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Sums of squares, moments and optimization

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A system of inequalities

$$-X^{12}+938X^9-56629X^6-54758X^{10}+109984X^7-55694X^4-110449X^8+ \\ 219494X^5-109513X^2+468X^{11}+110448X^3+468X-54756 \geq 0$$

might get easier to solve then you add a few other inequalities

$$X - 234 \geq 0$$
$$234 - X \geq 0$$

to it.

Notation for the whole week

- $X := (X_1, \ldots, X_n)$ variables
- $\mathbb{R}[X] := \mathbb{R}[X_1, \ldots, X_n]$ polynomial ring
- $f \in \mathbb{R}[X]$ an arbitrary polynomial
- $g \in \mathbb{R}[X]$ the polynomial defining. . .
- . . . the set $S := \{x \in \mathbb{R}^n \mid g(x) \geq 0\}$. . .
- . . . and the preorder $T := \sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 g$

Sums of squares

$$\mathbb{R}[X]^2 := \{p^2 \mid p \in \mathbb{R}[X]\}$$
$$\sum \mathbb{R}[X]^2 := \{p_1^2 + \cdots + p_s^2 \mid s \in \mathbb{N}, p_1, \ldots, p_s \in \mathbb{R}[X]\}$$
$$\mathbb{R}[X]^2 g := \{p^2 g \mid p \in \mathbb{R}[X]\}$$
$$\sum \mathbb{R}[X]^2 g := \{p_1^2 g + \cdots + p_s^2 g \mid p \in \mathbb{R}[X]\}$$
$$\sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 g := \left\{\sigma + \tau g \mid \sigma, \tau \in \sum \mathbb{R}[X]^2\right\}$$

and so on. . .

Sums of squares

$$\mathbb{R}[X]^2 := \{p^2 \mid p \in \mathbb{R}[X]\}$$
$$\sum \mathbb{R}[X]^2 := \{p_1^2 + \cdots + p_s^2 \mid s \in \mathbb{N}, p_1, \ldots, p_s \in \mathbb{R}[X]\}$$
$$\mathbb{R}[X]^2 g := \{p^2 g \mid p \in \mathbb{R}[X]\}$$
$$\sum \mathbb{R}[X]^2 g := \{p_1^2 g + \cdots + p_s^2 g \mid p \in \mathbb{R}[X]\}$$
$$T = \sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 g := \left\{\sigma + \tau g \mid \sigma, \tau \in \sum \mathbb{R}[X]^2\right\}$$

is a set of polynomials which are

for obvious reasons ≥ 0 on $S = \{x \in \mathbb{R}^n \mid g(x) \geq 0\}$.

Sums of squares

$$\mathbb{R}[X]^2 := \{p^2 \mid p \in \mathbb{R}[X]\}$$
$$\sum \mathbb{R}[X]^2 := \{p_1^2 + \cdots + p_s^2 \mid s \in \mathbb{N}, p_1, \ldots, p_s \in \mathbb{R}[X]\}$$
$$\mathbb{R}[X]^2 g := \{p^2 g \mid p \in \mathbb{R}[X]\}$$
$$\sum \mathbb{R}[X]^2 g := \{p_1^2 g + \cdots + p_s^2 g \mid p \in \mathbb{R}[X]\}$$
$$T = \sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 g := \left\{\sigma + \tau g \mid \sigma, \tau \in \sum \mathbb{R}[X]^2\right\}$$

is the preorder generated by g.

We call $P \subseteq \mathbb{R}[X]$ a preorder if $\mathbb{R}[X]^2 \subseteq P$, $P + P \subseteq P$ and $PP \subseteq P$.

Everything works for basic closed semialgebraic sets

$$S = \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}.$$

with modified T (best choice is not clear). We restrict us to

$$S = \{x \in \mathbb{R}^n \mid g(x) \geq 0\}.$$

- Principle ideas remain.
- Notation can be simplified.
- Technical problems disappear, and with them the names

Thomas Jacobi, Alexander Prestel and Mihai Putinar.

Positivstellensatz (Krivine).

$$f > 0 \text{ on } S \implies \exists q \in T : qf \in 1 + T$$

- The converse is trivial.
- Rediscovered by Stengle. Usually attributed to him.

Gilbert Stengle: A Nullstellensatz and a Positivstellensatz in semialgebraic geometry
Math. Ann. **207**, 87–97 (1974)

Jean-Louis Krivine: Anneaux préordonnés
J. Anal. Math. **12**, 307–326 (1964)

Positivstellensatz. $f > 0$ on $S \implies \exists q \in T : qf \in 1 + T$

Tentative proof. Suppose $-1 \notin T - Tf$. To show: The system of inequalities $g \geq 0, -f \geq 0$ has a solution in $\mathbb{R}^n$. Proceed in two steps:

- First, find a good candidate $x \in \mathbb{R}^n$ for such a solution.
- Second, show that x is a solution.

(Like in the proof of the intermediate value theorem.)

Positivstellensatz. $f > 0$ on $S \implies \exists q \in T : qf \in 1 + T$

Tentative proof. Suppose $-1 \notin T - Tf$. To show: The system of inequalities $g \geq 0, -f \geq 0$ has a solution in $\mathbb{R}^n$. Proceed in two steps:

- First, find a good candidate $x \in \mathbb{R}^n$ for such a solution.
- Second, show that x is a solution.

Second step gets harder when we enlarge the system of inequalities but first step easier.

While enlarging the system, keep its good property, namely that it is not unsolvable for silly reasons.

Positivstellensatz. $f > 0$ on $S \implies \exists q \in T : qf \in 1 + T$

Tentative proof. Suppose $-1 \notin T - Tf$. To show: The system of inequalities $g \geq 0, -f \geq 0$ has a solution in $\mathbb{R}^n$. Choose a maximal preordering $P \supseteq T - Tf$ such that $-1 \notin P$.

Hope:

- There is an $x \in \mathbb{R}^n$ such that $X_1 - x_1, \dots, X_n - x_n \in P \cap -P =: I$.
- I is an ideal of $\mathbb{R}[X]$, i.e., $0 \in I$, $I + I \subseteq I$ and $\mathbb{R}[X]I \subseteq I$.

Then, for any $p \in P$, $p(x) \in p(X) + I = p + I \subseteq P + P \subseteq P$.
Hence $p(x) \in P \cap \mathbb{R} = [0, \infty)$, i.e., $p(x) \geq 0$.
In particular, $g(x) \geq 0$ and $-f(x) \geq 0$.

Positivstellensatz. $f > 0$ on $S \implies \exists q \in T : qf \in 1 + T$

Tentative proof. Suppose $-1 \notin T - Tf$. To show: The system of inequalities $g \geq 0, -f \geq 0$ has a solution in $\mathbb{R}^n$. Choose a maximal preordering $P \supseteq T - Tf$ such that $-1 \notin P$. Is $I := P \cap -P$ an ideal of $\mathbb{R}[X]$, in other words, does $\mathbb{R}[X]I \subseteq I$ hold? Observation: $PI \subseteq I$ and $(-P)I \subseteq I$. Does $P \cup -P = \mathbb{R}[X]$ hold? Yes!

$$\forall p, q \in \mathbb{R}[X] : (-pq \in P \implies p \in P \text{ or } q \in P)$$

If $-pq \in P$, $p \notin P$ and $q \notin P$, then $-1 \in P + Pp$ and $-1 \in P + Pq$, i.e., there are $a, b, c, d \in P$ such that

$$\left.\begin{aligned} -1 = a + bp &\implies a + 1 = -bp \\ -1 = c + dq &\implies c + 1 = -dq \end{aligned}\right\} \implies -1 = ac + a + c - bdpq \in P.$$

Positivstellensatz. $f > 0$ on $S \implies \exists q \in T : qf \in 1 + T$

Tentative proof. Suppose $-1 \notin T - Tf$. To show: The system of inequalities $g \geq 0, -f \geq 0$ has a solution in $\mathbb{R}^n$. Choose a maximal preordering $P \supseteq T - Tf$ such that $-1 \notin P$. Is $I := P \cap -P$ an ideal of $\mathbb{R}[X]$, in other words, does $\mathbb{R}[X]I \subseteq I$ hold? Observation: $PI \subseteq I$ and $(-P)I \subseteq I$. Does $P \cup -P = \mathbb{R}[X]$ hold? Yes!

$$\forall p, q \in \mathbb{R}[X] : (-pq \in P \implies p \in P \text{ or } q \in P)$$

This observation shows even that I is a prime ideal: Given $p, q \in \mathbb{R}[X]$ such that $pq \in I$, even the four polynomials $-(\pm p)(\pm q)$ lie in $I \subseteq P$. By the observation, $p \notin P$ implies $q \in I$, and $-p \notin P$ implies the same. So $p \notin I \implies q \in I$, i.e., $p \in I$ or $q \in I$.

Positivstellensatz. $f > 0$ on $S \implies \exists q \in T : qf \in 1 + T$

Tentative proof. Suppose $-1 \notin T - Tf$. To show: The system of inequalities $g \geq 0, -f \geq 0$ has a solution in $\mathbb{R}^n$. Choose a maximal preordering $P \supseteq T - Tf$ such that $-1 \notin P$. Then $P \cup -P = \mathbb{R}[X]$ and $I := P \cap -P$ is a prime ideal.

Would suffice to find $x \in \mathbb{R}^n$ such that $X_1 - x_1, \dots, X_n - x_n \in I$.

Good case. $\exists N \in \mathbb{N} : N - \sum_{i=1}^n X_i^2 \in P$

This is true: Set $x_i := \sup\{a \in \mathbb{R} \mid X_i - a \in P\} \dots$

Bad case. $\forall N \in \mathbb{N} : \sum_{i=1}^n X_i^2 - N \in P$

This is wrong. **What to do?**

In the good case, $\mathbb{R} \to \mathbb{R}[X]/I$ was surjective, and therefore an isomorphism. Indeed, every $X_i + I$ was the image of some $x_i \in \mathbb{R}$. In the bad case, $\mathbb{R} \to \mathbb{R}[X]/I$ is not surjective. Problem: Not all the $X_i + I$ can be identified with a real number. Idea: They can however be identified with an element in an ordered field extension K of $\mathbb{R}$:

$$\mathbb{R} \to \mathbb{R}[X]/I \subseteq \mathsf{qf}(\mathbb{R}[X]/I) =: K.$$

The ordering $\leq$ on K is defined via P such that

$$\forall p \in \mathbb{R}[X] : (p + I \geq 0 \iff p \in P).$$

In the bad case, we now get at least a solution $y \in K^n \supseteq \mathbb{R}^n$ of the system $g \geq 0, -f \geq 0$: Setting

$$y := (X_1 + I, \dots, X_n + I) \in K^n,$$

we get for every $p \in P$,

$$p(y) = p\,(X_1 + I, \dots, X_n + I) = p(X) + I = p + I \geq 0,$$

in particular, $g(y) \geq 0$ and $-f(y) \geq 0$.

Have $y \in K^n$. Want $x \in \mathbb{R}^n$!

Suppose $F \subseteq K$ is a field extension. If a finite system of linear equations has a solution $y \in K^n$, then it has also a solution $x \in F^n$ because Gauss elimination works the same.

Suppose $F \subseteq K$ is an extension of ordered fields. If a finite system of polynomial inequalities has a solution $y \in K^n$, then it has also a solution $x \in F^n$? No! But true if F and K are real closed fields because Tarski's decision procedure works the same.

Artin and Schreier: Every ordered field K can be extended to a real closed field.

Alfred Tarski: A decision method for elementary algebra and geometry
The Rand Corporation (1948) work done before World War II

Emil Artin, Otto Schreier: Algebraische Konstruktion reeller Körper
Abh. math. Sem. Hamburg **5**, 85–99 (1926)

Remarks about the proof

- Distinction between good and bad case is not necessary.
- The sketched (standard) proof was found by Krivine, rediscovered by Prestel and is usually attributed to Prestel.
- The proof gives no information how to construct a certificate of positivity.

Alexander Prestel: Lectures on formally real fields
Monografias de Matemática **22**, Instituto de Matemática Pura e Aplicada, Rio de Janeiro

Jean-Louis Krivine: Anneaux préordonnés
J. Anal. Math. **12**, 307–326 (1964)

Notation for the whole week (recapitulation)

- $X := (X_1, \dots, X_n)$ variables
- $\mathbb{R}[X] := \mathbb{R}[X_1, \dots, X_n]$ polynomial ring
- $f \in \mathbb{R}[X]$ an arbitrary polynomial
- $g \in \mathbb{R}[X]$ the polynomial defining. . .
- . . . the set $S := \{x \in \mathbb{R}^n \mid g(x) \geq 0\}$. . .
- . . . and the preorder $T := \sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 g$

Positivstellensatz. $f > 0$ on $S \implies \exists q \in T : qf \in 1 + T$

For the rest of the week: Let S be compact.

Schmüdgen's Positivstellensatz. $f > 0$ on $S \implies f \in T$

Konrad Schmüdgen: The K-moment problem for compact semi–algebraic sets
Math. Ann. **289**, No. 2, 203–206 (1991)

Schmüdgen's Positivstellensatz. $f > 0$ on $S \implies f \in T$

- Converse trivially fails.
- The equivalence $f \geq 0$ on $S \iff f \in T$ fails, too:

$$1 - X^2 = \sigma + \tau(1 - X^2)^3 \implies (1 - X^2)(1 - \tau(1 - X^2)^2) = \sigma$$

- Correct formulation as equivalence:

$$f > 0 \text{ on } S \iff \exists \varepsilon > 0 : f \in \varepsilon + T$$

- Denominatorfree version of following formulation of the Positivstellensatz:

$$f > 0 \text{ on } S \iff \exists \varepsilon > 0 : \exists q \in T : qf \in \varepsilon + T$$

Weak version of Schmüdgen's Theorem. $\exists N \in \mathbb{N} : N + f \in T$

Proof (Wörmann). Step 1. Okay for $g = M - \sum_{i=1}^n X_i^2$, $M \in \mathbb{N}$.

The set $\{p \in \mathbb{R}[X] \mid \exists N \in \mathbb{N} : N \pm p \in T\}$ contains $\mathbb{R}$ and is closed under addition. Because of the two equalities

$$NN' + pp' = \frac{1}{2}((N + p)(N' + p') + (N - p)(N' - p'))$$
$$NN' - pp' = \frac{1}{2}((N - p)(N' + p') + (N + p)(N' - p')),$$

it is closed under multiplication. It contains every X_i because of

$$\frac{M+1}{2} \pm X_i = \frac{1}{2}\left((X_i \pm 1)^2 + \left(M - \sum_{j=1}^n X_j^2\right) + \sum_{j \neq i} X_j^2\right).$$

Weak version of Schmüdgen's Theorem. $\exists N \in \mathbb{N} : N + f \in T$

Proof (Wörmann). Step 1. Okay for $g = M - \sum_{i=1}^n X_i^2$, $M \in \mathbb{N}$.

Step 2. $\exists h \in T : \forall p \in \mathbb{R}[X] : \exists N \in \mathbb{N} : (1 + h)(N + p) \in T$

Positivstellensatz: $q\left(M - \sum_{i=1}^n X_i^2\right) = 1 + h \qquad (M \in \mathbb{N}, q, h \in T)$.

$$(1 + h)\left(M - \sum_{i=1}^n X_i^2\right) = q\left(M - \sum_{i=1}^n X_i^2\right)^2 \in T$$

$$\forall \sigma, \tau \in \sum \mathbb{R}[X]^2 : (1 + h)\left(\sigma + \tau\left(M - \sum_{i=1}^n X_i^2\right)\right) \in T$$

Weak version of Schmüdgen's Theorem. $\exists N \in \mathbb{N} : N + f \in T$

Proof (Wörmann). Step 1. Okay for $g = M - \sum_{i=1}^n X_i^2$, $M \in \mathbb{N}$.

Step 2. $\exists h \in T : \forall p \in \mathbb{R}[X] : \exists N \in \mathbb{N} : (1 + h)(N + p) \in T$

Step 3. Suffices to show that $M' - \sum_{i=1}^n X_i^2 \in T$ for some $M' \in \mathbb{N}$.

$$(1 + h)\left(M - \sum_{i=1}^n X_i^2\right) \in T \qquad M - \sum_{i=1}^n X_i^2 + Mh \in T$$

$$(1 + h)(N - Mh) \in T \qquad N + (N - M)h - Mh^2 \in T$$

w.l.o.g. $M \neq 0$ $\qquad (\lambda + \sqrt{M}h)^2 = \lambda^2 + 2\lambda\sqrt{M}h + Mh^2 \in T$

$- - - - - - - - - - -$

Add for good λ.

Schmüdgen's Positivstellensatz. $f > 0$ on $S \implies f \in T$

This ends Wörmann's proof of the

Weak version of Schmüdgen's Theorem. $\exists N \in \mathbb{N} : N + f \in T$

- Proof applied Positivstellensatz on $N - \sum_{i=1}^n X_i^2$ for some $N \in \mathbb{N}$.

- Apart from this, it was an effective construction.

Thorsten Wörmann: Strikt positive Polynome in der semialgebraischen Geometrie
Dissertation, Universität Dortmund (1998)

Schmüdgen's Positivstellensatz. $f > 0$ on $S \implies f \in T$

Proof. By the Positivstellensatz, $qf \in 1 + T$ for some $q \in T$.

By the weak version, $r + f \in T$ for some $r \in \mathbb{N}$ and
$N - q \in T$ for some $1 \leq N \in \mathbb{N}$.

$$\left(r - \frac{1}{N}\right) + f = \underbrace{\frac{1}{N}}_{\in T}\Big(\overbrace{\underbrace{(N-q)}_{\in T}\underbrace{(r+f)}_{\in T}}^{Nr+Nf-rq-qf} + \underbrace{(qf-1)}_{\in T} + \underbrace{rq}_{\in T}\Big) \in T$$

Iterate until $r = 0$ (or even $r < 0$).

Marshall Stone, Donald Dubois, Richard Kadison, Eberhard Becker

Jean-Louis Krivine: Anneaux préordonnés
J. Anal. Math. **12**, 307–326 (1964)

Schmüdgen's Positivstellensatz. $f > 0$ on $S \implies f \in T$

- In the proof, we applied the Positivstellensatz twice, on $N - \sum_{i=1}^n X_i^2$ for some $N \in \mathbb{N}$ and on f.

- Apart from this, it is an explicit construction.

- Original functional analytic proof of Schmüdgen and first algebraic proof of Wörmann apply the Positivstellensatz only on $N - \sum_{i=1}^n X_i^2$ but are for other reasons even less effective.

- When applying the Positivstellensatz on f, we know already that the bad case in its proof cannot occur.

- Nevertheless, the application on f is the bad one for applications in optimization.

It seems that Schmüdgen's Theorem is even less effective than the Positivstellensatz. But this is not true. In fact it is in between the Positivstellensatz and the very effective

Theorem of Pólya. Suppose f is homogeneous. If $f > 0$ on $[0, \infty)^n \setminus \{0\}$, then there exists a $k \in \mathbb{N}$ such that $(X_1 + \cdots + X_n)^k f$ has no negative coefficients.

Victoria Powers, Bruce Reznick: A new bound for Pólya's theorem with applications to polynomials positive on polyhedra
J. Pure Appl. Algebra 164, No.1–2, 221–229 (2001)
Jesús De Loera, Francisco Santos: An effective version of Pólya's theorem on positive definite forms
J. Pure Appl. Algebra **108**, No. 3, 231–240 (1996)
erratum ibid. 155, 309–310 (2001)
George Pólya: Über positive Darstellung von Polynomen
Vierteljahresschrift der Naturforschenden Ges. in Zürich **73**, 141–145 (1928)

Theorem of Pólya. Suppose f is homogeneous. If $f > 0$ on $[0,\infty)^n \setminus \{0\}$, then there exists a $k \in \mathbb{N}$ such that $(X_1 + \dots + X_n)^k f$ has no negative coefficients.

Proof. For $\alpha \in \mathbb{N}^n$, we set $|\alpha| := \alpha_1 + \dots + \alpha_n$ and $X^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. Set $\Delta := \{x \in [0,\infty)^n \mid \sum_{i=1}^n x_i = 1\}$.

$$(X_1 + \dots + X_n)^k f = \sum_{|\alpha|=k+d} \binom{k}{k_1 \dots k_n} (k+d)^d f \quad (\underbrace{\frac{\alpha}{k+d}}_{\in \Delta}) X^\alpha.$$

Theorem of Pólya. Suppose f is homogeneous. If $f > 0$ on $[0,\infty)^n \setminus \{0\}$, then there exists a $k \in \mathbb{N}$ such that $(X_1 + \dots + X_n)^k f$ has no negative coefficients.

Proof. For $\alpha \in \mathbb{N}^n$, we set $|\alpha| := \alpha_1 + \dots + \alpha_n$ and $X^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. Set $\Delta := \{x \in [0,\infty)^n \mid \sum_{i=1}^n x_i = 1\}$. Write $f = \sum_{|\alpha|=d} a_\alpha X^\alpha$, $a_\alpha \in \mathbb{R}$, and set

$$f_\varepsilon := \sum_{|\alpha|=d} a_\alpha (X_1)_\varepsilon^{\alpha_1} \cdots (X_n)_\varepsilon^{\alpha_n} \qquad \text{where} \qquad (X_i)_\varepsilon^{\alpha_i} := \prod_{j=0}^{\alpha_i - 1} (X_i - j\varepsilon).$$

$$(X_1 + \dots + X_n)^k f = \sum_{|\alpha|=k+d} \binom{k}{k_1 \dots k_n} (k+d)^d f_{\frac{1}{k+d}} (\underbrace{\frac{\alpha}{k+d}}_{\in \Delta}) X^\alpha.$$

But $f_\varepsilon := \sum_{|\alpha|=d} a_\alpha (X_1)_\varepsilon^{\alpha_1} \cdots (X_n)_\varepsilon^{\alpha_n} \to f$ uniformly on Δ for $\varepsilon \to 0$.

Remember that S is assumed to be compact.

Schmüdgen's Positivstellensatz. $f > 0$ on $S \implies f \in T$

Theorem of Pólya. Suppose f is homogeneous. If $f > 0$ on $[0,\infty)^n \setminus \{0\}$, then there exists a $k \in \mathbb{N}$ such that $(X_1 + \dots + X_n)^k f$ has no negative coefficients.

Weak version of Schmüdgen's Theorem. $\exists N \in \mathbb{N} : N + f \in T$

- Proof of the weak version was an effective construction apart from the application of the Positivstellensatz to $N - \sum_{i=1}^n X_i^2$.

- Proof of the strong version from the weak version was an effective construction apart from the application of the Positivstellensatz to f.

- Application of the Positivstellensatz to f bothers us the most.

Schmüdgen's Positivstellensatz. $f > 0$ on $S \implies f \in T$

Theorem of Pólya. Suppose f is homogeneous. If $f > 0$ on $[0,\infty)^n \setminus \{0\}$, then there exists a $k \in \mathbb{N}$ such that $(X_1 + \cdots + X_n)^k f$ has no negative coefficients.

Weak version of Schmüdgen's Theorem. $\exists N \in \mathbb{N} : N + f \in T$

$$\text{Weak version} \overset{\text{Pólya}}{\Longrightarrow} \text{Strong version}$$

???

Schmüdgen's Positivstellensatz. $f > 0$ on $S \implies f \in T$

Theorem of Pólya. Suppose f is homogeneous. If $f > 0$ on $[0,\infty)^n \setminus \{0\}$, then there exists a $k \in \mathbb{N}$ such that $(X_1 + \cdots + X_n)^k f$ has no negative coefficients.

Weak version of Schmüdgen's Theorem. $\exists N \in \mathbb{N} : N + f \in T$

- f might not be homogeneous.
- S versus $[0,\infty)^n \setminus \{0\}$
- How to get rid of the denominator $(X_1 + \cdots + X_n)^k$?
- X_i might not be in T.

Proof. Suppose $f > 0$ on S. To show: $f \in T$. By weak version, w.l.o.g. $X_1, \dots, X_n \in T$ and we find $N \in \mathbb{N}$ such that $N - (X_1 + \cdots + X_n) \in T$. W.l.o.g. $g \leq 1$ on $C := \{x \in [0,\infty)^n \mid x_1 + \cdots + x_n \leq N\} \supseteq S$. W.l.o.g. $f > 0$ on C by the lemma. Write $f = \sum_{i=0}^d F_i$ for homogeneous polynomials F_i of degree i (or $F_i = 0$). Then

$$F := \sum_{i=0}^d \left(\frac{X_1 + \cdots + X_n + Z}{N}\right)^{d-i} F_i \in \mathbb{R}[X, Z]$$

is homogeneous, and $F = \sum_{i=0}^d F_i = f > 0$ on

$$\Delta := \{(x_1, \dots, x_n, z) \in [0,\infty)^{n+1} \mid x_1 + \cdots + x_n + z = N\}.$$

Therefore $F > 0$ on $[0,\infty)^{n+1} \setminus \{0\}$. By Pólya's Theorem, there is $k \in \mathbb{N}$ such that $(X_1 + \cdots + X_n + Z)^k F$ has no negative coefficients. Finally, substitute $N - (X_1 + \cdots + X_n)$ for Z.

- The proof is an effective construction, in particular, it avoids the Positivstellensatz.
- Main idea was introduction of new coordinates lying in T and summing up to a natural number N (barycentric coordinates).
- The fact that these coordinates sum up to N allowed rewriting f as a homogeneous polynomial in the new coordinates and made the denominator from Pólya's Theorem harmless.

- Rewriting did not change the values of f on

$$S \hookrightarrow \Delta = \{(x_1, \dots, x_n, z) \in [0,\infty)^{n+1} \mid x_1 + \cdots + x_n + z = N\}.$$

This is good since positivity on S is conserved.

- But it did not even change the values of f on Δ.
This is bad since possible nonpositivity on Δ is kept.

- Therefore we were forced to establish positivity on S in advance by the lemma. But the lemma behaves bad with respect to degree complexity.

Lemma. Suppose $C \subseteq \mathbb{R}^n$ is compact and $g \leq 1$ on C. Then

$$f > 0 \text{ on } S \implies \exists s, k \in \mathbb{N} : f - s(1-g)^{2k} g > 0 \text{ on } C.$$

- Try to avoid the pretreatment of f, i.e., application of the lemma, and instead extend positivity from $S \hookrightarrow \Delta$ to Δ in the rewrite step.

- With the chosen $n+1$ (barycentric) coordinates not possible since we can only rewrite with respect to the only algebraic relation among them (which says that they sum up to N).

- Need other coordinates satisfying more algebraic relations. In addition, S must inside Δ be defined by an equation since any rewrite step which lets f invariant on S, lets f invariant on the Zariski–closure of S.

- Idea: Try to take g itself as an additional coordinate.

Proof. Suppose $f > 0$ on S. To show: $f \in T$. By weak version, w.l.o.g. $X_1, \dots, X_n \in T$ and we find $N \in \mathbb{N}$ such that $N - (X_1 + \cdots + X_n + g) \in T$.

$$\Delta := \{(x_1, \dots, x_n, y, z) \in [0,\infty)^{n+2} \mid x_1 + \cdots + x_n + y + z = N\}$$
$$V := \{(x_1, \dots, x_n, y, z) \in \Delta \mid y = g(x)\} \subseteq \Delta$$

Then $f > 0$ on V. For big $\lambda \in \mathbb{R}$, $h := f + \lambda(Y - g)^2 \in \mathbb{R}[X, Y, Z]$ is positive on Δ. Write $h = \sum_{i=0}^d F_i$ for homogeneous F_i of degree i. Then

$$F := \sum_{i=0}^{d} \left(\frac{X_1 + \cdots + X_n + Y + Z}{N} \right)^{d-i} F_i \in \mathbb{R}[X, Y, Z]$$

is homogeneous, and $F = \sum_{i=0}^d F_i = h > 0$ on Δ. Therefore $F > 0$ on $[0,\infty)^{n+2} \setminus \{0\}$. By Pólya's Theorem, there is $k \in \mathbb{N}$ such that $(X_1 + \cdots + X_n + Y + Z)^k F$ has no negative coefficients. Finally, substitute g for Y and $N - (X_1 + \cdots + X_n + g)$ for Z.

- The proof is again an effective construction, in particular, it avoids the Positivstellensatz.

- Degree of $h := f + \lambda(Y - g)^2$ depends only on $\deg f$ and $\deg g$ and not on geometric properties of f.

First proof:
Optimization of polynomials on compact semialgebraic sets
preprint

Second proof:
An algorithmic approach to Schmüdgen's Positivstellensatz
Journal of Pure and Applied Algebra **166**, 307–319 (2002)

Consequences of second proof:
On the complexity of Schmüdgen's Positivstellensatz
to appear in Journal of Complexity

Remember that S is assumed to be compact.

The S–moment problem

Given a family $(a_\alpha)_{\alpha\in\mathbb{N}^n}$ of real numbers, when is it true that they are the moments of some probability measure on S?

To be more precise, denote by $\mathcal{M}^1(A)$ the set of all probability measures on a subset A of $\mathbb{R}^n$. Then the question is:

For which real families $(a_\alpha)_{\alpha\in\mathbb{N}^n}$ is it true that

$$\exists \mu \in \mathcal{M}^1(S) : \forall \alpha \in \mathbb{N}^n : a_\alpha = \int X^\alpha d\mu$$

holds?

Schmüdgen's solution to the moment problem. Write $g = \sum_{\alpha\in\mathbb{N}^n} c_\alpha X^\alpha$, $c_\alpha \in \mathbb{R}$. For every real family $(a_\alpha)_{\alpha\in\mathbb{N}^n}$ are equivalent:

(1) $a_0 = 1$ and for all real families $(b_\alpha)_{\alpha\in\mathbb{N}^n}$ with finite support,

$$\sum_{\alpha,\beta\in\mathbb{N}^n} b_\alpha b_\beta a_{\alpha+\beta} \geq 0 \qquad \text{and} \qquad \sum_{\alpha,\beta,\gamma\in\mathbb{N}^n} b_\alpha b_\beta c_\gamma a_{\alpha+\beta+\gamma} \geq 0.$$

(2) $\exists \mu \in \mathcal{M}^1(S) : \forall \alpha \in \mathbb{N}^n : a_\alpha = \int X^\alpha d\mu$

Konrad Schmüdgen: The K-moment problem for compact semi–algebraic sets
Math. Ann. **289**, No. 2, 203–206 (1991)

Schmüdgen's solution to the moment problem. Write $g = \sum_{\alpha\in\mathbb{N}^n} c_\alpha X^\alpha$, $c_\alpha \in \mathbb{R}$. For every linear map $L : \mathbb{R}[X] \to \mathbb{R}$ are equivalent:

(1) $L(1) = 1$ and for all real families $(b_\alpha)_{\alpha\in\mathbb{N}^n}$ with finite support,

$$\sum_{\alpha,\beta\in\mathbb{N}^n} b_\alpha b_\beta L(X^{\alpha+\beta}) \geq 0 \qquad \text{and} \qquad \sum_{\alpha,\beta,\gamma\in\mathbb{N}^n} b_\alpha b_\beta c_\gamma L(X^{\alpha+\beta+\gamma}) \geq 0.$$

(2) $\exists \mu \in \mathcal{M}^1(S) : \forall \alpha \in \mathbb{N}^n : L(X^\alpha) = \int X^\alpha d\mu$

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(1) $L(1) = 1$ and for all real families $(b_\alpha)_{\alpha \in \mathbb{N}^n}$ with finite support,

$$L\left(\sum_{\alpha,\beta \in \mathbb{N}^n} b_\alpha b_\beta X^{\alpha+\beta}\right) \geq 0 \quad \text{and} \quad L\left(\sum_{\alpha,\beta,\gamma \in \mathbb{N}^n} b_\alpha b_\beta c_\gamma X^{\alpha+\beta+\gamma}\right) \geq 0.$$

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(2) $\exists \mu \in \mathcal{M}^1(S) : \forall p \in \mathbb{R}[X] : L(p) = \int p d\mu$

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Schmüdgen's solution to the moment problem. For every linear map $L : \mathbb{R}[X] \to \mathbb{R}$ are equivalent:

(1) $L(1) = 1$ and for all $p \in \mathbb{R}[X]$,

$$L(p^2) \geq 0 \qquad \text{and} \qquad L(p^2 g) \geq 0.$$

(2) $\exists \mu \in \mathcal{M}^1(S) : \forall p \in \mathbb{R}[X] : L(p) = \int p d\mu$

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Schmüdgen's solution to the moment problem. For every linear map $L : \mathbb{R}[X] \to \mathbb{R}$ are equivalent:

(1) $L(1) = 1$ and $L(T) \subseteq [0, \infty)$

(2) $\exists \mu \in \mathcal{M}^1(S) : \forall p \in \mathbb{R}[X] : L(p) = \int p d\mu$

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Schmüdgen's solution to the moment problem. For every map $L : \mathbb{R}[X] \to \mathbb{R}$ are equivalent:

(1) L is linear, $L(1) = 1$ and $L(T) \subseteq [0, \infty)$

(2) $\exists \mu \in \mathcal{M}^1(S) : \forall p \in \mathbb{R}[X] : L(p) = \int p d\mu$

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(1) L is linear, $L(1) = 1$ and $L(T) \subseteq [0, \infty)$

(2) $\exists \mu \in \mathcal{M}^1(S) : \forall p \in \mathbb{R}[X] : L(p) = \int p d\mu$

Proof sketch. For the nontrivial implication, it suffices to show

$$\forall p \in \mathbb{R}[X] : (p \geq 0 \text{ on } S \implies L(p) \geq 0)$$

by Stone–Weierstrass approximation and the Riesz Representation Theorem. Suppose $p \in \mathbb{R}[X]$ such that $p \geq 0$ on S. Then, for every $\varepsilon > 0$, $p + \varepsilon > 0$ on S implying $p + \varepsilon \in T$ by Schmüdgen's Theorem and $L(p) + \varepsilon = L(p + \varepsilon) \in L(T) \subseteq [0, \infty)$ by (1). Therefore $L(p) \geq 0$.

Optimization

We consider the problem of minimizing f on S. So we want to compute numerically the infimum (minimum if $S \neq \emptyset$)

$$f^* := \inf\{f(x) \mid x \in S\} \in \mathbb{R} \cup \{\infty\}$$

and, if possible, a minimizer, i.e., an element of the set

$$S^* := \{x^* \mid \forall x \in S : f(x^*) \leq f(x)\}.$$

Best known strategy for minimization:

Go downhill!

- Problem: local minima
- Remedy: convexity

Convexify the problem by brute force. Two ways to do so:

- Generalize from points to probability measures:

$$f^* = \inf\left\{\int f d\mu \mid \mu \in \mathcal{M}^1(S)\right\}$$

- Take a dual standpoint:

$$f^* = \sup\{a \in \mathbb{R} \mid f - a \geq 0 \text{ on } S\} = \sup\{a \in \mathbb{R} \mid f - a > 0 \text{ on } S\}$$

$$f^* = \inf\left\{\int f d\mu \mid \mu \in \mathcal{M}^1(S)\right\}$$

Schmüdgen's solution $\Downarrow$ to the moment problem

$$\boxed{f^* = \inf\{L(f) \mid L : \mathbb{R}[X] \to \mathbb{R} \text{ is linear}, L(1) = 1, L(T) \subseteq [0, \infty)\}}$$

$$f^* = \sup\{a \in \mathbb{R} \mid f - a \geq 0 \text{ on } S\} = \sup\{a \in \mathbb{R} \mid f - a > 0 \text{ on } S\}$$

Schmüdgen's $\Downarrow$ Positivstellensatz

$$\boxed{f^* = \sup\{a \in \mathbb{R} \mid f - a \in T\}}$$

Introduce finite–dimensional approximations $T_k \subseteq \mathbb{R}[X]_k$ of $T \subseteq \mathbb{R}[X]$.

$$\mathbb{R}[X]_k := \{p \mid p \in \mathbb{R}[X], \deg p \leq k\} \qquad \text{real vector space}$$

$$T_k := \sum \mathbb{R}[X]_d^2 + \sum \mathbb{R}[X]_e^2\, g \qquad \text{convex cone}$$

$$= \left\{\sigma + \tau g \mid \sigma, \tau \in \sum \mathbb{R}[X]^2, \deg \sigma \leq k, \deg(\tau g) \leq k\right\}$$

for arbitrary $k \in \mathcal{N} := \{s \in \mathbb{N} \mid s \geq \max\{\deg g, \deg f\}\}$.

Here $d := \max\{m \in \mathbb{N} \mid 2m \leq k\}$ and $e := \max\{m \in \mathbb{N} \mid 2m + \deg g \leq k\}$.

Warning: Never confuse T_k with $T \cap \mathbb{R}[X]_k \supseteq T_k$.

We saw that

$$f^* = \inf\{L(f) \mid L : \mathbb{R}[X] \to \mathbb{R} \text{ is linear}, L(1) = 1, L(T) \subseteq [0, \infty)\} \quad \text{and}$$
$$f^* = \sup\{a \in \mathbb{R} \mid f - a \in T\}.$$

In analogy to this, we set

$$P_k^* = \inf\{L(f) \mid L : \mathbb{R}[X]_k \to \mathbb{R} \text{ is linear}, L(1) = 1, L(T_k) \subseteq [0, \infty)\} \quad \text{and}$$
$$D_k^* = \sup\{a \in \mathbb{R} \mid f - a \in T_k\}$$

for every $k \in \mathcal{N}$.

$P_k^* \in \mathbb{R} \cup \{\pm\infty\}$ and $D_k^* \in \mathbb{R} \cup \{\pm\infty\}$ are the optimal values of the following pair of optimization problems. . .

(P_k)	minimize	$L(f)$	subject to	$L : \mathbb{R}[X]_k \to \mathbb{R}$ is linear,
				$L(1) = 1$ and
				$L(T_k) \subseteq [0, \infty)$
(D_k)	maximize	a	subject to	$a \in \mathbb{R}$ and
				$f - a \in T_k$

Theorem (Lasserre). $(D_k^*)_{k\in\mathcal{N}}$ and $(P_k^*)_{k\in\mathcal{N}}$ are increasing sequences that converge to f^* and satisfy $D_k^* \leq P_k^* \leq f^*$ for all $k \in \mathcal{N}$.

Proof. $P_k^* \leq f^*$ because $p \mapsto p(x)$ feasible for (P_k) for $x \in S$.

$D_k^* \leq P_k^*$: $L(f) - a = L(f) - aL(1) = L(f - a) \subseteq L(T_k) \subseteq [0, \infty)$

Clear: P_k^* and D_k^* increase. $\lim_{k\to\infty} D_k^* \to f^*$: If $a < f^*$, then $f - a \in T_k$ for some $k \in \mathcal{N}$ by Schmüdgen's Positivstellensatz. Then a is feasible for (D_k) whence $a \leq D_k^*$. Convergence of D_k^* implies convergence of P_k^*.

(P_k)	minimize	$L(f)$	subject to	$L : \mathbb{R}[X]_k \to \mathbb{R}$ is linear,
k–th primal relaxation				$L(1) = 1$ and
(primal relaxation of order k)				$L(T_k) \subseteq [0, \infty)$
(D_k)	maximize	a	subject to	$a \in \mathbb{R}$ and
k–th dual relaxation				$f - a \in T_k$

Theorem (Lasserre). $(D_k^*)_{k\in\mathcal{N}}$ and $(P_k^*)_{k\in\mathcal{N}}$ are increasing sequences that converge to f^* and satisfy $D_k^* \leq P_k^* \leq f^*$ for all $k \in \mathcal{N}$.

Jean Lasserre: Global optimization with polynomials and the problem of moments
SIAM J. Optim. **11**, No. 3, 796–817 (2001)

(P_k)	minimize	$L(f)$	subject to	$L : \mathbb{R}[X]_k \to \mathbb{R}$ is linear,
				$L(1) = 1$ and
				$L(T_k) \subseteq [0, \infty)$
(D_k)	maximize	a	subject to	$a \in \mathbb{R}$ and
				$f - a \in T_k$

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Theorem. There exists $C \in \mathbb{N}$ depending on f and g and $c \in \mathbb{N}$ depending on f such that

$$f^* - D_k^* \leq \frac{C}{\sqrt[c]{k}} \qquad \text{for big } k.$$

On the complexity of Schmüdgen's Positivstellensatz
to appear in Journal of Complexity

(P_k) minimize $L(f)$ subject to $L : \mathbb{R}[X]_k \to \mathbb{R}$ is linear, $L(1) = 1$ and $L(T_k) \subseteq [0, \infty)$

(D_k) maximize a subject to $a \in \mathbb{R}$ and $f - a \in T_k$

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Dependance on f can be made explicit. Proof hints to make dependance on g explicit for concrete g. Main idea: Second approach to Schmüdgen's Positivstellensatz via Pólya.

(P_k) minimize $L(f)$ subject to $L : \mathbb{R}[X]_k \to \mathbb{R}$ is linear, $L(1) = 1$ and $L(T_k) \subseteq [0, \infty)$

(D_k) maximize a subject to $a \in \mathbb{R}$ and $f - a \in T_k$

Theorem (Lasserre). $(D_k^*)_{k \in \mathcal{N}}$ and $(P_k^*)_{k \in \mathcal{N}}$ are increasing sequences that converge to f^* and satisfy $D_k^* \leq P_k^* \leq f^*$ for all $k \in \mathcal{N}$. How fast?

Theorem. There exists $C \in \mathbb{N}$ depending on f and g and $c \in \mathbb{N}$ depending on f such that

$$f^* - D_k^* \leq \frac{C}{\sqrt[c]{k}} \qquad \text{for big } k.$$

In practice: Convergence usually very fast, often $D_k^* = P_k^* = f^*$ for small k.

(P_k) minimize $L(f)$ subject to $L : \mathbb{R}[X]_k \to \mathbb{R}$ is linear, $L(1) = 1$ and $L(T_k) \subseteq [0, \infty)$

(D_k) maximize a subject to $a \in \mathbb{R}$ and $f - a \in T_k$

Schmüdgen's Positivstellensatz implies convergence of D_k^* and therefore of P_k^*.

What can we know from Schmüdgen's solution to the moment problem?
A priori nothing! But with additional compactness arguments involving Tychonoff's Theorem, the following. . .

(P_k) minimize $L(f)$ subject to $L : \mathbb{R}[X]_k \to \mathbb{R}$ is linear, $L(1) = 1$ and $L(T_k) \subseteq [0, \infty)$

Theorem. Suppose that L_k solves (P_k) nearly to optimality ($k \in \mathcal{N}$).

$$\forall d \in \mathbb{N} : \forall \varepsilon > 0 : \exists k_0 \in \mathcal{N} \cap [d, \infty) : \forall k \geq k_0 : \exists \mu \in \mathcal{M}^1(S^*) : \left\| \left(L_k(X^\alpha) - \int X^\alpha d\mu \right)_{|\alpha| \leq d} \right\| < \varepsilon.$$

Optimization of polynomials on compact semialgebraic sets
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(P_k) minimize $L(f)$ subject to $L : \mathbb{R}[X]_k \to \mathbb{R}$ is linear,
$L(1) = 1$ and
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$$\left\| \left(L_k(X^\alpha) - \int X^\alpha d\mu \right)_{|\alpha| \leq d} \right\| < \varepsilon.$$

In particular, if $S^* = \{x^*\}$ is a singleton, then

$$\lim_{k \to \infty} (L_k(X_1), \ldots, L_k(X_n)) = x^*.$$

(P_k) minimize $L(f)$ subject to $L : \mathbb{R}[X]_k \to \mathbb{R}$ is linear,
$L(1) = 1$ and
$L(T_k) \subseteq [0, \infty)$

(D_k) maximize a subject to $a \in \mathbb{R}$ and
$f - a \in T_k$

Theorem (Lasserre). If S has nonempty interior, then $D_k^* = P_k^*$.

- "Strong duality"
- "Weak duality" $D_k^* \leq P_k^*$ always holds.
- First proved by using duality theory from semidefinite programming since (P_k) and (D_k) can be translated into semidefinite programs and are as such dual to each other.

(P_k) minimize $L(f)$ subject to $L : \mathbb{R}[X]_k \to \mathbb{R}$ is linear,
$L(1) = 1$ and
$L(T_k) \subseteq [0, \infty)$

(D_k) maximize a subject to $a \in \mathbb{R}$ and
$f - a \in T_k$

Theorem (Lasserre). If S has nonempty interior, then $D_k^* = P_k^*$.

Sketch of Marshall's direct proof. It suffices to show that T_k is closed in $\mathbb{R}[X]_k$. For s big (see below), M_k is image of $\mathbb{R}[X]_d^s \times \mathbb{R}[X]_e^s \to \mathbb{R}[X]_k$:

$$(p_1, \ldots, p_s, q_1, \ldots, q_s) \mapsto \sum_{i=1}^s p_i^2 + \sum_{i=1}^s q_i^2 g.$$

This map is quadratically homogeneous and injective.

(P_k) minimize $L(f)$ subject to $L : \mathbb{R}[X]_k \to \mathbb{R}$ is linear,
$L(1) = 1$ and
$L(T_k) \subseteq [0, \infty)$

(D_k) maximize a subject to $a \in \mathbb{R}$ and
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Theorem (Lasserre). If S has nonempty interior, then $D_k^* = P_k^*$.

Optimization of polynomials on compact semialgebraic sets
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Murray Marshall: Optimization of polynomial functions
preprint
Jean Lasserre: Global optimization with polynomials and the problem of moments
SIAM J. Optim. **11**, No. 3, 796–817 (2001)

Remember that S is assumed to be compact.

Optimization

We consider the problem of minimizing f on S. So we want to compute numerically the infimum (minimum if $S \neq \emptyset$)

$$f^* := \inf\{f(x) \mid x \in S\} \in \mathbb{R} \cup \{\infty\}$$

and, if possible, a minimizer, i.e., an element of the set

$$S^* := \{x^* \mid \forall x \in S : f(x^*) \leq f(x)\}.$$

Introduce finite–dimensional approximations $T_k \subseteq \mathbb{R}[X]_k$ of $T \subseteq \mathbb{R}[X]$.

$$\begin{aligned} \mathbb{R}[X]_k &:= \{p \mid p \in \mathbb{R}[X], \deg p \leq k\} && \text{real vector space} \\ T_k &:= \sum \mathbb{R}[X]_d^2 + \sum \mathbb{R}[X]_e^2\, g && \text{convex cone} \\ &= \Big\{\sigma + \tau g \mid \sigma, \tau \in \sum \mathbb{R}[X]^2, \deg \sigma \leq k, \deg(\tau g) \leq k\Big\} \end{aligned}$$

for arbitrary $k \in \mathcal{N} := \{s \in \mathbb{N} \mid s \geq \max\{\deg g, \deg f\}\}$.

Here $d := \max\{m \in \mathbb{N} \mid 2m \leq k\}$ and $e := \max\{m \in \mathbb{N} \mid 2m+\deg g \leq k\}$.

(P_k)	minimize	$L(f)$	subject to	$L : \mathbb{R}[X]_k \to \mathbb{R}$ is linear, $L(1) = 1$ and $L(T_k) \subseteq [0, \infty)$
(D_k)	maximize	a	subject to	$a \in \mathbb{R}$ and $f - a \in T_k$

Denote the optimal values of these optimization problems by $P_k^* \in \mathbb{R} \cup \{\pm\infty\}$ and $D_k^* \in \mathbb{R} \cup \{\pm\infty\}$, respectively.

Theorem (Lasserre). $(D_k^*)_{k\in\mathcal{N}}$ and $(P_k^*)_{k\in\mathcal{N}}$ are increasing sequences that converge to f^* and satisfy $D_k^* \leq P_k^* \leq f^*$ for all $k \in \mathcal{N}$.
If S has nonempty interior, then $D_k^* = P_k^*$.

Theorem. Suppose that $S^* = \{x^*\}$ is a singleton and L_k solves (P_k) nearly to optimality ($k \in \mathcal{N}$). Then

$$\lim_{k\to\infty} (L_k(X_1), \ldots, L_k(X_n)) = x^*.$$

- (P_k) and (D_k) can be translated in a primal–dual pair of semidefinite programs which can be solved efficiently (Lasserre).

- Feasible solutions of the semidefinite program corresponding to (D_k) give rise to a lower bound a of f^* together with a certificate (advantage) in form of a representation of $f - a$ proving $f - a \in T_k$.

- Method converges from below to the infimum (advantage in many applications).

- Method converges to unique minimizers. Disadvantage: Possibly from outside the set.

- If there is a unique minimizer and it lies in the interior of S, then the method produces a sequence of intervals containing f^* whose endpoints converge to f^*.

Additional instruments for detecting optimality and extracting solutions

- If L is an optimal solution of (P_k), $x := (L(X_1), \ldots, L(X_n)) \in S$ and $L(f) = f(x)$, then $L(f) = P_k^* \leq f^* \leq f(x) = L(f)$, i.e., $L(f) = f(x) = f^*$ and therefore $x \in S^*$.

- If L is an optimal solution of (P_k) which comes from a measure μ on S (criteria of Curto and Fialkow for the truncated S–moment problem), then $L(f) = P_k^* \leq f^* \leq \int f d\mu = L(f)$, i.e., $L(f) = f^*$ and $\mu \in \mathcal{M}^1(S^*)$. In case that μ has finite support supp(μ), it seems that often (?) numerical linear algebra methods can obtain all elements of supp(μ) $\subseteq S^*$ from the moments $L(X^\alpha)$, $|\alpha| \leq k$ of the measure μ?

Raul Curto, Lawrence Fialkow: The truncated complex K–moment problem Trans. Am. Math. Soc. **352**, No. 6, 2825–2855 (2000)

How to solve the relaxations?

(P_k)	minimize	$L(f)$	subject to	$L : \mathbb{R}[X]_k \to \mathbb{R}$ is linear, $L(1) = 1$ and $L(T_k) \subseteq [0, \infty)$
(D_k)	maximize	a	subject to	$a \in \mathbb{R}$ and $f - a \in T_k$

- Optimization of a linear function on a convex set. No problem with local minima.

- When going downhill, we could hit the boundary. Therefore we need to be able to compute effectively a so called barrier function defined on the interior of the convex set.

- The cone $S\mathbb{R}_+^{s \times s}$ of positive semidefinite symmetric matrices has such a barrier function:
$$X \mapsto -\ln\det X$$

- Semidefinite programming is an extension of linear programming.
- Linear programming: Optimization of a linear function $\mathbb{R}^s \to \mathbb{R}$ on the intersection of the selfdual cone $[0,\infty)^s$ with an affine subspace of $\mathbb{R}^s$.
- Semidefinite programming: Optimization of a linear function $S\mathbb{R}^{s\times s} \to \mathbb{R}$ on the intersection of the selfdual cone $S\mathbb{R}_+^{s\times s}$ with an affine subspace.
- A lot of efficient semidefinite programming solvers are freely available.

Sums of squares and semidefinite matrices

Let v a column vector of length s whose entries generate the vector space $\mathbb{R}[X]_d$. Then $\sum \mathbb{R}[X]_d^2 = \{v^T G v \mid G \in S\mathbb{R}_+^{s\times s}\}$.

Proof. "$\subseteq$" Suppose $t \in \mathbb{N}$ and $p_1, \ldots, p_t \in \mathbb{R}[X]_d$. To show: $\sum_{i=1}^t p_i^2 = v^T G v$ for some $G \in S\mathbb{R}_+^{s\times s}$. Choose a real $t \times s$ matrix A such that $p_1, \ldots, p_t$ are the rows of Av. Then

$$\sum_{i=1}^t p_i^2 = (Av)^T Av = v^T(\underbrace{A^T A}_{\in S\mathbb{R}_+^{s\times s}})v.$$

Sums of squares and semidefinite matrices

Let v a column vector of length s whose entries generate the vector space $\mathbb{R}[X]_d$. Then $\sum \mathbb{R}[X]_d^2 = \{v^T G v \mid G \in S\mathbb{R}_+^{s\times s}\}$.

Proof. "$\supseteq$" If $G \in SR_+^{s\times s}$, then $G = A^T D A$ for a real (orthogonal) $s \times s$ matrix A and an $s \times s$ diagonal matrix with nonnegative entries. Then $G = (A^T\sqrt{D})(\sqrt{D}A) = (\sqrt{D}A)^T(\sqrt{D}A)$. Hence $v^T G v = (\sqrt{D}Av)^T(\sqrt{D}Av) = \sum_{i=1}^s p_i^2$ where $p_1, \ldots, p_s$ denote the entries of the column vector $\sqrt{D}Av$.

Shows also what we used for showing strong duality.

Sums of squares and semidefinite matrices

Remember that, for $k \in \mathcal{N}$,

$$T_k = \sum \mathbb{R}[X]_d^2 + \sum \mathbb{R}[X]_e^2 g$$

where $d := \max\{m \in \mathbb{N} \mid 2m \le k\}$ and
$e := \max\{m \in \mathbb{N} \mid 2m + \deg g \le k\}$.

Let v a column vector of length s whose entries generate the vector space $\mathbb{R}[X]_d$. Let w a column vector of length t whose entries generate the vector space $\mathbb{R}[X]_e$. Then

$$T_k = \{v^T G v + w^T H w g \mid G \in S\mathbb{R}^{s\times s}, H \in S\mathbb{R}^{t\times t}\}.$$

With little elaborations this gives the translation of (P_k) into a semidefinite program.

Translation into a semidefinite program

Remember that, for $k \in \mathcal{N}$,

$$T_k = \sum \mathbb{R}[X]_d^2 + \sum \mathbb{R}[X]_e^2 g$$

where $d := \max\{m \in \mathbb{N} \mid 2m \leq k\}$ and
$e := \max\{m \in \mathbb{N} \mid 2m + \deg g \leq k\}$.

We just outlined how (P_k) can be formulated as a semidefinite program. For (P_k) this is even easier. To express that a linear map $L : \mathbb{R}[X]_k \to \mathbb{R}$ satisfies $L(T_k) \subset [0, \infty)$, one writes down that the matrices representing the following bilinear forms are positive semidefinite:

$$\mathbb{R}[X]_d \times \mathbb{R}[X]_d \to \mathbb{R} : (p, q) \mapsto L(pq) \quad \text{and}$$
$$\mathbb{R}[X]_e \times \mathbb{R}[X]_e \to \mathbb{R} : (p, q) \mapsto L(pqg)$$

Implementations

- Henrion and Lasserre: GloptiPoly
  `http://www.laas.fr/~henrion/software/gloptipoly/`
- Prajna, Papachristodoulou, Parrilo: SOSTOOLS
  `http://control.ee.ethz.ch/~parrilo/sostools/`
- Both use the free SeDuMi solver by Jos Sturm
- But they need MATLAB and the MATLAB Symbolic Toolbox

Example: The maximum cut problem

Given a graph, i.e., an $n \in \mathbb{N}$ (number of nodes) and a set

$$E \subseteq \{(i, j) \in \{1, \dots, n\}^2 \mid i < j\}$$

(of edges), find the maximum cut value, i.e., the maximal possible number of edges that connect nodes with different signs when each node is assigned a sign $+$ or $-$.

$$\begin{aligned} \text{maximize} \quad & \sum_{(i,j)\in E} \frac{1}{2}(1 - x_i x_j) \\ \text{subject to} \quad & x_i^2 = 1 \text{ for all } i \in \{1, \dots, n\} \end{aligned}$$

- The maximum cut problem is NP–complete
- Solving the first relaxation is a polynomial time algorithm which overestimates the maximum cut value at most by a factor of ≈ 1.1382.
- The first algorithm turns out to be the famous algorithm of Goemans and Williamson. From no polynomial algorithm it is known that it has a better approximation ratio. Existence of such an algorithm with ratio < 1.0625 implies $P = NP$ (Hastad).
- Solving the second relaxation is a polynomial time algorithm which yields the exact value for all planar graphs (consequence of results of Seymour, Barahona, Mahjoub), and is conjectured to improve over the GW–algorithm.