithm used in this priprint in section 2 is more complicate than the proof of our proposition 7. The way that we use for introducing the class of functions is more conveniently and we can have almost all properties, that we use, of such functions by this way. We have also all the theory of quasi semi analytic germs (theorem 5, theorem 6). We prove also the Łojasiewicz inequalities for functions in this class by the same way that used in [10] for the Gevrey class.

The author thanks Professor A.J. Wilkie for his comments.

# 1 background .

Let $n$ be a positive integer, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $x = (x_1, \ldots, x_n)$ the canonical coordinates on $\mathbb{R}^n$. We use the standard notations: $\mid \alpha \mid = \alpha_1 + \ldots + \alpha_n$, $\alpha! = \alpha_1! \ldots \alpha_n!$, $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_1^{\alpha_n}}$, and the preorder on $\mathbb{N}^n$ is define by $\alpha = (\alpha_1, \ldots, \alpha_n) \le \beta = (\beta_1, \ldots, \beta_n) \iff \alpha_i \le \beta_i$, $\forall i = 1, \ldots, n$.
We say that a real function, m, of one real variable is $C^\infty$ for $t \gg 0$, if there is $b > 0$ such that $m$ is $C^\infty$ in the interval $[b, \infty[$.
In all the following; $m$ will be a $C^\infty$ function for $t \gg 0$; $m$, $m'$, $m'' > 0$, $lim_{t \to \infty} m'(t) = \infty$ and there is $\delta > 0$ such that $m''(t) \le \delta$ for $t \gg 0$. We put $M(t) = e^{m(t)}$. If $U \subset \mathbb{R}^n$ is an open subset, recall that $\mathcal{E}(U)$ denotes the algebra of $C^\infty$ on $U$.

# 2 Functions of the class M

**Definition 1** *A function $f \in \mathcal{E}(U)$ is said to be in the class $M$, if for each compact $K \subset U$, there are $C_K > 0$, $\rho_K > 0$, such that, $\forall x \in K$:*

$$\mid D^\alpha f(x) \mid \le C_K \rho_K^{|\alpha|} M(\mid \alpha \mid), \ \ for \quad \mid \alpha \mid \gg 0.$$

We let $C_U(M)$ be the collection of all $C^\infty$ functions on $U$ which are in the class $M$.

**Remark 1** *Let* $M_1(t) = cr^t M(t)$, *where* $c > 0, r > 0$; *we easily see that a function* $f \in \mathcal{E}(U)$ *is in the class* $M$ *if and only if* $f$ *is in the class* $M_1$; *hence the class does not change when* $m(t)$ *is replaced by* $m(t) + at + b$, $a, b \in I\!R$; *so we will suppose, in the following, that* $m(0) = 0$ *without changing the class.*

In the following, if $m : [b, \infty[ \to I\!R,\ m(b) = 0,\ b \geq 0$ and $m,\ m',\ m'' > 0$ in the interval $[b, \infty[$; we still denote by $m$ the extension of $m$ to $[0, \infty[$ by sitting $m(t) = 0$ if $t \leq b$. We see that this extension is convex

**Lemma 1** *For all* $j \in I\!N$, $j \gg 0$, *there exists* $C_j > 0$, $\rho_j > 0$, *with:*

$$M(p + j) \leq C_j \rho_j^p M(p),,\ \forall p \in I\!N,\ \ p \gg 0.$$

Proof.
There exists $\theta \in ]p, p+j[$ such that $m(p+j) - m(p) = jm'(\theta)$. Since $m'' \leq \delta$; there exists $C > 0$, with $m'(t) \leq \delta t + C$. We have $m'(\theta) \leq m'(p+j) \leq \delta p + (C + \delta j)$. Put $\rho_j = e^{j\delta},\ C_j = e^{j(\delta j + C)}$; then we have $M(p + j) \leq C_j \rho_j^p M(p)$.

**Lemma 2** $C_U(M)$ *is an algebra, closed under differentiation.*

Since $m$ is convex and $m(0) = 0$; we have, if $0 \leq j \leq n$, $m(n-j) \leq \frac{n-j}{n} m(n)$ and $m(j) \leq \frac{j}{n} m(n)$, hence $m(j) + m(n-j) \leq m(n)$ i.e $M(n-j)M(j) \leq M(n)$. Using this inequality and Leibnitz formula, we deduct the first statement of the lemma.
The second statement follows immediately from lemma 1.

The following theorem gives a one-dimensional characterization of functions of the class $M$ and can be considered as an extension of a result in [4].
Let $\Omega$ be an open subset of the sphere $S^{n-1} \subset I\!R^n$ ( $n > 1$) and $f \in \mathcal{E}(U)$. We suppose that the following condition, on $f$, is satisfied :
For each $\xi \in \Omega$ and each compact subset $K \subset U$, there exists a constant $C_{K,\xi} > 0$ such that:

$$\mid \frac{d^m}{dt^m} f(x + t\xi)_{|t=0} \mid \leq C_{K,\xi} M(m)\ \ \forall x \in K \ \text{ and } \forall m \in I\!N.$$

**Theorem 1** *Let* $f \in \mathcal{E}(U)$ *and suppose that the last condition is satisfied; then* $f \in C_U(M)$.

Proof.
Let $K \subset U$ be a fixed compact. For each $\xi \in \Omega$, we put:

$$\theta_m(\xi) = \ \underset{x \in K}{Sup} \ \frac{\mid \frac{d^m}{dt^m} f(x + t\xi)_{|t=0} \mid}{M(m)}, \quad m \in I\!N,$$

and

$$\theta(\xi) = \ \underset{m \in I\!N}{Sup} \ \theta_m(\xi)$$

$\theta$ is a lower semicontinuous function; by Baire's theorem, there exist $\Omega_1 \subset \Omega$ an open subset and a constant $C_1 > 0$ such that:

$$\forall \xi \in \Omega_1,\ \ \theta(\xi) \leq C_1.$$

We have:
$$\frac{\partial^m f}{\partial \xi^m}(x) := \frac{d^m}{dt^m} f(x+t\xi)_{|t=0} = \sum_{|\omega|=m} D^\omega f(x) \frac{m!}{\omega_1! \dots \omega_n!} \xi_1^{\omega_1} \dots \xi_n^{\omega_n}.$$

Since $\Omega_1$ is open in $S^{n-1}$, by a result in [5], there exists a constant $C_2 > 0$, such that:

$$\underset{|\xi|=1}{Sup} \mid \frac{\partial^m f}{\partial \xi^m}(x) \mid \leq C_2^m \underset{\xi \in \Omega_1}{Sup} \mid \frac{\partial^m f}{\partial \xi^m}(x) \mid .$$

In view of Bernstein's inequality; there exists a constant $C_3 > 0$, such that:

$$C_3^m \underset{|\omega|=m}{Sup} \mid D^\omega f(x) \mid \leq \underset{|\xi|=1}{Sup} \mid \frac{\partial^m f}{\partial \xi^m}(x) \mid .$$

Put $\rho = \frac{C_2}{C_3}$, we have:

$$Sup_m \underset{\substack{x \in K \\ |\omega|=m}}{Sup} \frac{\mid D^\omega f(x) \mid}{M(m)\rho^m} < \infty,$$

hence $f \in C_U(M)$.

**Remark 2** *If $M(t) = t^t$ i.e $m(t) = tlogt$; we have the analytic class. In the following we will consider $M$ such that the class $C_U(M)$ contains strictly the analytic class. We take then $m$ of the form:*

$$m(t) = tlogt + t\mu(t),$$

*where $\mu$ is increasing and $lim_{t\to\infty}\mu(t) = \infty$. In order to have $m''(t) \leq \delta$, we must suppose that $\mu(t) \leq at$ for $t \gg 0$ ($a > 0$). We suppose also that $\mu$ is in a Hardy field.*

**Proposition 1** *$C_U(M)$ is closed under composition.*

Proof.
Since $t \to t\mu(t)$ is convex; the proposition follows from [3]. Proposition 1 shows that we can define $C_X(M)$ by means of local coordinate system when $X$ is a real analytic manifold.

Let $t \to M(t)$ as above and for $s \in I\!R_+$, put:

$$\Lambda(s) = inf_{t \geq t_0} M(t) s^{-t}$$

where $t_0$ is a fixed positive real.
The infimum is reached at a point t where $m'(t) = logs$, and this point is unique since $m'$ is increasing and $lim_{t\to\infty} m'(t) = \infty$.
we define $s \to \omega(s)$ by $\Lambda(s) = e^{-\omega(s)}$, we have:

$$\begin{cases} s = e^{m'(t)} \\ \omega(s) = tm'(t) - m(t) \end{cases}$$

or

$$\begin{cases} s = ete^{\mu(t)+t\mu'(t)} \\ \omega(s) = t + t^2\mu'(t) \end{cases}$$

Since $\mu' > 0$, then $\omega > 0$ and $lim_{t\to\infty}\omega(s) = \infty$; we can easily inverse the last system; we have then:

$$\begin{cases} t = s\omega'(s) \\ m(t) = s\omega'(s)logs - \omega(s) \end{cases}$$

Since $m(t) = tlogt + t\mu(t)$, we have

$$\begin{cases} t = s\omega'(s) \\ \mu(t) = -log\omega'(s) - \frac{\omega(s)}{s\omega'(s)} \end{cases}$$

We see that $\omega$ is increasing and when $t \to \infty$, $s\omega'(s) \to \infty$ and $-log\omega'(s) - \frac{\omega(s)}{s\omega'(s)} \to \infty$, so $\omega'$ is decreasing and when $s \to \infty$, $\omega'(s) \to 0$.
For $s > 0$ let $\lambda(s) = inf_{n\in I\!N, n\geq t_0} M(n)s^{-n}$

**Lemma 3** *For $s \gg 0$, we have:*

$$e^{-\delta}\lambda(s) \leq \Lambda(s) \leq \lambda(s)$$

Proof.
Put $\alpha(t) = m(t) - tlogs$; we have $\Lambda(s) = e^{\alpha(t_0)}$ where $\alpha'(t_0) = 0$; then $\lambda(s) = e^{\alpha(n_0)}$ with $\mid n_0 - t_0 \mid < 1$. Note that $\alpha(n_0) - \alpha(t_0) = \alpha'((1-\theta)n_0 + \theta t_0)$, $0 < \theta < 1$, since $m'' \leq \delta$ and $\mid \alpha'((1-\theta)n_0 + \theta t_0) - \alpha(t_0) \mid \leq \delta$, we have $e^{-\delta}\lambda(s) \leq \Lambda(s)$, the second inequality is trivial.

**Proposition 2** *The following three statements are equivalent:*

*(i)* $\sum_n \frac{M(n)}{M(n+1)} = \infty$,

*(ii)* $\int_{s_0}^{\infty} \frac{\omega(s)}{s^2} ds = \infty$ , *for some* $s_0 > 0$,

*(iii)* $\int_{s_0}^{\infty} \frac{Log\lambda(s)}{s^2} ds = -\infty$ , *for some* $s_0 > 0$,

Proof
We have $m'(n) \leq m(n+1) - m(n) \leq m'(n+1)$, hence $\sum_n \frac{M(n)}{M(n+1)} = \infty \Longleftrightarrow \int_{t_0}^{\infty} e^{-m'(t)}dt = \infty$. Recall that by above, $\int_{t_0}^{\infty} e^{-m'(t)}dt = \int_{s_0}^{\infty} \frac{d(s\omega'(s))}{s}ds$. Since $\omega'(s) \to 0$ when $s \to \infty$ and it is decreasing then $\int_{s_0}^{\infty} \frac{d(s\omega'(s))}{s}ds = \infty \Longleftrightarrow \int_{s_0}^{\infty} \frac{\omega(s)}{s^2}ds = \infty$, which proves $(i) \Leftrightarrow (ii)$. By lemma 3, we have $\frac{-\omega(s)}{s^2} \leq \frac{Log\lambda(s)}{s^2}$, hence $(ii) \Leftrightarrow (iii)$.

**Definition 2** *We said that $C_U(M)$ is quasianalytic, if for any $f \in C_U(M)$ and any $x \in U$ the Taylor series $T_x f$ of $f$ at $x$ uniquely determines $f$ around $x$.*

By a well know result of Donjoy-Carleman, $C_U(M)$ is quasianalytic if and only if we have:

$$\sum_n \frac{M(n)}{M(n+1)} = \infty.$$

If the class is quasianalytic; proposition 2 tell us that the function $\omega(s)$ tend to $\infty$, when $s \to \infty$, rapidly as $s^q$, for all $q < 1$. Probably the converse of this statement is true.
In the case of the analytic class ($m(t) = t\log t$), we have $\omega(s) = s\omega'(s)$, hence $\omega(s) = Cs$. The converse is also true:

**Proposition 3** *If $\omega(s) \simeq s$ when $s \to \infty$; then any $f \in C_U(M)$ is analytic.*

Proof.
By hypothesis, there exist $c > 0$ and $A > 0$, such that, $\forall s \geq A$, we have $\omega(s) \geq C.s$; then:

$$\forall m \in I\!N,\ \forall s \geq A,\ \ e^{-\omega(s)} \leq \frac{c^{-m}}{s^m} m!.$$

Since $m'(t) \to \infty$ when $t \to \infty$; there exists $N_0 \in I\!N$, such that $e^{m'(t)} \geq A$, $\forall t > N_0$ (we can suppose $N_0 > t_0$). Let $r > N_0$ and put $s = e^{m'(r)}$; then $s \geq A$ and $\frac{M(r)}{s^r} \leq \inf_{n \geq t_0} \frac{M(n)}{s^n}$. By lemma 3, we have, $\forall m > N_0$:

$$\frac{M(n)}{s^n} \leq e^{\delta} e^{-\omega(s)} \leq e^{\delta} \frac{C^{-m}}{s^m} m!,$$

hence $M(m) \leq \frac{e^{\delta}}{C^m} m!$. This proves the result.

**Proposition 4** *Let $\mu(t) = loglogt$ i.e $m(t) = tlogt + tloglogt$; then the class $C_U(M)$ is quasi-analytic (recall that $M(t) = e^{m(t)}$).*

Proof .
We may show that $\int_{s_0}^{\infty} \frac{\omega(s)}{s^2} ds = \infty$. We have $s = e^{m'(t)} = etlogte^{\frac{1}{logt}} \sim etlogt$, and $\omega(s) = tm'(t) - m(t) = t + \frac{t}{logt} \sim t \sim \frac{s}{elogs}$, then $\frac{\omega(s)}{s^2} \sim \frac{1}{es(logs)}$ when $s \to \infty$, which shows the proposition.

From now on we take $m(t) = tlogt + t\mu(t)$, $\mu$ increasing, $\mu(t) \leq at$ for $t \gg 0$, $a > 0$ , and $lim_{t\to\infty}\mu(t) = \infty$.We suppose also that $\mu$ is in a Hardy field. Then the class $C_U(M)$ is an algebra, closed under differentiation and composition. We take also $\mu$ such that $C_U(M)$ is quasi analytic; for example, take $\mu(t) = loglogt$.

## 2.1 The ring of germs of quasi analytic functions

Let $r > 0$, we use the notation $\Delta_n(r) = \{x \in I\!R^n /\ \mid x_i \mid < r, \text{ for, } 1 \leq i \leq n\}$, if $x \in I\!R^n$, $x = (x', x_n)$, $x' \in I\!R^{n-1}$, and we put $C_{n,r}(M) = C_{\Delta_n(r)}(M)$. If $f \in \mathcal{E}(\Delta_n(r))$, we define, for $\rho > 0$,

$$\|f\|_{\rho,r,M} = Sup_m \ \underset{\substack{|\alpha|=m \\ x\in\Delta_n(r)}}{Sup} \ \frac{\mid D^{\alpha} f(x) \mid}{M(\mid \alpha \mid)\rho^{|\alpha|}} \in [0, \infty]$$

and we note $C_{n,\rho,r}(M) = \{f \in C_{n,r}(M) \,/\, \|f\|_{\rho,r,M} < \infty\}$. Clearly $C_{n,\rho,r}(M)$ is a Banach space, let $C_n(M)$ be the inductive limit of $C_{n,\rho,r}(M)$ when $r \to 0$, $\rho \to \infty$. We have an injection:

$$C_n(M) \to I\!R[[X_1, \dots, X_n]]$$

defined by $f \to T_0 f$.

In general, we will not distinguish notationaly between the germ of a function and a representative of the germ.

**Lemma 4** *The algebra $C_n(M)$ is local and its maximal ideal is generated by $(x_1, \dots, x_n)$.*

Proof.

Let $f \in C_n(M)$ such that $f(0) = a_0 \neq 0$; put $\rho =\mid a_0 \mid > 0$ and $\varphi(\xi) = \frac{1}{\xi + a_0}$. The function $\varphi$ is analytic in $\{\xi \in I\!R / \mid \xi \mid < \rho\}$. Put $g = f - a_0$; $g \in C_n(M)$ and $g(0) = 0$. There exists $\eta > 0$ such that $g([-\eta, \eta]^n) \subset \{\xi \in I\!R / \mid \xi \mid < \rho\}$. By proposition 1, $\varphi \circ g \in C_n(M)$, hence $\frac{1}{f} \in C_n(M)$. The algebra is then local and its maximal ideal is $\mathcal{M} = \{f \in C_n(M) / f(0) = 0\}$. Let $f \in \mathcal{M}$, then $f(x) = \sum_{j=1}^{n} x_j g_j(x)$, where $g_j(x) = \int_0^1 \frac{\partial f}{\partial x_j}(tx) dt$; we easily see that $g_j \in C_n(M), \forall j = 1, \dots, n$.

**Corollary 1** *If $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ divides $f \in C_n(M)$ in the ring of formal power series at $0 \in I\!R^n$, then $x^\alpha$ divides $f$ in $C_n(M)$.*

Proof.

It is an immediate consequence of the previous lemma.

**Proposition 5** *Let $f \in C_{n,\rho_1,r_1}(M) - \{0\}$ such that $f(0) = 0$. For each $\epsilon > 0$, there exist $r' > 0$, $\rho' > 0$, $r' < r_1$, ,$\rho' > \rho_1$, such that, for all $r < r'$ and $\rho > \rho'$, we have $\|f\|_{\rho,r,M} \leq \epsilon$.*

Proof.

By hypothesis, we have:

$$Sup_m \underset{\substack{|\omega|=m \\ x \in \Delta_n(r_1)}}{Sup} \frac{\mid D^\omega f(x) \mid}{M(\mid \omega \mid) \rho_1^m} < \infty.$$

Put

$$R = Sup_{m \neq 0} \underset{\substack{|\omega|=m \\ x \in \Delta_n(r_1)}}{Sup} \frac{\mid D^\omega f(x) \mid}{M(\mid \omega \mid) \rho_1^m}.$$

Since $f(0) = 0$ and $f \neq 0$, then $R \neq 0$. Let $\epsilon > 0$ ( we can suppose $\epsilon < 1$), there exists $\rho' > \rho_1$ such that for all $\rho > \rho'$, $(\frac{\rho_1}{\rho})^m \leq \frac{\epsilon}{R}$, $\forall m \in I\!N^*$. We have then:

$$Sup_{m \neq 0} \underset{\substack{|\omega|=m \\ x \in \Delta_n(r_1)}}{Sup} \frac{\mid D^\omega f(x) \mid}{M(\mid \omega \mid) \rho^m} \leq \epsilon.$$

Since $f(0) = 0$, there exists $r' < r_1$ such that for all $r \leq r'$, $\mid f(x) \mid \leq \epsilon$, $\forall x \in \Delta_n(r)$, hence $\|f\|_{\rho,r,M} \leq \epsilon$.

# 3 The implicit function theorem

It was proved in [7] that if the sequence $M(n) = M_n$ satisfies the following conditions:

(1) $$(\frac{M_q}{q!})^{\frac{1}{q-1}} \leq C(\frac{M_p}{p!})^{\frac{1}{p-1}}, \;\; 2 \leq q \leq p$$

and

(2) $$M_0 = M_1 = 1$$

where $C > 0$ is a constant; then the implicit function theorem holds in the ring $C_n(M)$.
Recall that we have $M(t) = e^{m(t)}$, $m(t) = t\log t + t\mu(t)$. We put $g(t) = t\mu(t)$. By remark 1, we can suppose $M(1) = 1$; we see that the condition (1) is satisfied if

(*) $$\forall p \geq q \geq 2, \;\; (p-1)g(q) \leq C(q-1)g(p).$$

for a constant $C > 0$.
We remark that ($\mu$ is increasing):

$$\forall p \geq 1, \;\;\; pg(p-1) \leq (p-1)g(p).$$

By repeating the processes, we prove (*). We deduct that the implicit function theorem holds in $C_n(M)$.

# 4 Algebraic Properties

It is well know that the Weierstrass preparation theorem does not hold in $C_n(M)$ [3]. We don't know if $C_n(M)$ is a noetherian ring ( $n > 1$). In this paragraph we will show that $C_n(M)$ has a weak noetherian property which we call topological noetherianity. This property will be enough for us to extend some well know properties of semianalytic germs to the case where the germs are defined by equations and inequations of elements in $C_n(M)$.

We shall use a very elementary version of resolution of singularities consisting of blowings-up of a neighborhood of $0 \in \mathbb{R}^n, n > 1$, say $V$, either with center an open subset, $W \subset \mathbb{R}^{n-p}$, $p < n$, such that $\{0\} \times W \subset V$, or with center $\{0\} \subset \mathbb{R}^n$.

## 4.1 Blowings-up

For each positive integer $r$, let $\mathbb{P}^{r-1}(\mathbb{R})$ denote the $(r-1)$ dimensional projective real space of lines through the origin in $\mathbb{R}^r$. Let $\sigma : \mathbb{R}^r - \{0\} \to \mathbb{P}^{r-1}(\mathbb{R})$ be the canonical surjection which associates to each $t \in \mathbb{R}^r - \{0\}$ the line, say $\sigma(t)$, in $\mathbb{R}^r$ passing by 0 and $t$. For each $i = 1, \ldots, r$, let $V_i = \{x = (x_1, \ldots, x_r) \,/\, x_i \neq 0\}$ and $U_i = \sigma(V_i)$; $U_i$ is a coordinate chart of $\mathbb{P}^{r-1}(\mathbb{R})$ with coordinates:

$$\varphi_i : U_i \to \mathbb{R}^{r-1}$$

given by $\varphi_i(\sigma(t)) = (\frac{t_1}{t_i}, \ldots, \frac{t_{i-1}}{t_i}, \frac{t_{t+1}}{t_i}, \ldots, \frac{t_r}{t_i})$.

**Definition 3** *Let $V$ be an open neighborhood of* 0 *in* $I\!R^r$. *Put*

$$Z = \{(x, \sigma(t)) \in V \times I\!P^{r-1}(I\!R) \,/\, x \in \sigma(t)\}$$

*and let $\pi : Z \to V$ denote the mapping $\pi(x, \sigma(t)) = x$. The mapping $\pi$ is called the blowing-up of $V$ with center* 0.

$\pi$ is proper, $\pi$ restricts to a homeomorphism on $V - \{0\}$ and $\pi^{-1}(0) = I\!P^{r-1}(I\!R)$.
We can cover $Z$ with coordinate charts

$$Z_i = Z \cap V \times \sigma(U_i)$$

with coordinates $\psi_i : Z_i \to I\!R^r$ given by:

$$\psi_i(x, \sigma(t)) = (\frac{t_1}{t_i}, \ldots, \frac{t_{i-1}}{t_i}, x_i, \frac{t_{t+1}}{t_i}, \ldots, \frac{t_r}{t_i}).$$

In these local coordinates, $\pi$ is given by:

$$\pi(y_1, \ldots, y_r) = (y_1 y_i, \ldots, y_{i-1} y_i, y_i, y_{i+1} y_i, \ldots, y_r y_i).$$

Let $n > r$ be an integer and $W$ an open subset of $I\!R^{n-r} = \{x = (x_1, \ldots, x_n) \in I\!R^n \,/\, x_1 = \ldots x_r = 0\}$. Let $w = (w_1, \ldots, w_{n-r})$ be the coordinates of a point in $I\!R^{n-r}$. The mapping $\tilde{\pi} = \pi \times id_W : \tilde{Z} = Z \times W \to V \times W$ is called the blowing-up of $V \times W$ with center $\{0\} \times W$.
We can cover $\tilde{Z}$ with coordinate charts:

$$\tilde{Z}_i = \tilde{Z} \cap V \times \sigma(U_i) \times W$$

with coordinates $\tilde{\varphi}_i : \tilde{Z}_i \to V \times W$ given by:

$$\tilde{\varphi}_i(x, \sigma(t), w) = (\frac{t_1}{t_i}, \ldots, \frac{t_{i-1}}{t_i}, x_i, \frac{t_{i+1}}{t_i}, \ldots, \frac{t_r}{t_i}, w).$$

We put $\tilde{\varphi}_i = (y_1, \ldots, y_r, w')$.
Recall that $\mathcal{E}_n$ is the ring of the germs at $0 \in I\!R^n$ of $C^\infty$ functions. Let $a \in \tilde{\pi}^{-1}(0) \cap \tilde{Z}_i$ and $f \in \mathcal{E}_n$; then the Taylor expansion of $f \circ \tilde{\pi}$ at $a$ is given by formal substitution of $w = w'$, $X_i = y_i$, and $X_l = y_i(y_l(a) + y_l)$, $l \neq i$, in the Taylor expansion of $f$ at 0. In particular if $\hat{f} \in I\!R[[X, W]]$ is a formal series, we will denote by $\hat{f} \circ \tilde{\pi}_a$ the formal series obtained by formal substitution of $w = w'$, $X_i = y_i$, and $X_l = y_i(y_l(a) + y_l)$, $l \neq i$, in the formal series $\hat{f}$.
We need the following lemma proved in [10] but for completeness we will give the proof.

**Lemma 5** *Let $\Omega \subset I\!N^n$, $n > 1$, be a finite set and put $\mathcal{F} = \{X^\alpha = X_1^{\alpha_1} \ldots X_n^{\alpha_n} \,/\, \alpha = (\alpha_1, \ldots, \alpha_n) \in \Omega\}$. Let $V$ be an open neighborhood of* 0 *in* $I\!R^n$. *Then there exist a real analytic manifold $Z$, $\pi : Z \to V$ a proper real analytic surjective mapping such that:*

*a) For all $a \in \pi^{-1}(0)$ there is $U$ a chart, $a \in U$, with coordinates $y = (y_1, \ldots, y_n)$ such that the set $\{\mu_\alpha \in I\!N^n \,/\, X^\alpha \circ \tilde{\pi}_a = y^{\mu_\alpha}\}$ is totally ordered by the product order on $I\!N^n$.*

*b) $\pi_{|U} : U \to V$ is a composition of a finite sequence of blowings-up.*

Proof.
We proceed by induction on $n \geq 2$. We can suppose that the cardinal of $\Omega$ is equal to 2. After making a finite number of blowings-up of $V$ with center the origin of $I\!R^2$, we can easily see that the lemma is true for $n = 2$. Suppose $n \geq 3$, after dividing each monomial by the common factors, we can also suppose that there is $r \in I\!N$, $r \leq n$, such that the monomials are of the form: $X_1^{\alpha_1} \dots X_r^{\alpha_r}$ and $X_{r+1}^{\alpha_{r+1}} \dots X_n^{\alpha_n}$ with $\alpha_n = min_{i=1,\dots,n}\alpha_i$ (after making a permutation on $(X_1, \dots, X_n)$).
We proceed by induction on $\alpha_n$; if $\alpha_n = 0$ we are done by the inductive hypothesis on $n$. Suppose $\alpha_n > 0$ and consider the two monomials $A = X_1^{\alpha_1} \dots X_r^{\alpha_r}$ and $B = X_{r+1}^{\alpha_{r+1}} \dots X_{n-1}^{\alpha_{n-1}}$; by the induction hypothesis on $n$ , if $V'$ is a neighborhood of $0 \in I\!R^{n-1}$, there exist a real analytic manifold $M$, $\pi : M \to V'$ a proper real analytic surjective mapping such the conditions a) and b) of lemma are satisfied. Let $a \in \pi^{-1}(0)$, there is $U'$ a chart, $a \in U'$, with coordinates $y = (y_1, \dots, y_{n-1})$ such that $A \circ \hat{\pi}_a = y_1^{\beta_1} \dots y_{n-1}^{\beta_{n-1}}$ and $B \circ \hat{\pi}_a = y_1^{\beta'_1} \dots y_{n-1}^{\beta'_{n-1}}$ with $(\beta_1, \dots, \beta_{n-1}) \leq (\beta'_1, \dots, \beta'_{n-1})$ or $(\beta'_1, \dots, \beta'_{n-1}) \leq (\beta_1, \dots, \beta_{n-1})$. Consider the two monomials $y_1^{\beta_1} \dots y_{n-1}^{\beta_{n-1}}$ and $y_1^{\beta'_1} \dots y_{n-1}^{\beta'_{n-1}} X_n^{\alpha_n}$ on $U' \times I\!R$. If $(\beta_1, \dots, \beta_{n-1}) \leq (\beta'_1, \dots, \beta'_{n-1})$ we are done. We suppose $(\beta'_1, \dots, \beta'_{n-1}) < (\beta_1, \dots, \beta_{n-1})$; after dividing by common factors, we are in the situation :$y_1^{\gamma_1} \dots y_{n-1}^{\gamma_{n-1}}$ and $X_n^{\alpha_n}$. If one of the $\gamma_i < \alpha_n$, then we use the second induction (induction on $\alpha_n$). Suppose $\gamma_i \geq \alpha_n$, $\forall i = 1, \dots, n-1$. We will blow up $U' \times I\!R$ with center $y_1 = X_n = 0$. Let $\tilde{\pi} : \tilde{U} \to U' \times I\!R$ this blowing-up. We can cover $\tilde{U}$ by two coordinate charts: $\tilde{U}_1$ and $\tilde{U}_2$, with respect to these charts, $\tilde{\pi}$ is given, respectively, by

$$\tilde{\pi}(y_1, \dots, y_n) = (y_1, y_2, \dots, y_{n-1}, y_n y_1)$$

and

$$\tilde{\pi}(y_1, \dots, y_n) = (y_n y_1, y_2, \dots, y_{n-1}, y_n)$$

In the chart $\tilde{U}_1$ ours monomials where of the form:

$$y_1^{\gamma_1 - \alpha} y_2 \dots y_{n-1}^{\gamma_{n-1}}, \;\; y_n^{\alpha_n}$$

By continuing, we will have $\gamma_1 - \alpha_n < \alpha_n$ and the inductive hypothesis on $inf \gamma_i$ will proves the lemma. In the second chart $\tilde{U}_2$, the result is true since $(\gamma_1, \dots, \gamma_{n-1}, \gamma_1, ) \geq (0, \dots, 0, \alpha_n)$.

**Proposition 6** *Let $\hat{f} \in I\!R[[X_1, \dots, X_n]]$ and $V \subset I\!R^n$ an open neighborhood of $0$. There exist $Z$ a real analytic manifold, $\pi : Z \to V$ a proper real analytic surjective mapping with:*
*$\forall a \in \pi^{-1}(0)$ admits a coordinate neighborhood $U$ with coordinates $y = (y_1, \dots, y_n)$, such that $\hat{f} \circ \hat{\pi}_a = y_1^{\alpha_1} \dots y_n^{\alpha_n} \hat{h}$, where $\hat{h} \in I\!R[[Y_1, \dots, Y_n]]$ a unit.*

Proof.
Let us remark that we can write $\hat{f}$ on the form:

$$\hat{f} = \sum_{\omega \in \Omega \subset I\!N^n} \hat{f}_\omega X^\omega,$$

where $\Omega \subset I\!N^n$ is a finite set and $\hat{f}_\omega \in I\!R[[X_1, \dots, X_n]]$ is a unit for each $\omega \in \Omega$. By lemma 5 there exist a real analytic manifold $Z$, $\pi : Z \to V$ a real analytic proper surjective mapping such that: for all $a \in \pi^{-1}(0)$ admits a coordinate neighborhood $U$ with coordinates $y = (y_1, \dots, y_n)$ and the set $\{\mu_\omega \,/\, X^\omega \circ \hat{\pi}_a = y^{\mu_\omega}\}$ is totally ordered. Let $\mu_{\omega_0}$ the least element. We have $\hat{f} \circ \hat{\pi}_a = \sum_{\omega \in \Omega \subset I\!N^n} \hat{f}_\omega \circ \hat{\pi}_a X^\omega \circ \hat{\pi}_a = y^{\mu_{\omega_0}} \sum_{\omega \in \Omega \subset I\!N^n} \hat{f}_\omega \circ \hat{\pi}_a y^{\mu_\omega - \mu_{\omega_0}}$, this proves the result.

**Proposition 7** *Let $f \in C_n(M)$; then there exist $V$ an open neighborhood of $0 \in I\!R^n$, $Z$ a real analytic manifold and $\pi : Z \to V$ a proper real analytic surjective mapping with:*
*$\forall a \in \pi^{-1}(0)$ admits a coordinate neighborhood $U$, with coordinates $y = (y_1, \dots, y_n)$, such that $f \circ \pi_{|U}(y) = y^\mu \varphi(y)$ where $\mu \in I\!N^n$, $\varphi \in C_U(M)$ and $\varphi(y) \neq 0$, $\forall y \in U$.*

Proof.
Choose $V$ an open neighborhood of $0 \in I\!R^n$ where $f$ is defined and the proposition 6 can be applied, there exist $Z$ a real analytic manifold, $\pi : Z \to V$ a proper real analytic surjective mapping, such that for each $a \in \pi^{-1}(0)$ admits a coordinate neighborhood $U$, with coordinates $y = (y_1, \dots, y_n)$ in which we have $T_0 f \circ \hat{\pi}_a = y^\mu \hat{h}$ where $\hat{h} \in I\!R[[y_1, \dots, y_n]]$ a unit. Since $f \circ \pi_{|U} \in C_U(M)$; then corollary 1 implies that $f \circ \pi_{|U} = y^\mu \varphi(y)$, $\varphi(0) \neq 0$; which proves the proposition.

# 5 Topological noetherianity

**Lemma 6** *Every decreasing sequence of germs: $f_1^{-1}(0) \supset f_2^{-1}(0) \supset \dots f_q^{-1}(0) \supset \dots$, with $f_j \in C_n(M)$ is stationary.*

Proof.
By induction on $n$; the lemma is trivially true for $n = 1$. Suppose $n > 1$ and the result holds for $n - 1$. According to the proposition 7, there exist $V$ an open neighborhood of $0 \in I\!R^n$, $Z$ a real analytic manifold and $\pi : Z \to V$ a proper real analytic surjective mapping such that for each $a \in \pi^{-1}(0)$ admits a coordinate neighborhood $U$, with coordinates system $y = (y_1, \dots, y_n)$ in which we have $f_1 \circ \pi_{|U}(y) = y^\mu \varphi(y)$ and $\varphi(y) \neq 0$, $\forall y \in U$. It is enough to prove that the sequence $(f_j \circ \pi)^{-1}(0)$ is stationary in a neighborhood for every point $a \in \pi^{-1}(0)$. We can then suppose that $f_1(y) = y_1^{\mu_1} \dots y_n^{\mu_n} \varphi(y)$, $\varphi(y) \neq 0, \forall y \in U$. Let $J = \{j = 1, \dots, n \,/\, \mu_j \neq 0\}$. For each $j \in J$ the sequence $(f_l^{-1}(0) \cap \{y \in U \,/\, y_j = 0\})_l$ is stationary by the inductive hypothesis; so then our sequence is stationary near $a$, which proves the lemma.

## 5.1 M-manifold

**Definition 4** *An n-dimensional manifold is a hausdorff space with countable basis in which each point has a neighborhood homeomorph to an open set in $I\!R^n$. A M-structure on a manifold $Z$ is a family $\mathcal{F} = \{(U_i, \varphi_i) \,/\, i \in I\}$ of homeomorphism $\varphi_i$, called local coordinate system, of open set $U_i \subset Z$ on open set $\tilde{U_i} \subset I\!R^n$ such that:*

*a) If $(U_i, \varphi_i), (U_j, \varphi_j) \in \mathcal{F}$, then each cartesian component of the map: $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \subset I\!R^n \to \varphi_j(U_i \cap U_j) \subset I\!R^n$ is in $C_{\varphi_i(U_i \cap U_j)}(M)$.*

*b) $Z = \cup_{i \in I} U_i$.*

A manifold with M-structure is called a M-manifold.
Let $Z$ be a M-manifold and $U \subset Z$ an open set. A function $\varphi$ defined in $U$ will be said to be in $C_U(M)$ if for every coordinate system, $(U_i, \varphi_i)$, the composite function $\varphi \circ \varphi_i^{-1} \in C_{\varphi_i(U_i \cap U)}(M)$. We shall some what denote $\varphi \circ \varphi_i^{-1}$ by $\varphi_{|U_i \cap U}$.
Let us remark that every real analytic manifold is a M-manifold.

Let $Y \subset Z$, we said that $Y$ is a smooth M-submanifold if $Y$ is covered by coordinate charts $U$ of $M$, each of which has local coordinates $z = (x, y)$; $x = (x_1, \ldots, x_m)$ , $y = (y_1, \ldots, y_p)$ in which $Y \cap U = \{y_1 = \ldots = y_p = 0\}$.

Let $Z$ be a M-manifold and $Y$ a closed M-submanifold of $Z$, we define the blowing-up $\pi : Z' \to Z$ with center $Y$: $Z'$ is a M-manifold and $\pi$ is a proper map in the class $M$ such that:

1) $\pi$ restricts to an isomorphism in the class $M$: $Z' - \pi^{-1}(Y) \to Z - Y$.

2) Let $U \subset Z$ be a coordinate chart with local coordinates in $U$ defined by $\varphi : U \to V \times W$, where $U$ , $W$ are open neighborhoods of the origin in $I\!R^p$ , $I\!R^{n-p}$, respectively, and $\varphi(U \cap Y) = \{0\} \times W$. Let $\pi_0 : V' \to V$ be the blowing-up of $V$ with center $\{0\}$. Then there is a isomorphism in the class $M$: $\varphi' : \pi^{-1}(U) \to V' \times W$ such that:
$$\pi_0 \times id_W \circ \varphi' = \varphi \circ \pi_{|\pi^{-1}(U)}.$$

**Definition 5** *Let $Z$ be a M-manifold. Let $U$ be an open subset of $Z$ and let $Y$ be a closed M-submanifold of $U$. Let $\pi : Z' \to Z$ denote the composition of the blowing-up $Z' \to U$ of $U$ with center $Y$ and the inclusion $U \to Z$. We call $\pi$ a local blowing-up of $Z$ with center $Y$.*

We will consider mappings, $\pi : \ Z' \to Z$ obtained as the composition of a finite sequence of local blowings-up; i.e. $\pi = \pi_1 \circ \pi_2 \circ \ldots \pi_k$, where, for each $i = 1, \ldots, k$, $\pi_i : Z_{i+1} \to Z_i$ is a local blowing-up of $Z_i$, and $Z_1 = Z$, $Z_{k+1} = Z'$.

# 6 Łojasiewicz's inequality

In the following, $Z$ will be a M-manifold, $dimZ = n$ and $W$ an open subset of $Z$. As an immediate consequence of proposition 7, we have:

**Proposition 8** *Let $f \in C_W(M)$; then each $a \in W$ admits an open neighborhood, $V$, for which there exist $Z'$ a M-manifold, and $\pi : Z' \to V$ a proper surjective mapping in the class $M$, such that:*

*i) $\forall b \in \pi^{-1}(a)$ admits a coordinate neighborhood $U$, with coordinates system $y = (y_1, \ldots, y_n)$ in which $f \circ \pi(y) = y^\mu \varphi(y)$, $\forall y \in U$, where $\varphi \in C_U(M)$ and $\varphi(y) \neq 0$, $\forall y \in U$.*

*ii) $\pi_{|U} : U \to V$ is a finite composition of local blowings-up.*

**Remark 3** *We require that the mapping $\pi : Z' \to V$ satisfy the following additional condition: $\forall b \in \pi^{-1}(a)$ admits a coordinate neighborhood $U_b$ for which there exist $q \in I\!N$, $\varphi' : U_b \to \varphi'(U_b) \subset V \times P^q(I\!R)$ an isomorphism in the class $M$, such that $\varphi'(U_b)$ is a M-submanifold defined by homogeneous polynomial equations (in homogeneous coordinates of $P^q(I\!R)$) whose coefficients are in $C_V(M)$.*

A local blowing-up has this property. We can easily see that the composition of two local blowings-up has also this property. By condition ii) of the last proposition, we see then that $\pi$ can be chosen as in the remark.

**Theorem 2** *Let $f \in C_W(M)$; let us denote by $V_W(f) = \{x \in W \,/\, f(x) = 0\}$. Let $g$ be any $C^\infty$ function on $W$ such that $g(x) = 0$, $\forall x \in V_W(f)$. Then, for every compact subset $K \subset W$, there exist $N > 0$, $C > 0$, such that:*

$$| g(x) |^N \leq C \mid f(x) |, \ \forall x \in K$$

Proof.
We can suppose that $| g(x) | \leq 1$, $\forall x \in K$. The question is local $W$, we may prove:
$\forall a \in W$ admits a coordinate neighborhood, $V_a$, for which there exist $N_a > 0$, $C_a > 0$, such that:

$$| g(x) |^{N_a} \leq C_a \mid f(x) |, \ \forall x \in V_a.$$

Hence we can cover $K$ by a finite $V_{a_i}$, $i = 1, \ldots, l$, and we take $N = max_i N_{a_i}$, $C = max_i C_{a_i}$.
Let $a \in W$, by proposition 8 , there exist $V_a$ a coordinate neighborhood of $a$, with coordinates system $x = (x_1, \ldots, x_n)$, centred at $a$ i.e $x_i(a) = 0, \forall i = 1, \ldots, n$, $Z'$ a M-manifold and $\pi : Z' \to V_a$ a proper surjective mapping in the class M, such that:

(*) $\forall b \in \pi^{-1}(a)$ admits a coordinate neighborhood, $U_b$, with coordinates system $y = (y_1, \ldots, y_n)$ centred at $b$ in which $f \circ \pi(y) = y^\mu \varphi(y), \forall y \in U_b$, where $\mu \in I\!N^n$, $\varphi \in C_{U_b}(M)$ and $\varphi(y) \neq 0$, $\forall y \in U_b$. Since $\pi$ is proper, there exists a finite set $\Lambda \subset I\!N$ such that $\bigcup_{\alpha\in\Lambda} U_{b_\alpha}$ is an open covering of $\pi^{-1}(a)$, $b_\alpha \in \pi^{-1}(a)$, $\forall \alpha \in \Lambda$ and $f \circ \pi(y) = y^{\mu_\alpha} \varphi_\alpha(y), \forall y \in U_{b_\alpha}$, $\varphi_\alpha(y) \neq 0$, $\forall y \in U_{b_\alpha}$ and $\varphi_\alpha \in C_{U_{b_\alpha}}(M)$.
Write $\mu_\alpha = (\mu_{\alpha 1}, \ldots, \mu_{\alpha n})$ and let $\Delta_\alpha$ be the set of those $i$ where $\mu_{\alpha i} > 0$ ($\Delta_\alpha$ may be empty for some $\alpha$). The assumption on $g$ implies that $g \circ \pi$ vanishes identically on each of the hyperplane in $U_{b_\alpha}$ $y_i = 0$ with $i \in \Delta_\alpha$. Hence $g \circ \pi$ is divisible by the product of those $y_i$ with $i \in \Delta_\alpha$. Then $g \circ \pi(y) = y^{\beta_\alpha} h_\alpha(y)$, $\forall y \in U_{b_\alpha}$, $\beta_\alpha = (\beta_{\alpha 1}, \ldots, \beta_{\alpha n})$ and $h_\alpha$ is a $C^\infty$ function on $U_{b_\alpha}$. Recall that $\beta_{\alpha j} > 0$ if $j \in \Delta_\alpha$.
Let $\Delta'_\alpha = \{j \in \Delta_\alpha \,/\, \beta_{\alpha j} < \mu_{\alpha j}\}$ and put $q_\alpha = max_{j \in \Delta'_\alpha} \frac{\mu_{\alpha j}}{\beta_{\alpha j}}$. We see that $(g \circ \pi(y))^{q_\alpha} = \psi_\alpha(y)(f \circ \pi)(y), \forall y \in U_{b_\alpha}$. where $\psi_\alpha$ is a $C^\infty$ function on $U_{b_\alpha}$.
If $r > 0$, we write $U_{b_\alpha}(r) := \{y \in U_{b_\alpha} \,/\, \sum_{i=1}^n y_i^2 \leq r\}$; since $\pi$ is proper, there exists $\rho > 0$ such that:

$$V_a(\rho) = \{x \in V_a \,/\, \sum_{i=1}^n x_i^2 \leq \rho\} \subset \cup_{\alpha\in\Lambda} \pi(U_{b_\alpha}(\rho).$$

Let $C_\alpha = sup_{y \in U_{b_\alpha}(\rho)} \mid \psi_\alpha(y) |$, $C = max C_\alpha$ and $N = max q_\alpha$. Then for all $x \in V_a(\rho)$, we have $| g(x) |^N \leq | f(x) |$, which proves the theorem.

Let us remark, by the previous proof, that the infimum of $\lambda > 0$ such that there exists $C > 0$ with: $| g(x) |^\lambda \leq C \mid f(x) |$, $\forall x \in K$, is a rational number.

**Theorem 3** *Suppose that $W \subset I\!R^n$ is an open set and $f \in C_W(M)$. Then for each compact subset $K \subset W$, we can find $N > 0$, $C > 0$, such that:*

$$C \mid f(x) | \geq d(x, V_W(f))^N, \quad \forall x \in K.$$

Proof.
We may prove that, for all $a \in W$ admits a neighborhood, $V_a$, and constants $N_a > 0$, $C_a > 0$, such that:

$$C_a \mid f(x) \mid \geq d(x, V_{V_a}(f))^{N_a}, \quad \forall x \in V_a.$$

Let $a \in W$, there exists $\pi : Z' \to V_a$ having the same properties as the proof of the previous theorem. We have then a finite covering of $\pi^{-1}(a) \subset \bigcup_{\alpha \in \Lambda} U_{b_\alpha}$ and $\forall \alpha \in \Lambda$, $f \circ \pi(y) = y^{\mu_\alpha} \varphi_\alpha(y), \forall y \in U_{b_\alpha}$. Then $V_{U_{b_\alpha}}(f \circ \pi)$ is equal to the union of those coordinate hyperplane $H_{\alpha i}$ defined by $y_i$ with $i \in \Delta_\alpha$. Let us define $\psi_{\alpha i}(y) = d(\pi(y), \pi\gamma_{\alpha i}(y))^2$, $y \in U_{b_\alpha}$, where $\gamma_{\alpha i}(y)$ denote the orthogonal projection from $U_{b_\alpha} \simeq I\!R^n$ to $H_{\alpha i}$. We see that $\psi_{\alpha i}$ is a $C^\infty$ function on $U_{b_\alpha}$. Let $\psi_\alpha = \prod_{i \in \Delta_\alpha} \psi_{\alpha i}$. Then:

- $\psi_\alpha$ is a $C^\infty$ function on $U_{b_\alpha}$,
- $\psi_\alpha(y) \geq d(\pi(y), V_{V_a}(f))^{2n_\alpha}$, $n_\alpha$ is the number of elements of $\Delta_\alpha$.

We have $V_{U_{b_\alpha}}(f \circ \pi) \subset V_{U_{b_\alpha}}(\psi_\alpha)$; by the previous theorem, there exist $\rho > 0$, $N_\alpha > 0$, $C_\alpha > 0$, such that, $\forall y \in U_{b_\alpha}(\rho)$:

$$C_\alpha \mid (f \circ \pi)(y) \mid \geq \mid \psi_\alpha(y) \mid^{N_\alpha} .$$

Let $N = max_{\alpha \in \Lambda} \frac{N_\alpha}{2n_\alpha}$ with $n_\alpha \neq 0$; then we have: $\forall x \in V_a(\rho)$, if $C = max_{\alpha \in \Lambda} C_\alpha$, $C \mid f(x) \mid \geq d(x, V_{V_a(\rho)})^N$.

# 7 Quasi semi-analytic sets

**Definition 6** *Let $A$ be a subset of a M-manifold $Z$. It is said to be quasi semi-analytic at point $a \in Z$, if there exist an open neighborhood $V$ of $a$ in $Z$ and a finite number of elements of $C_V(M)$, $g_i$ and $f_{ij}$, such that:*

$$A \cap V = \bigcup_i \{x \in V \ / \ g_i(x) = 0, \ \ f_{ij}(x) > 0, \ \forall j\ \}$$

If $A$ is quasi semi-analytic at every point on $Z$, we say that $A$ is quasi semi-analytic in $Z$.

**Remark 4** *i) The property " quasi semi-analytic" is preserved by locally finite union, locally finite intersection and the complement.*

*ii) If $A \subset Z$ is quasi semi-analytic set it is easy to see that for all $a \in Z$, there exists $V$ an open neighbourhood of $a$ in $Z$ such that $A \cap V$ is a finite disjoint union of sets of the form:*

$$\{x \in V \ / \ \varphi_0(x) = 0, \ \ \varphi_1(x) > 0, \ldots, \varphi_r(x) > 0\},$$

*where $\varphi_0, \varphi_1, \ldots, \varphi_r$ are in $C_V(M)$.*

**Theorem 4** *Let $A$ be a quasi semi-analytic set in $Z$, then for each $x \in Z$ admits a neighborhood $V$ such that $A \cap V$ has only a finite number of connected components.*

Proof.
We will use the notation of theorem 2 with $f = \prod_{i,j} g_i f_{ij}$. It is enough to prove that for each $\alpha \in \Lambda$, the number of connected components of $U_{b_\alpha} \cap \pi^{-1}(A)$ is finite. Since $f \circ \pi(y) = y^{\mu_\alpha} \varphi_\alpha(y)$ and $\varphi_\alpha(y) \neq 0$, $\forall y \in U_{b_\alpha}$, we can easily see that :

$$g_i \circ \pi(y) = y^{\mu_{\alpha_i}} \varphi_{\alpha_i}(y) \ , \ f_{ij} \circ \pi(y) = y^{\mu_{\alpha_{ij}}} \varphi_{\alpha_{ij}}(y), \ \forall y \in U_{b_\alpha},$$

where $\varphi_{\alpha_i}(y) \neq 0$ , $\varphi_{\alpha_{ij}}(y) \neq 0$, $\forall y \in U_{b_\alpha}, \forall i$ , $\forall j$.
This proves that $U_{b_\alpha} \cap \pi^{-1}(A)$ has only a finite number of connected components, which proves the theorem.
Let us give some notations and definitions. Let $U$ be an open subset of $Z$, $A \subset U$; we denote by $I_U(A) := \{ f \in C_U(M) \,/\, f(x) = 0 \,,\, \forall x \in A \}$; $I_U(A)$ is an ideal of $C_U(M)$. Let $F \subset U$; we say that $F$ is a global quasi analytic set in $U$, if there exist $h_1, \ldots, h_q \in C_U(M)$, such that $F = \{ x \in U \,/\, h_1(x) = 0, \ldots, h_q(x) = 0 \}$. We suppose that $U$ is a chart of $Z$, $a \in U$, with coordinates $x = (x_1, \ldots, x_n)$ centered at $a$. If $f \in C_U(M)$ we denote by $\nu_a(f)$ the maximum of $q \in I\!N$, such that the Taylor expansion of $f$ at $a$, $T_a f$, is in $\underline{m}^q$ ($\underline{m}$ is the maximal ideal of $I\!R[[X_1, \ldots, X_n]]$).

**Proposition 9** *Let $F$ be a global quasi analytic set in $U$. Let $k \in I\!N$ be the maximum of integers such that there exist $f_1, \ldots, f_k \in I_U(F)$ and a jacobien $\Delta = \frac{D(f_1,\ldots,f_k)}{D(x_{i_1},\ldots,x_{i_k})} \notin I_U(F)$. Put $\Gamma = \{ x \in U \,/\, f_1(x) = \ldots = f_k(x) = 0 \,,\, \Delta(x) \neq 0 \}$. Then $F - V(\Delta) := \{ x \in F \,/\, \Delta(x) \neq 0 \}$, is a submanifold of $U$, quasi semi-analytic; moreover $F - V(\Delta)$ is a union of some connected components of $\Gamma$.*

Proof.
Clearly we have $F - V(\Delta) \subset \Gamma$, in order to prove the proposition, it is enough to prove that for each $x \in F - V(\Delta)$, the germs of $\Gamma$ and $F - V(\Delta)$ at $x$ are the same. We may prove that the germ of $\Gamma$ at $x$, $\Gamma_x$, is contained in $(F - V(\Delta))_x$. Suppose, for a contradiction, that $\Gamma_x \not\subset (F - V(\Delta))_x$; then there exists $g \in I_U(F)$ such that $g_{|\Gamma_x} \neq 0$. By lemma 7, there exists $h \in \{1, \ldots, n\} - \{i_1, \ldots, i_k\}$ such that, if $g_1 = \frac{D(f_1,\ldots,f_k,g)}{D(x_{i_1},\ldots,x_{i_k},x_h)}_{|\Gamma_x}$, then $\nu_x(g_1) < \nu_x(g_{|\Gamma_x})$. By definition of $k$, we have $g_1 \in I_U(F)$ and also $g_{|\Gamma_x} \neq 0$. We continue with $g_1$ in place of $g$ and so on. At the end we find $g_q \in I_U(F)$ and $g_q(x) \neq 0$, which is a contradiction.

**Lemma 7** *Let $U$ be an open neighborhood of $0$ in $I\!R^n$, put*

$$S = \{ x \in U \,/\, f_1(x) = \ldots = f_k(x) = 0 \,,\, \Delta(x) = \frac{D(f_1, \ldots, f_k)}{D(x_1, \ldots, x_k)}(x) \neq 0 \},$$

*where $f_1, \ldots, f_k \in C_U(F)$. Suppose that $0 \in S$. Let $g \in C_U(M)$ such that $g_{|S} \neq 0$. Then there exists $h > k$, $h \leq n$, such that $\nu_0(g_{|S}) > \nu_0[\frac{D(f_1,\ldots,f_k,g)}{D(x_1,\ldots,x_k,x_h)}_{|S}]$.*

Proof.
Since the mapping $x = (x_1, \ldots, x_n) \to (f_1(x), \ldots, f_k(x), x_{k+1}, \ldots, x_n)$ is a local diffeomorphism near 0, we can suppose that $f_i(x) = x_i$, $\forall i = 1, \ldots, k$. The result is then abvious in this situation.
In the following we call $\Gamma = \{ x \in U \,/\, f_1(x) = \ldots = f_k(x) = 0 \,,\, \Delta(x) \neq 0 \}$ a quasi analytic

strate. Let $B \subset U$; $B$ is called quasi semi-analytic strate, if $B$ is the intersection of a quasi analytic strate with an open set of the form: $\{x \in U \,/\, \varphi_1(x) > 0, \ldots, \varphi_q(x) > 0\}$, where $\varphi_1, \ldots, \varphi_q \in C_U(M)$.
Let $U \subset Z$ be a chart of $Z$ with coordinates system $y = (y_1, \ldots, y_n)$. Let $B \subset U$, we say that $B$ is a quadrant if $B$ is defined by a system of some equalities $y_i = 0$ and some inequalities $\epsilon_j y_j > 0$ with $\epsilon_j = \pm 1$.

**Theorem 5** *Let $A \subset Z$ be a quasi semi-analytic set; then for each $a \in Z$ admits an open neighborhood, $V$, such that: $A \cap V = \bigcup_{j=1}^{s} \Lambda_j$, where, for each $j = 1, \ldots s$, $\Lambda_j$ is a submanifold of $V$, $\Lambda_i \cap \Lambda_j = \emptyset$ if $i \neq j$, and $\Lambda_j$ is a finite union of connected components of a quasi semi-analytic strate.*

Proof.
By remark 4, ii), it is enough to prove the theorem with a set of the form: $A = \{x \in U \,/\, \varphi_0(x) = 0, \varphi_1(x) > 0, \ldots, \varphi_q(x) > 0\}$, $\varphi_0, \ldots, \varphi_q \in C_U(M)$ and $U$ an open neighborhood of $a$ in $Z$. Let $F = \{x \in U \,/\, \varphi_0(x) = 0\}$; by proposition 9, there exists $f_0 \in C_U(M)$ , $f_0 \notin I_U(F)$, such that the set $F - V(f_0) = \{x \in F \,/\, f_0(x) \neq 0\}$ is a union of some connected components of a quasi analytic strate. Put $F_1 = \{x \in \,/\, \varphi_0^2(x) + f_0^2(x) = 0\}$; $F_1 \subset F$. We repeat the same thing with $F_1$ in place of $F$. Hence we construct a decreasing sequence $F \supset F_1 \supset \ldots$, $F_j = V(f_j)$ , $f_j \in C_U(M)$, such that for each $j \in I\!N$, $F_j - F_{j+1}$ is a union of some connected components of a quasi analytic strate. By lemma 6, there exist $s \in I\!N$, and an open neighborhood of $a$, $V$, such that, $\forall j > s$, $F_j \cap V = F_{j+1} \cap V$. For $j \leq s$, put $\tilde{\Gamma}_j = F_j - F_{j+1}$, then $V \cap F = \bigcup_{j=1}^{s} \tilde{\Gamma}_j \cap V$. We see then that:

$$A \cap V = \bigcup_{j=1}^{s} \Lambda_j$$

where $\Lambda_j = \{x \in \tilde{\Gamma}_j \cap V \,/\, \varphi_1(x) > 0, \ldots, \varphi_q(x) > 0\}$. By shrinking, if necessary, $V$, we see that $\Lambda_j$ has a finite number of connected components (theorem 4), which proves the theorem.

By the previous theorem, we define the topological dimension of $A$ at $a \in Z$, $dim_a A$, by the maximum of dimension of $\Lambda_j$, $j = 1, \ldots, s$. This definition is independent of the family $\Lambda_j$: $dim_a A = q$ if and only if $A$ contains an open set homeomorph to an open ball in $I\!R^q$, but not an open set homeomorph to an open ball in $I\!R^l$, $l > q$.

**Theorem 6** *Let $A \subset Z$ be a quasi semi-analytic set; then each connected component of $A$ is a quasi semi-analytic set. The closure of $A$ in $Z$, $\overline{A}$, is also a quasi semi-analytic set.*

Proof.
Let $\Gamma \subset A$ be a connected component of $A$. Let $a \in Z$ such that the germ of $\Gamma$ at $a$ is not empty. There exists a neighborhood of $a$ in $Z$, $V_a$, such that $A \cap V_a$ is a finite union of sets of the form:

$$\Lambda = \{x \in V_a \,/\, \varphi_0(x) = 0, \varphi_1(x) > 0, \ldots, \varphi_q(x) > 0\},$$

where $\varphi_0, \varphi_1, \ldots, \varphi_q \in C_{V_a}(M)$.
Clearly we can suppose that $A \cap V = \Lambda$. Let $f = \varphi_0.\varphi_1.\ldots.\varphi_q$; we keep the notation of the proof of theorem 2. Since $\pi^{-1}(\Gamma) \cap U_{b_\alpha}$ is open and closed in $\pi^{-1}(A) \cap U_{b_\alpha}$; $\pi^{-1}(\Gamma) \cap U_{b_\alpha}$ is a finite union of quadrants in $U_{b_\alpha}$. By remark 3, there exists $q \in I\!N$ such that $U_{b_\alpha}$ is isomorphic to a M-submanifold of $V_a \times I\!P^q(I\!R)$ defined by homogeneous polynomials with coefficients in $C_{V_a}(M)$.

By lemma 8, $\pi(\pi^{-1}(\Gamma)\cap U_{b_\alpha})$ is a quasi semi-analytic set. Since $\pi$ is proper, there exists $V'_a \subset V_a$ a neighborhood of $a$ such that $\pi^{-1}(V'_a) \subset \bigcup_{\alpha\in\Lambda} U_{b_\alpha}$; then $\pi[\bigcup_{\alpha\in\Lambda} U_{b_\alpha}]$ is a neighborhood of $a$ ($\pi$ is surjective) and $\bigcup_\alpha \pi(U_{b_\alpha}) \cap \Gamma = \bigcup_\alpha \pi(\pi^{-1}(\Gamma) \cap U_{b_\alpha})$, which proves the first statement.
We can choose, for each $\alpha \in \Lambda$, a closed neighborhood of a, $U'_{b_\alpha} \subset U_{b_\alpha}$ such that, $\pi^{-1}(a) \subset \bigcup_\alpha U'_{b_\alpha}$. Let

$$A_1 = \bigcup_\alpha \pi(U'_{b_\alpha} \cap \overline{\pi^{-1}(A)}).$$

We have $A_1 \subset V_a \cap \overline{A}$ and $V'_a \cap \overline{A} \subset A_1$.
Now since $U'_{b_\alpha} \cap \overline{\pi^{-1}(A)} = \overline{\pi^{-1}(A) \cap U'_{b_\alpha}}$, and $\pi^{-1}(A) \cap U'_{b_\alpha}$ is a finite union of quadrants, by lemma 8, $\pi(U'_{b_\alpha} \cap \overline{\pi^{-1}(A)})$ is a quasi semi-analytic set, hence $V_a \cap \overline{A}$ is also a quasi semi-analytic set since it coincide with $A_1$ in a neighbourhood of $a$ ($V'_a$).
Łojasiewicz's version of Tarski-Seidenberg theorem.

**Lemma 8** *[?] Let $U \subset Z$ be an open set. Put:*

$$A = \bigcup_{i=1}^{s} \{(x, t_1, \ldots, t_q) \in U \times I\!R^q \,/\, g_i(x, t_1, \ldots, t_q) = 0, f_{i,1}(x, t_1, \ldots, t_q) > 0, \ldots, f_{i,r}(x, t_1, \ldots, t_q) > 0\},$$

*where $g_i$, $f_{i,j} \in C_U(M)[t_1, \ldots, t_q]$, $\forall i$, $\forall j$. If $\pi : U \times I\!R^q \to U$ denote the projection; then $\pi(A)$ is a quasi semi-analytic set.*

# 8 Quasi subanalytic sets

Let $U \subset I\!R^2$ be an open neighborhood of the origin and $\varphi : U \subset I\!R^2 \to I\!R^3$ a mapping with components $\varphi_1, \varphi_2, \varphi_3 \in C_U(M)$. We suppose that there is no nontrivial formal relations between Taylor's series, $T_0\varphi_1, T_0\varphi_2, T_0\varphi_3$, of $\varphi_1, \varphi_2, \varphi_3$ at the origin. Let $r > 0$, such that the set $W = \{(x, y) \in I\!R^2 \ / \ x^2 + y^2 \leq r\} \subset U$. Then $A = \varphi(W) \subset I\!R^3$ is not quasi semianalytic at the origin in $I\!R^3$, whereas $A$ is the projection of the set $\{(x, y, t_1, t_2, t_3) \in U \times I\!R^3 \ / \ x^2 + y^2 \leq r, \ t_i = \varphi_i(x, y), \ i = 1, 2, 3\}$ which is a quasi semianalytic set relatively compact.
Thus the Tarski-Seidenberg theorem is false for quasi semianalytic sets.

**Definition 7** *Let $Z$ be a M-manifold and $A \subset Z$. We say that $A$ is quasi subanalytic in $Z$, if for each $a \in Z$, there exist $U$ an open neighborhood of $a$ in $Z$, $Z'$ a M-manifold and $\mathcal{A} \subset Z \times Z'$ a quasi semi-analytic set in $Z \times Z'$, relatively compact, such that $\pi(\mathcal{A}) = A \cap U$, where $\pi : Z \times Z' \to Z$ is the projection.*

From the properties of quasi semi-analytic sets, we can easily see that a locally finite union and intersection of quasi subanalytic sets is quasi subanalytic. The closure and each connected component of a quasi subanalytic set is quasi subnalytic; the projection of a relatively compact quasi subanalytic set is quasi subanalytic.
We will prove that the complement (and thus the interior) of a quasi subanalytic set is quasi subanalytic. Firstly we establish some measure properties of a quasi subanalytic set. By the work of Charbonnel [2] and Wilkie [11], we will show, first, that we have an uniform bound on the number of connected components of the fibers of a projection restricted to a relatively compact quasi subanalytic set; more precisely:

**Theorem 7** *Let $Z$ and $Z'$ two M-manifolds and $A$ be a relatively compact quasi subanalytic set in $Z \times Z'$. Let $\pi : Z \times Z' \to Z$ be the projection. Then the number of connected components of a fiber $\pi^{-1}(x) \cap A$ is bounded, $x \in Z$.*

Proof.
We proceed by induction on $dimZ$. If $dimZ = 0$, the result is true, since $A$ is relatively compact. Suppose that $dimZ \geq 1$ and the result is true for $n-1$. We can assume that $Z = \mathbb{R}^n$, $Z' = \mathbb{R}^p$ and $A$ relatively compact quasi semi-analytic in $\mathbb{R}^n \times \mathbb{R}^p$. We argue by induction on the maximum dimension of the fibers $A_x = \pi^{-1}(x) \cap A,\ x \in \mathbb{R}^n$. By lemma 6, it is enough to find a quasi analytic set $F \subset \mathbb{R}^n \times \mathbb{R}^p$ such that theorem is true for $A - F$. By theorem 5, we can suppose that $A$ is a connected component of a quasi semi-analytic strate

$$S = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^p \ / \ f_1(x,y) = \ldots = f_k(x,y) = 0,\ \delta(x,y) \neq 0,\ g_1(x,y) > 0, \ldots, g_q(x,y) > 0\}$$

where $\delta(x,y)$ is a jacobien of $(f_1, \ldots, f_k)$. Let $n - \beta$, $0 \leq \beta \leq n$, be the maximum rank of $\pi_{|S}$, then there exists a jacobien:

$$\delta_1(x,y) = \frac{D(f_1, \ldots, f_k)}{D(x_{i_1}, \ldots, x_{i_\beta}, y_{j_1}, \ldots, y_{j_\alpha})}$$

with $\alpha + \beta = k$, such that $\delta_1 \notin I(S)$. We take $F = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^p \ / \ \delta_1(x,y) = 0\}$ and put $S' = S - F$. The rank of $\pi_{|S'} : S' \to \mathbb{R}^n$ is constant and equal to $n - \beta$. For all $x \in \mathbb{R}^n$, $S'_x = \pi^{-1}(x) \cap S'$ is a submanifold of dimension $p - \alpha$.

We can suppose, for the proof, that $p - \alpha = 0$.
Indeed, if $p - \alpha \geq 1$, then each connected component, say $C$, of $\pi^{-1}(x) \cap S'$ is such that $\overline{C} - C \neq \emptyset$ (the projection :$\pi^{-1}(x) \cap S' \to \{y \in \mathbb{R}^p \,/\, y_{j_1} = \ldots = y_{j_\alpha} = 0\}$ is open). Let $\psi(x,y) = \sum_{j=1}^{q} g_j(x,y) + \delta(x,y)^2 + \delta(x,y)^2$; then $\psi(x,y) > 0$ on $C$ and $\psi(x,y) = 0$ if $(x,y) \in \overline{C} - C$. Put $S'' = \{(x,y) \,/\, grad(\psi_{|\pi^{-1}(x) \cap S'})(x,y) = 0\}$, $S''$ is a quasi semi-analytic set. Since $\psi$ is not constant on any connected component of $\pi^{-1}(x) \cap S'$, we have then, for all $x \in \mathbb{R}^n$, $dimS''_x < dimS'_x$ ($S''_x = S'' \cap \pi^{-1}(x)$). We remark that $\psi$ has a positive maximum on each connected component of $\pi^{-1}(x) \cap S'$ hence $S''_x \neq \emptyset$. By the inductive hypothesis on the dimension of the fibers, the theorem is true for $S''$ which implies the result for $S'$.
Suppose $p - \alpha = 0$, then for all $x \in \mathbb{R}^n$, $S'_x$ is a finite set. We consider two cases:
Case 1. $n - \beta < n$.
Let $\pi_1 : \mathbb{R}^n \to \mathbb{R}^{n-\beta} = \{x \in \mathbb{R}^n \,/\, x_{i_1} = \ldots = x_{i_\beta} = 0\}$ be the projection. The inductive hypothesis on $n$ implies that the theorem is true for the mapping $\pi_1 \circ \pi_{|S'}$, hence the theorem is true for $\pi_{|S'} : S' \to \mathbb{R}^n$.

Case 2. $n - \beta = n$.
Let $\pi' : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection on $x_n = 0$ and put $\tilde{\pi} = \pi' \circ \pi$; $\tilde{\pi}_{|S'} : S' \to \mathbb{R}^{n-1}$ is a submersion. For all $x' \in \mathbb{R}^{n-1}$, $\tilde{\pi}^{-1}(x') \cap S'$ is a disjointe union of a finite number of connected curves of class M; by the inductive hypothesis, on $n$, this number of curves is bounded when $x' \in \mathbb{R}^{n-1}$. In order to prove that the number of points in $\pi^{-1}(x) \cap S'$ is bounded ($x = (x', x_n)$), we will prove that each connected component of $\tilde{\pi}^{-1}(x') \cap S'$ does not contains two points of $\pi^{-1}(x) \cap S'$, which proves our result, since the number of connected component of $\tilde{\pi}^{-1}(x') \cap S'$

is bounded when $x' \in I\!R^{n-1}$.
Suppose, for a contradiction, that there exists $C$ a connected component of $\tilde{\pi}^{-1}(x') \cap S'$ such that $C$ contains $a, b \in \pi^{-1}(x) \cap S'$, $a \neq b$. The curve $C$ intersects $\pi^{-1}(x)$ in two points $a, b$. By the generated Roll's lemma [6], there exists $\xi \in C$ such that the tangent space to $C$ at $\xi$ contains a parallel vector to $\pi^{-1}(x) = I\!R^n$ hence the tangent space to $S'$ at $\xi$, $T_\xi S'$, contains a parallel vector to $\pi^{-1}(x) = I\!R^n$, but this is a contradiction since $T_\xi S'$ is transverse to $I\!R^n$.

**Definition 8** *Let $Z$ be a M-manifold and $A \subset Z$. We say that $A$ is Lebesgue measurable [resp. $A$ has measure null] if for any coordinate chart $U$ with coordinates system $\varphi = (x_1, \ldots, x_n)$; $\varphi(U \cap A)$ is Lebesgue measurable in $I\!R^n$ [resp. $\varphi(U \cap A)$ is of measure null].*

Using the last theorem and properties of the class of quasi subanalytic sets cited above, we prove:

**Theorem 8** *Let $A$ be a quasi subanalytic set; the following conditions are equivalent:*

1) *$A$ has non interior point.*
2) *$\overline{A}$ has non interior point.*
3) *$A$ has measure null.*
4) *$\overline{A}$ has measure null.*

Proof.
The proof use theorem 7 and it is the same as in [9].

**Definition 9** *Let $Z'$ be a M-manifold. A mapping $f : A \subset Z \to Z'$ is quasi subanalytic if its graph, $\Gamma_f$, is quasi subanalytic in $Z \times Z'$.*

We will use the following result:

**Proposition 10** *[?] Let $f : A \subset Z \to Z'$ be a quasi subanalytic mapping, then the set of points in $A$ where $f$ is not continuous has no interior points.*

In the following we will show that the dimension of a quasi semi-analytic set is well behaved.

**Lemma 9** *Let $A \subset Z$ be a non empty quasi semi-analytic set, then $dim\,(\overline{A} - A) < dim\, A$.*

Proof.
Recall that, by theorem 6, $(\overline{A} - A)$ is quasi semi-analytic. Suppose, for a contradiction, that $dim\,(\overline{A} - A) := n - k \geq dim\, A := n - l$. We can suppose that $Z = I\!R^n$ and $A$ is relatively compact. Let $\Lambda$ be a connected component of a quasi semi-analytic strate $S \subset I\!R^n$ such that $\Lambda \subset (\overline{A} - A)$ and $dim\, \Lambda = dim\,(\overline{A} - A)$. We have:

$$S = \{x \in I\!R^n \,/\, f_1(x) = \ldots = f_k(x) = 0,\, \delta(x) = \frac{D(f_1, \ldots, f_k)}{D(x_{i_1}, \ldots, x_{i_k})}(x) \neq 0,\; g_1(x) > 0, \ldots, g_q(x) > 0\,\},$$

note that we have by hypothesis $k \leq l$.
Let $\pi_{n-k} : I\!R^n \to I\!R^{n-k} = \{x \in I\!R^n \,/\, x_{i_1} = \ldots = x_{i_k} = 0\}$ be the projection; $\pi_{n-k|\Lambda} : \Lambda \to I\!R^{n-k}$

is a local diffeomorphism. Let $a \in \Lambda$ and put $a' = \pi_{n-k}(a)$. There exist balls in $I\!R^n$ and $I\!R^{n-k}$: $B_n(a,r)$, $B_{n-k}(a',r)$ such that $\pi_{n-k|\Lambda\cap B_n(a,r)} : \Lambda \cap B_n(a,r) \to B_{n-k}(a',r)$ is a diffeomorphism. Let $g : B_{n-k}(a',r) \to \Lambda \cap B_n(a,r)$ be the inverse mapping of $\pi_{n-k|\Lambda\cap B_n(a,r)}$.
Let $B = \{x' \in B_{n-k}(a',r) \,/\, \exists x \in A \cap B_n(a,r),\ \text{with}\ \pi_{n-k}(x) = x'\}$; $B$ is a quasi subanalytic set. Clearly, we have $B_{n-k}(a', \frac{r}{2}) \subset \overline{B}$; hence, by theorem 8, $int(B) \neq \emptyset$, this implies that $k = l$.
Put $\pi_k : I\!R^n \to I\!R^k = \{x \in I\!R^n \,/\, x_j = 0,\ \forall j \notin \{i_1, \ldots, i_k\}\}$. For each $p = 1, 2, \ldots$, let:

$$B_p = \{x' \in B \,/\, \exists y_1, \ldots, \exists y_p \in I\!R^k,\ y_i \neq y_j \text{ if } i \neq j,\ y_i \in \pi_k[A \cap \pi_{n-k}^{-1}(x')]\}.$$

We have

$$\ldots B_{\nu+1} \subset B_\nu \subset \ldots \subset B_2 \subset B_1 = B.$$

By theorem 7, there exists $\mu \in I\!N^*$ such that $int(B_\mu) \neq \emptyset$ and $int(B_{\mu+1}) = \emptyset$; We have then $int(\overline{B_{\mu+1}}) = \emptyset$, hence $int(B_\mu) \cap B - \overline{B_{\mu+1}} \neq \emptyset$. Then there exists a ball $B' \subset B_\mu - B_{\mu+1}$. For each $x' \in B'$, $\pi_k^{-1}(x') \cap A$ contains exactly $\mu$ elements; we can then construct $\mu$ functions $h_1, \ldots, h_\mu : B' \subset I\!R^{n-k} \to I\!R^k$ such that, $\forall j = 1, \ldots, \mu$, $\Gamma_{h_j}$ is quasi subanalytic and $\forall x' \in B'$, $\pi_k[A \cap \pi_{n-k}^{-1}(x')] = \{h_1(x'), \ldots, h_\mu(x')\}$. By construction, we have, $\forall j = 1, \ldots, \mu$, $\forall x' \in B'$, $,h_j(x') \neq \pi_k \circ g(x')$.
By proposition 10, there is a ball $B" \subset B$ such that the restriction of all $h_1, \ldots, h_\mu$ is continuous on $B"$ and there exists $c > 0$ such that $\forall x' \in B"$, $\mid h_j(x') - \pi_k(g(x')) \mid > c,$ $\forall j = 1, \ldots, \mu$; but this is a contradiction with the fact that for all $x' \in B"$, $g(x') \in \Lambda \subset (\overline{A} - A)$; hence the lemma.

# 9 Theorem of the complement

**Theorem 9** *Let $Z$ be a M-manifold and let $B \subset Z$ be a quasi subanalytic set. Then $Z - B$ is quasi subanalytic.*

Proof.
We can assume that $Z = I\!R^n$ and $B$ is relatively compact. We argue by induction on $n$. There exists $A \subset I\!R^n \times I\!R^p$ a relatively compact semianalytic set such that $\pi(A) = B$, where $\pi : I\!R^n \times I\!R^p \to I\!R^n$ is the projection. By theorem 5, we can assume that $A$ is a connected component of a quasi semianalytic strate:

$$S = \{(x,y) \in I\!R^n \times I\!R^p \ /\ f_1(x,y) = \ldots = f_k(x,y) = 0,\ \delta(x,y) \neq 0,\ g_1(x,y) > 0, \ldots, g_q(x,y) > 0\}$$

As in the proof of the theorem 7 (which we keep its notations), it is enough to find a quasi analytic set $F \subset I\!R^n \times I\!R^p$ such that $A - F \neq \emptyset$ and the theorem is true for $\pi(A - F)$. We take $F$ as in the proof of theorem 7 and put $A' = A - F \subset S' = S - F$. We proceed by induction on the maximum dimension of the fibers $\pi^{-1}(x) \cap A'$. Recall that we have $dim(\pi^{-1}(x) \cap S') = p - \alpha$, $\forall x \in I\!R^n$.
Suppose that $p - \alpha = 0$, then $dim\, S' = n - \beta \leq n$. We consider two situations:

Case 1. $\beta > 0$.
Let $\pi_1 : I\!R^n \to I\!R^{n-\beta} = \{x \in I\!R^n \,/\, x_{i_1} = \ldots = x_{i_\beta} = 0\}$ be the projection. The inductive hypothesis shows that the theorem is true in $I\!R^{n-\beta}$. Put $\pi' = \pi_1 \circ \pi$; the number of points in

$S' \cap \pi'^{-1}(u)$ is bounded when $u \in I\!R^{n-\beta}$. Therefore the number of points in $\pi(A') \cap \pi_1^{-1}(u)$ is bounded. By lemma 10 , the complement of $\pi(A')$ in $I\!R^n$ is quasi subanalytic.

Case 2. $\beta = 0$.
We have then $dim\, S' = n$. Let $Q = \overline{A'} - A'$; by lemma 9, $dim\, Q < n$, hence, by the first case, $I\!R^n - \pi(Q)$ is quasi subanalytic. We have $I\!R^n - \pi(A') = (I\!R^n - \pi(\overline{A'}) \cup (\pi(Q) - \pi(A') \cap \pi(Q))$. By case 1, $I\!R^n - \pi(A') \cap \pi(Q)$ is quasi subanalytic, hence $I\!R^n - \pi(A')$ is quasi subanalytic.

If $p - \alpha > 0$, we have see that there exists $S'' \subset S'$, $dim\, S'' < dim\, S'$, $S''$ is quasi semianalytic such that $\pi(S'') = \pi(S')$, by using the inductive hypothesis on the maximum dimension of the fibers $\pi^{-1}(x) \cap A'$, we deduce that $I\!R^n - \pi(A')$ is quasi subanalytic.

**Lemma 10** *Suppose that, in $I\!R^n$, the complement of every quasi subanalytic set is quasi analytic. Let $A \subset I\!R^n \times I\!R^p$ be a relatively compact quasi subanalytic set. Suppose that the number of points in the fibers $A \cap \pi^{-1}(x)$, $x \in I\!R^n$, is bounded, where $\pi : I\!R^n \times I\!R^p \to I\!R^n$ is the projection. Then $I\!R^n \times I\!R^p - A$ is quasi subanalytic.*

Proof.
The proof is the same as in [1, lemma 3.9].

# References


[1] E. Bierstone, P.D. Milman, Semianalytic and subanalytic sets. Publ. Math. I.H.E.S. 67. 5-42 (1989)

[2] J.-Y. Charbonnel, Sur certains sous-ensembles de l'espace euclidien. Ann. Ins. Fourier. Grenoble 41 3 (1991), 679-717.

[3] C.L Childress, Weierstrass division in a quasianalytic local rings. Can. J. Math., Vol. XXVIII, N.5, 1976,pp.938-953.

[4] E. M. Dyn'kin, Pseudoanalytic extention of smooth functions, Amer.Math. Soc. Transl. (2), 115 (1980), pp. 33-58.

[5] M. Klimek, Pluripotentiel theory. London Math. Soc. Monographs. Clarendon Press.

[6] A.G. Khovanskii, Real analytic varieties with the finiteness property and complex abelian integrals, Funct.Ana.an appl. 18, 199-207 (1984).

[7] H. Komatsu, The implicit function theorem for ultradifferentiables mappings, Proc. Japan Acad., Ser. A, 55 (1979), pp. 69-72.

[8] S. £ojasiewicz, Ensembles semi-analytiques, Priprint, Ecole Polytechnique, Paris (1965).

[9] S. Maxwell, A general model completeness result for expansions of the real ordered field. Annals of Pure and Applied logic 95 (1998) 185-227.

[10] J.-Cl. Tougeron, Sur les ensembles semi-analytiques avec conditions Gevrey au bord. Ann. Sci. E.N.S.,4$^{\underline{eme}}$ séries T.27. (1994) 173-208.

[11] A.J. Wilkie, A theorem of the complement and some new o-minimal structures. Sel.math.,New ser. 5 (1999) 397-421.