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Description of basic semialgebraic sets

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DESCRIPTION OF BASIC SEMIALGEBRAIC SETS

C. ANDRADAS

1. Introduction

Semialgebraic sets are rather ubiquitous in mathematics and nature. The reason is that since they are described by polynomials on one hand they are "easy" to manipulate and on the other hand they are adequate to model almost any geometric object since polynomials are dense in compact ambiences.

Let $V \subset \mathbb{R}^n$ be an algebraic set, that is, the set of solutions of a finite system of polynomial equations. (Although most of results appearing below hold true over any real closed field for simplicity we will work over the real numbers) Recall that a subset $S \subset V$ is called semialgebraic if it is of the form

$$S = \bigcup_{i=1}^{t} \{x \in V \mid f_i(x) = 0, g_{i1} > 0, \ldots, g_{in_i} > 0\}$$

for some polynomial functions f_i, g_{ij} on V. The pieces $\{x \in V \mid g_{i1} > 0, \ldots, g_{in_i} > 0\}$, i.e., the solution set of a system of strict inequalities are called *open basic semialgebraic*. Open basic semialgebraic are, somehow, the building stones of semialgebraic sets, so that they have deserved some attention in the last years. In particular we have the following two main questions:

(P1): Recognize whether a given open semialgebraic set is basic.

(P2): Given a basic open semialgebraic set find a minimal (in the sense of shorter) description.

Both questions have precise answers, mainly due to the work of L. Bröcker in the 80's, and also C. Scheiderer by means of the theory of spaces of orderings, cf. [AnBrRz]. To be more precise, we denote by $A := \mathbb{R}[V]$ the ring of polynomials function on V. We will assume that V is irreducible so that A is a domain and we will denote by $K := \mathbb{R}(V)$ its field of fractions. In particular K is a finitely generated extension of $\mathbb{R}$ of transcendence degree $d = \dim V$. we have:

Theorem 1.1. ([Br1]) *An open semialgebraic set S is basic if and only if for any 4-element fan $F \subset \mathrm{Spec}_r A$ we have $\#(F \cap S) \neq 3$.*

Theorem 1.2. ([Br1]) *Set* $d = \dim(V)$*. Any basic open semialgebraic set can be written with* $s(d)$ *inequalities.*

All the above results share several characteristics: they are beautiful and they are non constructive. In the late 90's, Acquistapace et al, [AcBgVe], showed that question P1 is indeed algorithmically decidable and produce a theoretical algorithm to decide it. However it is very unpractical. Their work is inspired in previous work of [AAB] and [AnRz1,2,3]. Concerning question P2, not many attempts have been taken toward giving explicit algorithms for finding explicit inequations of minimal length. Only some work of A. Bernig in the 2 dimensional case gave explicit construction of the 2 inequalities to describe open, convex, basic semialgebraic sets, and recently [AnVe] have given an algorithm for basic semilinear sets (i.e. described by linear inequalities), showing that in this case basicness and convexity are equivalent. The main attempt toward finding a constructive minimal description was given by Buresi and Mahé, [BuMa,] but they only could find explicit upper bounds for the degrees of the polynomials appearing in it.

For simplicity we will assume that V is nonsingular. The reader may even think of semialgebraic subsets of the affine space $\mathbb{R}^d$. In this case $A = \mathbb{R}[X_1, \dots, X_d]$.

2. Changing inequalities

Suppose that we want to manipulate a system of strict inequalities in a similar way to what we made in Linear Algebra to transform a linear system in a triangular one, that is, defining some admissible operations to transform the initial system into an equivalent one and hoping that the latter is simpler (in some way) than the older.

Here is the first "rule of thumb", that can be checked by direct inspection: *the systems*

$$\left.\begin{array}{rcl} h_1 & > & 0 \\ h_2 & > & 0 \end{array}\right\} \qquad\qquad \left.\begin{array}{rcl} h_1 + h_2 & > & 0 \\ h_1 h_2 & > & 0 \end{array}\right\}$$

are equivalent.

The second rule states that the system does not change when equations are multiplied by polynomials of precise positivity point sets. Again a direct inspection shows that *the systems*

$$\left.\begin{array}{rcl} h_1 & > & 0 \\ h_2 & > & 0 \\ \cdots & & \\ h_{n-1} & > & 0 \\ h_n & > & 0 \end{array}\right\} \qquad\qquad \left.\begin{array}{rcl} h_1 u_0 & > & 0 \\ h_2 u_1 & > & 0 \\ \cdots & & \\ h_{n-1} u_{n-2} & > & 0 \\ h_n u_{n-1} & > & 0 \end{array}\right\}$$

are equivalent, where $u_0 > 0$ everywhere and $\{u_j > 0\} \supset \{h_1 > 0, \ldots, h_j > 0\}$.

Combining these two operations we get the following one which collect both at once (the u_j's are as above): *the systems*

$$\left.\begin{array}{rcl} h_1 & > & 0 \\ h_2 & > & 0 \\ \cdots & & \\ h_{n-2} & > & 0 \\ h_{n-1} & > & 0 \\ h_n & > & 0 \end{array}\right\} \qquad \left.\begin{array}{rcl} h_1 & > & 0 \\ h_2 & > & 0 \\ \cdots & & \\ h_{n-2} & > & 0 \\ h_{n-1}u_{n-2} + h_n u_{n-1} & > & 0 \\ h_{n-1}h_n u_{n-1} & > & 0 \end{array}\right\}$$

are equivalent.

Repeating the rule with the rows $n-1$ and $n-2$ and so on, we get that the initial system of inequalities is equivalent to

$$\left.\begin{array}{rcl} h_1u_0 + h_2u_1 + \cdots + h_nu_{n-1} & > & 0 \\ h_1(h_2u_1 + \cdots + h_nu_{n-1}) & > & 0 \\ \cdots & & \\ h_{n-2}(h_{n-1}u_{n-2} + h_nu_{n-1}) & > & 0 \\ h_{n-1}h_nu_{n-1} & > & 0 \end{array}\right\} \qquad (*)$$

Now, if $u_0, u_1, \ldots, u_{n-1}$ can be chosen such that $h_1u_0 + h_2u_1 + \cdots + h_nu_{n-1} > 0$ is trivial, for instance $1 = h_1u_0 + h_2u_1 + \cdots + h_nu_{n-1}$, then we get that our system is equivalent to one with one less equation, namely $\{g_1 > 0, \ldots, g_{n-1} > 0\}$ with $g_k = h_k(h_{k+1}u_k + \cdots + h_nu_{n-1})$.

This will be the strategy to shorten the system of inequalities: try to find $u_0, u_1, \ldots, u_{n-1}$ such that $h_1u_0 + h_2u_1 + \cdots + h_nu_{n-1}$ becomes a strict positive function (for instance 1). Of course the hard point is to choose the u_i's and for that we need to know how the polynomials which are positive on a certain basic semialgebraic set look like. The tool for that will be the use of Pfister forms and the elements represented by them.

3. Pfister's form attached to a semialgebraic set

Let K be any field of characteristic different from 2. A quadratic form of dimension n, over K is a homogeneous polynomial of degree 2 in n variables:

$$\varphi = \sum_{1 \le i \le j \le n} a_{ij} Y_i Y_j$$

The value of φ in the vector $w = (w_1, \ldots, w_n)$ is just $\varphi(w) = \sum_{1 \le i \le j \le n} a_{ij} w_i w_j$.

We can write φ in matrix form in a unique way $\varphi(Y) = Y^t M_\varphi Y$, where $Y^t = (Y_1, \ldots, Y_n)$ and M is a symmetric $n \times n$ matrix with $m_{ii} = a_{ii}$ and $m_{ij} = m_{ji} = (1/2)a_{ij}$. Two forms (of the same dimension) φ

and ψ are said equivalent or congruent and we will write $\varphi \equiv \psi$ if there is change of variable, given say by the matrix P, such that $\psi(Y) = \varphi(PY)$. In terms of the matrices that means that $M_\psi = P^t M_\varphi P$ for a nonsingular matrix P. Any quadratic form is equivalent to a diagonal one, that is one of the form

$$\psi(Y) = \sum_i a_i Y_i^2$$

, whose matrix diagonal with entries $a_1, \dots, a_n$. Therefore we will assume always that we are dealing with diagonal forms and we just denote $\varphi =< a_1, \dots, a_n >$. The form is called degenerated if its determinant is zero. Finally, if K is an ordered field the signature of $\varphi =< a_1, \dots, a_n >$ is defined as the difference between the number of positive elements minus the number of negative elements. If two forms φ and ξ are equivalent they have the same rank and signature.

Given $\varphi =< a_1, \dots, a_n >$ and $\psi =< b_1, \dots, b_m >$, We consider two operations. The *addition*

$$\varphi \perp \psi =< a_1, \dots, a_n, b_1, \dots, b_m >$$

which is defined on the vector space $K^n \times K^m$, and the *product*

$$\varphi \otimes \psi = a_1\psi \perp \cdots \perp a_n\psi$$

which is defined over the space $K^m \times \cdots \times K^m$ (n times).

A *Pfister* form φ is one of type

$$\varphi =< 1, f_1 > \otimes \cdots \otimes < 1, f_n >$$

and will be represented in sort as $\varphi =\ll f_1, \dots, f_n \gg$. Notice that it has dimension 2^n and that $\varphi =< 1 >\perp \varphi'$ for a $2^n - 1$ dimensional form φ' which is called the *pure part* of φ. If we denote by $\varphi_i =\ll f_1, \dots, f_i \gg$ we have

$$\varphi = \varphi_n = \varphi_{n-1} \otimes < 1, a_n >= \cdots =< 1 >\perp < a_1 >\perp a_2\varphi_1 \perp \cdots \perp a_n\varphi_{n=1}$$

Let us illustrate the relationship between forms and semialgebraic sets with an example. Consider $V = \mathbb{R}^d$ and let $K = \mathbb{R}(X_1, \dots, X_d)$ be the field of fractions of the polynomial ring. Let $\varphi =< f_1, \dots, f_n >$ be a form over K, i.e. $f_i = g_i/h_i$ with $g_i, h_i \in \mathbb{R}[X_1, \dots, X_d]$, and take a point $x \in \mathbb{R}^d$ at which $f_1, \dots, f_n$ can be evaluated. Then, specializing at x we get the form $\varphi_x =< f_1(x), \dots, f_n(x)$, now defined over $\mathbb{R}$. Then it is easy to see that the set of points x at which the form φ_x has a prescribed rank and signature is a semialgebraic subset of $\mathbb{R}^d$ described as boolean combination of equalities and inequalities of the polynomials g_i and h_i. For instance the set S of points such that the form $\varphi =< 1 - x^2 - y^2, x, y >$ has rank 2 and signature 1 is the set

$$S = \{x \in \mathbb{R}^2 \mid 1-x^2-y^2 > 0, xy < 0, \} \cup \{x \in \mathbb{R}^2 \mid 1-x^2-y^2 < 0, x > 0, y > 0\}.$$

Conversely, given a semialgebraic set we can construct a form which defines it in terms of rank and signature. The most important example is the following.

Let $S = \{f_1 > 0, \dots, f_n > 0\} \subset \mathbb{R}^d$ be a basic open semialgebraic set, with $f_i \in A$. Consider the Pfister form

$$\varphi =\ll f_1, \dots, f_n \gg := < 1, f_1 > \otimes \cdots \otimes < 1, f_n >$$

For any $x \in \mathbb{R}^d$ with $f_1(x) \cdots f_n(x) \neq 0$ we have

$$\text{signature}(\varphi(x)) = \begin{cases} 2^n & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$$

Therefore

$$S = \{x \in \mathbb{R}^d \mid \text{rank}(\varphi(x) = \text{signature}(\varphi(x)) = 2^n\}$$

so that the form φ acts as a characteristic function for the set S up to the set of zeros of the f_i's.

A word of warning is needed in this relationship between forms and sets: let $\varphi =< f_1, \dots, f_n >$ be as above and set $S_\varphi = \{x \in \mathbb{R}^d \mid \text{rank}(\varphi(x)) = r, \text{signature}(\varphi(x)) = s\}$. Assume that $\varphi \equiv \psi =< g_1, \dots, g_n >$ (as forms over K). Then there is a nonsingular $n \times n$ matrix P with entries in K such that $M_\varphi = P^t M_\psi P$. Thus, for any point $x \in \mathbb{R}^d$ outside the set H of zeros of the denominators of the entries of φ, ψ, and P and the determinant of P, by specialization at x we get and equation of matrices over $\mathbb{R}$:

$$M_{\varphi_x} = (P_x)^t M_{\psi_x} P_x$$

which shows that $\varphi(x)$ and $\psi(x)$ have the same rank and signature. In particular S_φ and S_ψ coincide up to the hypersurface H. We denote this phenomena saying that S_φ and S_ψ are *generically equal*. More precisely, we say that two semialgebraic subsets $S, T \subset V$ are generically equal if there is a proper algebraic subset $H \subset V$ (hence of smaller dimension sinve we are assuming V irreducible) such that $S \backslash H = T \backslash H$. Therefore we may summarize the argument above by saying that *equivalent forms give rise to generically equal semialgebraic sets.*

In particular if $\ll f_1, \dots, f_n \gg \equiv \ll g_1, \dots, g_n \gg$ the sets $\{f_1 > 0, \dots, f_n > 0\}$ and $\{g_1 > 0 \dots, g_n\}$ are generically equal, and if $\ll f_1, \dots, f_n \gg \equiv \ll 1, g_2, \dots, g_n \gg$, the set $\{f_1 > 0, \dots, f_n > 0\}$ is generically equal to $\{g_2 > 0 \dots, g_n\}$, reducing in one the number of inequalities. To advance in this situation we need some algebra of forms.

4. A little algebra of forms

In this section we collect some properties of forms. Proofs of them can be found for instance in [Lm]. We start by tha follwing elementary but fundamental Witt's cancellation theorem

Theorem 4.1. *Assume that* $\varphi \perp \psi \equiv \varphi \perp \rho$. *Then* $\psi \equiv \rho$.

Definition 4.2. *An element* $b \in K^*$ *is represented by the form* $\rho =< a_1, \ldots, a_n >$ *if there is a vector* $v \in K^n$ *such that* $b = \rho(v) = \sum a_i v_i^2$. *We denote by* G_ρ *the set of elements represented by* ρ. *The form is called isotropic if* 0 *is represented by* ρ, *that is,* $0 = \sum a_i v_i^2$ *for some* $v \in K^n$, $v \neq 0$. *The form is called anisotropic if it is not isotropic.*

Obviuosly, if $b = \sum a_i v_i^2 \in G_\rho$ and $c \in K^*$ we get $bc^2 = \sum a_i(v_i c)^2 \in G_\rho$. Here is a list of elementary properties:

Proposition 4.3. a) *For any* $b \in K^*$ *we have that* $b \in G_\rho$ *iff* $\rho =< b > \perp \psi$ *for some* ψ *of dimension* $n-1$.
b) ρ *is isotropic iff there is some* a *such that* $\rho =< a, -a > \perp \psi$ *for some* ψ *of dimension* $n-2$.
c) *for any* $a \in K^*$ *we have* $< a, -a >=< 1, -1 >$. *In particular, any isotropic form is universal, i.e., represents all the elements of* K.
d) *If* $c \in K^*$ *is represented by* $< a, b >$ *then* $< a, b >= c < 1, ab >$
e) *In particular, if* $c \in K^*$ *is represented by* $< 1, a >$ *then* $< 1, a >= c < 1, a >$

Property *e)* is taken as motivation of the following important notion:

Definition 4.4. *A form* ρ *is called multiplicative if for any* $b \in G_\rho$, $\rho \equiv b\rho$.

In particular *e)* above shows that 2 dimensional Pfister forms are multiplicative. In fact we have the following fundamental property

Proposition 4.5. *Assume that* ρ *is multiplicative. Then* $\rho \otimes \ll 1, a \gg$ *is also multiplicative. In particular Pfister forms are multiplicative.*

As a consequence we get that if ρ is a Pfister form, G_ρ is a multiplicative subgroup of K^*. Indeed, the proposition shows that G_ρ is multiplicatively closed. On the other hand, $\frac{1}{\rho(v)} = \rho(\frac{v}{\rho(v)})$, showing that inverse of elements of G_ρ are again in G_ρ.

The rules to manipulate systems of inequalities mentioned in section 1 have a well translation in terms of Pfister forms. Indeed, notice that if $\varphi =\ll, f_1, \ldots, f_n \gg$ is a form over $\mathbb{R}(V)$ and $g \in \mathbb{R}[V]$ is represented by φ then $g \geq 0$ over the semialgebraic set $S = \{x \in V \mid f_1(x) \geq$

$0, \ldots, f_n(x) \geq 0\}$. The converse, however, is not true: if a polynomial $g \in \mathbb{R}[V]$ is nonnegative on S then by the positivstellensatz we get that g is represented by some multiple $\varphi \perp \cdots \perp \varphi$ of φ.

Proposition 4.6. (1) $\ll a, b \gg = \ll a+b, ab \gg$
(2) *Let φ be a Pfister form and $b \in G_\varphi$. Then $\varphi \otimes \ll a \gg = \varphi \otimes \ll ab \gg$.*

Proof. a) $\ll a, b \gg = < 1, a, b, ab > = < 1, a+b, (a+b)ab, ab > = \ll a+b, ab \gg$, where the first equality follows from *d)* above.

b) Since φ is multiplicative and b is represented by φ we have $\varphi \otimes \ll a \gg = \varphi \perp a\varphi = b\varphi \perp ab\varphi = \varphi \perp ab\varphi = \varphi \otimes \ll ab \gg$

□

Combining these two operations we get the translation of the system $(*)$ above to Pfister forms, cf. [BCR, Lemma 6.4.14]:

Proposition 4.7. *Assume that u_1 is a square in K^* and that for $i = 2, \ldots, n$, $u_i \in K$ is represented by the form $\varphi_{i-1} = \ll a_1, \ldots, a_{i-1} \gg$. For $i = 1, \ldots, n$, set $b_i = f_i u_i + f_{i+1} u_{i+1} + \cdots + f_n u_n$ and assume that these are nonzero, $b_i \in K^*$. Then*

$$\varphi = \ll a_1, \ldots, a_n \gg = \ll b_1, a_1 b_2 \ldots, a_{n-1} b_n \gg$$

Notice that an element of K^* has the shape of b_1 above if and only if it is represented by the pure form φ' of φ. Thus we have:

Corollary 4.8. *Let $\varphi = \ll a_1, a_2, \ldots, a_n \gg$ and assume that b is represented by the pure part φ' of φ. Then $\varphi = \ll b, c_2, \ldots, c_n \gg$ for some $c_2, \ldots, c_n \in K$.*

Thinking in terms of systems of inequalities we are saying that we can always replace our original system $f_1 > 0, \ldots, f_n > 0\}$ by a new one which generically equivalent to it that contains the inequality $g_1 > 0$ where g_1 is any polynomial represented by φ'. In particular, if 1 is represented by φ' we get an spurious inequality and therefore, after clearing denominators, a new system of inequalities, generically equivalent to the former, and with one less inequality. This is exactly the content of the following result.

5. d inequalities suffice

The following theorem due to Tsen and Lang is crucial for our purposes. For a proof see [BCR, chapter 6].

Theorem 5.1. *Let K be an extendion of transcendence degree d over an algebraically closed field F. Then any homogeneous polynomial of* *degre k in more than k^d variables has a non trivial zero. In particular any quadratic form over K of dimension greater than d is isotropic.*

Let us come back to our geometrical setting. Let $V \subset \mathbb{R}^m$ be a non-singular real algebraic variety of dimension d. In particular the function field $K = \mathbb{R}(V)$ has transcendence degree d over $\mathbb{R}$.

Theorem 5.2. *Given $\varphi = \ll f_1, \ldots, f_{d+1} \gg$ then $1 = \varphi'(x)$ for some vector $x \in K^{2^{d+1}-1}$.*

Proof. Assume that φ is anisotropic over K. By Tsen/Lang Theorem φ is isotropic over $K(i)$. Now, consider two new variables T_1, T_2 over K and consider the form φ over the field $F = K(i)(T_1, T_2)$. Obviously it is also isotropic, whence universal over this field. In particular the element $\beta = T_1 + iT_2$ is represented by some element $z \in F$. Since F has degree 2 over $K(T_1, T_2)$ we have that $F = K(T_1, T_2)[\beta]$, so that we write $\beta = u + iv$ with $u, v \in K(T_1, T_2)$. Thus, we have

$$\beta = \varphi(u + \beta v) = \varphi(u) + \beta^2\varphi(v) + 2\beta B_\varphi(u, v),$$

where B_φ stands for the symmetric bilinear form associated to φ. Thus we get the equation

$$\beta^2\varphi(v) + (2B_\varphi(u, v) - 1)\beta + \varphi(u) = 0.$$

On the other hand, the irreducible polynomial of β over $K(T_1, T_2)$ is

$$\beta^2 + 2T_1\beta + (T_1^2 + T_2^2) = 0.$$

so that we get

$$\varphi(u) = \varphi(v)(T_1^2 + T_2^2).$$

Thus we have

$$(T_1^2 + T_2^2) = \frac{\varphi(u)}{\varphi(v)}$$

and since G_φ is multiplicative we get that $T_1^2 + T_2^2$ is represented by φ. By the lemma above this means that the $\varphi = < 1, 1 > \perp \psi$ for some form ψ, and by Witt's cancellation theorem we get $\varphi' = < 1 > \perp \psi$, so that 1 is represented by φ' as claimed.

In case φ is isotropic we have $\varphi = < 1, -1 > \perp \psi$ and by the cancellation theorem we get that $\varphi' = < -1 > \perp \psi$ so that -1 is represented by φ'. By the proposition above we have that $\varphi = \ll -1, c_2, \ldots, c_n \gg = < 1, -1 > \otimes \ll c_2, \ldots, c_n \gg$. It follows that φ' contains the subform $< c_i, -c_i >$ for all i, and therefore φ' is isotropic. In particular it is universal and $1 \in G_{\varphi'}$ as claimed. □

Theorem 5.3. *Any basic open semialgebraic subset $S = \{f_1 > 0, \ldots, f_n > 0\} \subset V$ can be described generically by d inequalities.*

Remark 5.4 The above theorem holds not only generically but also as a true equality of sets. The proof is essentially the same, but working with forms over rings with many units, that is, rings in which any function with no zeros is a unit, as for instance the ring of regular functions on V. In this settings the arithmetic of Pfister forms behaves as in teh case of fields and proposition ?? also holds. Proofs are however more involved and technical and we have preferred to keep ourselves at the level of fields.

The following example shows that the bound of d inequalities is optimal:

Example 5.5 Set $V = \mathbb{R}^d$. The semialgebraic set $S = \{x_1 > 0, x_2 > 0, \dots, x_d > 0\}$ is not generically equal to any open semialgebraic set described with less than d inequalities. We proof this assertion by induction on d. Assume that $\{x_1 > 0, x_2 > 0\}$ is generically equal to $\{f > 0\}$, $f \in \mathbb{R}[x_1, x_2]$. Evaluating f in the lines $x_2 = tx_1$ with $t > 0$ we get that f has different sign at $x_1 = -\infty$ and $x_1 = \infty$, so that must be of odd degree. But along the lines $x_2 = tx_1$ with $t < 0$, f has equal sign at $x_1 = -\infty$ and $x_1 = \infty$, and so it has even degree, contradiction. Assume by induction the result for $d-1$, and suppose that $\{x_1 > 0, x_2 > 0, \dots, x_d > 0\}$ coincides generically with $\{f_1 > 0, \dots, f_{d-1} > 0\}$. For each $j = 1, \dots, d-1$ we have $f_j = x_d^{m_j} g_j$ for some exponent m_j (possibly zero). Since the hyperplane $\{x_d = 0\}$ is part of the boundary of S, not all these exponent can be even. Reorder the f_j's so that $f_1, f_2, \dots, f_{r-1}$ have exponent odd and $f_r, \dots, f_{d-1}$ have exponent even. Then, consider the set

$$S' = \{(x_1, \dots, x_{d-1}) \in \mathbb{R}^{d-1} \mid g_1(x_1, \dots, x_{n-1}, 0) g_2(x_1, \dots, x_{n-1}, 0) > 0, \dots, g_1(x_1, \dots, x_{n-1}, 0$$

It is easy to check that S' is generically equal to $\{x_1 > 0, \dots, x_{d-1} > 0\} \subset \mathbb{R}^{d-1}$, and by induction we get a contradiction.

6. Some considerations about the effectiveness

What can be said about the degrees of the shorter description? It is well known that they may increase (and in fact must): for instance, take the regular n-polygon described by the n linear inequalities of its sides. Then the product of all these linear factors must divide the product of the two equations of the shorter generic description. Thus one of them must have degree al least $n/2$. On the other hand, Buresi and Mahé [BuMa] proved that there exists an upper bound for the degrees of a generic minimal description in terms of the dimension d and degrees and number of the initial description.

His approach is to keep track as close as possible of the construction of the vector $z \in K^{2^n-1}$ that gives the representation of 1 by φ'. By a simple consideration on the degrees, that the solution $z \in K(i)^{2^{d+1}}$ to the equation

$$\sum_{\varepsilon=0,1} f_1^{\varepsilon_1} \cdots f_{d+1}^{\varepsilon_{d+1}} X^2_{(\varepsilon_1,\ldots,\varepsilon_{d+1}} = 0,$$

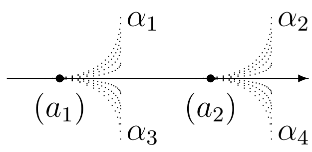
guaranteed by Tsen-Lang theorem can be taken in $A[i] = A \otimes_R R[i]$ of degree (i.e. in all its components) $\leq d\alpha$, where α is the sum of the degrees of the f_i's. The construction of vectors representing the product of two represented elements (i.e. the propoerty of Pfister forms of being multiplicative) can be done effectively by means of a Pfister construction, that allow to track up the polynomials appearing in the process, so that in the end we get

Theorem 6.1. *Given* $f_1, \ldots, f_{d+1}$ *as above, we can construct* $g_1, \ldots, g_d$ *such that* $\{f_1 > 0, \ldots, f_{d+1} > 0\}$ *and* $\{g_1 > 0, \ldots, g_d > 0\}$ *are generically equal and* $\deg(g_i) \leq (3^d + 1)d\alpha$

A. Bernig gave, in his diplomarbeit [Be], a constructive procedure for finding the two polynomials of the shortest description for open, convex, basic semialgebraic subsets of the plane $\mathbb{R}^2$. Very recently M. Groetschel and M. Henk have given an algorithm to find a shorter representation of polyhedra in $\mathbb{R}^d$ starting from a linear representation, but that is still far from being optimal (that is with d polynomials.

7. Fans and basicness

Let us pass to the question of deciding when a given semialgebraic set is basic. As pointed out above the main result in this direction is Bröcker's theorem, using fans. So, Let us try to understand what fans are. We start with an example. Consider a line H in $\mathbb{R}^2$ and two points $a_1, a_2 \in H$. Now, at each point take one of the two half branches of H at it, say ξ_1 at the point a_1 and ξ_2 at a_2. Again at each a_i consider the two ultrafilters α_k of open semialgebraic sets of $\mathbb{R}^2$ "adherent" to ξ_i, i.e., those whose elements contain ξ_i in its closure. This way we obtain four ultrafilters $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ with the property that we cannot separate one of them from the other three, that is, if a polynomial is positive in three of them, it is also positive in the fourth. Here, of course, we say that a $f \in \mathbb{R}[X, Y]$ is positive in the ultrafilter α if is positive in some $S \in \alpha$. In this situation we say that $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are a fan of $\mathbb{R}(X, Y)$. The following picture summarizes the situation:



Of course the same construction can be made in $\mathbb{R}^d$ starting with any algebraic hypersurface $H \subset \mathbb{R}^d$, two points $a_1, a_2 \in H$ and two ultrafilters ξ_1, ξ_2 of open semialgebraic subsets of H with a_1 and a_2 in its closure. Again the resulting $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are "inseparable" in the sense above and build a fan of $\mathbb{R}(X_1, \dots, X_d)$.

Let's make this definition more precise. By Artin-Lang Theorem, ultrafilters of open semialgebraic subsets of an irreducible algebraic set V correspond to orderings of the function field $R(V)$ of this set, and it is in this context of orderings of a field, where the original definition of fans takes place.

Definition 7.1. *We say that four orderings $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are a* 4-element fan *of K if any element of K which is positive in three of them is also positive in the fourth. Thinking in the orderings of K as mappings assigning a sign $+1$ or -1 to the elements of K, this property is stated by saying that the product of three of the α's is the fourth:*

$$\alpha_1\alpha_2\alpha_3 = \alpha_4.$$

More generally, a finite subset F of orderings of K are a *fan* if for any three $\alpha_1, \alpha_2, \alpha_3 \in F$ their product is also an ordering of F. Thus, 4-element fans are the simplest (non-trivial) examples of fans. Since we are interested in the geometric situation, the field K will always be the field $R(W)$ of rational functions of an algebraic subvariety $W \subset V$. In this situation we will say that the orderings (resp. fans) are supported in the subvariety W or simply, when no confusion is possible that they are orderings in W.

The set of all orderings in all possible subvarieties of V is the real spectrum $\mathrm{Spec}_r R[V]$ of the ring of polynomial functions on V, or simply the real spectrum of V, and will be also denoted by $\widetilde{V}$. In particular, $\widetilde{V}$ contains the orderings supported at the points $x \in V$ which are identified with the points, so that it contains V. Also it contains the orderings (ultrafilters) centered at infinity, so that $\widetilde{V}$ results a compactification of V. The set of all fans in all subvarieties of V constitutes the family of fans of $\mathrm{Spec}_r R[V]$. Notice in particular, that a fan F is always a set of orderings supported in the same subvariety $W \subset V$. Given any ordering $\alpha \in \widetilde{V}$ and a polynomial $f \in R[V]$ we say that

$\alpha \in \{f > 0\}$ if $f|_W$ is positive in α, and analogously for $f < 0$. In case f vanishes on W we say that $\alpha \in \{f = 0\}$. Translating this back to the geometric language, if we identify α with its corresponding ultrafilter, we are just saying that $\alpha \in \{f > 0\}$ if and only if $\{f > 0\}$ is in the ultrafilter α.

In general fans have remarkable combinatorial properties:

Proposition 7.2. *Let F be a fan in $\widetilde{V}$.*

a) *for any $f \in R[V]$ then $\{f > 0\} \cap F$ is either trivial (i.e. , or F) or contains exactly half of the elements of F.*
b) *$\#F = 2^k$ for some k.*
c) *If $S = \{f_1 > 0, \dots, f_n > 0\}$, then either $\#(F \cap S) \geq 2^{k-n}$ or $(F \cap S) = \emptyset$. In particular if there is a fan f with 2^n elements such that $\#(F \cap S) = 1$, then S cannot be written with less than n inequalities.*

Fans are a very important class of families of sets of orderings since they are like building stones for more general subspaces and also serve as a testing family for some properties like for instance the property of a semialgebraic set to be basic as we will see in the next section. Although combinatorially they are very simple, from the point of view of its geometric translation, arbitrary fans can be very weird. However, there is an easy way to produce a special type of fans in the spirit of the example quoted at the beginning of the section: take a flag of subvarieties $W_k \subset W_{k-1} \subset \cdots \subset W_1 \subset W_0 = V$, where $\dim W_j = d - j$, $1 \leq k \leq d-1$, and take two orderings (i.e. two ultrafilters) in W_k. Then pulling back these ultrafilters along the flag of subvarieties as in the example above,i.e., considering in W_j all the ultrafilters adherent to the ultrafilters in W_{j+1}, we get a fan with 2^{k+1} elements. In the example at the beginning of the section we have done this for the case $k = 1$ to produce 4-element fans. From the algebraic point of view we are considering a discrete valuation ring B of $R(V)$ of rank k and lifting through it two fixed orderings of its residue field by giving all possible signs to the regular parameters of B. With this valuation point of view is immediate to check that the set of orderings obtained is indeed a fan. We call such fans *geometric* because of its obvious geometric meaning. The nice fact is that *when the ground field is the field of real numbers $\mathbb{R}$, geometric fans are enough to check most properties of semialgebraic sets since they are dense in the set of all fans*, cf. [AnRz1].

8. Characterization of Basicness

Let's go back to the characterization of basic semialgebraic sets. We denote by $\partial_Z S$ the Zariski closure of the boundary, $\overline{S} \setminus S$, of S. A necessary condition for an open semialgebraic set to be basic is that $S \cap \partial_Z S = \emptyset$. Indeed, if $S = \{f_1 > 0, \dots, f_n > 0\}$ then $\partial_Z S \subset \{\prod f_i = 0\}$. Therefore we assume from now on that S verifies this topological condition, which in particular implies that it is open. We have the following celebrated theorem of Bröcker:

Theorem 8.1. [Br1]) *Assume that $S \cap \partial_Z S = \emptyset$. Then S is basic if and only if for any 4-elements fan F of $\widetilde{V}$, $\#(S \cap F) \neq 3$.*

One of the directions of the if and only if condition in the theorem is immediate from the properties of fans enumerated above: if S is basic its intersection with F must have cardinality $2^0 = 1$, $2^1 = 2$ or $2^2 = 4$. However the converse is not at all obvious although intuitively we are saying that if S fails to be basic this failure can be detected in $\widetilde{V}$ in the minimal possible spot where basicness is not trivial: 4-elements fans. Remark that in the Theorem we are considering all fans in all possible subvarieties W of V. If instead the statement holds only for fans in V, i.e., supported in the whole ambient variety V that is consisting of orderings in $R(V)$, we get that S is basic as a subset of the space of orderings of $R(V)$, or, again by applying Artin-Lang theorem, up to a subset of codimension 1. In this case we say that S is *generically basic*. Therefore Bröcker's theorem asserts that $S \subset V$ is basic if and only $S \cap W$ is generically basic for any subvariety $W \subset V$ (including V itself). Therefore, to check basicness we only need to check generic basicness although apparently we need to do it over an infinite family of subvarieties. However, only a finite family is required:

Proposition 8.2. *If S fails to be basic either it fails to be generically basic (in V) or $S \cap W$ is not basic for one of the irreducible components W of the set $\partial_Z S \cup \mathrm{Sing} X$.*

Proof. (Sketch) Assume that S is not basic but $S \cap W$ is basic for any irreducible component of $\partial_Z S \cup \mathrm{Sing} X$. In particular $S \cap Y$ is basic for any subvariety Y of this set. Thus, our assumption on S implies that there is a subvariety Y not contained in $\partial_Z S \cup \mathrm{Sing} X$ such that $S \cap Y$ is not generically basic, i.e. there is a fan $\overline{F}$ in Y such that $\#(S \cap \overline{F}) = 3$. But, since $Y \not\subset \mathrm{Sing}(X)$, the localization of $R[V]$ at the ideal of Y is a discrete valuation ring B, and we can pull back $\overline{F}$ to a 4-elements fan in V by assigning suitable signs to the regular parameters of B in such a way that $\#(S \cap F) = 3$. Therefore S is not generically basic. □

This proposition shows that the first (and main) step in proving basicness is to show generic basicness for V. Then we have to prove basicness for $S \cap W$ in the irreducible components W of $\partial_Z S \cup \operatorname{Sing} X$ which again yields to check generic basicness on $S \cap W$ and basicness on some smaller subvarieties. This way we produce a finite cascade of subvarieties so that generic basicness of S on them implies basicness of S.

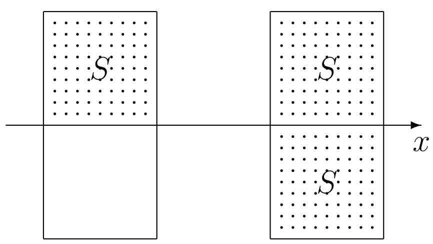
So, let us concentrate on showing generic basicness. Suppose now that the ground field is $\mathbb{R}$, the field of real numbers, what we assume from now on. Then, using the density of geometric fans, we get

Theorem 8.3. *S is generically basic if and only if $\#(S \cap F) \neq 3$ for any 4–elements geometric fan F in V.*

Now remember that geometric 4-elements fans appear attached to a hypersurface W and that, by the proposition above, after resolving singularities, W must be a component of $\partial_Z S$. Then, the condition that S contains only three of the elements of the fan means that W enters into S. We make this idea more precise: set $S^* = \operatorname{Int}(\overline{S})$. The set $\partial_Z S^* = \overline{(\overline{S^*} \setminus S^*)}^{Z}$ is called the generic Zariski boundary of S, and any irreducible component of codimension 1 of it is called a *wall* of S. We will say that $\partial_Z S^*$ *crosses* S if they intersect in a piece of dimension $d-1$, or equivalently if S contains some regular point of some wall. Then,

Theorem 8.4. (Universal obstruction to basicness, [AnRz2]) *S is generically basic if and only if in any birational model of V the generic Zariski boundary $\partial_Z S^*$ does not cross S.*

Here is the paradigmatic example of a non basic semialgebraic set. Consider the set S of the picture below. Notice that the 4-element fan constructed over the horizontal axis at the beginnng of section 5 has three elements in S and therefore it follows from Bröcker's theorem that S is not basic. On the other hand, the axis itself (that is, the subvariety to which the fan is attached) produces a wall of S that enters into S.



The above result shows that these is always the situation: an irreducible component of $\partial_Z S^*$ must enter into S although this component may appear only after blowing-up V. However this result can be improved yielding to a decision method to check basicness if we impose some extra condition on the walls of S.

9. Geometric decision method

Given a subvariety W of V. We define the *shadow* of S in W as $S_W := \overline{S} \cap W$. We assume that V is compact (which is not a restriction since any real algebraic variety can be easily compactified).

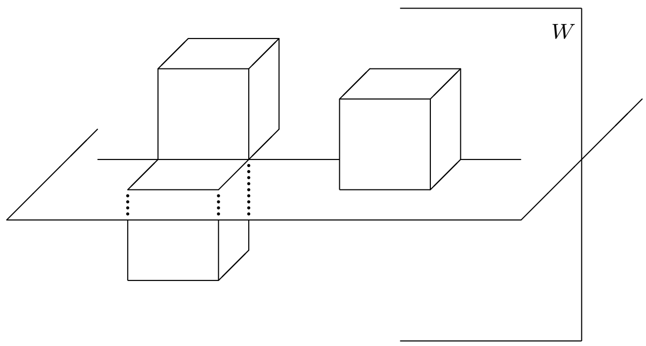
Theorem 9.1. ([AcBgVe]) *Assume that V is non-singular and that the Zariski boundary of S is at normal crossings. Then S is not generically basic if and only if there is a wall W of S such that either W crosses S or the shadow of S into W is not generically basic.*

Proof. (Sketch) If S is not generically basic there is a 4-elements geometric fan F in V such that $\#(F \cap S) \neq 3$. Let B be the rank 1 discrete valuation ring attached to F. If B is centered at a hypersurface in V then it must be a wall W of S and we get that W crosses S. If the center Z of B in V has codimension greater than 1 then it is contained in the intersection of at least two walls, say W_1, W_2, whose local equations $x_1 = 0$, $x_2 = 0$ are part of a system of parameters of Z. Then, we can lift the orderings induced by F in Z first to W_2 giving different signs to x_2 and then to W_1 keeping fix the sign of x_1 so that we get a 4-elements fan which can be read also in W_1 and produces an obstruction for the shadow of S in W_1 to be basic. □

The above result can be generalized to check s–basicness. A generically basic semialgebraic set S is called *generically s–basic* if it coincides, up to a proper algebraic set, with a set of the form $\{f_1 > 0, \ldots, f_s > 0\}$. Then we have:

Theorem 9.2. ([AcBgVe]) *Assume that M is non-singular and that the Zariski boundary of S is at normal crossings. Then S is generically s–basic if and only if for any wall N of S, the shadow S_N is $(s-1)$–basic.*

Example 9.3 In the following picture the union of any two cubes is a basic set, while the union of the three is not generically basic, since its shadow on the wall W is not generically basic, as one can easily see.



The simplest example for Theorem 3.5 is the n-ant $= \{x \in \mathbb{R}^n \mid x_1 > 0, \ldots, x_n > 0\} \subset \mathbb{R}^n$. The theorem shows that it is n-basic and for each wall $x_i = 0$ its shadow on it is a $n-1$-ant which is $n-1$ basic.

10. A concrete example

We finish this note with a concrete example: showing that the set S of quartic polynomials with no real roots is not basic, cf. [Dz]. So,

$$S = \{(a_1, \ldots, a_4) \mid t^4 + a_1 t^3 + \cdots + a_4 \text{ has no real root } \} \subset \mathbb{R}^4.$$

We will check that the intersection with the plane $H = \{a_1 = a_3 = 0\}$ of biquadratic polynomials is not basic. Indeed, the polynomial $t^4 + a_2 t^2 + a_4$ has no real root if and only if $s^2 + a_2 s + a_4$ has either no real roots or its two roots negative. So that $S \cap H = \{a_2^2 - 4a_4 < 0\} \cup \{a_2 > 0, a_4 > 0\}$. This set is not basic: consider the two half-branches ξ_1, ξ_2 of the parabola $a_2^2 - 4a_4 = 0$ at the origin, and let F be the 4-elements fan $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ of ultrafilters of $\mathbb{R}^2$ adherent to these half branches. We have that $\#(F \cap (S \cap H)) = 3$.

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