UNIFORM ASYMPTOTIC BOUND ON THE NUMBER OF ZEROS OF ABELIAN INTEGRALS

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Abstract

We give a uniform asymptotic bound for the number of zeros of complete Abelian integrals in domains bounded away from infinity and the singularities.

0. Introduction.

In [Aea], V.I. Arnol'd posed the question of the number of limit cycles which can arise by perturbing an integrable planar vector field of degree d by a small polynomial perturbation of the same degree. He indicates the relation of this question to the determination of the number of zeros of complete Abelian integrals parameterized by a value of a polynomial of degree not greater than d, which is just the Hamiltonian in the case that the vector field is a Hamiltonian vector field (which is the case treated in the present article). It is pointed out that in this case the number of zeros of these integrals (in an interval which does not contain a critical value of the Hamiltonian, for example), is always finite and could be bounded by a number which depends only on the degree d (this key result is due to Khovanskii and Varchenko [Kh], [Var]). It is noted that no effective estimates of this bound are known.

The main result of this article is formulated in Theorem 0.2 below (proved as Corollary 4.5 and Corollary 4.7 in the text). It was initially obtained in the author Ph.D. thesis [Gr], and gives a (doubly exponential in d) bound on the number of zeros of complete Abelian integrals parameterized by the value of a Hamiltonian, in an interval whose distance from the critical values of the Hamiltonian is bounded away from zero, and which is contained in a fixed bounded interval. The bound is uniform in the sense that it holds for all Hamiltonians in a dense open subset of the space of polynomials of degree d. In fact, the bound is stated for the number of zeros of the integrals in domains in $\mathbb{C}$, which satisfy certain natural restrictions, and which are bounded away from the critical values of the Hamiltonian and the infinity. This uniformity of the asymptotic bound (currently in domains bounded away from the singularities and infinity) appears to overcome the corresponding difficulty in the methods suggested in [NY1], [GI].

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We now give a more precise description of the problem. Consider a polynomial $H(x,y) \in \mathbb{R}[x,y]$ of degree d, such that its highest homogeneous part is a product of pairwise different (up to multiplication by a nonzero constant) linear factors in $\mathbb{C}[x,y]$. Suppose that for some $a,b \in \mathbb{R}$, (a,b) does not contain critical values of H, and let γ_{t_0} denote a compact component of $H^{-1}(t_0)$ for some $t_0 \in (a,b)$. For each $t_1 \in [a,b]$, let γ_{t_1} denote the compact component of $H^{-1}(t_1)$ which is obtained from γ_{t_0} by an isotopy as t varies from t_0 to t_1. Let $\sigma(t) : (a,b) \to \mathbb{R}^2$ denote an analytically embedded segment, such that $\sigma(t) \in \gamma_t \ \forall t \in (a,b)$, and σ is transversal to γ_t for all $t \in (a,b)$. Let $P(x,y), Q(x,y)$ be polynomials of degree not greater than d, and consider the following ε-dependent perturbation of the Hamiltonian vector field determined by H:

$$0.1) \qquad \dot{x} = \frac{\partial H}{\partial y} + \varepsilon Q, \qquad \dot{y} = -\frac{\partial H}{\partial x} - \varepsilon P.$$

For $\varepsilon = 0$ every trajectory which starts at $\sigma(t)$, $t \in (a,b)$, returns to $\sigma(t)$. For any $[a',b'] \in (a,b)$, there exists a small enough nonzero $\varepsilon_0 > 0$, such that the map from $[-\varepsilon_0, \varepsilon_0] \times \sigma[a',b']$ to $\sigma(a,b)$, defined for any $-\varepsilon_0 \le \varepsilon \le \varepsilon_0$ as the corresponding Poincare return map from $\sigma[a',b']$ to $\sigma(a,b)$, indeed exists, and is moreover analytic. Fix ε, $-\varepsilon_0 \le \varepsilon \le \varepsilon_0$; for any $t \in [a',b']$ suppose that the Poincare map sends $\sigma(t)$ to $\sigma(t')$. Then the *displacement map* $\delta(\varepsilon,t)$ is equal by definition to $t - t'$ (so that $\delta(0,t) \equiv 0$ as a function of t). So $\delta(\varepsilon,t)$ is an analytic function on $[-\varepsilon_0,\varepsilon_0] \times [a',b']$. Observe that for any fixed $-\varepsilon_0 \le \varepsilon \le \varepsilon_0$, the zeros of $\delta(\varepsilon,\cdot)$ correspond to periodic trajectories of the perturbed vector field, crossing $\sigma[a',b']$.

It is well known (cf. [R], pg. 72-73, for example) that the ε-derivative of $\delta(\varepsilon,t)$ on the line $\varepsilon = 0$ is given by

$$0.2) \qquad -\int_{\gamma_t} P(x,y)dx + Q(x,y)dy$$

(the orientation of γ_t being given by the flow of the Hamiltonian vector field). Observe that as ε varies, starting from $\varepsilon = 0$, zeros of $\delta(\varepsilon,\cdot)$ may bifurcate from $\{0\} \times \sigma[a',b'] \subset \mathbb{R}^2$ at point $(0,\sigma(t))$ only if $(0,t)$ is a nonsmooth point of the (semianalytic) set $\{(\varepsilon,t) \in [-\varepsilon_0,\varepsilon_0] \times [a',b'] : \delta(\varepsilon,t) = 0\}$. Such points must be the critical points of the map $\delta(t,\varepsilon)$ on the line $\varepsilon = 0$; since the derivative of $\delta(t,\varepsilon)$ on the line $\varepsilon = 0$ is given by 0.2), $(0,t)$ is such a critical point iff 0.2) is zero. Therefore the number of bifurcation points is bounded by the number of zeros of the integral 0.2) on (a,b).

In other words, if 0.2) is not identically zero as a function of t and its number of zeros on (a,b) is N, for any $[a',b'] \subset (a,b)$ there exists $\varepsilon_0 > 0$, such that for any ε, $0 < |\varepsilon| \le |\varepsilon_0|$, the number of limit cycles (i.e. isolated periodic trajectories) of the vector field 0.1) which cross $\sigma[a',b']$ is bounded by N.

Since the time [Aea] was published, numerous partial results became available regarding effective estimates of the number of zeros of Abelian integrals, especially in low degrees; we refer the reader to the survey in [I2]. For the general case, however, there still exist no effective estimates. Below, we formulate the main result of this article. Since we consider the Abelian integrals in the complex plane instead of the real line, we give first some additional background.

For any polynomial $H(x,y) \in \mathbb{C}[x,y]$, there exists a finite set $\Sigma_H \subset \mathbb{C}$, whose points are called the atypical points of $H(x,y)$, such that $H : \mathbb{C}^2 - H^{-1}(\Sigma_H) \to \mathbb{C} - \Sigma_H$ is a (smooth, not holomorphic in general) locally trivial (fiber) bundle. The integral 0.2) can be then analytically continued along any path not passing through the atypical points (which are generically just the critical values of $H(x,y)$). It is therefore a (transcendental) multivalued analytic function on $\mathbb{C} - \Sigma_H$. This fact generalizes as follows. Any element of the homology group of a fiber over a point which is not atypical, can be naturally continued along any path not passing through the atypical points, the resulting homology class being dependent only on the homotopy class of the path connecting the initial and final points. The result of such (multivalued) continuation is called a continuously varying cycle, since in the case that the initial element of the homology group is realized by a map of S^1 into the fiber over the initial point, the continuation along a path not passing through the atypical points is realized by any isotopy of S^1 which is contained in the preimage of the path and respects the fibers. The integral of $Pdx + Qdy$ over any continuously varying cycle is a multivalued analytic function.

Below, $\mathcal{H}^d \cong \mathbb{C}^{(d+1)(d+2)/2}$ denotes the subspace of $\mathbb{C}[x,y]$ consisting of polynomials of degree not greater than d. The bound is given for polynomials $H(x,y)$ whose highest homogeneous part is a product of pairwise different (up to multiplication by a nonzero constant) linear factors. Equivalently, these are polynomials of degree d, whose level curves intersect the line at infinity at precisely d points. We say in this case that the polynomial $H(x,y)$ is regular at infinity. For such polynomials, the only atypical points are their critical values.

Definition 0.1. Let $\{t_1, .., t_N\} \subset \mathbb{C}$ be a finite set of points. A domain $U \subset \mathbb{C} - \{t_1, .., t_N\}$ is called a *simple domain in* $\mathbb{C} - \{t_1, .., t_N\}$ if there exist N nonintersecting rays $r_1, .., r_N$, issuing from the points $t_1, .., t_N$ respectively, such that $U \subset \mathbb{C} - \cup_i r_i$.

Theorem 0.2 [Gr]. *There exists a universal constant $c > 0$, such that the following holds.*

Let $H \in \mathcal{H}^d$ be regular at infinity. Let $P, Q \in \mathcal{H}^d$. Let $\rho > 0$ be any positive number, and let U be a simple domain in $\mathbb{C} - \Sigma_H$, contained in the unit disc, whose distance to Σ_H is at least ρ. Then the number of zeros in U of the integral $\int_{\gamma(t)} P(x,y)dx + Q(x,y)dy$, where $\gamma(t)$ is any continuously varying cycle in the locally trivial bundle determined by H, is not greater than

$$\left(\frac{2}{\rho}\right)^{2^{d^c}}.$$

Remark 0.3. The *existence* of a bound for the number of zeros in *any* simple domain in $\mathbb{C} - \Sigma_H$, follows from Theorem 5 in [Kh].

Remark 0.4. Since the title of [IY] mentions a double exponential estimate as well, the following comment might be helpful. In [IY], the polynomial $H(x,y)$, of some degree d_0, is *fixed*, the polynomials P, Q being any polynomials of degree d. In this setting, it is now known (Khovanskii and Petrov, yet unpublished) that the bound is in fact *linear* in d.

We now outline the contents of the article. In section 1 we deal with linear differential equations whose coefficients are meromorphic and depend on parameters, and which are not singularly perturbed along any holomorphic arc in the parameter space. We say that such equations depend *regularly* on parameters (this terminology is due to the author; we have not found an analogous notion in the literature). It is shown that solutions of such equations admit a uniform (over compact subsets in the parameter space) bound for the number of zeros in domains of the type considered in Theorem 0.5. I thank S. Yakovenko for the suggestion to use his theorem ([Y]) to simplify some of the arguments. To exclude a possible linguistic confusion, we remark that the notion of regular dependence on parameters and the notion of a polynomial regular at infinity are not related.

In section 2, we point out that if the meromorphic coefficients of a linear differential equation are quotients of *integral* polynomials of known degree and height, the bound of section 1 can be made effective. We rely on a theorem of Renegar which gives effective estimates for elimination of quantifiers in the first order theory of the reals.

In section 3, we construct such equations for the Abelian integrals, relying on a result of Gavrilov ([Ga],[N]) and a quantitative version of a theorem of Ilyashenko [I1], the parameters being the coefficients of the polynomial H and (roughly speaking) the coefficients of the polynomials P and Q. In section 4, we show that the linear differential equations constructed indeed depend regularly on parameters. Theorem 0.2 then follows from the result of section 2 and from an additional argument, which shows that Picard-Fuchs equations stay regularly dependent on parameters also after algebraic parameter dependent changes of variable.

Remark 0.5. One may show using similar arguments [Gr], that the number of zeros in the unit disc of (components of) solutions of the linear differential system $\dot{x} = (A_0 + A_1 t + ... + A_d t^d)x$, where $A_0, .., A_d$ are some $n \times n$ matrices over $\mathbb{C}$, of (say l_∞) norm not greater than 1, is bounded by $2^{2^{(nd)^c}}$, where $c > 0$ is again some universal constant. This improves the bound which follows for this case from [NY2], Theorem 1.

This article is in essence a presentation of the major part of the results of [Gr]. We have tried to make the exposition here clearer. In particular, some of the arguments in the thesis are made simpler. The construction of the Picard-Fuchs system 3.15) for the Abelian integrals, while being based on the same elementary idea, differs somewhat from the one given in the thesis, where there was a gap in the proof of the corresponding theorem. Recently, another such construction appeared [N] (see also [NY3]), giving a much better information on the system than that which is provided by our argument. However, if we used this construction here (together with Proposition 3.2 for the quantitative estimates), it would not improve the doubly exponential bound of Theorem 0.2. The reason for this is that the estimate of Theorem 2.1, which we use in the proof of Theorem 0.2, is doubly exponential.

Finally, an important remark about notations. Instead of introducing each time a new universal constant when such is needed, we write $O(x)$ to denote $c_1 x + c_2$, where $c_1, c_2 > 0$

are some *universal* constants (here x is any positive number). Thus, $O(1)$ simply denotes a universal constant.

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1. Linear differential equations which depend regularly on parameters.

Below, we formulate both the definitions and the statements for polydiscs instead of more general domains, as no greater generality is needed. We denote by $\mathcal{O}(T)$ the set of holomorphic functions on the set (open or closed) T, and by $\mathcal{M}(T)$ the set of meromorphic functions on T.

Let $U \times W \subset \mathbb{C} \times \mathbb{C}^p$, $p \geq 1$, be a polydisc, and let the coordinates on $\mathbb{C} \times \mathbb{C}^p$ be denoted by $(t, \lambda_1, .., \lambda_p) = (t, \lambda)$. Below we think of $\lambda_1, .., \lambda_p$ as parameters. Let $f(t, \lambda) \in \mathcal{M}(U \times W)$ be a meromorphic function; we consider it as a parameter dependent meromorphic function of t. To emphasize its domain of definition, we will sometimes say that $f(t, \lambda)$ is a parameter dependent meromorphic function on $U \times W$.

We say that $f(t, \lambda)$ depends *holomorphically* on λ at $\lambda_0 = (\lambda_{10}, .., \lambda_{p0}) \in W$, if $f(t, \lambda) = a(t, \lambda)/b(t, \lambda)$ where $a(t, \lambda), b(t, \lambda) \in \mathcal{O}(U \times W)$, such that $b(t, \lambda_0)$ is not identically zero (it is suitable to recall here that a meromorphic function on a polydisc is always a quotient of two holomorphic functions).

Definition 1.1. A *holomorphic arc* in the parameter space, passing through $\lambda_0 \in W$, is a germ of a holomorphic map from a neighbourhood of $0 \in \mathbb{C}$ into $\mathbb{C}^p$, mapping 0 to λ_0.

Definition 1.2. A parameter dependent meromorphic function $f(t, \lambda)$ on a polydisc $U \times W \subset \mathbb{C} \times \mathbb{C}^p$ will be said to depend *regularly* on parameters at $\lambda = \lambda_0 \in W$, if $f(t, \lambda(\varepsilon))$ depends holomorphically on ε at $\varepsilon = 0$, for every holomorphic arc $\lambda(\varepsilon)$ passing through λ_0, on which $f(t, \lambda)$ is defined. We will say that $f(t, \lambda)$ depends regularly on λ in W, if it depends regularly on λ at any point of W.

Remark 1.3. We say that $f(t, \lambda)$ is defined on $\lambda(\varepsilon)$ if it is possible to write $f(t, \lambda)$ as a quotient $a(t, \lambda)/b(t, \lambda)$, $a(t, \lambda), b(t, \lambda) \in \mathcal{O}(U \times W)$, where $b(t, \lambda(\varepsilon)) \not\equiv 0$.

Remark 1.4. As it was noted, in the case $dim\lambda = 1$, the notions of holomorphic dependence and regular dependence at a fixed point of the parameter space coincide. This is not the case for $dim\lambda > 1$. Indeed, consider $\lambda_1\lambda_2/(\lambda_1^2 + \lambda_2^2 t)$, which depends regularly on λ at $\lambda = 0$, but not holomorphically. In section 4 we verify that the coefficients of Picard-Fuchs equations constructed in section 3 depend regularly on parameters; due to complexity of computations, we do not have a counterexample which would show that at the same time they do not depend holomorphically on parameters, though it is likely that they do not (when the dimension of the parameter space is greater than one). In another related situation linear differential equations which depend regularly, but not holomorphically, on parameters, do arise naturally. Such are, in general, the linear differential equations for the components of a linear differential system with a polynomial system matrix, with parameters being the coefficients of the polynomial entries ([Gr]).

Remark 1.5. Definition 1.2, put slightly differently, is: for *any* representation of $f(t,\lambda)$ as $a(t,\lambda)/b(t,\lambda)$, $a(t,\lambda), b(t,\lambda) \in \mathcal{O}(U \times W)$, and for any holomorphic arc $\lambda = \lambda(\varepsilon)$ passing through λ_0 such that $b(t,\lambda(\varepsilon)) \not\equiv 0$, $a(t,\lambda(e))/b(t,\lambda(\varepsilon))$ depends holomorphically on ε. It is in fact equivalent to the following alternative definition: for *some* representation of $f(t,\lambda)$ as $a(t,\lambda)/b(t,\lambda)$, $a(t,\lambda), b(t,\lambda) \in \mathcal{O}(U \times W)$, and for any holomorphic arc $\lambda = \lambda(\varepsilon)$ passing through λ_0 such that $b(t,\lambda(\varepsilon)) \not\equiv 0$, $a(t,\lambda(e))/b(t,\lambda(\varepsilon))$ depends holomorphically on ε. The equivalence of both definitions is a corollary of Lemma 1.6 below.

Lemma 1.6. *Let $f(t,\lambda)$ be a parameter dependent meromorphic function on a polydisc $U \times W \subset \mathbb{C} \times \mathbb{C}^p$, and suppose that for the holomorphic arc $\lambda = \lambda(\varepsilon)$, $f(t,\lambda(\varepsilon))$ is defined and does not depend holomorphically on ε at $\varepsilon = 0$. Then there exists $m > 0$, such that for any holomorphic arc $\lambda = \widehat{\lambda}(\varepsilon)$, for which $\lambda(\varepsilon) - \widehat{\lambda}(\varepsilon) = o(\varepsilon^m)$, $f(t,\widehat{\lambda}(\varepsilon))$ does not depend holomorphically on ε at $\varepsilon = 0$ as well.*

Proof. By assumption $f(t,\lambda(\varepsilon))$ is defined and does not depend holomorphically on ε at $\varepsilon = 0$. Then there exists a representation $f(t,\lambda) = a(t,\lambda)/b(t,\lambda)$, $a(t,\lambda), b(t,\lambda) \in \mathcal{O}(U \times W)$, such that $b(t,\lambda(\varepsilon)) \not\equiv 0$ and such that, writing $a(t,\lambda)$ and $b(t,\lambda)$ as the power series (convergent in some small polydisc around $(t_0,\lambda_0) \in U \times W$) $a_0(\lambda) + a_1(\lambda)t + ...$ and $b_0(\lambda) + b_1(\lambda)t + ...$ respectively, the following holds.
All $b_j(\lambda(\varepsilon))$, $j = 0, 1, 2, ..$, have a zero of order at least $k \geq 1$ at $\varepsilon = 0$, while for some i, $a_i(\lambda(\varepsilon))$ has there a zero of a smaller order (if at all). Clearly, for any holomorphic arc $\widehat{\lambda}(\varepsilon)$ for which $\lambda(\varepsilon) - \widehat{\lambda}(\varepsilon) = o(\varepsilon^k)$, the order of vanishing of $a_i(\widehat{\lambda}(\varepsilon))$ and of $b_j(\widehat{\lambda}(\varepsilon))$, $j = 0, 1, 2, ..$ stays the same. We conclude that $f(t,\widehat{\lambda}(\varepsilon))$ is defined and does not depend holomorphically on ε at $\varepsilon = 0$. $\square$

Lemma 1.6 will be often used as follows. Suppose that a given parameter dependent meromorphic function, restricted to a certain holomorphic arc in the parameter space, does not depend holomorphically on the arc parameter. Then there exists another holomorphic arc, such that the restriction of the meromorphic function to this arc again does not depend holomorphically on the arc parameter, and which (possibly unlike the original arc) is in a general position (so that it does not lie on certain exceptional subsets of the parameter space $\mathbb{C}^p$).

One may characterize regular dependence on parameters as follows.

Proposition 1.7. *Let $f(t,\lambda) = a(t,\lambda)/b(t,\lambda)$, $a(t,\lambda), b(t,\lambda) \in \mathcal{O}(U \times W)$. Denote by $S \subset W$ the set $\{\lambda \in W : b(\cdot,\lambda) \equiv 0\}$, and let K be any compact subset of U with nonempty interior. For a fixed $\lambda \notin S$, denote by $K(\rho,\lambda)$ the set obtained by removing from K the points whose distance from the (discrete) set $\{t \in U : b(t,\lambda) = 0\}$ is smaller than ρ. Then the following are equivalent:*
i) $f(t,\lambda)$ depends regularly on λ in W,

ii) for any compact set $F \subset W$, and any $\rho > 0$

$$1.1) \qquad sup_{\lambda \in F-S}\ sup_{t \in K(\rho,\lambda)} \frac{|a(t,\lambda)|}{|b(t,\lambda)|} \ < \ \infty$$

whenever defined (i.e. whenever the set $\{(t,\lambda) : \lambda \in F - S, t \in K(\rho,\lambda)\}$ is nonempty).

Proof. We prove the proposition when $a(t,\lambda), b(t,\lambda)$ are polynomials (with complex coefficients), and then comment how it is to be modified in the general case. So let $a(t,\lambda), b(t,\lambda) \in \mathbb{C}[t,\lambda]$, and suppose that i) holds but ii) does not hold.

In this case there exists a compact polydisc $K \times F \subset U \times W$ and a positive number $\rho > 0$, such that

$$sup_{\lambda \in F-S}\ sup_{t \in K(\rho,\lambda)} \frac{|a(t,\lambda)|}{|b(t,\lambda)|} \ = \ \infty.$$

Let $D_r \subset F$ be the set of points at (Euclidean) distance from S, $r > 0$. Denote by M_r the subset of D_r, for which

$$1.2) \qquad sup_{t \in K(\rho,\lambda)} \frac{|a(t,\lambda)|}{|b(t,\lambda)|}$$

is defined and is equal to its maximal value on D_r. If at all there are points on D_r for which $K(\rho,\lambda)$ is nonempty, M_r will be a nonempty set. Denote by Γ the set $\cup_{r>0} M_r$.

Recall now the notion of a first order formula in the sense of elementary mathematical logic. Taking the language to be the language of ordered rings (i.e. the language is the set of 5 symbols $\{0, 1, +, \cdot, >\}$), it is not difficult to write a first order formula defining the set Γ (using the real and imaginary parts of the coefficients of the given polynomials $a(t,\lambda), b(t,\lambda)$). By the Tarski-Seidenberg principle (by which we mean here the fact that the first order theory of the reals eliminates quantifiers), Γ is then a semialgebraic subset of $\mathbb{R}^{2p} \cong \mathbb{C}^p$. If Γ is bounded away from S, it means that for some $r_0 > 0$, $K(\rho,\lambda)$ is empty for all $\lambda \in D_r$, $r < r_0$. One cannot have

$$sup_{\lambda \in F-S}\ sup_{t \in K(\rho,\lambda)} \frac{|a(t,\lambda)|}{|b(t,\lambda)|} \ = \ \infty,$$

since $|b(t,\lambda)|$ is bounded away from zero on $\cup_{\lambda \in D_r, r \geq r_0} K(\rho,\lambda)$ (and since $|a(t,\lambda)|$ is bounded on the compact set $K \times F$). We conclude that $\Gamma \subset F$ has a limit point on $S \cap F$. By the Curve Selection lemma there exists a real analytic curve $\lambda(\varepsilon) : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^{2p} \cong \mathbb{C}^p$, such that $\lambda((-\varepsilon_0, \varepsilon_0) - \{0\}) \subset F$ and $\lambda(0) \in S$. This real analytic curve defines a holomorphic arc passing through $\lambda(0)$, which we also denote by $\lambda(\varepsilon)$. It follows that

$$1.3) \qquad sup_{t \in K(\rho,\lambda(\varepsilon))} \frac{|a(t,\lambda(\varepsilon))|}{|b(t,\lambda(\varepsilon))|}$$

is not bounded as ε tends to 0. This implies that $a(t,\lambda(\varepsilon))/b(t,\lambda(\varepsilon))$ does not depend holomorphically on ε at $\varepsilon = 0$, contradicting the initial assumption that i) is true. Therefore i) implies ii).

To show the converse, suppose that i) does not hold. Then there exists a holomorhic arc $\lambda(\varepsilon)$, for which $b(t,\lambda(\varepsilon)) \not\equiv 0$ and

$$\frac{a(t,\lambda(\varepsilon))}{b(t,\lambda(\varepsilon))} \ = \ \frac{1}{\varepsilon^s} \frac{\alpha(t,\lambda(\varepsilon))}{\beta(t,\lambda(\varepsilon))},$$

for some $s \geq 1$, such that for some $t_0 \in U$, $\alpha(t_0, \lambda(0)) \neq 0$ and $\beta(t_0, \lambda(0)) \neq 0$. For small enough $\rho > 0$, and since K has a nonempty interior, $K(\rho, \lambda)$ is nonempty for all $\lambda \in F - S$, where F is a compact polydisc in W. Then

$$sup_{t\in K(\rho,\lambda(\varepsilon))} \frac{|a(t,\lambda(\varepsilon))|}{|b(t,\lambda(\varepsilon))|} = sup_{t\in K(\rho,\lambda(\varepsilon))} \frac{1}{|\varepsilon|^s} \frac{|\alpha(t,\lambda(\varepsilon))|}{|\beta(t,\lambda(\varepsilon))|}$$

is unbounded as ε tends to 0, and therefore ii) does not hold. Thus ii) implies i).

Regarding the changes needed to be done when $a(t, \lambda), b(t, \lambda)$ are not necessarily polynomials, one has to replace the semialgebraic category with the subanalytic one (cf. [BM], for example). The proof that the set $\Gamma \subset \mathbb{R}^{2p} \cong \mathbb{C}^p$ is subanalytic, is more cumbersome now. One may either use a language suitable for dealing with subanalytic sets (so that they (or a subclass of them) become definable), or, what is essentially the same, one may replace the logical operations in the formula defining the semialgebraic Γ above, by corresponding set-theoretic operations (being careful not to project unbounded subanalytic sets). The other details are virtually unchanged. $\square$

We are now coming to the issue which motivated the considerations above. Let

$$1.4) \quad y^{(n)} + c_{n-1}(t, \lambda)y^{(n-1)} + ... + c_0(t, \lambda)y = 0$$

be a parameter dependent linear differential equation with meromorphic coefficients in the polydisc $U \times W \subset \mathbb{C} \times \mathbb{C}^p$.

Definition 1.8. The linear differential equation 1.1 will be said to depend regularly on parameters at $\lambda = \lambda_0 \in W$ (respectively, in W) if its coefficients depend regularly on parameters at λ_0 (respectively, in W).

As shown below (Theorem 1.9), Proposition 1.7 implies that the number of zeros of solutions of such equations in domains satisfying some natural restrictions, is uniformly bounded over compact sets in the parameter space. It is the quantitative version of this assertion, formulated in section 2, which then allows to give asymptotic bounds on the number of zeros of Abelian integrals.

Note, that if there exists a holomorphic arc $\lambda(\varepsilon)$, such that for some i, $0 \leq i \leq n-1$ $c_i(t, \lambda(\varepsilon))$ is defined and does not depend holomorphically on ε (at $\varepsilon = 0$), but other coefficients are not necessarily defined on $\lambda(\varepsilon)$), then by Lemma 1.6 there exists another arc $\widehat{\lambda}(\varepsilon)$, $\widehat{\lambda}(0) = \lambda(0)$, (a perturbation of the original arc $\lambda(\varepsilon)$)) along which all the coefficients of 1.4) are defined, and $c_i(t, \widehat{\lambda}(\varepsilon))$ does not depend holomorphically on ε. Then the equation 1.4), restricted to the arc $\widehat{\lambda}(\varepsilon)$, takes the form

$$1.5) \quad \varepsilon^s y^{(n)} + \alpha_{n-1}(t, \varepsilon)y^{(n-1)} + .. + \alpha_0(t, \varepsilon)y = 0,$$

with $s \geq 1$ and $\alpha_i(t, \varepsilon) \in \mathcal{O}(U' \times W')$ for all i, in some polydisc $U' \times W' \subset \mathbb{C} \times \mathbb{C}$, such that $(t_0, 0) \in U' \times W'$ for some $t_0 \in U$ (in fact for any t_0 outside of a certain discrete subset of U, there will exist a polydisc in which 1.4) takes the form 1.5)). Parameter dependent differential equations such as 1.5), are usually called *singularly perturbed.* So we may rephrase Definition 1.8 as follows: we say that 1.4) depends regularly on λ at

λ_0, if along no holomorphic arc passing through λ_0, and on which 1.4) is defined, is 1.4) singularly perturbed.

Let us write the equation 1.4) in the form

$$1.6) \qquad y^{(n)} + \frac{a_{n-1}(t,\lambda)}{b(t,\lambda)} y^{(n-1)} + ... + \frac{a_0(t,\lambda)}{b(t,\lambda)} y = 0$$

where $a_i(t,\lambda) \in \mathcal{O}(U \times W)$, $i = 0,..,n-1$, $b(t,\lambda) \in \mathcal{O}(U \times W)$ (again, this is always possible since $U \times W$ is a polydisc). We denote by $S \subset W$ the set

$$S = \{\lambda : b(\cdot,\lambda) \equiv 0\}.$$

For each fixed $\lambda \in W - S$, we denote by $\Sigma_\lambda \subset \mathbb{C}$ the set of true singular points of 1.6) (i.e. the points where some solution of 1.6 is not holomorphic); $\Sigma_\lambda \subset Z_\lambda$ where $Z_\lambda = \{t \in U : b(t,\lambda) = 0\}$.

Recall from section 0, that a *simple domain in* $\mathbb{C} - \Sigma_\lambda$ is any domain V contained in a simply connected domain $G \subset \mathbb{C} - \Sigma_\lambda$ obtained by removing from $\mathbb{C}$ nonintersecting rays which initiate at the points of Σ_λ and go to infinity (in other words, G is obtained by making straight line cuts at the points of Σ_λ).

Theorem 1.9. *Suppose that 1.6) depends regularly on λ in W. Then for any compact sets $K \subset U$, $F \subset W$, and for any $\rho > 0$, there exists $N \geq 0$, such that the following holds.*

Fix $\lambda \in F - S$, and let $y(t)$ satisfy 1.6) for that value of the parameter. Then the number of zeros of $y(t)$ in any simple domain V in $\mathbb{C} - \Sigma_\lambda$, $V \subset K$, for which $dist(V, \Sigma_\lambda) \geq \rho$, is not greater than N.

In other words, the number of zeros of solutions of 1.6) in simple domains in $\mathbb{C} - \Sigma_\lambda$, contained in K and bounded away by ρ from Σ_λ, is uniformly bounded over $\lambda \in F - S$.

Theorem 1.9 is in fact the corollary of Proposition 1.7 and the following theorem from [Y], which gives a bound on the variation of argument of a solution of a linear differential equation on a line segment, in terms of a bound on the magnitude of the coefficients of the equation on that segment. In a sense, this is the generalization of the fact that the number of oscillations of the solutions of $y'' + Ky = 0$ on the segment $[0,1]$ is bounded by K (in fact by $\sqrt{K}/2\pi$) for $K \geq 1$, as for $K > 0$ this equation describes a linear harmonic oscillator of frequency $\sqrt{K}/2\pi$.

Theorem 1.10 [Y]. *Let the linear differential equation*

$$y^{(n)} + a_{n-1}(t) y^{(n-1)} + .. + a_0(t) y = 0$$

have coefficients which are holomorphic on a finite line segment $I \subset \mathbb{C}$ of length l, their modulus being bounded there by $C \geq 1$. Then the variation of the argument of any non-trivial solution of this equation on I is bounded by

$$Var\ Arg\ y(t)|_I \ \leq\ \pi(n+1)(1 + lC/log(3/2)).$$

The fact that the domain V in Theorem 1.9 is simple, permits us to decompose it into a (uniformly bounded) number of pieces, each of which is contained in a polygonal domain, such that the sum of the lengths of the segments which constitute the boundary of these polygonal domains, is uniformly bounded, and the cardinality of the set of segments is uniformly bounded as well. We then combine Proposition 1.7 and Theorem 1.10 to prove Theorem 1.9. The formal argument is given below.

Lemma 1.11. *Let Σ, Z be finite subsets of a bounded rectangular domain $U \subset \mathbb{C}$. Let V be a simple domain in $\mathbb{C} - \Sigma$, such that $\overline{V} \subset U$, and such that $dist(V, \Sigma) \geq \rho$, $dist(V, \partial U) \geq \rho$. Then there exist linear segments $\gamma_1, .., \gamma_N$, all contained in U, $N \leq O(|\Sigma|^2)$, such that*
i) for all $i = 1, .., N$, $dist(\gamma_i, Z) \geq \rho/O(|Z|)$, $dist(\gamma_i, \partial V) \geq \rho/2$;
ii) let f be a multivalued analytic function in $\mathbb{C} - \Sigma$; given a branch $\widetilde{f}$ on V, there exist branches $\widetilde{f}_1, .., \widetilde{f}_N$ defined on $\gamma_1, .., \gamma_N$, such that the number of zeros of $\widetilde{f}$ in V is not greater than the sum of absolute values of the variations of arguments of $\widetilde{f}_1, .., \widetilde{f}_N$ on $\gamma_1, .., \gamma_N$, respectively, divided by 2π.

Proof. Since V is a simple domain in $\mathbb{C} - \Sigma$, there exist $|\Sigma|$ nonintersecting rays initiating from points of Σ and going to infinity, $\Gamma_1, .., \Gamma_{|\Sigma|}$, such that $\mathbb{C} - \cup_i \Gamma_i$ is a simply connected domain (not containing points from Σ) and $V \subset \mathbb{C} - \cup_i \Gamma_i$. Remove now from $\mathbb{C} - \cup_i \Gamma_i$ squares of diameter 2ρ centered at points of Σ, obtaining the region G. Let $\widehat{V}$ be the rectangular domain such that $\widehat{V} \subset U$, $dist(\widehat{V}, \partial U) = \rho$. It is not difficult to represent $G \cap \widehat{V}$ as the union of at most $O(|\Sigma|^2)$ quadrilateral domains. Now slightly perturb the boundary of these domains, so that the resulting domains $\widehat{V}_1, .., \widehat{V}_M$ are polygonal, contain the unperturbed domains, and their boundaries constitute a set of N segments, $N \leq O(|\Sigma|^2)$, which we call $\gamma_1, .., \gamma_N$, such that $dist(\gamma_i, Z) \geq \rho/C|Z|$, and $dist(\gamma_i, \partial U) \geq \rho/2$, $i = 1, .., N$, where C is some universal constant (it is again not difficult, though tedious, to construct such a perturbation).
Since the number of zeros of any given branch $\widetilde{f}$ of f on V is not greater than the sum of the numbers of zeros of corresponding branches $\widetilde{f}_1, .., \widetilde{f}_M$ on $\widehat{V}_1, .., \widehat{V}_M$, respectively, it is also not greater than the sum of the absolute values of variations of arguments of the branches on $\gamma_1, .., \gamma_N$, divided by 2π, by the argument principle. □

Proof of Theorem 1.9. Without limiting generality, suppose that $K \subset U_1 \subset \overline{U_1} \subset U$, where U_1 is a bounded rectangular domain for which $dist(K, \partial U_1) \geq \rho$ (this makes the proof shorter, as it allows us to use Lemma 1.11 right away).

Fix $\lambda \in F - S$, and let $V \subset K$ be a simple domain in $\mathbb{C} - \Sigma_\lambda$. Denote by $\Sigma'_\lambda, Z'_\lambda$ the sets $\Sigma_\lambda \cap U_1, Z_\lambda \cap U_1$, respectively. Note that the cardinality of Z'_λ is uniformly bounded over $\lambda \in F - S$, since F and U_1 are compact and $b(t, \lambda) \in \mathcal{O}(U \times W)$. Denote this bound by N_1.

By Lemma 1.11, there exist linear segments $\gamma_1, .., \gamma_N$, $N \leq O(N_1^2)$, all contained in U_1, $dist(\gamma_i, Z'_\lambda) \geq \rho/CN_1$, $dist(\gamma_i, \partial U_1) \geq \rho/2$ (C being a universal constant which we assume to be larger than 2), such that the following holds: the number of zeros of a solution of 1.6) in V is not greater than the sum of the absolute values of variations of arguments of some solutions of 1.6) on $\gamma_1, .., \gamma_N$.

From Theorem 1.10, this sum of absolute values of variations of argument is bounded by

$$1.7) \qquad N\pi(n+1)\left(1+\frac{diam(U_1)}{log(3/2)}\; max_{t\in\cup_i\gamma_i}\; max_j\; \frac{|a_j(t,\lambda)|}{|b(t,\lambda)|}\right).$$

Since $dist(\gamma_i, \partial U_1) \geq \rho/2$, we conclude that not only for Z'_λ $dist(\gamma_i, Z'_\lambda) \geq \rho/CN_1$, but also for Z_λ $dist(\gamma_i, Z_\lambda) \geq \rho/CN_1$ (recall $C \geq 2$). Since $N \leq O(N_1^2)$ and $\overline{U_1} \subset U$ is compact, from Proposition 1.7 we conclude that 1.7) is uniformly bounded over $\lambda \in F - S$, thereby proving the theorem. $\square$

2. An asymptotic bound on the number of zeros of solutions of linear differential equations which depend regularly on parameters.

The (parameter dependent) Picard-Fuchs equations obtained in section 3 below for the Abelian integrals are of the form

$$2.1) \qquad y^{(n)} + \frac{p_{n-1}(t,\lambda)}{q(t,\lambda)}y^{(n-1)} + ... + \frac{p_0(t,\lambda)}{q(t,\lambda)}y = 0,$$

where $p_i(t,\lambda)$, $i = 0,..,n-1$ and $q(t,\lambda)$ are polynomials in t, λ with *integral* coefficients. Moreover their degree and height (by which we mean the maximum of the moduli of their coefficients), as well as the order n of 2.1), will admit asymptotic bounds depending only on the degree of the Hamiltonian H (see section 0). In this section we show how one may obtain a quantitative version of Theorem 1.9 in this case.

So suppose that $p_i(t,\lambda)$, $i = 0,..,n-1$ and $q(t,\lambda)$ in the equation 2.1) are polynomials with integral coefficients of degree d and height M. To avoid problems with notation, it is always assumed below that $d \geq 2$, $M \geq 2$.

We keep notations of section 1: $S \subset \mathbb{C}^p$, $p = dim(\lambda)$, denotes the subset of the parameter space where $q(\cdot,\lambda)$ becomes identically zero. For $\lambda \notin S$, Z_λ denotes the (now finite) set of zeros of $q(\cdot,\lambda)$, and $\Sigma_\lambda \subset Z_\lambda$ denotes the set of true singular points of the linear differential equation 2.1) for that value of parameter.

Below, $||\lambda||$ always denotes the l_∞ norm of λ considered as a vector in $\mathbb{R}^{2p} \cong \mathbb{C}^p$, and ρ denotes some positive number between 0 and 1.

Theorem 2.1. *Suppose that the linear differential equation 2.1) depends regularly on λ (in $\mathbb{C}^p$). Then for any $\lambda \notin S$, $||\lambda|| \leq 1$, a solution of 2.1) cannot have more than*

$$n\left(\frac{M}{\rho}\right)^{d^{O(p^3)}}$$

zeros in any simple domain in $\mathbb{C} - \Sigma_\lambda$, whose distance from Σ_λ is at least ρ, and which is contained in the unit disc.

Let us remind that according to our convention, $O(p^3)$ stands for $c_0 + c_1 p^3$, where $c_0, c_1 > 0$ are some *universal* (i.e. not dependent on λ or other characteristics of the problem) constants.

In the proof of Theorem 1.9 the existence of a first-order formula for the set Γ, in the language of ordered rings, is rather clear, so we do not write it explicitly. On the contrary, in the proof of Theorem 2.1, we write an explicit first-order formula for the maximum of the moduli of the coefficients of 2.1) on certain subset of $\mathbb{C}\times\mathbb{C}^p$. We then obtain the value of this maximum from a theorem due to Renegar, regarding the complexity of quantifier elimination in the first order theory of the reals. A crucial point here is the integrality of the polynomials $p_i(t,\lambda)$, $i=0,..,n-1$ and $q(t,\lambda)$.

As we will be writing first-order formulas, the reader, strictly speaking, should recall the necessary definitions and background of elementary mathematical logic. We give, though, an intuitive explanation of what a logical formula defining a semialgebraic set is.

A *quantifier free formula* in the language of ordered rings (this *language* is just the following set of symbols: $\{0,1,+,\cdot,>\}$), is an expression such as

$$2.2)\qquad x+z>0\ \vee\ xyz^5-5y^2+xz-7>0\ \wedge\ \neg(xyz=0)$$

(strictly speaking, we should have written $1+1+1+1+1$ instead of 5, $y\cdot y$ instead of y^2, etc.). The formulas $x+z>0$, $xyz^5-5y^2+xz-7>0$ and $xyz=0$ are sometimes called the *atomic predicates* of the quantifier free formula 2.2), and below we refer to them as such. To decipher 2.2), recall that $\wedge$ is the logical 'and', $\vee$ is the logical 'or', and $\neg$ is the logical 'not'. So the formula 2.2) in fact defines a subset of $\mathbb{R}^3$, coordinatized by (x,y,z)[1], such that

$$x+z>0\ \ or\ \ \big(xyz^5-5y^2+xz-7>0\ \ and\ \ (not\ \ xyz=0)\ \big)$$

(recall rules of precedence). In other words, the subset defined by the formula contains points (x,y,z) for which either $x+z>0$, *or* the following holds: $xyz^5-5y^2+xz-7>0$, *and* $xyz=0$ is *not* true.

A (quantifier free) formula defining a subset of points $(x,y)\in\mathbb{R}^2$ for which the truthfulness of $x-y=5$ implies (in the sense of mathematical logic) the truthfulness of $y>7$ (i.e. the set $\{(x,y): x-y\neq 5\}\cup\{(x,y): y>7\}$) will be written as

$$x-y=5\ \rightarrow\ y>7.$$

Here $e\rightarrow f$ is a shorthand for $(\neg e)\vee f$.

Let us now write a formula defining the set of $(x,y,z,w)\in\mathbb{R}^4$ such that there exists $u>0$ for which $xyu+zw+5>0$ and such that for all $v\neq 0$, $xv+yzw\neq 5$. A possible variant is

$$(\exists u\ (u>0\wedge xyu+zw+5>0))\wedge(\forall v\ (v\neq 0\ \rightarrow\ xv+yzw\neq 5)).$$

Here of course $\exists$ stands for the quantifier 'there exists', and $\forall$ stands for the quantifier 'for all' ($\forall$ may be regarded as a shorthand for $\neg\exists\neg$). Thus this formula is not quantifier free. Note that the quantifiers range over variables which take values in $\mathbb{R}$. This is why the formula above is called a *first order* formula (had we allowed quantification over subsets of the reals we would get a second order formula, etc.).

[1] or a subset of any other Euclidean space three of whose coordinates are labeled by x,y,z.

By the Tarski-Seidenberg principle, for every first order formula in the language of ordered rings, there exists a quantifier free formula defining the same set in a corresponding Euclidean space. One concludes, therefore, that the set defined by the above first-order formula is in fact semialgebraic, which is not clear *a priori*. This is precisely the argument we used in the proof of Proposition 1.7 to show that the set Γ is semialgebraic.

Theorem 1.1 from [Re] establishes a quantitative version of the Tarski-Seidenberg principle, as well as establishing complexity bounds for the quantifier elimination algorithm which the author constructs. Cited below is an immediate corollary of that theorem. Let us first set the framework.

Consider the formula

$$2.3) \qquad (Q_1 x_1)...(Q_t x_t)\ P(y, x_1, .., x_t),$$

where $y = (y_1, .., y_l) \in \mathbb{R}^l$, $x_m = (x_{m1}, .., x_{mn_i}) \in \mathbb{R}^{n_i}$, $m = 1, .., t$, each Q_i stands either for $\exists$ or $\forall$, and where $P(y, x_1, .., x_m)$ denotes the a quantifier free formula with atomic predicates

$$g_k(y, x_1, .., x_t)\ s_k\ 0$$

$k = 1, .., K$, where each $g_k(y, x_1, .., x_t) \in \mathbb{Z}[y, x_1, .., x_t]$, s_k stand for either of $=, >, <, \geq, \leq, \neq$ (though strictly speaking, only the symbols $=, >$ are allowed in the language of ordered rings). Suppose now that the sum of the degrees of the different polynomials among g_k, $k = 1, .., K$, is not greater than $D \geq 2$ (which we call the *total degree* of the quantifier free formula $P(y, x_1, .., x_t)$), and suppose that the height of the polynomials g_k does not exceed $A \geq 2$ for all $k = 1, .., K$ (in which case we say that the *height* of $P(y, x_1, .., x_t)$ is not greater than M).

Theorem 2.2 [Re]. *The subset of $\mathbb{R}^l$ defined by the formula 2.3), can be also defined by the quantifier free formula*

$$2.4) \qquad \bigvee_{i=1}^{I} \bigwedge_{j=1}^{J}\ h_{ij}(y)\ s_{ij}\ 0,$$

such that $h_{ij}(y) \in \mathbb{Z}[y]$ for all $1 \leq i \leq I, 1 \leq j \leq J$, the total degree of 2.4) is not greater than

$$D^{2^{O(t)} l \Pi_k n_k},$$

and its height is not greater than

$$A^{O(l) + D^{2^{O(t)} \Pi_k n_k}}.$$

The proof of Theorem 2.1 consists of repeating the proof of Theorem 1.9 together with using Theorem 2.2 to estimate the value of 1.7).

Proof of Theorem 2.1. We follow here the proof of Theorem 1.9, taking $U = \mathbb{C}$, taking K to be the origin centered square of diameter $2\sqrt{2}$, and taking U_1 to be the origin centered square of diameter $4\sqrt{2}$. We prove the theorem for simple domains in $\mathbb{C}-\Sigma_\lambda$, whose distance to Σ_λ is at least ρ, and which are contained in the square K. In the proof of Theorem 1.9, the estimate for the number of zeros in any such domain is given by 1.7)

$$N\pi(n+1)\left(1+\frac{diam(U_1)}{log(3/2)}\; max_{t\in\cup_i\gamma_i}\; max_j\; \frac{|p_j(t,\lambda)|}{|q(t,\lambda)|}\right),$$

where $N\le O(|Z'_\lambda|^2)$, $Z'_\lambda=Z_\lambda\cap U_1$, and γ_i, $i=1,..,N$ are certain segments lying at a distance of at least $\rho/O(|Z'_\lambda|)$ from Z_λ.

Since $|Z'_\lambda|\le|Z_\lambda|\le d$, to obtain the sought bound for the number of zeros from 1.7) it remains to estimate

$$2.5)\qquad max_{t\in\cup_i\gamma_i}\; max_j\; \frac{|p_j(t,\lambda)|}{|q(t,\lambda)|}.$$

Pick an integer (universal constant) $C>0$, and let R be the nearest integer to $1/\rho$, $R\ge 1/\rho$, so that the distance of the segments $\gamma_1,..,\gamma_N$ to Z_λ, which is at least $\rho/O(|Z'_\lambda|)$, is bounded by $1/(CRd)$ from below. Clearly 2.5) is bounded by the maximum over $j=0,..,n-1$ of

$$sup_{\lambda\in\{\lambda:||\lambda||\le1\}-S}\; sup_{t\in U_1(1/(CRd),\lambda)}\; \frac{|p_j(t,\lambda)|}{|q(t,\lambda)|},$$

which by Proposition 1.7 is finite (recall from section 1 that $U_1(1/(CRd),\lambda)$ denotes a subset of points of U_1 whose distance to Z_λ is at least $1/(CRd)$).

We now construct a first order formula for the one point subset of $\mathbb{R}$ whose only point is the value of that maximum. To estimate this value using Theorem 2.2, all polynomials entering the formula must have integral coefficients; this is why ρ was replaced by $1/R$.

For a fixed $\lambda\notin S$, the set $U_1(1/(CRd),\lambda)\subset\mathbb{C}\cong\mathbb{R}^2$ is defined by the following formula, which we denote by $F_1(t_{re},t_{im},\lambda_{re},\lambda_{im})$:

$$\forall(z_{re},z_{im})\quad (q_{re}(z_{re},z_{im},\lambda_{re},\lambda_{im})=0\wedge\; q_{im}(z_{re},z_{im},\lambda_{re},\lambda_{im})=0$$
$$\to\; CRd((t_{re}-z_{re})^2+(t_{im}-z_{im})^2)\ge1)\quad\wedge\quad t_{re}^2\le4\;\wedge\; t_{im}^2\le4.$$

The polynomials q_{re} and q_{im} which appear in the formula F_1, are the real and imaginary parts of the polynomial q. Observe that though their degrees are not greater than the degree of q, their height will be in general larger than that of q (for example, the height of t^3 is 1, but each of the real and imaginary parts of

$$(t_{re}+\sqrt{-1}t_{im})^3=(t_{re}^3-3t_{re}t_{im}^2)+\sqrt{-1}(-t_{im}^3+3t_{re}^2t_{im})$$

has height 3). To save on space, below we do not write the formulas explicitly in terms of the real and the imaginary parts of the variables, but use, when possible, a shorthand form, from which the reconstruction of the formula is more or less immediate. For example, such a shorthand form of the formula F_1 is (recall that by our conventions $||\cdot||$ denotes the l_∞ norm)

$$\forall z\;(q(z,\lambda)=0\to\; CRd|t-z|^2\ge1)\quad\wedge\quad ||t||\le2.$$

The height of each of q_{re}, q_{im} is bounded by $2^{2d}(d+1)^{p+1}M$. By Theorem 2.2, there is a quantifier free formula $F_2(t,\lambda)$ equivalent to the formula $F_1(t,\lambda)$ above, its total degree being not greater than

$$d^{2^{O(1)}\cdot(2p+2)\cdot 2} = d^{O(p)},$$

and its height (recall that $d \geq 2$, $M \geq 2$ and C is an integral *universal* constant) is not greater than

$$max(CRd, 2^{2d}(d+1)^{p+1}M)^{O(2p+2)+d^{2^{O(1)}\cdot 2\cdot 2}} \leq (MR)^{d^{O(1)}(p+1)^2}.$$

Writing $q(t,\lambda) = q_0(\lambda) + q_1(\lambda)t + .. + q_d(\lambda)t^d$, the (quantifier free) formula $F_3(\lambda)$ for the set $\{\lambda : ||\lambda|| \leq 1\} - S$ is given by

$$||\lambda|| \leq 1 \quad \wedge \quad \neg\left(\bigwedge_{i=1}^{d} q_i(\lambda) = 0\right),$$

its total degree being not greater than d, and its height being not greater than $2^{2d}(d+1)^p M$.

Fix $j \in \{0,..,n-1\}$, and let v_j denote the value of

$$2.6) \qquad sup_{\lambda\in\{\lambda:||\lambda||\leq 1\}-S}\ sup_{t\in U_1(1/(CRd),\lambda)}\ \frac{|p_j(t,\lambda)|}{|q(t,\lambda)|}.$$

Since by our assumption, 2.1) depends regularly on parameters, such $v_j \in \mathbb{R}$ exists. We write now a first order formula for the subset $\mathbb{R} - [-v_j, v_j]$ of $\mathbb{R}$:

$$\forall(t,\lambda)\ \left(F_2(t,\lambda) \wedge F_3(t,\lambda)\ \rightarrow\ s^2|q(t,\lambda)|^2 > |p_j(t,\lambda)|^2\right).$$

The total degree of the quantifier free part of this first order formula is not greater than $d^{O(p)}$, and its height is not greater than $(MR)^{d^{O(1)}(p+1)^2}$. Therefore, using theorem 2.2 once more, there exists an equivalent quantifier free formula, whose total degree is not greater than

$$(d^{O(p)})^{2^{O(1)}\cdot 1\cdot(2p+2)} = d^{O(p^2)},$$

and its height is not greater than

$$((MR)^{d^{O(1)}(p+1)^2})^{O(1)+(d^{O(p)})^{2^{O(1)}\cdot(2p+2)}} \leq ((MR)^{d^{O(1)}(p+1)^2})^{d^{O(p^2)}} \leq (MR)^{d^{O(p^3)}}.$$

We may write now a first order formula for the value v_j itself (i.e. for the one point set $\{v_j\} \subset \mathbb{R}$). Omitting the simple details, we obtain by Theorem 2.2, a quantifier free formula for the value v_j

$$2.4) \qquad \bigvee_{i=1}^{I}\bigwedge_{l=1}^{L}\ h_{il}(s)\ s_{il}\ 0,$$

of total degree and height being bounded again by $d^{O(p^2)}$, $(MR)^{d^{O(p^3)}}$. Since v_j is the only point in the set defined by 2.4), v_j must be in fact a root of one of the polynomials h_{il}, which are polynomials (in one variable) with *integral* coefficients. Since the degree of this polynomial is bounded by $d^{O(p^2)}$, and its height is bounded by $(MR)^{d^{O(p^3)}}$, we conclude that v_j, for all $j = 0, .., n-1$, is bounded by

$$d^{O(p^2)} \cdot (MR)^{d^{O(p^3)}} \;\leq\; (MR)^{d^{O(p^3)}} \;\leq\; \left(\frac{M}{\rho}\right)^{d^{O(p^3)}}$$

(though $R \geq 1/\rho$, this is not a mistake, because of the $O(\cdot)$ notation). We thus conclude that a bound on the number of zeros may be given by

$$O(d^2) \cdot (n+1) \cdot O\left(\left(\frac{M}{\rho}\right)^{d^{O(p^3)}}\right) \;=\; n\left(\frac{M}{\rho}\right)^{d^{O(p^3)}}. \quad \square$$

3. Construction of Picard-Fuchs equations for Abelian integrals.

In this section, we construct a linear differential equation, satisfied by the Abelian integral

$$3.1) \qquad y(t) = \int_{\gamma(t)} P(x,y)dx + Q(x,y)dy,$$

where P, Q are polynomials, and $\gamma(t)$ is a continuously varying cycle (see below) in the locally trivial bundle determined by the mapping $H(x,y) : \mathbb{C}^2 \to \mathbb{C}$, where $H(x,y)$ is also a (generic) polynomial. Such linear differential equations are known to exist and are usually called Picard-Fuchs equations. More precisely, suppose that the degree of the polynomials H, P, Q is not greater than d, and denote the coefficients of $H(x,y)$ by the tuple λ, $dim(\lambda) = (d+1)(d+2)/2 \leq O(d^2)$. Then we construct a parameter-dependent linear differential equation, with coefficients being quotients of integral polynomials in t, λ, μ, such that for λ not lying in an exceptional codim 1 constructible subset of $\mathbb{C}^{dim(\lambda)}$, 3.1) is a solution of that equation for some value of μ. [2] Here, $dim(\mu) = (d-1)^2 \leq O(d^2)$ as well. In section 4 we will show that the linear differential equation we construct (for any fixed d) depends regularly on the parameters λ, μ, i.e. is not singularly perturbed along any arc in the parameter space on which it is defined. We then show that these equations moreover remain regularly dependent on parameters also after any rather general algebraic parameter-dependent change of variable. This property is a direct consequence of the algebro-geometric origin of these equations, and is (of course) false for general regularly dependent on parameter linear differential equations.

As in Introduction, $\mathcal{H}^d$ denotes the space of polynomials in two variables of degree not greater than d, coordinatized by tuples λ of polynomials coefficients, so that $\mathcal{H}^d \cong \mathbb{C}^{dim(\lambda)}$, $dim(\lambda) = (d+1)(d+2)/2$. $H(\lambda)$ will denote, for $\lambda \in \mathcal{H}^d$, the polynomial whose tuple of coefficients is λ. We write either $\lambda \in \mathcal{H}^d$ or $H \in \mathcal{H}^d$, depending on the context.

We first give the necessary background, explaining the precise meaning of the integral 3.1).

[2] To be precise, this will be true for a constructible subset of codimension zero in the space of the coefficients of the polynomials P, Q.

It is known, and is not difficult to prove, that for each $H \in \mathcal{H}^d$, there exists a finite set $\Sigma_H \subset \mathbb{C}$, whose points are called the atypical values of H, such that $H : \mathbb{C}^2 - H^{-1}(\Sigma_H) \to \mathbb{C} - \Sigma_H$ is a (smooth) locally trivial bundle. When the projective curve defined by $H(x,y) = 0$, H being a polynomial of degree d, intersects the (complex) line at infinity at precisely d points, we say that the polynomial H is regular at infinity. Being H regular at infinity depends only on its highest homogeneous part. The set of atypical points of a polynomial regular at infinity coincides with the set of its critical values.

Choose $t_0 \in \mathbb{C}$ which is not an atypical value of H, and let $\gamma(t_0)$ be a homology class in the first homology group of the level curve $\{(x,y) : H(x,y) = t_0\}$. It is a basic fact, implied solely by being $H : \mathbb{C}^2 - H^{-1}(\Sigma_H) \to \mathbb{C} - \Sigma_H$ a smooth locally trivial bundle, that this homology class admits a natural continuation along any path in $\mathbb{C} - \Sigma_H$ initiating at t_0, depending only on the homotopy class in $\mathbb{C} - \Sigma_H$ of the path joining t_0 and the given end point. This continuation is given, roughly speaking, by trivializing the locally trivial bundle $H : \mathbb{C}^2 - H^{-1}(\Sigma_H) \to \mathbb{C} - \Sigma_H$ at points of the path forming an increasing sequence (with respect to the path parameter), and transporting a representative of the class from a fiber to a neighbouring fiber by means of these trivializations.

Take now a polynomial (or holomorphic) form $P(x,y)dx + Q(x,y)dy$ on $\mathbb{C}^2$, and let $\gamma(t_0)$ denote, as above, an element in the first homology group of a point $t_0 \in \mathbb{C} - \Sigma_H$. Denote by $\gamma(t)$, $t \in \mathbb{C} - \Sigma_H$, the continuation of $\gamma(t_0)$ to t along some path lying in $\mathbb{C} - \Sigma_H$ ($\gamma(t)$, as explained above, is multivalued, depending only on the homotopy class of the path connecting t_0 to t in $\mathbb{C} - \Sigma_H$). The key fact is, that the integral 3.1), called sometimes a complete Abelian integral, defines in fact an *analytic* multivalued function on $C - \Sigma_H$ ([AGV], Chapter 10).

In fact, the same considerations apply not only when t varies, but also when both t and λ vary. Consider the map

$$L : \mathbb{C}^2 \times \mathcal{H}^d \to \mathbb{C} \times \mathcal{H}^d,$$

given by $(x, y, \lambda) \mapsto (H(\lambda)(x,y), \lambda)$. It defines a locally trivial bundle over the open subset $T \subset \mathbb{C} \times \mathcal{H}^d$, consisting of pairs (t, λ), for which $H(\lambda)$ is regular at infinity and has degree d, and t is not a critical value of $H(\lambda)$. Fix $(t_0, \lambda_0) \in T$ and let $\gamma(t_0, \lambda_0)$ be some element from the first homology group of $L^{-1}(t_0, \lambda_0) \cong \{(x,y) : H(\lambda_0)(x,y) = t_0\}$. As before, for any holomorphic form $P(x,y)dx + Q(x,y)dy$ (or even $P(x,y,\lambda)dx + Q(x,y,\lambda)dy$, where P, Q depend (holomorphically) on λ), the integral

$$3.2) \qquad \int_{\gamma(t,\lambda)} P(x,y)dx + Q(x,y)dy,$$

where by $\gamma(t, \lambda)$ we denote the multivalued continuation of $\gamma(t_0, \lambda_0)$, will be an analytic multivalued function on T.

Now fix $H \in \mathcal{H}^d$. Polynomial forms on $\mathbb{C}^2$ are the same as regular forms on the quasiprojective variety $\mathbb{C}^2$. The regular 1-forms on $\mathbb{C}^2$, $\Omega^1(\mathbb{C}^2)$, carry a natural (H dependent) structure of $\mathbb{C}[t]$-module, where $\mathbb{C}[t]$ being the ring of polynomials in t or, equivalently, regular functions on $\mathbb{C}$ (this structure is given simply by $t \cdot \omega = H(x,y) \cdot \omega$; it is useful because it commutes with integration: $\int_{\gamma(t)} t \cdot \omega = t \cdot \int_{\gamma(t)}$, where $\cdot$ on the left is the product operation of the module, and $\cdot$ on the right is the usual product in $\mathbb{C}$). The submodule *generated* by $d\alpha$, $\alpha \in \Omega^0(\mathbb{C}^2)$ (i.e. α is a polynomial in two variables), will be denoted by $d\Omega^0(\mathbb{C}^2) = d\Omega^0$. The following theorem is essentially contained in [I1].

Theorem 3.1. *Let $H(x, y)$ be a polynomial regular at infinity. Let $\omega \in \Omega^1(\mathbb{C}^2)$. Then the following are equivalent:*
i) for all continuously varying cycles $\gamma(t)$ in the locally trivial bundle determined by H, $\int_{\gamma(t)} \omega \equiv 0$,
ii) $\omega \in d\Omega^0(\mathbb{C}^2)$.

For our purposes we will need also the following quantitative assertion, from whose proof one may also extract the proof of Theorem 3.1.

Proposition 3.2. *Let $H(x, y)$ be a polynomial of degree $d \geq 2$.*
i) The 1-form $\alpha(x, y)dx + \beta(x, y)dy \in \Omega^1(\mathbb{C}^2)$ belongs to the submodule $d\Omega^0(\mathbb{C}^2)$ if and only if there exist $A, B \in \mathbb{C}[x, y]$ such that

$$\alpha dx + \beta dy = dA + BdH.$$

ii) Suppose that H is regular at infinity, and suppose that $\alpha dx + \beta dy \in d\Omega^0(\mathbb{C}^2)$, $deg(\alpha)$, $deg(\beta) \leq D$, $D \geq 1$. Then there exist polynomials $A, B \in \mathbb{C}[x, y]$ of degree not greater than $Dd^{O(1)}$, such that $\alpha dx + \beta dy = dA + BdH$.

Proof. Let $\omega \in d\Omega^0$. Then, by definition, $\omega = \sum_i p_i(t) \cdot dq_i$, $p_i(t) \in \mathbb{C}[t]$ and $q_i \in \mathbb{C}[x, y]$ for every i. It suffices therefore to show that $t^k \cdot dq$ is of the form $dA + BdH$ for all k and polynomials $q(x, y)$. But

$$t^k \cdot dq(x, y) = H(x, y)^k dq(x, y) = d(H(x, y)^k q(x, y)) - kH(x, y)^{k-1} q(x, y) dH(x, y).$$

As for the bound on the degrees of A, B, we repeat the argument from the proof in [I1] of Theorem 3.1, but with quantitative estimates.

So suppose that $H \in \mathcal{H}^d$ is regular at infinity and has degree d, and suppose $\alpha(x, y)dx + \beta(x, y)dy \in \Omega^1(\mathbb{C}^2)$ belongs to the submodule $d\Omega^0(\mathbb{C}^2)$. Without limiting generality, we assume that the leading coefficient of H w.r.t. y is cy^d, $c \neq 0$ (otherwise perform a linear change of coordinates to make it such). As shown above, $\omega = dA + BdH$ for some polynomials A, B. Now $dA + BdH = d(A + BH) - HdB$. For all continuously varying cycles $\gamma(t)$ in the locally trivial bundle determined by H,

$$3.3) \qquad \int_{\gamma(t)} \omega = \int_{\gamma(t)} (d(A + BH) - HdB) = \int_{\gamma(t)} d(A + BH) - t \int_{\gamma(t)} dB \equiv 0,$$

(note that this establishes Theorem 3.1 in one direction).

Fix now any $x_0 \in \mathbb{C}$. The roots of $H(x_0, y) - t = 0$ will be all different for all but finitely many (in fact, at most $d - 1$) values of t. Take $t_0 \in \mathbb{C}$ for which these roots are all distinct and number the roots as $y_1(t_0), .., y_d(t_0)$. Continuing them analytically, we get branches of analytic multivalued (algebraic) functions $y_1(t), .., y_d(t)$.

Consider now, for any t which is not a critical value of H,

$$3.4) \qquad r(t) = \left(\int_{x_0, y_d(t)}^{x_0, y_1(t)} \omega \quad \dots \quad \int_{x_0, y_d(t)}^{x_0, y_{d-1}(t)} \omega \right).$$

where the integration is along paths connecting $y_d(t)$ to $y_i(t)$, $i = 1, .., d-1$. Such paths exist since the affine curve $\{(x,y) : H(x,y) - t = 0\}$ is connected; indeed, this affine curve is smooth since t is not a critical value of H, and if it was disconnected, it would imply that the corresponding projective curve has a singularity on the complex line at infinity – but this would contradict the fact that H is regular at infinity. Moreover, after making a choice for $y_1(t), .., y_d(t)$ (i.e. numbering in some way the roots of $H(x_0, y) - t = 0$), the integrals in 3.4) become well defined, since the integral of ω on the cycle formed by any two paths connecting $y_d(t)$ to $y_i(t)$, is zero by 3.3).

Consider also

$$Y(t) = \begin{pmatrix} y_1(t) - y_d(t) & \dots & y_{d-1}(t) - y_d(t) \\ \vdots & \vdots & \vdots \\ y_1^{d-1}(t) - y_d^{d-1}(t) & \dots & y_{d-1}^{d-1}(t) - y_d^{d-1}(t) \end{pmatrix},$$

which is invertible at all points where all roots $y_1(t), .., y_d(t)$ are distinct. $r(t)Y(t)^{-1}$ is holomorphic univalued except perhaps for the (isolated) points t where some of the roots $y_1(t), .., y_d(t)$ become equal, or which are critical values of H. Since the growth of $r(t)Y(t)^{-1}$ at these points and the infinity is polynomial, $r(t)Y(t)^{-1}$ defines in fact a vector of rational functions in t. It is not difficult to see that these rational functions are in fact polynomials ([Gr], App. C). To estimate the degree of these polynomials, note that a path from $(x_0, y_i(t))$ to $(x_0, y_d(t))$ on $\{H(x,y) - t = 0\}$ may be constructed as follows.

Consider the projection of $\{H(x,y) - t = 0\} \in \mathbb{C}^2$ to the x-plane, and denote the set of critical values of this projection by Q. Note that the action of the fundamental groop $\pi_1(\mathbb{C} - Q, x_0)$ on the x_0-fiber of the projection is transitive (since the curve is connected and smooth). Since the fundamental group is generated by the elementary loops (i.e. paths which start from x_0, go to the vicinity of a point in Q, encircling it and returning to x_0 via the same path), and since the action is transitive and the fiber consists of d points, there is a path from $(x_0, y_i(t))$ to $(x_0, y_d(t))$ which projects to a loop based at x_0, composed of traversing at most $d-1$ elementary loops. We may deform this projection so that it consists of at most $6(d-1)$ straight line segments contained in an origin centered disc which contains all the points of Q and lift it to another path, which we denote by Γ, from $(x_0, y_i(t))$ to $(x_0, y_d(t))$ on $\{H(x,y) - t = 0\}$. Though the length of Γ is not expressible by a semialgebraic formula, we may estimate the radius of an origin centered ball in $\mathbb{C}^2 \cong \mathbb{R}^4$, which contains Γ, and we may also estimate the cardinality of the decomposition of Γ to connected smooth pieces such that the tangent to all the points in a given piece lies in some fixed orthant in $\mathbb{R}^4$. Indeed we may write a first-order formula for the points where Γ is either not smooth or the tangent goes from one orthant to another orthant, eliminate the quantifiers by Renegar's theorem, and conclude that the cardinality of such a decomposition is not greater than $d^{O(1)}$.

Consider now the rate of growth of the maximal norm (in $\mathbb{C}^2 \cong \mathbb{R}^4$) of the critical points of the projection of $\{(x,y) : H(x,y) = t\}$ to the x-plane. These correspond to the points of the set $\{(x,y) : H(x,y) = t,\ H_y(x,y) = 0\}$. If for $t \to t_0$, t_0 finite, one of the points in this set tends to ∞, this would mean that the projective curves defined by $H(x,y) = t$ and $H_y(x,y) = 0$ have a common point at infinity, therefore the projective curves defined by $H_x(x,y) = 0$ and $H_y(x,y) = 0$ have a common point at infinity, contradicting the assumption that H is regular at infinity. So the correspondence

κ sending $t \in \mathbb{C} \cong \mathbb{R}^2$ to the maximal modulus of the critical points of the projection is a semialgebraic function which is bounded on compact subsets of its domain. Its graph may be given by a quantifier free formula whose total degree is independent of H and is bounded by $d^{O(1)}$. From Renegar's theorem, for example, one deduces then that its value at $t \in \mathbb{C}$ is bounded by $C_H max(2,|t|)^{d^{O(1)}}$, where $C_H > 0$ is a constant depending on H.

By the construction of Γ, the radius of the origin centered ball containing Γ, is bounded by the maximal distance from the origin of points of $\{(x,y) : H(x,y) = t\}$ projected to the origin centered disc on the x-plane, which has a radius equal to the maximal modulus of the critical values of the projection plus 1, say. This maximal distance exists for all $t \in \mathbb{C}$, since the leading coefficient of H w.r.t. y is cy^d, $c \neq 0$, by the assumption we made. Once again, this maximal distance is a semialgebraic function of $t \in \mathbb{C} \cong \mathbb{R}^2$ which is bounded on compact sets, and its value at $t \in \mathbb{C}$ is bounded by $C_H max(2,|t|)^{d^{O(1)}}$, where $C_H > 0$ is a constant depending on H.

Therefore the length of Γ is not greater than $d^{O(1)} \cdot C_H max(2,|t|)^{d^{O(1)}}$, and since the value of the integrals in 3.4) is bounded by the length of Γ times the bound on $\sqrt{\|a(x,y)|^2 + |\beta(x,y)|^2}$ along Γ, where $\alpha(x,y)dx + \beta(x,y)dy = \omega$, we conclude finally that the degree of the entries of $r(t)Y(t)^{-1}$ is not greater than $Dd^{O(1)}$.

Writing

$$r(t)Y(t)^{-1} = (p_1(t),..,p_{d-1}(t)) = p(t),$$

we get

$$3.5) \qquad \int_{x_0,y_i(t)}^{x_0,y_d(t)} \omega - \sum_{l=1}^{d-1} p_l(t)(y_i^l(t) - y_d^l(t)) \ \equiv \ 0,$$

$i = 1,..,d-1$ (this is possible exactly since $\omega \in d\Omega^o(\mathbb{C}^2)$).

Now, 3.5) can be written as

$$\int_{x_0,y_i(t)}^{x_0,y_d(t)} \left(\omega - d\left(\sum_{l=1}^{d-1} p_l(H(x,y))y_i^l\right)\right) \ \equiv \ 0,$$

which means, putting $\widehat{\omega} = \omega - d\left(\sum_{l=1}^{d-1} p_l(H(x,y))y^l\right)$, that the integral

$$3.6) \qquad F(u,v) = \int_{(x_0,y_i(H(u,v)))}^{(u,v)} \widehat{\omega}$$

is well defined and holomorphic on $\mathbb{C}^2 - H^{-1}(\Sigma_H)$. Estimating the growth of $F(u,v)$ as u,v tend to infinity, we conclude that $F(u,v)$ is in fact a polynomial of degree $d^{O(1)}$. Indeed, fix $u,v \in \mathbb{C}^2$, and consider the path going from (u,v) to $(x_0,y_i(H(u,v)))$ on $\{(x,y) : H(x,y) - H(u,v) = 0\}$, which is constructed as the composition of the following two paths. The first path is the lift to $\{(x,y) : H(x,y) - H(u,v) = 0\}$ of the segment lying in the x-plane which connects u to x_0, such that the lifted path starts at (u,v) and ends at $(x_0,y_j(H(u,v))$ for some j. Compose now this first path with the second path, which connects $(x_0,y_j(H(u,v))$ to $(x_0,y_i(H(u,v))$, and which was constructed above. We again conclude that the length of the path going from (u,v) to $(x_0,y_i(H(u,v)))$ on $\{(x,y) : H(x,y) - H(u,v) = 0\}$ is not greater than $d^{O(1)} \cdot max(2,|H(u,v)|)^{d^{O(1)}}$, and it is contained in an origin centered ball of radius $max(2,|H(u,v)|)^{d^{O(1)}}$. When (u,v) tends to a point in $\mathbb{C}^2$, these estimates stay bounded, and therefore 3.6) defines a holomorphic function on $\mathbb{C}^2$. When (u,v) tend to infinity, the estimates imply that 3.6) defines in fact a polynomial of degree $Dd^{O(1)}$ (since the degree of the form $\widehat{\omega}$ is not greater than $Dd^{O(1)}$).

Writing $F(x,y)$ instead of $F(u,v)$, denoting $\widehat{\omega} = \widehat{\alpha}dx + \widehat{\beta}dy$, one has the identity

$$H_y F_x - H_x F_y = X_H(F) = \widehat{\omega}(X_H) = H_y\widehat{\alpha} - H_x\widehat{\beta},$$

which implies $H_y(F_x - \widehat{\alpha}) = H_x(F_y - \widehat{\beta})$. Since H is regular at infinity, H_x and H_y have no common factor, implying therefore by unique factorization in $\mathbb{C}[x,y]$, that

$$\frac{F_y - \widehat{\beta}}{H_y} = \frac{F_x - \widehat{\alpha}}{H_x}$$

is in fact a polynomial, denoted by B. It follows that $\widehat{\omega}$ is of the form $dF - BdH$, implying the same for ω. Together with the bound $d^{O(1)}$ for the degrees of F and B, this proves the proposition. $\square$

Our construction of the Picard-Fuchs equations depends on the following result of Gavrilov ([Ga], see Proposition 1 and Remark after Lemma 1 in [N]). For us, the degree of $P(x,y)dx + Q(x,y)dy \in \Omega^1(\mathbb{C}^2)$ is the maximum of the degrees of $P(x,y)$ and of $Q(x,y)$.

Theorem 3.3. *Let $H(x,y)$ be a polynomial of degree d, regular at infinity, and let $\widetilde{H}$ denote its highest homogeneous part. Then the factor module $\Omega^1(\mathbb{C}^2)/d\Omega^0(\mathbb{C}^2)$ is generated by any set of $(d-1)^2$ forms $\omega_1, .., \omega_{(d-1)^2} \in \Omega^1(\mathbb{C}^2)$, for which the polynomials g_i, defined by*

$$d\omega_i = g_i dx \wedge dy, \qquad i = 1, .., (d-1)^2,$$

constitute a monomial basis for the complex vector space $\mathbb{C}[x,y]/ < \widetilde{H}_x, \widetilde{H}_y >$. Moreover, for any $\omega \in \Omega^1(\mathbb{C}^2)$, there exist polynomials $a_i(t) \in \mathbb{C}[t]$, $i = 1, .., (d-1)^2$, of degree at most $(deg(\omega) - deg(\omega_i))/d$, such that $\omega = \sum_{i=1}^k a_i(t) \cdot \omega_i$.

Lemma 3.4. *There exists a constructible subset $\mathcal{G}^d \subset \mathcal{H}^d$ of codimension zero, and a set of $(d-1)^2$ monomials $g_1, .., g_{(d-1)^2} \in \mathbb{C}[x,y]$, their degrees being not greater than $(d-1)^2 - 1$, such that each $H \in \mathcal{G}^d$ is regular at infinity and has degree d, and for each $H \in \mathcal{G}^d$ $g_1, .., g_{(d-1)^2}$ constitute a basis for $\mathbb{C}[x,y]/ < \widetilde{H}_x, \widetilde{H}_y >$.*

Proof. Observe that for any polynomial H of degree d, regular at infinity, the dimension of $\mathbb{C}[x,y]/ < \widetilde{H}_x, \widetilde{H}_y >$ is $(d-1)^2$ (cf., for example, [Br] Proposition 2.4).

We assume some familiarity with monomial orderings and leading terms diagrams (cf. [CLO]). We choose a monomial ordering. Then, for each $H \in \mathcal{H}^d$, a basis $g_1, .., g_{(d-1)^2}$, of $C[x,y]/ < \widetilde{H}_x, \widetilde{H}_y >$ may be constructed as follows. Let $\Delta_{\widetilde{H}}$ denote the leading terms diagram of $< \widetilde{H}_x, \widetilde{H}_y >$; a basis for $\mathbb{C}[x,y]/ < \widetilde{H}_x, \widetilde{H}_y >$ may then be taken to consist of the $(d-1)^2$ monomials in the complement of $\Delta_{\widetilde{H}}$ ([CLO], Pr. 1, pg. 228, Pr. 8, pg. 232).

Now, there is only a finite number of diagrams with the complement consisting of $(d-1)^2$ monomials; denote them by $\Delta_1, .., \Delta_l$. The set $\mathcal{N}_j \in \mathcal{H}^d$ of polynomials H which are regular at infinity and have degree d, and for which the leading terms diagram of the ideal $< \widetilde{H}_x, \widetilde{H}_y >$ is Δ_j, is constructible. The sets $\mathcal{N}_1, .., \mathcal{N}_l$ constitute a partition of the set of polynomials of degree d regular at infinity, which is an open subset of $\mathcal{H}^d$. Therefore for some k, $1 \leq k \leq l$, $\mathcal{N}_k$ must be of codimension zero. We take $\mathcal{G}^d = \mathcal{N}_k$.

For every $H \in \mathcal{G}^d$, the set of monomials in the complement of Δ_k, denoted $g_1, .., g_{(d-1)^2}$, will constitute a basis of $\mathbb{C}[x,y]/ < \widetilde{H}_x, \widetilde{H}_y >$. Note the degrees of these monomials cannot be larger than $(d-1)^2 - 1$ (because of the structure of leading terms diagrams). $\square$

By Theorem 3.3 and Lemma 3.4, there exist monomial forms ω_l, $l = 1, .., (d-1)^2$, of degree not greater than $(d-1)^2$, which generate the $\mathbb{C}[t]$-module $\Omega^1(\mathbb{C}^2)/d\Omega^0(\mathbb{C}^2)$ for any $H \in \mathcal{G}^d$.

We need the following standard fact (10.2.4, pg.284 [AGV]), which enables us to write the derivatives of complete Abelian integrals as complete Abelian integrals again.

Proposition 3.5 (Gelfand-Leray derivative). *Let $H \in \mathbb{C}[x,y]$ and let $\gamma(t)$ denote a continuously varying cycle in the locally trivial bundle determined by the polynomial H. Let ω be a holomorphic 1-form on $\mathbb{C}^2$, and suppose there exists a 1-form α, holomorphic on $\mathbb{C}^2$, such that $dH \wedge \alpha = d\omega$. Then*

$$\frac{d}{dt}\int_{\gamma(t)} \omega \;=\; \int_{\gamma(t)} \alpha.$$

To construct Picard-Fuchs (linear differential) equations, we first construct a linear differential system for $(\int_{\gamma(t)} \omega_1, .., \int_{\gamma(t)} \omega_{(d-1)^2})^T$.

Fix $H \in \mathcal{H}^d$, $1 \leq l \leq (d-1)^2$. By Proposition 3.5, one can find polynomials $\alpha_l, \beta_l \in \mathbb{C}[x,y]$ of degree not greater than $d(d-1)$, so that

$$3.7) \qquad \frac{d}{dt}\int_{\gamma(t)} (xH_x + yH_y)^2 \omega_l \;=\; \int_{\gamma(t)} \alpha_l dx + \beta_l dy,$$

holds for any continuously varying cycle $\gamma(t)$ in the locally trivial bundle determined by H. In fact, we could have taken any polynomial in the ideal $< H_x, H_y >$ instead of $xH_x(x,y) + yH_y(x,y)$. What matters for us, however, is that for *homogeneous* H of degree d, $H(x,y)/d = xH_x(x,y) + yH_y(x,y)$.

Let now $H \in \mathcal{G}^d$. Note that $H \in \mathcal{G}^d$ has degree d and is regular at infinity. By Theorem 3.3 and i) of Proposition 3.2 , there exist polynomials $A_l, B_l \in \mathbb{C}[x,y]$, and polynomials $c_{lm}(t) \in \mathbb{C}[t]$, $m = 1, .., (d-1)^2$, so that the integrand on the left hand side of 3.7) can be written as

$$3.8) \quad (xH_x(x,y) + yH_y(x,y))^2 \omega_l = \sum_{m=1}^{(d-1)^2} c_{lm}(H(x,y))\omega_m + dA_l(x,y) + B_l(x,y)dH(x,y).$$

By Theorem 3.3, the degree of $c_{lm}(t) \in \mathbb{C}[t]$ is not greater than $(2d + (d-1)^2)/d$, i.e. is not greater than d for $d \geq 2$. By ii) of Proposition 3.2, A_l, B_l can be taken to be of degree not greater than $d^{O(1)}$.

Observe that 3.8) defines in fact a linear system over $\mathbb{C}(\lambda)$, λ being the coefficients of $H \in \mathcal{H}^d$

$$3.9) \qquad M(\lambda)u \; = \; w(\lambda),$$

the unknowns u being the coefficients of $c_{lm}(t)$ and of A_l, B_l, $1 \le l, m \le (d-1)^2$. Thus $dim(u) \le d^{O(1)}$. The entries of the matrix $M(\lambda)$ and the vector $w(\lambda)$ are integral polynomials in λ of degree $d^{O(1)}$ and height not greater than $2^{d^{O(1)}}$.

Since 3.9) is solvable for all $\lambda \in \mathcal{G}^d$, and since $\mathcal{G}^d$ open, there exists a constructible set $\mathcal{G}'^d \subset \mathcal{G}^d$ of codimension zero, and polynomials $p_{lmr}(\lambda) \in \mathbb{Z}[\lambda]$, $l, m = 1, .., (d-1)^2$, $r = 0, .., d$, and $q(\lambda) \in \mathbb{Z}[\lambda]$, such that $\forall \lambda \in \mathcal{G}'^d$, $q(\lambda) \neq 0$, and

$$3.10) \qquad c_{lm}(t) = \sum_{r=0}^{d} \frac{p_{lmr}(\lambda)}{q(\lambda)} t^r.$$

By using Cramer's rule for a suitable subsystem of 3.9), we conclude that $deg(q(\lambda)) \le d^{O(1)}$ and its height is not greater than $2^{d^{O(1)}}$, and the same bounds hold for $p_{lmr}(\lambda)$. Here we use that $dim(\lambda) \le O(d^2)$, and the fact that for any two integral polynomials $a, b \in \mathbb{Z}[\lambda]$

$$3.11) \qquad height(a \cdot b) \; \le \; (1 + min(deg(a), deg(b)))^{dim(\lambda)} \cdot height(a) \cdot height(b).$$

Integrating now both sides of 3.8), we get by Theorem 3.1 that

$$3.12) \qquad \int_{\gamma_\lambda(t)} (xH_x + yH_y)^2 \, \omega_l = \int_{\gamma_\lambda(t)} \sum_{m=1}^{(d-1)^2} c_{lm}(H)\omega_m = \sum_{m=1}^{(d-1)^2} c_{lm}(t) \int_{\gamma_\lambda(t)} \omega_m,$$

for any continuously varying cycle $\gamma_\lambda(t)$ in the locally trivial bundle determined by $H(\lambda)$. Using 3.10), we write 3.12) in the matrix form

$$3.13) \qquad \left(\int_{\gamma_\lambda(t)} (xH_x + yH_y)^2 \, \omega_m \right)_m \; = \; \frac{K(t,\lambda)}{q(\lambda)} \left(\int_{\gamma_\lambda(t)} \omega_m \right)_m,$$

where, to save space, $\left(\int_{\gamma_\lambda(t)} \omega_m \right)_m$ denotes the column vector $\left(\int_{\gamma_\lambda(t)} \omega_1, .., \int_{\gamma_\lambda(t)} \omega_{(d-1)^2} \right)^T$, and where $q(\lambda)$ and the entries of $K(t, \lambda)$ are integral polynomials in t, λ of degree and height bounded by $d^{O(1)}, 2^{d^{O(1)}}$, respectively.

Observe that likewise, the right hand side of 3.7), may be written using Theorem 3.3, Theorem 3.1, and Proposition 3.2, as

$$\left(\int_{\gamma_\lambda(t)} \alpha_m dx + \beta_m dy \right)_m \; = \; \frac{L(t,\lambda)}{w(\lambda)} \left(\int_{\gamma_\lambda(t)} \omega_m \right)_m,$$

where, as in 3.13), $w(\lambda)$ and the entries of $L(t, \lambda)$ are again integral polynomials in t, λ of degree and height bounded by $d^{O(1)}, 2^{d^{O(1)}}$.

The equality 3.7) then implies

$$\frac{d}{dt}\ \frac{K(t,\lambda)}{q(\lambda)}\left(\int_{\gamma_\lambda(t)}\omega_m\right)_m = \frac{L(t,\lambda)}{w(\lambda)}\left(\int_{\gamma_\lambda(t)}\omega_m\right)_m .$$

If the matrix $K(t,\lambda)$ was invertible over $\mathbb{C}(t,\lambda)$, we could have written, for $\lambda\in\mathcal{G}''^d\subset\mathcal{G}^d$, $\mathcal{G}''^d$ being again a constructible subset of $\mathcal{H}^d$ of codimension zero, the equality

$$3.14)\qquad \left(\int_{\gamma_\lambda(t)}\omega_m\right)_m' = K(t,\lambda)^{-1}\left(\frac{L(t,\lambda)q(\lambda)}{w(\lambda)} - K'(t,\lambda)\right)\left(\int_{\gamma_\lambda(t)}\omega_m\right)_m .$$

valid for all continuously varying cycles $\gamma_\lambda(t)$ in the locally trivial bundle determined by $H(\lambda)$, $\lambda\in\mathcal{G}''^d$.

The equality 3.14) would then define a linear differential system of dimension $(d-1)^2$, which would be satisfied, for any fixed $\lambda\in\mathcal{G}''^d$, by $\left(\int_{\gamma_\lambda(t)}\omega_m\right)_m$. We write this system as

$$3.15)\qquad I' = \frac{A(t,\lambda)}{a(t,\lambda)}I,$$

where $a(t,\lambda)$ and the entries of the $(d-1)^2\times(d-1)^2$ matrix $A(t,\lambda)$ are polynomials in $\mathbb{Z}[t,\lambda]$. Their degree and height can be again shown, using 3.11), to be bounded by $d^{O(1)}, 2^{d^{O(1)}}$, respectively. We note, that since the Abelian integrals may be analytically continued to all points of the open set $T\subset\mathbb{C}\times\mathbb{C}^{dim(\lambda)}$, 3.15) would be in fact satisfied by $\left(\int_{\gamma_\lambda(t)}\omega_m\right)_m$ for all $\lambda\in\mathcal{H}^d$ for which $H(\lambda)$ is regular at infinity and has degree d, and for which the matrix $A(t,\lambda)/a(t,\lambda)$ is defined (as a rational function in t). Thus, the restriction $\lambda\in\mathcal{G}''^d$ is unnecessary.

We now show that the assumption we made is indeed true:

Proposition 3.6. *The matrix $K(t,\lambda)$ (3.13)) is invertible over the field $\mathbb{C}(t,\lambda)$.*

Proof. It is for proving this proposition that the integrand $(xH_x+yH_y)^2\omega_m$ in 3.7) was chosen. Take any polynomial $H\in\mathcal{G}^d$. By Lemma 3.4 its highest homogeneous part $\widetilde{H}$ also belongs to $\mathcal{G}^d$.

Denote the coefficient tuple of $\widetilde{H}$ by λ_0. Since $H\in\mathcal{G}^d$, its highest homogeneous part $\widetilde{H}$ is regular at infinity and has degree d. Hence, any cycle $\gamma_(t_0,\lambda_0)$, $t_0\neq 0$, can be naturally continued to a neighbourhood of (t_0,λ_0). From Theorem 12.1, pg 317 [AGV], we conclude that there exists an origin centered V ball in $\mathbb{C}^2$, holomorphic 1-forms $\alpha_1,..,\alpha_{(d-1)^2}$ on V, and cycles $\gamma_1(t_0,\lambda_0),..,\gamma_{(d-1)^2}(t_0,\lambda_0)\in H_1(\{H(\lambda_0)=t_0\}\cap V,\mathbb{Z})$ for some $t_0\neq 0$ close enough to $0\in\mathbb{C}$ (i.e. close enough to the critical point of $H(\lambda_0)$), for which

$$det\left(\int_{\gamma_j(t_0,\lambda_0)}\alpha_i\right)_{ij} \neq 0.$$

Instead of the holomorphic forms $\alpha_1,..,\alpha_{(d-1)^2}$ one may take polynomial forms $\beta_1,..,\beta_{(d-1)^2}$, since on the union of (images of representatives of) $\gamma_1(t_0,\lambda_0),..,\gamma_{(d-1)^2}(t_0,\lambda_0)$, which is a compact subset of V, one may approximate $\alpha_1, .., \alpha_m$ by polynomial forms to any given accuracy (true because V is a ball).

Since $\lambda_0 \in \mathcal{G}^d$, by Theorem 3.3 and Theorem 3.1, integral of any polynomial form is a linear combination over $\mathbb{C}[t]$ of integrals of the polynomial forms $\omega_1, .., \omega_{(d-1)^2}$ we conclude that

$$det \left(\int_{\gamma_j(t_0,\lambda_0)} \omega_i \right)_{ij} \neq 0.$$

Now, in a neighbourhood of (t_0, λ_0) the matrix $K(t,\lambda)/q(\lambda)$ will be equal, as a matrix with *meromorphic* entries, to the holomorphic matrix

$$\left(\int_{\gamma_j(t,\lambda)} (xH(\lambda)_x + yH(\lambda)_y)^2 \, \omega_i \right)_{ij} \cdot \left(\int_{\gamma_j(t,\lambda)} \omega_i \right)_{ij}^{-1}.$$

But this holomorphic matrix is equal, for (t_0, λ_0) to

$$\left(\int_{\gamma_j(t_0,\lambda_0)} (xH(\lambda_0)_x + yH(\lambda_0)_y)^2 \, \omega_i \right)_{ij} \cdot \left(\int_{\gamma_j(t_0,\lambda_0)} \omega_i \right)_{ij}^{-1} = \frac{t_0^2}{d} \cdot Id.$$

precisely since $xH(\lambda_0)_x + yH(\lambda_0)_y = H(\lambda_0)/d$; note that the right hand side is an invertible matrix (recall $t_0 \neq 0$). But this means that in a neighbourhood of (t_0, λ_0), $K(t,\lambda)/q(\lambda)$ is equal (as a meromorphic matrix) to a holomorphic invertible matrix, implying that $K(t,\lambda)$ is indeed invertible over $\mathbb{C}(t,\lambda)$. $\square$

We now show how a linear differential equation for $\int_{\gamma_\lambda(t)} \omega_m$ may be derived from the system 3.15), $m = 1, .., (d-1)^2$. Taking the derivative of both sides of 3.15), then multiplying by $a(t,\lambda)$, and taking into account that $a(t,\lambda)I' = A(t,\lambda)I$, we get

$$a^2(t,\lambda)I'' = \left(A^2(t,\lambda) + a(t,\lambda)A'(t,\lambda) - a'(t,\lambda)A(t,\lambda)\right) I.$$

Taking further derivatives and proceeding in the same manner, we get

$$3.16) \quad a^j(t,\lambda)I^{(j)} = A_j(t,\lambda)I,$$

where $A_0(t,\lambda) = Id$, and

$$A_{j+1}(t,\lambda) = a(t,\lambda)A_j'(t,\lambda) + A_j(t,\lambda)\left(A(t,\lambda) - ja'(t,\lambda)Id\right),$$

$j = 0, 1, 2, ..$. Using 3.11), it may be checked that

$$3.17) \quad deg(A_j) \leq jd^{O(1)}, \quad height(A_j) \leq 2^{jd^{O(1)}}$$

(where $deg(A_j), height(A_j)$ mean the maximal degree and height of the entries of A_j). Bounds of the same form hold for the degree and height of $a(t,\lambda)^j$.

Take now the m-th component of I, $1 \leq m \leq (d-1)^2$. Then for any $j = 0, 1, 2, ..$, denoting by α_{jm} the m-th row of the matrix A_j in 3.16), we get

$$3.18) \quad a^j(t,\lambda)I_m^{(j)} = \alpha_{jm}(t,\lambda)I.$$

Clearly there exists k_m, $1 \leq k_m \leq (d-1)^2$, such that the vectors $\alpha_{0m}(t,\lambda),..,\alpha_{k_m m}(t,\lambda)$ are linearly dependent over $\mathbb{C}(t,\lambda)$, but $\alpha_{0m}(t,\lambda),..,\alpha_{(k_m-1)m}(t,\lambda)$ are not. Then one may solve uniquely

$$3.19) \qquad \alpha_{k_m m}(t,\lambda) \;=\; \sum_{l=0}^{k_m-1} w_{lm}(t,\lambda)\alpha_{lm}(t,\lambda)$$

for the rational functions $w_{lm}(t,\lambda)$, $l=0,..,k-1$. Multiplying 3.19) by the (column) vector I and using 3.18), we get the parameter dependent linear differential equation

$$3.20) \qquad I_m^{(k_m)} - \sum_{l=0}^{k_m-1} w_{lm}(t,\lambda)I_m^{(l)} = 0.$$

Proposition 3.7. *For all $m=1,..,(d-1)^2$, $k_m=(d-1)^2$. There exists a constructible set $V_m \subset \mathcal{H}^d$ of codimension zero, such that for any $\lambda \in V_m$, the coefficients of 3.20) are defined, and the space of solutions of 3.20) is spanned by $\int_{\gamma_\lambda(t)} \omega_m$.*

Proof. Recall that in the proof of Proposition 3.6, we have shown that for a homogeneous polynomial $H(\lambda_0) \in \mathcal{G}^d$, there are cycles $\gamma_1(t_0,\lambda_0),..,\gamma_{(d-1)^2}(t_0,\lambda_0) \in H_1(\{H(\lambda_0) = t_0\} \cap V, \mathbb{Z})$ for some $t_0 \neq 0$ close enough to $0 \in \mathbb{C}$ such that

$$det \left(\int_{\gamma_j(t_0,\lambda_0)} \omega_i \right)_{ij} \;\neq\; 0,$$

and therefore (since $(t_0,\lambda_0) \in T$),

$$3.21) \qquad det \left(\int_{\gamma_j(t,\lambda)} \omega_i \right)_{ij} \;\neq\; 0,$$

in some neighbourhood U of (t_0,λ_0). By Theorem 3.4 in [AGV], there is $\lambda_1 \in \mathcal{G}^d$ close enough to λ_0, so that $(t_0,\lambda_1) \in U$, and such that the monodromy representation of the fundamental group $\pi_1(\mathbb{C} - \Sigma_{\lambda_1}, t_0)$ on the homology group of the fiber $\{(x,y) : H(\lambda_1)(x,y) = t_0\}$ is irreducible.

It is not difficult to show that the homology group (with coefficients in $\mathbb{Z}$) of any fiber over a point in T is isomorphic to $\mathbb{Z}^{(d-1)^2}$. Without limiting generality, we may assume that $\gamma_1(t_0,\lambda_1),..,\gamma_{(d-1)^2}(t_0,\lambda_1)$ is in fact a basis of $H_1(\{(x,y) : H(\lambda_1)(x,y) = t_0\}, \mathbb{Z})$. Suppose that $\int_{\gamma_1(t,\lambda_1)} \omega_m,..,\int_{\gamma_{(d-1)^2}(t,\lambda_1)} \omega_m$ are linearly dependent over $\mathbb{C}$ (as functions of t), so that there exist $\alpha_1,..,\alpha_{(d-1)^2}$, not all zero, for which $\sum_{l=1}^{(d-1)^2} \alpha_l \int_{\gamma_l(t,\lambda_1)} \omega_m = 0$ for all t. This means that there exists a nonzero element

$$\delta = \sum_{l=1}^{(d-1)^2} \alpha_l \gamma_l(t_0,\lambda_1) \in H_1(\{(x,y) : H(\lambda_1)(x,y) = t_0\}, \mathbb{C}) \cong \mathbb{C}^{(d-1)^2},$$

and a nonzero linear form $\gamma \mapsto \int_\gamma \omega_m$ on $H_1(\{(x,y) : H(\lambda_1)(x,y) = t_0\}, \mathbb{C})$, such that δ stays in the kernel of this form under all monodromy transformations. Therefore the monodromy invariant subspace of all elements which remain in the kernel of $\gamma \mapsto \int_\gamma \omega_m$ under all monodromy transformations is nonzero. Since at least one of $\int_{\gamma_1(t_0,\lambda_1)} \omega_m$,..,

$\int_{\gamma_{(d-1)^2}(t_0,\lambda_1)} \omega_m$ is nonzero (otherwise the determinant 3.21) would be zero at (t_0, λ_1)), this invariant subspace is nontrivial, and therefore the monodromy representation on $H_1(\{(x,y) : H(\lambda_1)(x,y) = t_0\}, \mathbb{C})$ is reducible, contradicting the fact cited above. We conclude that $\int_{\gamma_1(t,\lambda_1)} \omega_m, .., \int_{\gamma_{(d-1)^2}(t,\lambda_1)} \omega_m$ are linearly independent over $\mathbb{C}$.

We consider now the action of the fundamental group $\pi_1(T, (t_0, \lambda_1))$ on the homology of the fiber over (t_0, λ_1). Choosing a basis for this lattice, the monodromy representation acts by linear transformations which are represented by integral matrices. Since also the inverse transformations are so represented, these matrices must have determinant equal to 1 or -1, in fact 1 in our case. Therefore the Wronskian of $\int_{\gamma_1(t,\lambda)} \omega_m, .., \int_{\gamma_{(d-1)^2}(t,\lambda)} \omega_m$ is a holomorphic function on T, not identically zero. By considering its rate of growth as (t, λ) tends to infinity and to points on ∂T along all complex lines parallel to the axes of $\mathbb{C} \times \mathbb{C}^{dim(\lambda)}$, one concludes that the restriction of the Wronskian to any such complex line is a rational function (in one complex variable). This implies, however, that the Wronskian is in fact a rational function in t, λ. Thus, there exists a constructible set $V'_m \subset \mathcal{H}^d$ of codimension zero, such that $\forall \lambda \in V'_m$ the Wronskian is defined and is not identically zero as a function of t. We conclude that there exists a constructible subset $V_m \subset \mathcal{H}^d$ of codimension zero, for which $\int_{\gamma_1(t,\lambda)} \omega_m, .., \int_{\gamma_{(d-1)^2}(t,\lambda)} \omega_m$ is a set of linearly independent solutions for 3.20).

It follows that $k_m \geq (d-1)^2$; since $k_m \leq (d-1)^2$ as well, $k_m = (d-1)^2$. $\square$

Again, since $\int_{\gamma_\lambda(t)} \omega_m$ continue analytically to all of $T \subset \mathbb{C} \times \mathbb{C}^{dim(\lambda)}$, it in fact solves 3.20) for all $\lambda \in \mathcal{H}^d \cong \mathbb{C}^{dim(\lambda)}$ for which $H(\lambda)$ is regular at infinity and has degree d, and for which the coefficients of 3.20) are defined.

Now, although 3.20) is a parameter-dependent equation for the integral $\int_{\gamma_\lambda(t)} \omega_m$, we are interested in fact in a linear differential equation for the integral of an arbitrary 1-form of degree d. To obtain such equation we proceed as follows. By Theorem 3.3 and Theorem 3.1 again, we know that for $\lambda \in \mathcal{G}^d$, the integral of any such form can be written as the integral of a linear combination over $\mathbb{C}$ of $\omega_1, .., \omega_{(d-1)^2}$ (where the coefficients depend on the parameter λ). Let $\mu_1, .., \mu_{(d-1)^2}$ be the coefficients of this linear combination, which we view again as parameters. Consider now the system 3.15) augmented by the following equation obtained by differentiating both sides of $I_0 = \mu_1 I_1 + .. + \mu_{(d-1)^2} I_{(d-1)^2}$, multiplying by $a(t, \lambda)$, and then simplified using 3.15):

$$I'_0 \;=\; (\mu_1, .., \mu_{(d-1)^2}) \frac{A(t, \lambda)}{a(t, \lambda)} I.$$

We call the new system obtained the *augmented system* (for degree d). It depends on the parameters (λ, μ), and is satisfied by (the column vector)

$$\left(\int_{\gamma_\lambda(t)} \sum_{l=1}^{(d-1)^2} \mu_l \omega_l, \int_{\gamma_\lambda(t)} \omega_1, .., \int_{\gamma_\lambda(t)} \omega_{(d-1)^2} \right)^T$$

for any $\lambda \in \mathcal{H}^d$, for which the system is defined and $H(\lambda)$ is regular at infinity and has degree d.

We may now construct a linear differential equation for I_0 from the augmented system in the same way by which we constructed the linear differential equations satisfied by the components of the system 3.15). Its order is *a priori* less or equal to $(d-1)^2+1$. For generic λ, μ this equation has a set of $(d-1)^2$ linearly independent solutions. Clearly it has also the solution given by a nonzero constant. For a generic λ, μ, this constant cannot be a linear combination of the former solutions by an argument similar to the argument used in the proof of Proposition 3.7. Therefore the order of the equation for I_0 is $(d-1)^2+1$. For further reference, we write it explicitly as

$$3.22) \quad I_0^{((d-1)^2+1)} + \frac{p_{(d-1)^2}(t,\lambda,\mu)}{q(t,\lambda,\mu)} I_0^{((d-1)^2)} + .. + \frac{p_0(t,\lambda,\mu)}{q(t,\lambda,\mu)} I_0 = 0.$$

$q(t,\lambda), p_n(t,\lambda) \in \mathbb{Z}[t,\lambda,\mu]$, $n = 0,..,(d-1)^2$. Again, it is satisfied by $\int_{\gamma_\lambda(t)} \sum_{l=1}^{(d-1)^2} \mu_l \omega_l$ whenever 3.22) is defined for the given values of λ, μ, and $H(\lambda)$ is regular at infinity and has degree d.

Proposition 3.8. *The degree and the height of the polynomials $q, p_n \in \mathbb{Z}[t,\lambda,\mu]$ in 3.22), $n = 0,..,(d-1)^2$, are bounded by $d^{O(1)}, 2^{d^{O(1)}}$, respectively.*

Proof. A computation by Cramer's rule, taking into consideration the bounds for the degree and height of a^j, A_j, $j \leq (d-1)^2+1$, which are, according to 3.17),

$$((d-1)^2+1)d^{O(1)} = d^{O(1)}, \qquad 2^{((d-1)^2+1)d^{O(1)}} = 2^{d^{O(1)}}$$

respectively. □

4. Picard-Fuchs equations depend regularly on parameters.

Our first aim in this section is to show that the parameter dependent linear differential equation 3.22) depends regularly on the parameters (λ, μ). In fact, singularly perturbed linear differential equations admit parameter dependent solutions with special properties, as stated in Lemma 4.1 below. This allows to exclude the possibility that such equations appear as restrictions of Picard-Fuchs equations to holomorphic arcs in general position in the parameter space. We then use Theorem 2.1 (with the bounds given in Proposition 3.8) to obtain a bound for the number of zeros of an Abelian integral.

Lemma 4.1 *Let*

$$4.1) \quad \varepsilon^s y^{(n)} + a_{n-1}(t,\varepsilon) y^{(n-1)} + ... + a_0(t,\varepsilon) y = 0,$$

be a linear differential equation such that $s \geq 1$, $a_i(t,\varepsilon)$ are holomorphic in the polydisc $U \times W \subset \mathbb{C} \times \mathbb{C}$, $i = 1,..,n$, and for some k, $1 \leq k \leq n-1$, $a_k(t,0) \not\equiv 0$ (i.e. 4.1) is, in our terminology, singularly perturbed). Then for any $t_0 \in U$ (with a possible exception of a discrete subset of U), there exists a parameter dependent solution of 4.1), $y(t,\varepsilon) \in \mathcal{O}(U \times (W - \{0\}))$, for which $(y(t_0,\varepsilon),..,y^{(n-1)}(t_0,\varepsilon)) \in \mathcal{O}(W-\{0\})^n$ is a vector with constant entries, such that for any other $t_1 \in U$ outside of the exceptional, at most countable, subset $E \subset U$, $y(t_1,\varepsilon) \in \mathcal{O}(W-\{0\})$ has an essential singularity at $\varepsilon = 0$.

Proof. Let $z(t,\varepsilon)$ be a parameter dependent solution holomorphic on $U\times W$ (and not just on $U\times(W-\{0\})$). Let t_0 be such that $a_k(t_0,0)\neq 0$.

We first show that $z(t,\varepsilon)$ is uniquely determined by the $n-1$ functions

$$y^{(i)}(t_0,\varepsilon)\in\mathcal{O}(W),$$

$i=0,..,k-1,k+1,..,n-1$. Indeed, if not, then there exists such a parameter dependent solution, not equal identically to zero, with $z^{(i)}(t_0,\varepsilon)\equiv 0$ for $i=0,..,k-1,k+1,..,n-1$.

Since $z(t,\varepsilon)$ is not identically zero, we then must have that $z^{(k)}(t_0,\varepsilon)\in\mathcal{O}(W)$ is not identically zero. Write $z^{(k)}(t_0,\varepsilon)=\varepsilon^q m(\varepsilon)$, $q\geq 0$, $m(0)\neq 0$, $m(\varepsilon)\in\mathcal{O}(W)$. Suppose $q>0$. $z(t,0)$ is the solution of 4.1) at $\varepsilon=0$. Since $a_k(t_0,0)\neq 0$, $z(t,0)$ satisfies a linear differential equation of order not smaller than k and not greater than $n-1$, with initial conditions at t_0 being $z^{(i)}(t_0,0)=0\ \forall i\neq k$, $z^{(k)}(t_0,0)=(0)^q m(0)=0$, $(q>0)$. Thus $z(\cdot,0)$ equals identically to zero. This implies that $z(t,\varepsilon)=\varepsilon z_1(t,\varepsilon)$, $z_1(t,\varepsilon)$ being another parameter dependent solution satisfying $z_1^{(i)}(t_0,\varepsilon)\equiv 0$ for $i=0,..,k-1,k+1,..,n-1$, $z_1^{(k)}(t_0,\varepsilon)=\varepsilon^{q-1}m(\varepsilon)$. Continuing in the same manner, we finally get the parameter dependent solution $z_q(t,\varepsilon)\in\mathcal{O}(U\times W)$, for which $z_q^{(i)}(t_0,\varepsilon)\equiv 0$, $i=0,..,k-1,k+1,..,n-1$, $z_q^{(k)}(t_0,\varepsilon)=m(\varepsilon)$. Substituting $z_q(t,\varepsilon)$ into the original equation, putting $t=t_0$, we get:

$$e^s z_q^{(n)}(t_0,\varepsilon)+a_k(t_0,\varepsilon)z_q^{(k)}(t_0,\varepsilon)=0,$$

which cannot be true, since for $\varepsilon=0$ it implies that $a_k(t_0,0)\cdot z_q^{(k)}(t_0,0)=a_k(t_0,0)\cdot m(0)\neq 0$ is zero.

The set of parameter dependent solutions of 4.1), holomorphic in $U\times W$ has the obvious structure of $\mathcal{O}(W)$ module. By the above, this module can be identified with a submodule of $\mathcal{O}(W)^{n-1}$ (recall that $U\times W$ is connected).

Suppose there exist n such solutions, $y_1(t,\varepsilon),..,y_n(t,\varepsilon)\in\mathcal{O}(U\times W)$, linearly independent for at least one value of ε in W, and identify them with the elements $\alpha_1,..,\alpha_n\in\mathcal{O}(W)^{n-1}$. Then there exist $s_1,..,s_n\in\mathcal{O}(W)$, not all zero, such that $\sum s_i\alpha_i=0$ (there are such $s_i\in\mathcal{M}(W)$, since $\alpha_1,..,\alpha_n$, considered as elements of the $\mathcal{M}(W)$ vector space $\mathcal{M}(W)^{n-1}$, are of course linearly dependent; now just multiply by a common denominator). Thus for all $\varepsilon\in W$, besides maybe a discrete set of points where all $s_1,..,s_n$ vanish, these n parameter dependent solutions are linearly dependent over $\mathbb{C}$. Therefore, their (parameter dependent) Wronskian, which is a holomorphic function on $U\times W$, vanishes on an open subset of $U\times W$, implying it in fact vanishes everywhere, and there can not exist such a set of parameter dependent solutions.

Take now a maximal set of parameter dependent solutions of 4.1), $y_1(t,\varepsilon),..,y_j(t,\varepsilon)$, $j<n$, holomorphic on $U\times W$, which are linearly independent for some parameter value in W. Let us denote this parameter value by ε_0, and choose a tuple of initial conditions which are linearly independent from $(y_i(t_0,\varepsilon_0),..,y_i^{(n-1)}(t,\varepsilon))$, $i=1,..,j$, for example $(\iota_0,..,\iota_{n-1})\in\mathbb{C}^{(n-1)}$. Since $j<n$, such a tuple exists. Let $y(t,\varepsilon)$ be a parameter dependent solution, holomorphic this time only on $U\times(W-\{0\})$, given by setting all $y^{(l)}(t_0,\varepsilon)\in\mathcal{O}(U\times(W-\{0\}))$, $l=0,..,n-1$, equal identically to ι_l, $l=0,..,n-1$, respectively (it is not difficult to show, by utilizing the holomorphic dependence on parameter in $W-\{0\}$ of 4.1), that the solution obtained will be holomorphic on $U\times(W-\{0\})$). If

$y(t,\varepsilon)$ was holomorphic on all of $U \times W$, we would then get a contradiction to maximality of $y_1(t,\varepsilon),..,y_j(t,\varepsilon)$ with respect to linear independence. If for some $q > 0$ $\varepsilon^q y(t,\varepsilon)$ is holomorphic on $U \times W$, it would be another parameter-dependent solution of 4.1) which would again violate the maximality of $y_1(t,\varepsilon),..,y_j(t,\varepsilon)$ with respect to linear independence. Considering t-dependent Laurent expansions of $y(t,\varepsilon)$ around $\varepsilon = 0$, one sees that $y(t,\varepsilon)$ indeed has the properties stated in the lemma. $\square$

Our argument relies on a basic fact about the behaviour of Abelian integrals in a neighbourhood of a singularity (a much more precise information regarding this behaviour is available, but it will not be needed here). A similar, but not identical, proposition can be found in [AGV], Chapter 10. Below, a *meromorphic arc* in $\mathbb{C}^n$ means a (germ of) holomorphic mapping from a punctured neighbourhood of $0 \in \mathbb{C}$ into $\mathbb{C}^n$ with at most a pole at the origin. Thus, a holomorphic arc is also a meromorphic arc, but the converse is not necessarily true.

Proposition 4.2. *Let $(t(\varepsilon),\lambda(\varepsilon),\mu(\varepsilon))$ be a meromorphic arc in the space $\mathbb{C}\times\mathbb{C}^{dim(\lambda)}\times\mathbb{C}^{dim(\mu)}$, such that the arc $(t(\varepsilon),\lambda(\varepsilon))$ lies in $T \subset \mathbb{C}\times\mathbb{C}^{dim(\lambda)}$ for all $\varepsilon \neq 0$, and such that*

$$4.2) \qquad det\left(\int_{\gamma_j(t(\varepsilon),\lambda(\varepsilon))} \omega_i\right)_{ij} \not\equiv 0$$

as a function of t,ε. Then, for any continuously varying cycle $\gamma(t(\varepsilon),\lambda(\varepsilon))$ in the locally trivial bundle induced on a punctured neighbourhood of $\varepsilon = 0$ by the locally trivial bundle over T, the Abelian integral $\int_{\gamma(t(\varepsilon),\lambda(\varepsilon))}\sum_{l=1}^{(d-1)^2}\mu_l(\varepsilon)\omega_l$ can be written in a neighbourhood of the origin as a sum of finitely many terms

$$4.3) \qquad \sum_{r,s}\varepsilon^{\sigma_r} log^s(\varepsilon) h_{rs}(\varepsilon),$$

where $h_{rs}(\varepsilon)$ are holomorphic, and σ_r are certain complex numbers.

Proof. Since $(t(\varepsilon),\lambda(\varepsilon)) \in T$ $\forall \varepsilon \neq 0$,

$$\left(\int_{\gamma_j(t(\varepsilon),\lambda(\varepsilon))} \omega_i\right)_{ij}^{-1}$$

is in fact a matrix which is analytic multivalued in some punctured neighbourhood of $\varepsilon = 0$. Since the monodromy of the derivative (w.r.t. ε) of the matrix

$$\left(\int_{\gamma_j(t(\varepsilon),\lambda(\varepsilon))} \omega_i\right)_{ij}$$

is the same as the monodromy of the matrix itself, and since both are analytic (multivalued) in a punctured neighbourhood of $\varepsilon = 0$, $K(\varepsilon)$ is in fact holomorphic in this punctured neighbourhood. The rate of growth of the Abelian integrals as $\varepsilon \to 0$ in any given sector, is at most polynomial in $1/|\varepsilon|$ (one may estimate the growth by estimating the length of transported cycles and the diameter of an origin centered ball containing them, as it was done in the proof of Proposition 3.2; the details are now different, however, and considerably more tedious - see Extended remark 4.3 below). Hence $K(\varepsilon)$ has at most a pole at $\varepsilon = 0$. The solution space of the system

$$4.4) \quad I'_\varepsilon \;=\; K(\varepsilon)\; I$$

is spanned by

$$\left(\int_{\gamma_j(t_1,\lambda(\varepsilon))} \omega_i\right)_i .$$

Since the components of these solutions have at most a polynomial rate of growth as $\varepsilon \to 0$ in any given sector, $\varepsilon = 0$ is a regular singular point of 4.4). It is known that solutions of linear differential systems near a regular singular point are of the form 4.3) (indeed, the components of such systems satisfy linear differential equations for which the $\varepsilon = 0$ is a regular singularity; according to Theorems 3.1 and 5.2 in Chapter 4 of [CL], any solution of such equation has the form 4.3) in the neighbourhood of $\varepsilon = 0$). Therefore also the integral

$$\int_{\gamma(t(\varepsilon),\lambda(\varepsilon))} \sum_{l=1}^{(d-1)^2} \mu_l(\varepsilon)\omega_l \;=\; \sum_{l=1}^{(d-1)^2} \mu_l(\varepsilon) \int_{\gamma(t(\varepsilon),\lambda(\varepsilon))} \omega_l$$

can be written in the form 4.3) in a neighbourhood of $\varepsilon = 0$. $\square$

Extended remark 4.3. In the proof we used the fact that the growth of $\int_{\gamma_j(t_(\varepsilon),\lambda(\varepsilon))} \omega_i$, as $\varepsilon \to 0$, is at most polynomial in $1/|\varepsilon|$. Though this fact appears to be well-known, it is hard to find a reference for the exact statement; we therefore sketch a proof. Take $\tau \in [0,1] \mapsto \varepsilon(\tau)$ be a path in the punctured neighbourhood of $\varepsilon = 0$. A possible way to obtain a continuously varying cycle in the locally trivial bundle which is induced over the punctured neighbourhood of zero E, starting from a cycle γ_0 in the homology of the fiber $\{(x,y) : H(\lambda(\varepsilon_0))(x,y) = t(\varepsilon_0)\}$, $\varepsilon_0 \in E$, is as follows.

Take a path, say piecewise real analytic, from $\varepsilon_0 \in E$ to $\varepsilon_1 \in E$, $\tau \in [0,1] \mapsto \varepsilon(\tau) \in E$, denoted by $\varepsilon(\tau)$. Take the cycle γ_0 in the homology of $\{(x,y) : H(\lambda(\varepsilon_0))(x,y) = t(\varepsilon_0)\}$, realized as a real analytic mapping of a circle into $\{(x,y) : H(\lambda(\varepsilon_0))(x,y) = t_1\}$. Project γ_0 on the x-axis, obtaining δ_0, and then construct a continuous deformation $\delta(\tau)$, $\delta(0) = \delta_0$, so that $\forall\tau \in [0,1]$, $\delta(\tau)$ never intersects the (τ-dependent) critical values of the projection of $\{(x,y) : H(\lambda(\varepsilon(\tau)))(x,y) = t(\varepsilon(\tau))\}$. One can then naturally lift $\delta(\tau)$ to $\gamma(\tau)$, a continuously varying cycle along the path $\varepsilon(\tau)$ in the locally trivial bundle over E. One shows that a construction exists (its precise description being the most tedious part of the proof), such that the length of $\delta(\tau)$, and consequently of $\gamma(\tau)$, is controlled, in a certain precise sense, by the following quantities. One is the length of the locus L traversed by the critical values of the projection to the x-plane of $\{(x,y) : H(\lambda(\varepsilon(\tau)))(x,y) = t(\varepsilon(\tau))\}$, as τ varies from 0 to 1. The second is the complexity of the map sending $\tau \in (0,1)$ to the collection of the critical values of the projection. In the nondegenerate case, this complexity is simply the number of self intersections of the locus L.

Observe now that to construct a representative for the transport of γ_0 from the fiber over ε_0 to the fiber over ε_1, $|\varepsilon_1| \leq |\varepsilon_0|$, along any path which does not wind around $\varepsilon = 0$, it is sufficient to consider a path composed of at most four line segments, each of which has a distance of at least $|\varepsilon_1|/\sqrt{2}$ from the origin (since each nonwinding path from ε_0 to ε_1 is homotopic to such a path). Parameterizing the line segments in an origin centered disc $D \subset E$, by pairs of their endpoints in $D \times D$, one then shows, using local finiteness properties of subanalytic sets and maps (cf. for example [BM]), that the cardinality of the decomposition of L into smooth connected pieces, such that the tangent to each piece lies in some fixed quadrant in $\mathbb{R}^2 \cong \mathbb{C}$, is uniformly bounded over all linear segments lying in D (we include also segments which pass through $\varepsilon = 0$, and for which L may have infinite length). One also shows that the maximal modulus of the critical values of the projection for paths $\varepsilon(\tau)$ which are line segments going from ε_0 to ε_1, $|\varepsilon_1| \leq |\varepsilon_0|$, lying at a distance of at least $|\varepsilon_1|/\sqrt{2}$ from the origin, is bounded by $C/|\varepsilon_1|^\alpha$, for some $C > 0, \alpha > 0$. The conclusion is that for such paths, the length of L is bounded by $C/|\varepsilon_1|^\alpha$ as well ($C > 0, \alpha > 0$ being now some other constants). The complexity of the locus L for each linear segment lying in D may be also shown to be uniformly bounded over all such segments, using similar arguments. One concludes that the length of $\gamma(\tau)$, for each path going from ε_0 to ε_1 and which does not wind around the origin, is bounded by $C/|\varepsilon_1|^\alpha$ for some constants $C > 0, \alpha > 0$.

One also shows that the cycle $\gamma(\tau)$ is contained in an origin centered ball in $\mathbb{C}^2$ of radius at most $C/|\varepsilon_1|^\alpha$ for some, possibly other, constants $C > 0, \alpha > 0$. Consequently the growth of the integral $\int_{\gamma(t(\varepsilon),\lambda(\varepsilon))} \omega$, $\omega \in \Omega^1(\mathbb{C}^2)$, as $\varepsilon \to 0$ (in a given sector), is at most polynomial in $1/|\varepsilon|$. (End of Extended remark 4.3.)

Theorem 4.4. *The Picard-Fuchs equation 3.22) depends regularly on the parameters* λ, μ *in* $\mathbb{C}^{dim(\lambda)} \times \mathbb{C}^{dim(\mu)} \cong \mathbb{C}^{(d+1)(d+2)/2} \times \mathbb{C}^{(d-1)^2}$.

Proof. Suppose not. Then there exists a holomorphic arc $(\lambda(\varepsilon), \mu(\varepsilon))$ in the parameter space $\mathbb{C}^{dim(\lambda)} \times \mathbb{C}^{dim(\mu)}$, such that the equation 3.22), restricted to $(\lambda(\varepsilon), \mu(\varepsilon))$, (is defined and) becomes singularly perturbed. Let the set of points of $T \subset \mathbb{C} \times \mathcal{H}^d$ where 4.2) vanishes, be denoted again by Z. Let the set of points of $T \times \mathbb{C}^{dim(\mu)}$, where the Wronskian of

$$4.5) \quad 1, \int_{\gamma_1(t,\lambda)} \sum_{l=1}^{(d-1)^2} \mu_l \omega_l, \; .. \; , \int_{\gamma_{(d-1)^2}(t,\lambda)} \sum_{l=1}^{(d-1)^2} \mu_l \omega_l$$

vanishes, be denoted by Z'. One concludes from the proofs of Proposition 4.2 and Proposition 3.7 that both Z and Z' are intersections with T of proper algebraic subsets of $\mathbb{C} \times \mathbb{C}^{dim(\lambda)}$ and $\mathbb{C} \times \mathbb{C}^{dim(\lambda)} \times \mathbb{C}^{dim(\mu)}$. Therefore, the set $Z'' \subset \mathbb{C} \times \mathbb{C}^{dim(\lambda)} \times \mathbb{C}^{dim(\mu)}$ defined as the union of the complement of $T \times C^{dim(\mu)}$ in $\mathbb{C} \times \mathbb{C}^{dim(\lambda)} \times \mathbb{C}^{dim(\mu)}$, with $(Z \times \mathbb{C}^{dim(\mu)}) \cup Z'$, is a closed constructible set of codimension at least 1 (recall T is constructible and open). Let $S \subset \mathbb{C}^{dim(\lambda)} \times \mathbb{C}^{dim(\mu)}$ denote the constructible set, such that $(\lambda, \mu) \in S$ if $(t, \lambda, \mu) \in Z''$ for all $t \in \mathbb{C}$. Since Z'' is closed and has codimension at least 1, S is closed and has codimension at least 1 as well.

If the arc $(\lambda(\varepsilon), \mu(\varepsilon))$ intersects S at infinitely many points ε_n, $\varepsilon_n \to 0$, this means (since S is constructible and closed) that the arc lies on S (at least when the parameter of the arc is restricted to a sufficiently small disc). Since S has codimension at least 1, using Lemma 1.6 it is possible to find a nearby holomorphic arc not lying on S, such that 3.22), restricted to this new arc, (is defined and) stays singularly perturbed. Restricting the arc parameter to lie in a sufficiently small disc, this arc may intersects S only at $\varepsilon = 0$. So we may assume that if the arc $(\lambda(\varepsilon),\mu(\varepsilon))$ intersects S, it intersects it only at $\varepsilon=0$. If the arc did not intersect S, 3.22) restricted to $(\lambda(\varepsilon),\mu(\varepsilon))$ would have a linearly independent set of $(d-1)^2+1$ parameter dependent solutions, holomorphic on a polydisc (recall that S is closed). From the proof of Lemma 4.1 it would then follow that 3.22), restricted to $(\lambda(\varepsilon),\mu(\varepsilon))$, is not singularly perturbed. Therefore $(\lambda(0),\mu(0))\in S$.

Note that being 3.22) defined on the original arc (which may lie on S), does not imply (at least *a priori*) that the Wronskian of 4.5) is not identically zero when restricted to that arc. This is the reason that the set Z' appears in the definition of the exceptional set S. The reason that the set Z appears in this definition, is to let us use Proposition 4.2 in the argument below.

Take now a polydisc $U\times W\subset\mathbb{C}\times\mathbb{C}$, such that the coefficients of 3.22) are holomorphic on $U\times(W-\{0\})$, and on which the equation 3.22), restricted to $(\lambda(\varepsilon),\mu(\varepsilon))$, takes the form 4.1). According to Lemma 4.1, there exists then a parameter dependent solution $y(t,\varepsilon)\in\mathcal{O}(U\times(W-\{0\}))$ of this equation, such that for some $t_0,t_1\in U$, $y^{(j)}(t_0,\varepsilon)\in\mathcal{O}(W-\{0\})$, $j=0,..,(d-1)^2$, are constants, and $y(t_1,\varepsilon)\in\mathcal{O}(W-\{0\})$ has an essential singularity at $\varepsilon=0$. We may assume that W, t_0 and t_1 are such that the arcs $(t_0,\lambda(\varepsilon),\mu(\varepsilon))$ and $(t_1,\lambda(\varepsilon),\mu(\varepsilon))$, where $\varepsilon\in W$, intersect Z'' only at $\varepsilon=0$ (indeed, the arc $(\lambda(\varepsilon),\mu(\varepsilon))$ intersects S only at $\varepsilon=0$, and Lemma 4.1 shows that we may choose $t_0,t_1\in U$ almost arbitrarily). We may then write, for all $\varepsilon\in W-\{0\}$ (since for $\varepsilon\neq0$ the integrals are defined and the Wronskian is not identically zero),

$$4.6)\qquad y(t_0,\varepsilon)=\;c_0(\varepsilon)\cdot1+\sum_{i=1}^{(d-1)^2}c_i(\varepsilon)\int_{\gamma_i(t_0,\lambda(\varepsilon))}\sum_{l=1}^{(d-1)^2}\mu_l(\varepsilon)\omega_l.$$

Fixing a basis of the homology of the fiber over $(t_0,\lambda(\varepsilon_0))$ for some $\varepsilon_0\in W$, $\varepsilon_0\neq0$, $c_i(\varepsilon)$ are then certain branches of analytic multivalued functions on $W-\{0\}$. To compute $c_i(\varepsilon)$, $i=0,..,(d-1)^2$, fix $t=t_0$. $(c_1(\varepsilon),..,c_{(d-1)^2}(\varepsilon))^T$ is then given by the product of the inverse of the Wronskian *matrix* of 4.5), and of $(y(t_0,\varepsilon),..,y^{((d-1)^2)}(t_0,\varepsilon))^T$. Note that the latter does not depend on ε. Since the arcs $(t_0,\lambda(\varepsilon),\mu(\varepsilon))$ and $(t_1,\lambda(\varepsilon),\mu(\varepsilon))$ intersect Z'' only at $\varepsilon=0$, we conclude from Cramer's rule and Proposition 4.2, that $c_i(\varepsilon)$ can be written as ratio of finite sums of the form $\sum_{i,j}\ \varepsilon^{\sigma_i}log^j(\varepsilon)h_{ij}(\varepsilon)$. Now, since the arc $(\lambda(\varepsilon),\mu(\varepsilon))$ intersects S only at $\varepsilon=0$, the set of points $(t,\varepsilon)\in U\times W$, mapped by $(t,\lambda(\varepsilon),\mu(\varepsilon))$ to points of Z'', is of (complex) codimension 1 (consequently not separating $U\times W$). By analytic continuation, 4.6) holds therefore also if we replace t_0 by t_1, implying

$$y(t_1,\varepsilon)=\;c_0(\varepsilon)\cdot1+\sum_{i=1}^{(d-1)^2}c_i(\varepsilon)\int_{\gamma_i(t_1,\lambda(\varepsilon))}\sum_{l=1}^{(d-1)^2}\mu_l(\varepsilon)\omega_l.$$

Using Proposition 4.2 again, we then conclude that $y(t_1,\varepsilon)$ can be written as a ratio of sums $\sum_{i,j}\ \varepsilon^{\sigma_i}log^j(\varepsilon)h_{ij}(\varepsilon)$ as well. But this is impossible, since $y(t_1,\varepsilon)$ has an essential singularity at $\varepsilon=0$. Indeed, it is not difficult to show that a ratio of sums $\sum_{i,j}\ \varepsilon^{\sigma_i}log^j(\varepsilon)h_{ij}(\varepsilon)$ can never be equal to a holomorphic function in a punctured neighbourhhood of the origin with an essential singularity at the origin.

We conclude that 3.22) must depend regularly on the parameters λ,μ in $\mathbb{C}^{dim(\lambda)}\times\mathbb{C}^{dim(\mu)}$. $\square$

Recall now that for $H \in \mathcal{H}^d$, Σ_H denotes the set of its atypical points (which are just the critical values for the polynomials H we consider below).

Corollary 4.5. *Let $\lambda \in \mathcal{H}^d$, $d \geq 2$, be such that $H(\lambda)$ is regular at infinity, and suppose that $||\lambda|| \leq 1$. Then for any polynomials $P, Q \in \mathcal{H}^d$, the integral $\int_{\gamma(t)} P(x,y)dx + Q(x,y)dy$ can have not more than*

$$\left(\frac{2}{\rho}\right)^{2^{d^{O(1)}}}$$

zeros in any simple domain in $\mathbb{C} - \Sigma_H$, whose distance from Σ_H is not smaller than ρ, $0 < \rho < 1$, and which is contained in the unit disc. Here $\gamma(t)$ is any continuously varying cycle in the locally trivial bundle determined by $H(\lambda)$.

Proof. The coefficients of 3.22) are ratios of integral polynomials of degree and height not greater than $d^{O(1)}, 2^{d^{O(1)}}$, respectively. The order of 3.22) is $(d-1)^2 + 1$. For λ, μ in a certain constructible set $V \subset \mathcal{H}^d$ of codimension zero, 3.22) is defined and its solutions contain all integrals of the form $\int_{\gamma(t)} \sum_{l=1}^{(d-1)^2} \mu_l \omega_l$. Let $(\lambda, \mu) \in V$ and suppose $||\lambda|| \leq 1$, $||\mu|| \leq 1$. The bound for their number of zeros in any simple domain of $\mathbb{C} - \Sigma_H$, whose distance from Σ_H is not smaller than ρ, and which is contained in the unit disc, is then given by Theorem 2.1 as (note that the dimension of the parameter space is $dim(\lambda) + dim(\mu) \leq (d+1)(d+2)/2 + (d-1)^2 \leq O(d^2)$)

$$((d-1)^2 + 1)\left(\frac{2^{d^{O(1)}}}{\rho}\right)^{d^{O(O(d^2)^3)}} \leq \left(\frac{2}{\rho}\right)^{2^{d^{O(1)}}}.$$

One may omit now the restriction $||\mu|| \leq 1$, since the zeros of $\int_{\gamma(t)} \sum_{l=1}^{(d-1)^2} \mu_l \omega_l$ are the same as the zeros of $\int_{\gamma(t)} \sum_{l=1}^{(d-1)^2} (\mu_l/||\mu||)\omega_l$. Since any integral $\int_{\gamma(t)} \sum_{l=1}^{(d-1)^2} \mu_l \omega_l$ has a natural analytic continuation to $T \times \mathbb{C}^{dim(\mu)}$, we immediately conclude (using Rouche theorem, for example), that for any μ and $||\lambda|| \leq 1$, such that $H(\lambda)$ is regular at infinity and has degree d, the same bound for the number of zeros holds.

Observe that for $\lambda \in \mathcal{G}^d$ any integral $\int_{\gamma(t)} P(x,y)dx + Q(x,y)dy$ can be written as $\int_{\gamma(t)} \sum_i \mu_i \omega_i$ for some $\mu \in \mathbb{C}^{(d-1)^2}$ (Theorem 3.3), implying the bound for such integrals as well. Now, the set $\mathcal{G}^d$ is of codimension zero, and any integral $\int_{\gamma(t)} \sum_{l=1}^{(d-1)^2} Pdx + Qdy$ continues analytically to $T \times \mathbb{C}^{dim(\mu)}$. This implies that the bound holds for the number of zeros of any integral $\int_{\gamma(t)} P(x,y)dx + Q(x,y)dy$, where $\gamma(t)$ is a continuously varying cycle in the locally trivial bundle determined by $H(\lambda)$, $||\lambda|| \leq 1$, for which $H(\lambda)$ is regular at infinity and has degree d. $\square$

Note that for general linear differential equations which depend regularly on parameters, making a parameter dependent algebraic change of variable in general will not produce an equation which depends regularly on parameters. For example,

$$\frac{dy}{dt} - \frac{y}{t(t-\varepsilon)} = 0$$

becomes, after putting $t = \varepsilon\tau$:

$$\frac{dy}{d\tau} - \frac{y}{\varepsilon\tau(\tau-1)} = 0.$$

On the contrary, Picard-Fuchs equations stay regularly dependent on parameters also after a parameter dependent algebraic change of variable. This is of course the consequence of their algebro-geometric origin, or, more to the point, the consequence of the fact that the rate of growth of Abelian integrals along holomorphic curves in the new coordinates, stays at most polynomial. We now prove a precise claim of this sort.

By a rational map from $\mathbb{C}^m$ to $\mathbb{C}^n$ we mean a map whose domain is a dense subset of $\mathbb{C}^m$ and whose components are rational functions.

Let $\beta(\kappa) : \mathbb{C}^{dim(\kappa)} \to \mathbb{C}^{dim(\lambda)} \times \mathbb{C}^{dim(\mu)}$ be a dominant rational map (i.e. a rational map which maps its domain to a dense subset of the target space). We may write the linear differential equation 3.22) in the new coordinate and parameters, obtaining (the derivative is with respect to τ)

$$4.7) \quad I_0^{((d-1)^2+1)} + \frac{a_{(d-1)^2}(\tau,\kappa)}{b(\tau,\kappa)} I_0^{((d-1)^2)} + .. + \frac{a_0(\tau,\kappa)}{b(\tau,\kappa)} I_0 = 0,$$

where $b(t,\kappa), a_i(\tau,\kappa) \in \mathbb{C}[\tau,\kappa]$, $i = 0,..,(d-1)^2$.

Proposition 4.6. *The linear differential equation 4.7) depends regularly on κ in $\mathbb{C}^{dim(\kappa)}$.*

Proof. We proceed as in the proof of Theorem 4.4. Suppose 4.7) does not depend regularly on parameters. Then there exists a holomorphic arc $\kappa(\varepsilon)$ in the parameter space $\mathbb{C}^{dim(\kappa)}$, such that the equation 4.7), restricted to $\kappa(\varepsilon)$, is (defined and) singularly perturbed.

Let the set $K \subset \mathbb{C} \times \mathbb{C}^{dim(\kappa)}$ be the closed constructible subset obtained as the union of the points where the rational map $(\tau,\kappa) \mapsto (\alpha(\tau,\kappa),\beta(\kappa))$ is undefined, and of the preimage, under this map, of the set $Z'' \subset \mathbb{C}\times\mathbb{C}^{dim(\lambda)} \times \mathbb{C}^{dim(\mu)}$ which was constructed in the proof of Theorem 4.4. Since the derivative of $\alpha(\tau,\kappa)$ is not identically zero and $\beta(\kappa)$ is dominant, K is of codimension at least 1. Let $S \subset \mathbb{C}^{dim(\kappa)}$ denote the constructible set, such that $\kappa \in S$ if $(\tau,\kappa) \in K$ for all $\tau \in \mathbb{C}$. It is closed and of codimension at least 1 as well. As in the proof of Theorem 4.4, it may be assumed that the arc $\kappa(\varepsilon)$ intersects S only at $\varepsilon = 0$.

Take now a polydisc $U \times W \subset \mathbb{C}\times\mathbb{C}$, such that the coefficients of 4.7) are holomorphic on $U \times (W - \{0\})$, and on which the equation 4.7), restricted to $\kappa(\varepsilon)$, takes the form 4.1). Again, Lemma 4.1 implies that there exists a parameter dependent solution $y(\tau,\varepsilon) \in \mathcal{O}(U\times(W-\{0\}))$ of this equation, such that for some $\tau_0, \tau_1 \in U$, $y^{(j)}(\tau_0,\varepsilon) \in \mathcal{O}(W-\{0\})$, $j = 0,..,(d-1)^2$, are constants, and $y(\tau_1,\varepsilon) \in \mathcal{O}(W-\{0\})$ has an essential singularity at $\varepsilon = 0$.

We proceed now as in the proof of Theorem 4.4. We may again assume that W, τ_0 and τ_1 are such that the arcs $(\tau_0,\kappa(\varepsilon))$ and $(\tau_1,\kappa(\varepsilon))$, where $\varepsilon \in W$, intersect K only at $\varepsilon = 0$. We then write, for all $\varepsilon \in W - \{0\}$ (since for $\varepsilon \neq 0$ the integrals are defined and the Wronskian is not identically zero),

$$4.8)\qquad y(\tau_0,\varepsilon) \;=\; c_0(\varepsilon)\cdot 1+\sum_{j=1}^{(d-1)^2} c_j(\varepsilon)\int_{\gamma_j(\alpha(\tau_0,\kappa(\varepsilon)),\beta_\lambda\circ\kappa(\varepsilon))}\sum_{l=1}^{(d-1)^2}(\beta_\mu\circ\kappa(\varepsilon))_l\omega_l.$$

The coefficients $c_i(\varepsilon)$, $i=0,..,(d-1)^2$, are given, as before, by the product of the inverse of the Wronskian matrix of

$$1,\ \int_{\gamma_1(\alpha(\tau_0,\kappa(\varepsilon)),\beta_\lambda\circ\kappa(\varepsilon))}\sum_{l=1}^{(d-1)^2}(\beta_\mu\circ\kappa(\varepsilon))_l\omega_l,\ ..,\ \int_{\gamma_{(d-1)^2}(\alpha(\tau_0,\kappa(\varepsilon)),\beta_\lambda\circ\kappa(\varepsilon))}\sum_{l=1}^{(d-1)^2}(\beta_\mu\circ\kappa(\varepsilon))_l\omega_l,$$

and of $(y(\tau_0,\varepsilon),..,y^{((d-1)^2)}(\tau_0,\varepsilon))^T$, the latter being a constant vector in $\mathbb{C}^{(d-1)^2+1}$. From Proposition 4.2 we conclude that also in this case $c_i(\varepsilon)$ can be written as ratio of finite sums of the form $\sum_{i,j}\ \varepsilon^{\sigma_i}log^j(\varepsilon)h_{ij}(\varepsilon)$.

Since the arcs $(\tau_0,\kappa(\varepsilon))$ and $(\tau_1,\kappa(\varepsilon))$, where $\varepsilon\in W$, intersect K only at $\varepsilon=0$, 4.8) holds also if we replace τ_0 by τ_1, implying

$$y(\tau_1,\varepsilon) \;=\; c_0(\varepsilon)\cdot 1+\sum_{j=1}^{(d-1)^2} c_j(\varepsilon)\int_{\gamma_j(\alpha(\tau_1,\kappa(\varepsilon)),\beta_\lambda\circ\kappa(\varepsilon))}\sum_{l=1}^{(d-1)^2}(\beta_\mu\circ\kappa(\varepsilon))_l\omega_l.$$

Again, one concludes that $y(\tau_1,\varepsilon)\in\mathcal{O}(W-\{0\})$ can be written as a ratio of sums $\sum_{i,j}\ \varepsilon^{\sigma_i}log^j(\varepsilon)h_{ij}(\varepsilon)$. This is impossible, since $y(\tau_1,\varepsilon)$ has an essential singularity at $\varepsilon=0$.

We conclude that 4.7) depends regularly on κ in $\mathbb{C}^{dim(\kappa)}$. □

We use this result only to derive Corollary 4.7 below, though more general statements can be made. Together with Corollary 4.5, it proves Theorem 0.2.

Corollary 4.7. *The condition $||\lambda||\le 1$ in Corollary 4.5 may be removed.*

Proof. Indeed, let $\alpha(\tau,\kappa)=\tau$ and let $\beta(\kappa_i)=\lambda_i$ for all i, $2\le i\le dim(\lambda)+dim(\mu)$, $\beta(\lambda_1)=1/\kappa_1$. Clearly β is then a dominating rational map, and $\alpha(\tau,\kappa)$ has a nonzero derivative w.r.t. τ. By Proposition 4.6, the equation 4.7) depends then regularly on κ. This implies (since the degree and the height of 4.7) are the same then as of 3.22)) that Corollary 4.5 holds also for all λ, for which $||(1/|\lambda_1|,\lambda_2,..,\lambda_{dim(\lambda)})||\le 1$. Considering not just the transformation above, but all transformations of the form $\lambda=(\kappa_1^{s_1},..,\kappa_{dim(\lambda)}^{s_{dim(\lambda)}})$, for different choices of $s_i\in\{-1,+1\}$, $i=1,..,dim(\lambda)$, we conclude that the condition $||\lambda||\le 1$ may be indeed removed. □

REFERENCES

[Aea]. Arnol'd V.I. et al, Some unsolved problems in the theory of differential equations and mathematical physics. *Uspekhi Mat. Nauk* 44 (1989), no. 4(268),191–202; translation in *Russian Math. Surveys* 44 (1989), no. 4, 157–171.

[AGV]. Arnol'd, V. I.; Gusein-Zade, S. M.; Varchenko, A. N. *Singularities of differentiable maps. Vol. II. Monodromy and asymptotics of integrals.* Monographs in Mathematics, 83. Birkhauser Boston, Inc., Boston, MA, 1988.
[BM]. Bierstone, E.; Milman, P. Semianalytic and subanalytic sets. *Inst. Hautes Études Sci. Publ. Math.* No. 67 (1988), 5–42.
[Br]. Broughton, S. A. Milnor numbers and the topology of polynomial hypersurfaces. *Invent. Math.* 92 (1988), no. 2, 217–241.
[CL]. Coddington, E. A.; Levinson, N. *Theory of ordinary differential equations.* McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.
[CLO]. Cox, D.; Little, J.; O'Shea, D. *Ideals, varieties, and algorithms. An introduction to computational algebraic geometry and commutative algebra.* Second edition. Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1997.
[Ga]. Gavrilov, L. Petrov modules and zeros of Abelian integrals. *Bull. Sci. Math.* 122 (1998), no. 8, 571–584.
[GI]. Glutsuk A.; Il'yashenko Y. An estimate of the number of zeros of Abelian integrals for special Hamiltonians of arbitrary degree. *Preprint* arXiv:math.DS/0112156v1.
[Gr]. Grigoriev, A. *Ph.D. thesis*, the Weizmann Institute of Science, December 2001.
[I1]. Il'yashenko, Yu. Appearance of limit cycles in perturbation of the equation $\frac{dy}{dz} = -\frac{R_z}{R_w}$, where $R(z, w)$ is a polynomial. *USSR Math. Sbornik*, 78 (1969), 360-373.
[I2]. Il'yashenko, Yu. Centennial history of Hilbert's 16th problem. *Bull. Amer. Math. Soc. (N.S.)* 39 (2002), no. 3, 301–354.
[IY]. Il'yashenko, Y.; Yakovenko, S. Double exponential estimate for the number of zeros of complete Abelian integrals and rational envelopes of linear ordinary differential equations with an irreducible monodromy group. *Invent. Math.* 121 (1995), no. 3, 613–650.
[Kh]. A. Khovanskii. Real analytic manifolds with the property of finiteness, and complex abelian integrals, *Funktsional. Anal. i Prilozhen.* 18 (1984), no. 2, 40-50.
[N]. Novikov, D. Modules of the abelian integrals and the Picard-Fuchs systems. *Nonlinearity* 15 (2002), no. 5, 1435–1444.
[NY1]. Novikov, D.; Yakovenko, S. Tangential Hilbert problem for perturbations of hyperelliptic Hamiltonian systems. *Electron. Res. Announc. Amer. Math. Soc.* 5 (1999), 55–65 (electronic).
[NY2]. Novikov, D.; Yakovenko, S. Trajectories of polynomial vector fields and ascending chains of polynomial ideals. *Ann. Inst. Fourier (Grenoble)* 49 (1999), no. 2, 563–609.
[NY3]. Novikov, D.; Yakovenko, S. Redundant Picard-Fuchs system for abelian integrals. *J. Differential Equations* 177 (2001), no. 2, 267–306.
[R]. Roussarie, R. *Bifurcation of planar vector fields and Hilbert's sixteenth problem.* Progress in Mathematics, 164. Birkhuser Verlag, Basel, 1998.
[Re]. Renegar, J. On the computational complexity and geometry of the first-order theory of the reals. III. Quantifier elimination. *J. Symbolic Comput.* 13 (1992), no. 3, 329–352.
[Var]. A.N. Varchenko. Estimation of the number of zeros of an abelian integral depending on a parameter, and limit cycles, *Funktsional. Anal. i Prilozhen.* 18 (1984), no.2, 14-25.
[Y]. Yakovenko S. On functions and curves defined by ordinary differential equations , to appear in *Proceedings of the Arnoldfest* (Ed. by Bierstone, Khesin, Khovanskii, Mars-

den), Fields Institute Communications, 1998.