SMR1537/2

Summer School and Conference on
Real Algebraic Geometry and its Applications

(4 - 22 August 2003)

Topological properties of real algebraic varieties

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1 Basic notions

The subject of the course is centered around Hilbert's 16-th problem (see [16]). The first part of this problem contains several questions concerning the topology of real algebraic varieties, and more precisely, the topology of plane algebraic curves and spatial algebraic surfaces. We will discuss only a few topics belonging to topology of real algebraic varieties. An extensive information on the subject can be found in the surveys [7, 68, 74, 12].

1.1 Definitions

Let us start with definitions. A *real algebraic curve* of degree m in the real projective plane $\mathbb{R}P^2$ is a real homogeneous polynomial of degree m in three variables considered up to a constant factor. Let A be such a polynomial. Then, the set of points defined by the equation $A(x_0, x_1, x_2) = 0$ in the real projective plane $\mathbb{R}P^2$ is called the *real point set* of the curve A and is denoted by $\mathbb{R}A$.

In a similar way one defines a *real algebraic hypersurface* of degree m in a real projective space $\mathbb{R}P^n$ as a real homogeneous polynomial of degree m in $n+1$ variables considered up to a constant factor. We consider only *nonsingular* hypersurfaces, which means that corresponding polynomials do not have critical points in $\mathbb{C}^{n+1}\setminus\{0\}$.

The real point set $\mathbb{R}A$ of a nonsingular plane projective real algebraic curve A is a union of disjoint circles smoothly embedded in $\mathbb{R}P^2$. A circle can be positioned in $\mathbb{R}P^2$ either one-sidedly or two-sidedly. A two-sidedly positioned circle is called an *oval*. An oval divides $\mathbb{R}P^2$ into two parts. The part homeomorphic to a disk is called the *interior* of the oval. All the connected components of the real point set of a nonsingular curve of an even degree in $\mathbb{R}P^2$ are ovals. The real point set of a nonsingular curve of an odd degree in $\mathbb{R}P^2$ contains exactly one one-sidedly positioned connected component. The topological type of the pair $(\mathbb{R}P^2, \mathbb{R}A)$ is

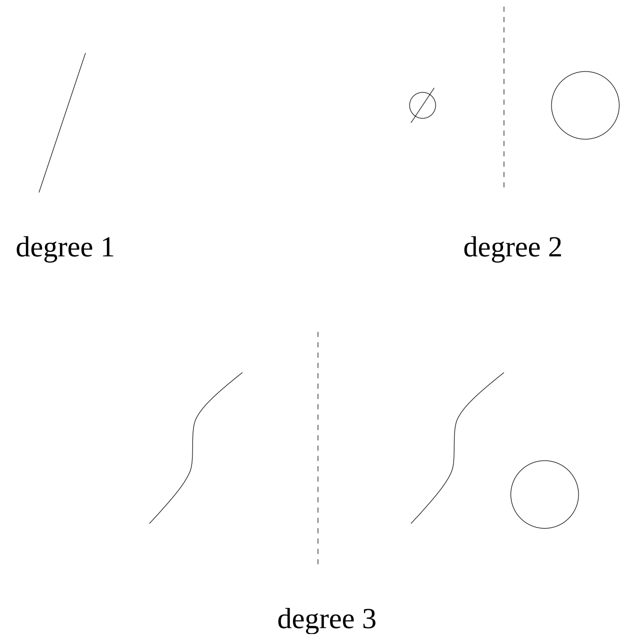

Figure 1: Real schemes of nonsingular curves of degrees 1, 2 and 3 in $\mathbb{R}P^2$

defined by the scheme of disposition of the connected components of $\mathbb{R}A$. This scheme is called the *real scheme* of the curve A.

1.2 Harnack theorem

Topology of real algebraic varieties, as a separated field, was founded in the XIX-th century in works of A. Harnack and F. Klein. In 1876 A. Harnack [14] formulated the problem of topological classification of nonsingular curves of a given degree in $\mathbb{R}P^2$: *given a positive integer m, describe the real point sets of the nonsingular curves of degree m in $\mathbb{R}P^2$ up to homeomorphism.* (Note that the topological type of $\mathbb{R}A$, where A is a nonsingular algebraic curve in $\mathbb{R}P^2$ is determined by the number of connected components of $\mathbb{R}A$.) In the same paper Harnack gave a complete answer to this problem.

Theorem 1.1 (Harnack theorem, [14]) *Let m be a positive integer. Then*

- *the number of connected components of the real point set of a curve of degree m in $\mathbb{R}P^2$ is at most $\frac{(m-1)(m-2)}{2}+1$;*
- *if m is even (resp., odd), then for any integer l verifying $0 \le l \le \frac{(m-1)(m-2)}{2}+1$ (resp., $1 \le l \le \frac{(m-1)(m-2)}{2}+1$) there exists a nonsingular curve of degree m in $\mathbb{R}P^2$ whose real point set has exactly l connected components.*

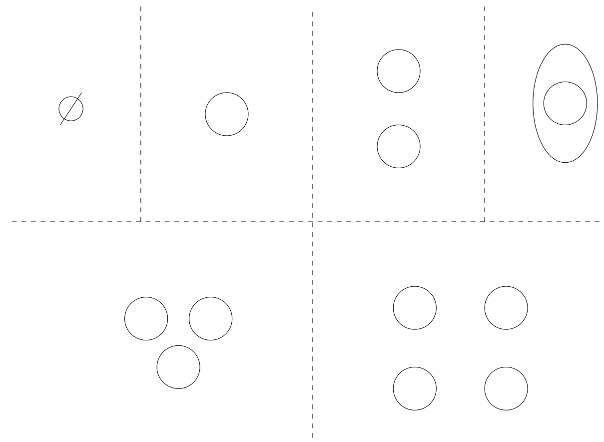

Figure 2: Real schemes of nonsingular curves of degree 4 in $\mathbb{R}P^2$

The first statement in the Harnack theorem is called the *Harnack inequality.* The nonsingular curves of degree m in $\mathbb{R}P^2$ whose real point set has $\frac{(m-1)(m-2)}{2}+1$ connected components are called *maximal* (or *M-curves*). We discuss the proof of the Harnack theorem in Section 2 (for the Harnack inequality) and in Section 3 (for the second statement).

The next natural question is to describe the real schemes of nonsingular curves of a given degree in $\mathbb{R}P^2$. This question was included in 1900 by D. Hilbert in the 16-th problem (see [16]). For small degrees (less than or equal to 5) the answer is shown in Figures 1, 2 and 3. This answer was known to Hilbert and can be obtained using the Harnack inequality and the fact that the number of intersection points of a curve of degree m and a line is at most m (if this number of intersection points is finite). The latter statement is a particular case of the Bézout theorem.

To prove that any real scheme of Figures 1-3 is realizable by a curve of the corresponding degree one can take an appropriate collection of lines and ellipses intersecting transversally and slightly perturb their union (*i.e.*, slightly change the coefficients of the product of the polynomials of degrees 1 and 2 defining these lines and ellipses) in order to obtain the desired picture.

For example, to obtain a maximal curve of degree 4 in $\mathbb{R}P^2$, we can take two ellipses intersecting in 4 points and consider a polynomial $E_1 \cdot E_2 + \varepsilon(x_0^4 + x_1^4 + x_2^4)$, where E_1 and E_2 are polynomials defining two ellipses, and ε is a sufficiently small positive number. If the signs of E_1 and E_2 are appropriately chosen, then this polynomial defines a curve whose real point set consists of 4 ovals (see Figure 4).

A classification of real schemes of nonsingular curves of degree 6 in $\mathbb{R}P^2$ is more difficult to obtain. This classification was completed by D. A. Gudkov (see [13]) in

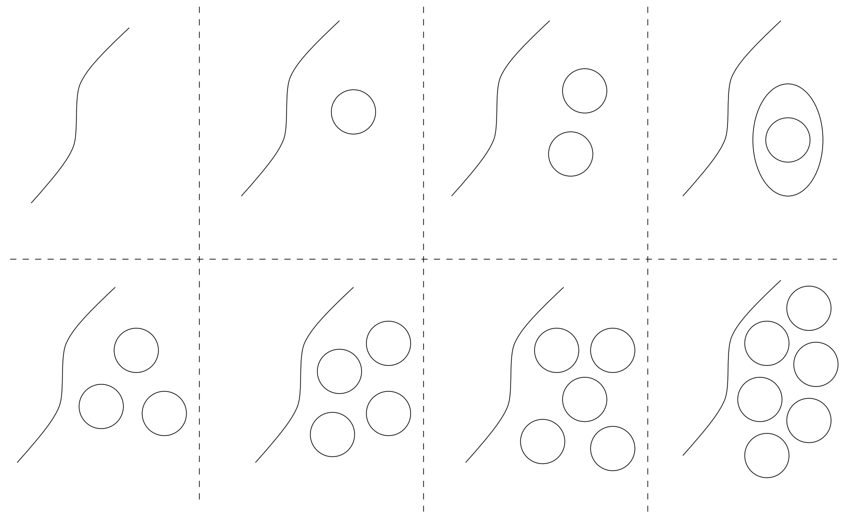

Figure 3: Real schemes of nonsingular curves of degree 5 in $\mathbb{R}P^2$

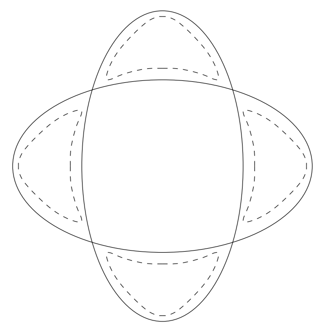

Figure 4: Maximal curve of degree 4 in $\mathbb{R}P^2$

1969. Only 3 real schemes are realizable by maximal curves of degree 6 in $\mathbb{R}P^2$. These schemes are shown in Figure 5.

In 1979 O. Viro [64] obtained a classification of real schemes of nonsingular curves of degree 7 in $\mathbb{R}P^2$. Despite of the efforts to classify the real schemes of nonsingular curves of degree 8 (see, for example, [6, 33, 43, 47, 48, 54, 55, 56, 64, 66]), the classification is still not completed.

A question similar to the first part of Hilbert's 16-th problem concerns a classification of real schemes realizable by real pseudo-holomorphic curves of a given degree. A Riemann surface M embedded in $\mathbb{C}P^2$ is a *real pseudo-holomorphic curve*, if it is a J-holomorphic curve in some tame almost complex structure J on $\mathbb{C}P^2$ (see [10]) such that $\mathrm{Conj}_* \circ J = J^{-1} \circ \mathrm{Conj}_*$ and $\mathrm{Conj}(M) = M$, where $\mathrm{Conj} : \mathbb{C}P^2 \to \mathbb{C}P^2$ is the involution of complex conjugation. The fixed point set $\mathbb{R}M \subset \mathbb{R}P^2$ of Conj restricted to a real pseudo-holomorphic curve M is called the *real point set* of M.

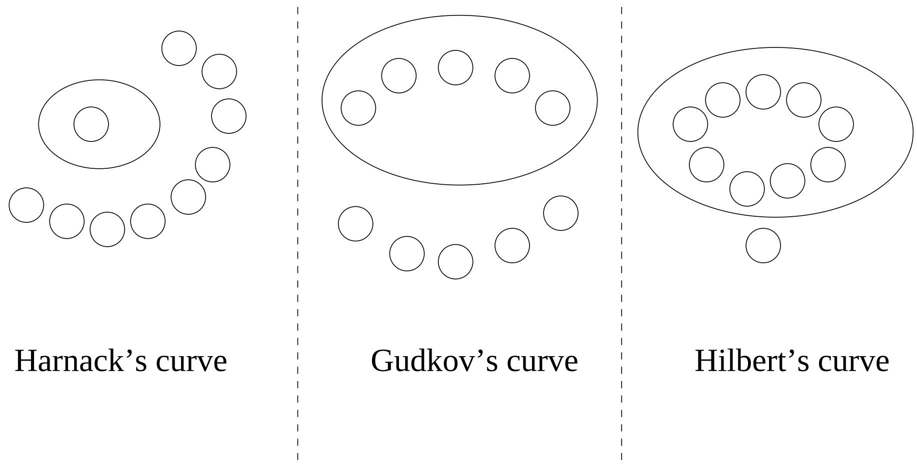

Figure 5: Real schemes of maximal curves of degree 6 in $\mathbb{R}P^2$

A real pseudo-holomorphic curve M in $\mathbb{C}P^2$ is *of degree* m if M realizes the class $m[\mathbb{C}P^1] \in H_2(\mathbb{C}P^2)$. It is interesting to compare isotopy classifications of real point sets of real algebraic curves and of real pseudo-holomorphic curves of the same degree. The classifications coincide for the curves of degree ≤ 7. Recently, S. Orevkov [44] obtained an isotopy classification of the real point sets of maximal real pseudo-holomorphic curves of degree 8 (exactly as in the algebraic case, a real pseudo-holomorphic curve of degree m is called *maximal* if its real point set has $\frac{(m-1)(m-2)}{2} + 1$ connected components). As it was already mentioned above, the corresponding classification of real algebraic curves is not yet known.

The question on a topological classification (*i.e.*, a classification of the real point sets up to homeomorphism) of nonsingular surfaces of a given degree in $\mathbb{R}P^3$ was also included by Hilbert in the first part of the 16-th problem. At the end of the XIX-th century the answer was known for surfaces of degrees ≤ 3. The topological classification of surfaces of degree 4 in $\mathbb{R}P^3$ was completed by V. Kharlamov [27] in 1976. Kharlamov also obtained finer classifications of surfaces of degree 4 (see [28] and [29]). A topological classification of surfaces of degree ≥ 5 in $\mathbb{R}P^3$ is unknown.

Several last decades were the years of an intensive progress in topology of real algebraic varieties. Many important restrictions on the topology of real algebraic varieties were obtained using the methods of modern topology of manifolds and complex algebraic geometry (see [1, 51, 52, 53, 23, 24, 25, 26, 8, 42], and also the surveys [12, 74, 2, 68, 7]). Some of the restrictions on the topology of real algebraic curves are presented in Section 2.

The above mentioned progress in topology of real algebraic varieties concerns also the methods of constructions. We discuss them in Section 3.

Exercises

1. Using the Harnack inequality and the Bézout theorem show that all the real schemes of nonsingular curves in $\mathbb{R}P^2$ of degrees less than or equal to 5 are among the schemes represented in Figures 1, 2, and 3.

2. Construct an M-curve of degree 5 in $\mathbb{R}P^2$.

3. Construct a nonsingular surface A of degree 3 in $\mathbb{R}P^3$ such that the real point set of A has two connected components.

2 Restrictions on the topology of algebraic curves in $\mathbb{R}P^2$

2.1 Space of curves

A generic homogeneous polynomial of degree m in three variables has $\frac{(m+1)(m+2)}{2}$ coefficients. Since a curve of degree m in $\mathbb{R}P^2$ is a real homogeneous polynomial of degree m in three variables considered up to a constant factor, the space $\mathbb{R}\mathcal{C}_m$ of all (not necessarily nonsingular) curves of degree m in $\mathbb{R}P^2$ is naturally identified with a real projectif space $\mathbb{R}P^N$, where $N = \frac{(m+1)(m+2)}{2} - 1 = \frac{m(m+3)}{2}$. Denote by $\mathcal{D}$ the *discriminant* in $\mathbb{R}\mathcal{C}_m$, *i.e.*, the subset of $\mathbb{R}\mathcal{C}_m$ formed by the points corresponding to singular curves. Two nonsingular curves of degree m in $\mathbb{R}P^2$ such that the corresponding points in $\mathbb{R}\mathcal{C}_m$ belong to the same connected component of $\mathbb{R}\mathcal{C}_m \setminus \mathcal{D}$ are called *rigidly isotopic*. Note that rigidly isotopic curves have the same real scheme. One can ask for a classification of nonsingular curves of a given degree in $\mathbb{R}P^2$ up to rigid isotopy. The answer is known only for degrees less than or equal to 6 (for a rigid isotopy classification of nonsingular curves of degree 6 in $\mathbb{R}P^2$ see [40]).

Pick a point $(x_0 : x_1 : x_2)$ in $\mathbb{R}P^2$. The fact that a real homogenous polynomial of degree m in three variables defines a curve passing through the point $(x_0 : x_1 : x_2)$ imposes a linear condition on the coefficients of the polynomial. Thus, the curves of degree m in $\mathbb{R}P^2$ passing through $(x_0 : x_1 : x_2)$ form a hyperplane in $\mathbb{R}\mathcal{C}_m$. Hence, for any $\frac{m(m+3)}{2}$ points in $\mathbb{R}P^2$ there exists a curve of degree m which passes through these points. In addition, if $\frac{m(m+3)}{2}$ chosen points in $\mathbb{R}P^2$ are in a general position, there exists a unique curve of degree m passing through these points.

2.2 Proof of the Harnack inequality

We present, first, original Harnack's proof.

Proof of the Harnack inequality. Consider a nonsingular curve A of degree m in $\mathbb{R}P^2$, and suppose that the number of connected components of $\mathbb{R}A$ is greater than or equal to $\frac{(m-1)(m-2)}{2}+2$. Choose $\frac{(m-1)(m-2)}{2}+1$ ovals among the connected components of $\mathbb{R}A$. Pick one point on each of the chosen ovals, and pick $m-3$ points on some connected component of $\mathbb{R}A$ different from the chosen ovals. There exists a curve B of degree $m-2$ in $\mathbb{R}P^2$ which passes through these $\frac{(m-2)(m+1)}{2}$ points. The number of the intersection points of A and B is at least $2(\frac{(m-1)(m-2)}{2}+1)+m-3 = m(m-2)+1$ which contradicts to the Bézout theorem (note that A is nonsingular, and hence, irreducible). $\square$

Another proof of the Harnack inequality was given in the same year 1876 by F. Klein (see [30]). Klein's proof uses the *complex point set* of a real curve.

If A is a real homogeneous polynomial in three variables, then the set of points defined by the equation $A(x_0, x_1, x_2) = 0$ in the complex projective plane $\mathbb{C}P^2$ is called the *complex point set* of the curve A and is denoted by $\mathbb{C}A$. If A is a nonsingular curve of degree m, the complex point set $\mathbb{C}A$ of A is a topological compact connected orientable (in fact, naturally oriented) surface of genus $g = \frac{(m-1)(m-2)}{2}$, *i.e.*, is homeomorphic to a sphere with $\frac{(m-1)(m-2)}{2}$ handles.

Klein's proof of the Harnack inequality. Denote by l the number of connected components of $\mathbb{R}A$. Let $\widetilde{\mathbb{C}A}$ be the quotient of $\mathbb{C}A$ under the involution of complex conjugation. Notice that $\widetilde{\mathbb{C}A}$ is a connected surface whose boundary consists of l circles. Glue a disc to $\widetilde{\mathbb{C}A}$ along each of the boundary components, and denote the resulting surface by S. The Euler characteristic $\chi(S)$ of S is equal to $\frac{1}{2}\chi(\mathbb{C}A) + l = (1 - \frac{(m-1)(m-2)}{2}) + l$, where $\chi(\mathbb{C}A) = 2 - 2\frac{(m-1)(m-2)}{2}$ is the Euler characteristic of $\mathbb{C}A$. Since S is a connected surface, its Euler characteristic is at most 2. Thus, $l \leq \frac{(m-1)(m-2)}{2}+1$. $\square$

2.3 Further restrictions

We mention here several general restrictions on the topology of real algebraic curves (an extensive list of restrictions can be found, for example, in the surveys [7, 68]). To formulate the restrictions introduce additional definitions and notations. An oval of a nonsingular curve of an even degree in $\mathbb{R}P^2$ is called *even* (resp. *odd*), if it lies inside of even (resp. odd) number of other ovals of the curve. Denote by p (resp., n) the number of even (resp., odd) ovals of a nonsingular curve of an even degree in $\mathbb{R}P^2$.

Theorem 2.1 (Petrovsky inequalities, see [45, 46]) *For a nonsingular curve of degree $2k$ in $\mathbb{R}P^2$, on has the inequalities*

$$p - n \leq \frac{3k(k-1)}{2} + 1, \quad n - p \leq \frac{3k(k-1)}{2}.$$

A nonsingular algebraic curve A in $\mathbb{R}P^2$ is said to be of *type I* if its real point set $\mathbb{R}A$ divides the complex point set $\mathbb{C}A$ of A into two parts; otherwise, the curve is of *type II*. This division of curves into types is due to Klein [30].

A nonsingular curve A of degree m in $\mathbb{R}P^2$ is called an $(M-i)$-curve if the real point set $\mathbb{R}A$ of A has $\frac{(m-1)(m-2)}{2} + 1 - i$ connected components.

Some restrictions on the topology of real algebraic curves have the form of congruences. We mention three of them. The first one was proved by F. Klein (see [30]).

Theorem 2.2 (Klein congruence, see [30]) *Let A be a nonsingular curve of type I in $\mathbb{R}P^2$. If A is an $(M-i)$-curve, then $i \equiv 0 \mod 2$.*

Proof. Let us consider again the quotient $\widetilde{\mathbb{C}A}$ of $\mathbb{C}A$ under the involution of complex conjugation (as in Klein's proof of the Harnack inequality). Since A is of type I, the quotient $\widetilde{\mathbb{C}A}$ is an orientable surface. Assume that A is an $(M-i)$-curve. Then the boundary of $\widetilde{\mathbb{C}A}$ consists of $\frac{(m-1)(m-2)}{2} + 1 - i$ circles. Glue a disc to $\widetilde{\mathbb{C}A}$ along each of the boundary components to obtain a surface S. The Euler characteristic $\chi(S)$ of S is equal to $(1 - \frac{(m-1)(m-2)}{2}) + (\frac{(m-1)(m-2)}{2} + 1 - i) = 2 - i$. Since S is a compact orientable surface without boundary, its Euler characteristic is even. Thus, $i \equiv 0 \mod 2$. □

The next congruence we would like to mention was proved by V. I. Arnold [1] in 1971. The paper [1] opened a new period in the development of topology of real algebraic varieties.

Theorem 2.3 (Arnold congruence, see [1]) *For a nonsingular curve of degree $2k$ and of type I in $\mathbb{R}P^2$, on has the congruence*

$$p - n \equiv k^2 \mod 4.$$

The last congruence we mention is the Gudkov-Rokhlin congruence. It was conjectured by D. A. Gudkov [11] on the base of the classification of curves of degree 6 and the available examples of curves of higher degrees, and was proved by V. A. Rokhlin [51].

Theorem 2.4 (Gudkov-Rokhlin congruence, see [51]) *For an M-curve of degree $2k$ in $\mathbb{R}P^2$, on has the congruence*

$$p - n \equiv k^2 \mod 8.$$

Note that the Gudkov-Rokhlin congruence and the Bézout theorem show, in particular, that an M-curve of degree 6 in $\mathbb{R}P^2$ should have one of the real schemes shown in Figure 5.

A lot of general restrictions on the topology of real algebraic varieties (in particular, generalizations to higher dimensions of theorems presented above; one can mention, for example, the generalized Harnack inequality and its extremal properties) were obtained by I. G. Petrovsky and O. A. Oleinik [41], V. I. Arnold [1], V. A. Rokhlin [51, 52] and V. Kharlamov [25, 26, 27].

We finish the section by a presentation of a restriction which, at the moment, is not generalized to higher dimensions: the Rokhlin-Mishachev formulas for complex orientations.

For a curve A of type I, the natural complex orientations of two halves of $\mathbb{C}A \setminus \mathbb{R}A$ induce on $\mathbb{R}A$ two opposite orientations which are called *complex orientations*. They were introduced by V. A. Rokhlin in [52].

Let A be a nonsingular algebraic curve in $\mathbb{R}P^2$. A pair of ovals of $\mathbb{R}A$ is *injective* if one of them is inside of the other one. A collection of ovals is called a *nest* if any two of them form an injective pair. The number of ovals in the nest is called the *depth* of the nest. Assume that A is of type I. An injective pair of ovals of $\mathbb{R}A$ is *positive* (resp., *negative*) if the complex orientations of the ovals are induced (resp., are not induced) from some orientation of the annulus bounded by the ovals. Suppose that the degree of A is odd. Pick an oval of $\mathbb{R}A$, and consider the Möbius band which is the complement in $\mathbb{R}P^2$ of the interior of the oval. The oval is called *positive* (resp., *negative*) if the integer homology class realized in the Möbius band by the oval equipped with a complex orientation differs in sign (resp., coincides) with the class of the doubled one-sidedly positioned connected component.

Theorem 2.5 (Rokhlin-Mishachev formulas, see [52, 39, 53]) *Let k be a positive integer. Then*

- *for a nonsingular curve of degree $2k$ and of type I in $\mathbb{R}P^2$, on has the formula*

$$2(\Pi^+ - \Pi^-) = l - k^2,$$

where l is the number of ovals of $\mathbb{R}A$, and Π^+ and Π^- are the numbers of positive and negative injective pairs in $\mathbb{R}A$, respectively;

- *for a nonsingular curve of degree $2k-1$ and of type I in $\mathbb{R}P^2$, one has the formula*

$$2(\Pi^+ - \Pi^-) + \Lambda^+ - \Lambda^- = l - k(k-1),$$

where Λ^+ and Λ^- are the numbers of positive and negative ovals in $\mathbb{R}A$, respectively.

Exercises

1. Let A be a nonsingular curve of degree m in $\mathbb{R}P^2$. Prove that

- the depth of any nest in $\mathbb{R}A$ does not exceed $[m/2]$,
- the sum of the depths of any two disjoint nests in $\mathbb{R}A$ does not exceed $[m/2]$,
- the sum of the depths of any five disjoint nests in $\mathbb{R}A$ does not exceed m if no oval of one nest contains inside all the ovals of the other four nests.

2. Prove that an M-curve is always of type I.
3. Deduce the Arnold congruence from the Rokhlin-Mishachev formulas.

3 Combinatorial patchworking

Until the latest 1970-s, all the constructions of real algebraic curves were based on the method of small perturbations of a product of curves which have smaller degrees and intersect transversally. One can mention here the classical methods of construction proposed by A. Harnack [14], D. Hilbert [15], L. Brusotti [4], and A. Wiman [75]. To construct several curves of degree 6 in $\mathbb{R}P^2$ Gudkov (see [13]) extended a little bit the classical scheme of construction of curves: he combined the method of small perturbations with the use of quadratic transformations of $\mathbb{R}P^2$.

In 1979, O. Viro proposed a principally new method of construction of real algebraic varieties (see [64, 65, 67, 69, 70] and also [50]). It provides a nice interaction of real algebraic geometry, toric geometry and combinatorics, and gives rise to various generalizations and applications [3, 6, 20, 17, 21, 22, 43, 57, 58, 59, 60, 62]. Almost all the constructions in topology of real algebraic varieties since 1979 use the Viro method.

We discuss here a particular case of the Viro method: the *combinatorial patchworking*. It is a powerful construction which gives a possibility to build real algebraic hypersurfaces in a simple combinatorial fashion: one can patchwork them from pieces which essentially are hyperplanes. One of the applications of the combinatorial patchworking is a construction of counter-examples (see [17, 18, 21]) to a conjecture formulated by V. Ragsdale [49] in 1906.

Let m be a positive integer number, and T the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(m,0)$, and $(0,m)$. Take a triangulation τ of T with integer vertices (*i.e.*, vertices having integer coordinates). Suppose that a distribution of signs at the vertices of τ is given. The sign (plus or minus) at the vertex with coordinates (i,j) is denoted by $\sigma_{i,j}$.

Take the copies

$$T_x = s_x(T), \;\; T_y = s_y(T), \;\; T_{xy} = s_x \circ s_y(T)$$

of T, where s_x, s_y are reflections with respect to the coordinate axes. Denote by T_* the square $T \cup T_x \cup T_y \cup T_{xy}$. Extend the triangulation τ to a symmetric triangulation of T_*, and the distribution of signs $\sigma_{i,j}$ to a distribution at the vertices of the extended triangulation by the following rule: passing from a vertex to its mirror image with respect to a coordinate axis we preserve its sign if the distance from the vertex to the axis is even, and change the sign if the distance is odd.

If a triangle of the triangulation of T_* has vertices of different signs, select a midline joining the middle points of the edges having endpoints of opposite signs. Denote by L the union of the selected segments. It is a piecewise-linear curve contained in T_*. Glue by $s_x \circ s_y$ the opposite sides of T_*. The resulting quotient space $\tilde{T}$ is homeomorphic to $\mathbb{R}P^2$. Denote by $\tilde{L}$ the image of L in $\tilde{T}$.

Let us introduce an additional assumption: the triangulation τ of T is *convex*. This means that there exists a convex piecewise-linear function $\nu : T \longrightarrow \mathbb{R}$ whose domains of linearity coincide with the triangles of τ. Sometimes, such triangulations are also called coherent (see [9]) or regular (see [76]).

Theorem 3.1 (O. Viro, see, for example, [70]) *If the triangulation τ of T is convex, then there exist a nonsingular real algebraic curve A of degree m in $\mathbb{R}P^2$ and a homeomorphism $\mathbb{R}P^2 \to \tilde{T}$ mapping the set of real points $\mathbb{R}A$ of A onto $\tilde{L}$.*

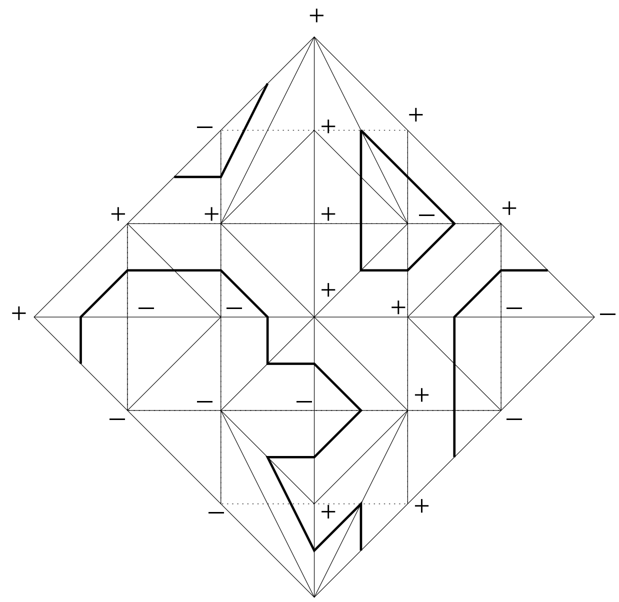

Figure 6: Combinatorial patchworking of a curve of degree 3 in $\mathbb{R}P^2$

The described construction is called the *combinatorial patchworking*. An example of a combinatorial patchworking of a curve of degree 3 in $\mathbb{R}P^2$ is shown in Figure 6.

A curve A obtained by the combinatorial patchworking is called a *T-curve*. A polynomial representing A can be written down explicitly: if t is positive and sufficiently small, the polynomial $\sum_{(i,j)\in V} \sigma_{i,j} x_0^i x_1^j x_2^{m-i-j} t^{\nu(i,j)}$ (where V is the set of vertices of τ, and ν is a convex function certifying the convexity of τ) defines a curve with the properties described in Theorem 3.1.

The combinatorial patchworking can be naturally generalized to higher dimensions. The construction starts with the simplex T in $\mathbb{R}^n$ having the vertices $(0,0,0,\ldots,0,0)$, $(m,0,0,\ldots,0,0)$, $(0,m,0,\ldots,0,0)$, $\ldots$, $(0,0,0,\ldots,0,m)$. One chooses a convex triangulation τ of T such that all the vertices of τ have integer coordinates, and a distribution of signs $\sigma : V \to \{+,-\}$ at the set V of vertices of τ. Take the union T_* of 2^n symmetric copies of T under the compositions of reflections with respect to coordinate hyperplanes. Extend τ to a symmetric triangulation τ_* of T_*, and σ to a distribution of signs at the vertices of τ_* using the same rule as above (replacing the words *axis* by the words *hyperplane*). If an n-simplex ξ of τ_* has vertices of different signs, select the convex hull of the middles of edges of ξ which have vertices of opposite signs. Let L be the union of all the selected hyperplane pieces, $\tilde{T}$ the quotient space (homeomorphic to $\mathbb{R}P^n$) of T_* under the identification of the points on the boundary which are symmetric with respect to the origin, and $\tilde{L}$ the image of L in $\tilde{T}$.

Theorem 3.2 (O. Viro, see, for example, [70]) *If the triangulation τ of T is convex, then there exist a nonsingular hypersurface A of degree m in $\mathbb{R}P^n$ and a homeomorphism $\mathbb{R}P^n \to \tilde{T}$ mapping the set of real points $\mathbb{R}A$ of A onto $\tilde{L}$.*

Such a hypersurface A in $\mathbb{R}P^n$ is called a T-*hypersurface.*

3.1 Construction of Harnack's curves

Using the combinatorial patchworking one can easily construct maximal curves of any degree in $\mathbb{R}P^2$. We present here a combinatorial patchworking of maximal curves whose real schemes coincide with the real schemes of curves constructed by Harnack in [14] (the maximal curves having these real schemes are sometimes called the *simplest Harnack curves*).

Let m again be a positive integer number, and T the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(m,0)$, and $(0,m)$.

A distribution of signs at the integer points of T is called a *Harnack* one if

- any two integer points (i_1, j_1) and (i_2, j_2) of T such that $i_1 \equiv i_2 \mod 2$ and $j_1 \equiv j_2 \mod 2$ have the same sign;
- if $m > 1$, the product of signs at the points $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$ is negative.

A triangulation τ of T is called *primitive* if all the triangles of τ are of area 1/2. (Notice that for any primitive triangulation τ of T all the integer points of T are vertices of τ.)

Proposition 3.3 (see [17, 18]) *If $m = 2k$ is even (resp., $m = 2k+1$ is odd), the combinatorial patchworking applied to an arbitrary primitive convex triangulation of T and a Harnack distribution of signs produces an M-curve with the real scheme shown in Figure 7 (resp., Figure 8).*

Proof. An integer point (i,j) of T is called *interior* if it lies strongly inside of T, *even* if both i and j are even, and *odd* if at least one of the coordinates i and j is odd. Permuting the quadrants and switching if necessary all the signs to the opposite ones (pluses for minuses and minuses for pluses), we can suppose that the distribution of signs is as follows:

all the even points get "−", and all the odd points get "+".

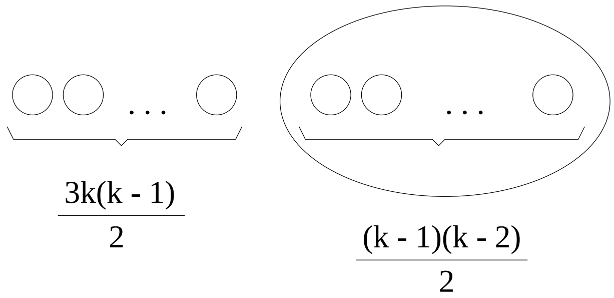

Figure 7: Real scheme of the simplest Harnack curve of degree $2k$ in $\mathbb{R}P^2$

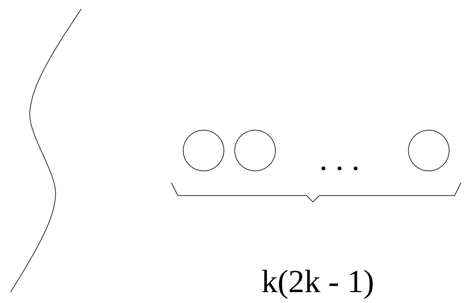

Figure 8: Real scheme of the simplest Harnack curve of degree $2k+1$ in $\mathbb{R}P^2$

Any even interior point (i,j) of T gives rise to an oval of the curve L. This oval encircles (i,j) and is contained in the star of (i,j). Any odd interior point (i,j) of T also gives rise to an oval of L. This oval encircles the symmetric copy $\varphi(i,j)$ of (i,j) equipped with "+", and is contained in the star of $\varphi(i,j)$. We have found $\frac{(m-1)(m-2)}{2}$ ovals and, thus, the curve L can have only one connected component more. This component does exist, because L intersects the coordinate axes. It finishes the proof in the case of odd m. If $m = 2k$ is even, it remains to notice that the union of the segments

$$\{i - j = -m,\ i \leq 0,\ j \geq 0\} \cup$$
$$\{-m \leq i \leq 0,\ j = 0\} \cup \{i = 0,\ -m \leq j \leq 0\}$$

contains only minuses, and its image in $\tilde{T}$ is not contractible. This means that $\frac{(k-1)(k-2)}{2}$ ovals corresponding to the even interior points of T (and encircling minuses) are situated inside of the oval intersecting the coordinate axes. $\square$

The distribution of signs mentioned in the proof of Proposition 3.3 (all the even points get "$-$", and all the odd points get "+") is called the *standard Harnack distribution.*

Now we are able to prove that for any positive integer m and for any integer l verifying $\frac{1-(-1)^m}{2} \le l \le \frac{(m-1)(m-2)}{2} + 1$ there exists a nonsingular curve of degree m in $\mathbb{R}P^2$ whose real point set has exactly l connected components.

Proof of the second part of the Harnack theorem. Pick a maximal curve A_0 of degree m in $\mathbb{R}P^2$ (the existence of such a curve is proved in Proposition 3.3). It is easy to find a nonsingular curve A_1 of degree m in $\mathbb{R}P^2$ such that $\mathbb{R}A$ is empty (resp., is connected) in the case of even m (resp., odd m). (For example, one can take the curve given by the polynomial $x_0^m + x_1^m + x_2^m$.) Let a_0 and a_1 be the points in $\mathbb{R}\mathcal{C}_m$ corresponding to A_0 and A_1. One can choose a smooth path $\gamma : [0,1] \to \mathbb{R}\mathcal{C}_m$ such that $\gamma(0) = a_0$ and $\gamma(1) = a_1$ which intersects the discriminant $\mathcal{D}$ transversally at nonsingular points of $\mathcal{D}$. For any $t \in [0,1]$ denote by A_t the curve corresponding to the point $\gamma(t)$. Near any point of intersection of the chosen path and $\mathcal{D}$ the numbers of connected components of the real point sets of curves A_t differ at most by 1. Thus, any intermediate integer value between $\frac{(m-1)(m-2)}{2} + 1$ and $\frac{1-(-1)^m}{2}$ is realized by the number of connected components of the real point set of certain nonsingular curve of degree m in $\mathbb{R}P^2$. $\square$

Exercises

1. Let $ABCD$ be a square. Find a convex triangulation of ABD and a convex triangulation of BCD such that the union of these triangulations produces a non-convex triangulation of $ABCD$.

2. Let k be a positive integer. Construct a T-curve A of degree $2k$ (resp., $2k-1$) in $\mathbb{R}P^2$ such that $\mathbb{R}A$ is empty (resp., is connected).

3. Using the combinatorial patchworking construct nonsingular curves of degree 6 in $\mathbb{R}P^2$ whose real schemes are shown in Figure 5.

4 Tropical geometry

4.1 Dequantization of positive real numbers

As it was noticed by O. Viro [71] the combinatorial patchworking is directly related to Maslov's dequantization of positive real numbers.

Consider a family of semi-rings $\{S_h\}$, $h \in [0, +\infty)$. As a set any semi-ring S_h coincides with $\mathbb{R}$. The addition and multiplication operations in S_h are defined by the following formulas:

$$
\begin{aligned}
a \oplus_h b &= \begin{cases} h\log(e^{a/h} + e^{b/h}), & \text{si } h > 0, \\ \max\{a, b\}, & \text{si } h = 0 \ ; \end{cases} \\
a \odot_h b &= a + b.
\end{aligned}
$$

These operations depend continuously on h. Each semi-ring S_h, $h > 0$ is isomorphic to the semi-ring $\mathbb{R}_+$ of positive real numbers equipped with the ordinary operations of addition and multiplication: the map $x \mapsto h\log x$ is an isomorphism between $\mathbb{R}_+$ and S_h. However, S_0 is not isomorphic to $\mathbb{R}_+$. It is an idempotent semi-ring which is called $(\max, +)$-*semi-ring.* It is one of the so-called *tropical* semi-rings. The operations in S_0 are called the *tropical addition* and the *tropical multiplication.*

The passage from the positive values of h to $h = 0$ in the family S_h is called *Maslov's dequantization of the positive real numbers* (see [34, 35]). This deformation (as well as the combinatorial patchworking) gives rise to an important relation between algebraic geometry and piecewise-linear geometry.

4.2 Tropical curves

Let I be a finite collection of integer points in $\mathbb{R}^2$. Consider a tropical polynomial $P(x, y) = \sum_{(i,j)\in I} a_{i,j}x^iy^j$, where $a_{i,j}$ are real numbers, and the operations of addition and multiplication are tropical. By the definition of the tropical operations, $P(x, y)$ is the maximum of the linear functions $ix + jy + a_{i,j}$, where this time the operations are standard.

The *tropical curve* $\mathcal{T}(P)$ defined by P is the corner locus of P, *i.e.*, the subset of $\mathbb{R}^2$ where the maximum of the functions $ix + jy + a_{i,j}$ is realized by at least two of them (cf., for example, [36] and [63]). The tropical curve $\mathcal{T}(P)$ is determined by the set I and the function $\nu : I \to \mathbb{R}$, $\nu(i, j) = a_{i,j}$. We say that $\mathcal{T}(P)$ is *associated with* (I, ν). The definition of tropical curves can be generalized to higher dimensions. Here, we restrict ourselves by the case of tropical curves in $\mathbb{R}^2$.

A tropical curve in $\mathbb{R}^2$ is a union of segments and rays which are called *edges.* The common points of different edges are called *vertices.* Notice that any edge of a tropical curve has a rational slope. Let $\mathcal{T}$ be a tropical curve in $\mathbb{R}^2$. Equip each edge of $\mathcal{T}$ with a positive integer weight in the following way. Any connected component of the complement of $\mathcal{T}$ in $\mathbb{R}^2$ corresponds to a point of I. The weight of an edge separating two connected components of the complement of $\mathcal{T}$ is the (integer) length of the segment joining the corresponding two integer points of I. For any vertex v of $\mathcal{T}$ one has $\lambda_1 s_1 + \ldots + \lambda_k s_k = 0$, where $s_1, \ldots, s_k$ are the primitive integer

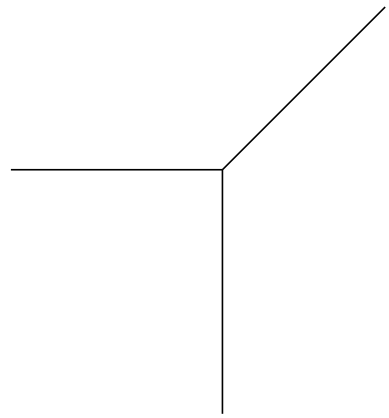

Figure 9: Tropical line

vectors pointed outside of v along the edges of $\mathcal{T}$, and λ_i is the weight of the edge corresponding to s_i.

The convex hull $\Delta(I)$ of I is called the *Newton polygon* of P. Let m be a positive integer. If the Newton polygon of P coincides with the triangle T having the vertices $(0,0)$, $(m,0)$ and $(0,m)$, then we say that the tropical curve $\mathcal{T}(P)$ defined by P is *plane projective of degree* m. A *tropical line*, *i.e.*, a plane projective tropical curve of degree 1, is represented on Figure 9.

Let $\mathcal{T}$ be the tropical curve associated with a pair (I,ν). The function ν defines a subdivision of $\Delta(I)$ in the following way. Consider the overgraph Υ_ν of ν, *i.e.*, the convex hull of the set $\{(i,j,k) \in \mathbb{R}^3 : \ (i,j) \in I \ , \ k \geq \nu(i,j)\}$. The polyhedron Υ_ν is naturally projected onto $\Delta(I)$. The faces of Υ_ν which project injectively, define a subdivision of $\Delta(I)$. Denote this subdivision by $S(I,\nu)$. The tropical curve $\mathcal{T}$ does not determine uniquely the pair (I,ν) (and even the polygon $\Delta(I)$). However, once the polygon $\Delta(I)$ is fixed, the tropical curve $\mathcal{T}$, whose edges are equipped with the weights, determines uniquely the subdivision $S(I,\nu)$. A plane projective tropical curve of degree m is called *nonsingular* if the corresponding subdivision of the triangle T with the vertices $(0,0)$, $(m,0)$ and $(0,m)$ is primitive. Notice that all the weights at the edges of a nonsingular tropical curve are equal to 1.

Tropical curves have many properties in common with algebraic curves. For example, one can prove the following analog of the Bézout theorem (see, *e.g.* [63]): *let* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *be two plane projective tropical curves of degrees* m_1 *and* m_2*, respectively, such that* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *are in a general position with respect to each other; then the number of intersection points (counted with certain multiplicities) of* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *is equal to* $m_1 m_2$. The multiplicities of intersection points are defined as follows.

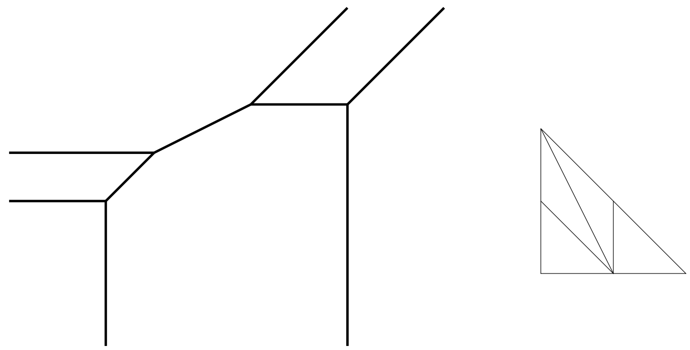

Figure 10: Tropical curve of degree 2 and corresponding subdivision

Consider an intersection point of an edge e_1 of $\mathcal{T}_1$ and an edge e_2 of $\mathcal{T}_2$. Let (a_i, b_i) be a primitive vector along e_i, and λ_i be the weight of e_i ($i = 1, 2$). Then the multiplicity of the intersection point is equal to $\lambda_1\lambda_2|a_1b_2 - a_2b_1|$.

Observe also that for any two generic points in $\mathbb{R}^2$ there exists exactly one tropical line passing through these points. This resemblance between tropical and algebraic curves has an important generalization which is discussed in Section 5.

4.3 Combinatorial patchworking and real tropical curves

We will discuss a real version of definitions given in the previous section. Let I again be a finite collection of integer points in $\mathbb{R}^2$, and $P(x, y) = \sum_{(i,j)\in I} a_{i,j}x^iy^j$ a tropical polynomial with real coefficients. Assume that we are given a distribution of signs $\sigma : I \to \{+, -\}$ at the points of I. The polynomial P and the distribution σ define a *real tropical curve* $\mathcal{T}(P, \sigma)$ in the following way. Let I_+ (resp., I_-) be the subset of I formed by the points equipped with "+" (resp., "-"). Put $P_\pm(x, y) = \sum_{(i,j)\in I_\pm} a_{i,j}x^iy^j$, and denote by $\Gamma_\pm$ the graph of the tropical polynomial $P_\pm$. The *real tropical curve* $\mathcal{T}(P, \sigma)$ is the projection of $\Gamma_+ \cap \Gamma_-$ to $\mathbb{R}^2$. Clearly, $\mathcal{T}(P, \sigma) \subset \mathcal{T}(P)$.

Choose now a collection I of integer points in such a way that $\Delta(I)$ coincides with the triangle T having the vertices $(0, 0)$, $(m, 0)$ and $(0, m)$. Pick such a tropical polynomial $P(x, y) = \sum_{(i,j)\in I} a_{i,j}x^iy^j$ that the corresponding function $\nu : I \to \mathbb{R}$ is generic, *i.e.*, the subdivision $S(I, \nu)$ is a triangulation. Assume again that we are given a distribution of signs $\sigma : I \to \{+, -\}$ at the points of I. Notice that the triangulation $S(I, \nu)$ of T and the restriction of σ to the vertices of $S(I, \nu)$ form initial data for the combinatorial patchworking.

What is a relation between the piecewise-linear curve L produced by the combinatorial patchworking starting from these initial data and the real tropical curve

$\mathcal{T}(P,\sigma)$? Denote by $\mathrm{Int}(T)$ the interior of T, and put $L_0 = L \cap \mathrm{Int}(T)$. From the definition of $\mathcal{T}(P,\sigma)$ it is easy to see that the topological pairs $(\mathrm{Int}T, L_0)$ and $(\mathbb{R}^2, \mathcal{T}(P,\sigma))$ are homeomorphic. In fact, using Maslov's dequantization one can prove in this way Theorem 3.1 (see [71]).

Exercises

1. Let I be a finite collection of integer points in $\mathbb{R}^2$, and $\nu : I \to \mathbb{R}$ a function. Denote by $\mathcal{T}$ the tropical curve associated with the pair (I,ν). Consider a collection I' such that $I' = I + (c_1, c_2)$, where $(c_1, c_2) \in \mathbb{Z}^2 \subset \mathbb{R}^2$. Define a function $\nu' : I' \to \mathbb{R}$ by $\nu'(i,j) = \nu(i - c_1, j - c_2)$. Prove that the tropical curve associated with (I', ν') coincides with $\mathcal{T}$.

2. Let Δ be a convex polygon with integer vertices in $\mathbb{R}^2$, and I the collection of integer points of Δ. Choose a function $\nu : \Delta \to \mathbb{R}$ defining a primitive triangulation of Δ. Prove that the tropical curve associated with the pair (I,ν) is homotopically equivalent to a bouquet of k circles, where k is the number of interior integer points of Δ.

3. Let I be the collection formed by four points $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$. The tropical curve associated with (I,ν), where $\nu : I \to \mathbb{R}$ is an arbitrary function, is said to be of *bidegree* $(1,1)$. Prove that for any three generic points in $\mathbb{R}^2$ there exists exactly one tropical curve of bidegree $(1,1)$ passing through these points.

5 Enumeration of curves

M. Kontsevich [32] proposed to use tropical curves in order to count algebraic curves passing through a given collection of points on a complex surface. This program was realized by G. Mikhalkin [37, 38].

5.1 Complex nodal curves

Let m be a positive integer. Pick an integer δ verifying $0 \le \delta \le \frac{(m-1)(m-2)}{2}$, and choose a collection U of $\frac{m(m+3)}{2} - \delta$ points in the complex projective plane $\mathbb{C}P^2$. If $\delta = 0$, then, under the condition that the collection U is sufficiently generic, there exists exactly one nonsingular curve of degree m in $\mathbb{C}P^2$ passing through all the points of U (cf. Section 2.1). In general, consider curves of degree m in $\mathbb{C}P^2$ which pass through $\frac{m(m+3)}{2} - \delta$ points of U and have δ nondegenerate double points. If U is sufficiently generic, then the number of these curves is finite and does not depend on U. Denote by $N_m(\delta)$ (resp., $N_m^{\mathrm{irr}}(\delta)$) the number of curves (resp., of irreducible curves) of degree m in $\mathbb{C}P^2$ which pass through $\frac{m(m+3)}{2} - \delta$ given generic points in $\mathbb{C}P^2$ and have δ nondegenerate double points.

The numbers $N_m^{\mathrm{irr}}(\delta)$ determine the numbers $N_m(\delta)$ and *vice versa* (see, for example, [5]). The numbers $N_m^{\mathrm{irr}}(\delta)$ are the *Gromov-Witten invariants* of $\mathbb{C}P^2$ (see [31]). A recursive formula for the numbers $N_m^{\mathrm{irr}}(\delta)$, where $\delta = \frac{(m-1)(m-2)}{2}$, was found by M. Kontsevich (see [31]). L. Caporaso and J. Harris [5] gave an algorithm which allows one to calculate the numbers $N_m(g)$ for an arbitrary δ.

Mikhalkin proposed a new formula for the numbers $N_m(\delta)$ (see [37]). This formula has an immediate generalization to the case of an arbitrary projective toric surface (see [37]). Mikhalkin's theorem is based on a reformulation of the enumerative problem presented above into an enumerative problem concerning tropical curves.

5.2 Correspondence theorem

To formulate Mikhalkin's correspondence theorem, introduce additional definitions.

Let m be a positive integer, and $\mathcal{T}$ a plane projective tropical curve of degree m. The curve $\mathcal{T}$ is called *nodal* if the corresponding subdivision τ of the triangle T having the vertices $(0,0)$, $(m,0)$, and $(0,m)$ verifies the following properties:

- any polygon of τ is either triangle or a parallelogram,
- any integer point on the boundary of T is a vertex of τ.

Assume that $\mathcal{T}$ is nodal. Then, the *rank* of $\mathcal{T}$ is the difference diminished by 1 between the number of vertices of τ and the number of parallelograms in τ. The *multiplicity* $\mu(\mathcal{T})$ of $\mathcal{T}$ is the product of areas of all the triangles in τ (we normalize the area in such a way that the area of a triangle whose only integer points are the vertices is equal to 1).

Let n be a natural number, and $\mathcal{U}$ a generic set of n points in $\mathbb{R}^2$. Consider the collection $\mathcal{C}(\mathcal{U})$ of nodal plane projective tropical curves of degree m and of rank n which pass through all the points of $\mathcal{U}$, and denote by $\mathcal{N}_n(\mathcal{U})$ the number of curves in $\mathcal{C}(\mathcal{U})$ counted with their multiplicities.

Theorem 5.1 (G. Mikhalkin, [38]). *Let $\mathcal{U}$ be a generic set of $n = \frac{m(m-3)}{2} - \delta$ points in $\mathbb{R}^2$, where $0 \le \delta \le \frac{(m-1)(m-2)}{2}$ is an integer. Then $\mathcal{N}_n(\mathcal{U})$ is equal to $N_m(\delta)$.*

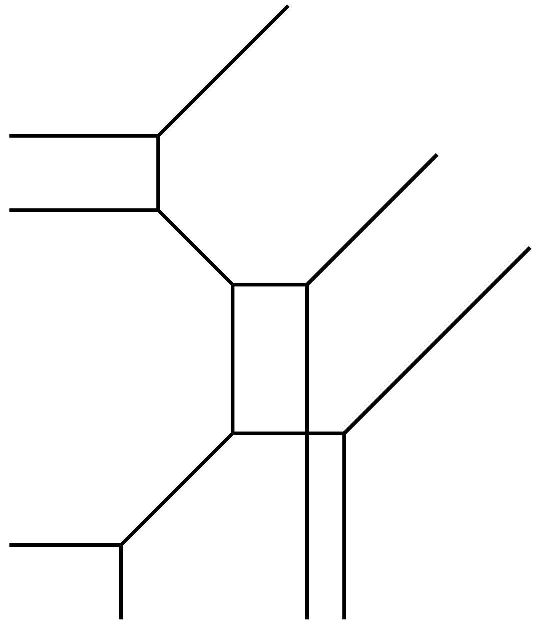

Figure 11: Tropical curve of degree 3 and of rank 8

Theorem 5.1 is a particular case of Mikhalkin's theorem which is valid in more general setting of projective toric surfaces. Mikhalkin's proof of Theorem 5.1 provides a bijection between the multi-set $\mathcal{C}(\mathcal{U})$ and the set of complex curves of degree m which pass through n given generic points in $\mathbb{C}P^2$ and have δ nondegenerate double points. Another approach establishing such a bijection was proposed by E. Shustin [61].

In addition, Mikhalkin [37] found a combinatorial algorithm which gives a possibility to calculate the number of tropical curves in question.

5.3 Welschinger invariant

Mikhalkin's correspondence also gives a possibility to enumerate real curves passing through specific configurations of real points in $\mathbb{R}P^2$ (as well as on other projective toric surfaces). Of course, in the real case the result depends on the chosen point configuration in $\mathbb{R}P^2$. Fortunately, another important discovery was made recently by J.-Y. Welschinger [72, 73]. He found a way of attributing weights to real rational curves which makes the number of curves counted with the weights to be independent of the configuration of points in $\mathbb{R}P^2$ and produce lower bounds for the number of real curves in question.

For given positive integer m and integer δ verifying $0 \le \delta \le \frac{(m-1)(m-2)}{2}$, choose a collection U of $\frac{m(m+3)}{2} - \delta$ generic points in $\mathbb{R}P^2$. Denote by $N_m^{\text{irr, even}}(\delta, U)$ (resp., $N_m^{\text{irr, odd}}(\delta, U)$) the number of real irreducible curves of degree m passing through all the points of U and having even (resp., odd) number of solitary nodes (*i.e.*, double points locally given by $x^2 + y^2 = 0$) among δ nondegenerate double points. Define the *Welschinger number* as $W_m(\delta, U) = N_m^{\text{irr, even}}(\delta, U) - N_m^{\text{irr, odd}}(\delta, U)$.

Theorem 5.2 (J.-Y. Welschinger [72, 73]). *If $\delta = \frac{(m-1)(m-2)}{2}$ (i.e., if the considered curves are rational), then $W_m(\delta, U)$ does not depend on the choice of the (generic) set U.*

In fact, Theorem 5.2 is a particular case of Welschinger's theorem which is formulated in a symplectic setting. The general statement and the proof can be found in [72, 73].

The number $W_m(\frac{(m-1)(m-2)}{2}, U)$ is called the *Welschinger invariant* and is denoted by W_m. Clearly, $N_m^{\text{irr}}(\frac{(m-1)(m-2)}{2}, U) \geq |W_m|$.

Welschinger's theorem gives rise to another type of applications of Mikhalkin's correspondence. This approach already gave some results, in particular, the existence of real rational curves passing through given points in $\mathbb{R}P^2$ (see [19]).

Consider the following question: *fix a positive integer m; whether for any generic $3m - 1$ points in the real plane there always exists a real rational curve of degree m which passes through these points* ? (The number $N_m^{\text{irr}}(\frac{(m-1)(m-2)}{2})$ of complex rational curves (see [31]) is even for every $m \geq 3$, so the existence of required real curves does not immediately follow from the computation in the complex case.) It is shown in [19] with the use of Mikhalkin's theorem, that Welschinger's bound implies the following statement.

Proposition 5.3 (see [19])). *For any positive integer m, through any $3m-1$ generic points in $\mathbb{R}P^2$ there can be traced at least $m!/2$ real rational curves of degree m.*

As a corollary, the aforementioned question is answered in the affirmative.

5.4 Hilbert-type inequalities

One of the exercises to Section 2 is, in fact, known as Hilbert's theorem.

Theorem 5.4 (Hilbert's theorem). *Let A be a nonsingular curve of degree m in $\mathbb{R}P^2$. Then,*

- *the sum of the depths of any two disjoint nests in $\mathbb{R}A$ does not exceed $[m/2]$,*

- *the sum of the depths of any five disjoint nests in* $\mathbb{R}A$ *does not exceed* m *if no oval of one nest contains inside all the ovals of the other four nests.*

Proof. The statements of Theorem 5.4 are corollaries of the Bézout theorem. To prove the first statement it suffices to choose one point inside of the most interior oval of each nest, trace the line passing through the chosen two points, and notice that this line intersects each oval of the union of two nests at least in two points. On the other hand, according to the Bézout theorem, the line intersects $\mathbb{R}A$ at most in m points.

To prove the second statement, we proceed in a similar way using an auxiliary conic. Choose one generic point inside of the most interior oval of each nest, trace the conic passing through the chosen five points, and notice that this conic intersects each oval of the union of five nests at least in two points. On the other hand, according to the Bézout theorem, the conic intersects $\mathbb{R}A$ at most in $2m$ points. □

The proof of Hilbert's theorem uses the facts that for any two points in $\mathbb{R}P^2$ there is a line passing through them, and for any five points in $\mathbb{R}P^2$ there is a conic passing through them. A natural question arises: is it possible to use auxiliary curves of higher degrees and to generalize in this way Hilbert's theorem for bigger numbers of nests? If we try to perform this plan straightforwardly and consider nine nests of a curve of degree m in $\mathbb{R}P^2$, we meet a difficulty: the real point set of the cubic passing through the chosen 9 points (we choose one generic point inside of the most interior oval of each nest) does not need to be connected. Thus, we cannot affirm that the auxiliary cubic intersects all the ovals of the union of nine nests.

To prove Hilbert-type inequalities, one can try to use auxiliary curves with connected real point set, or can try to make sure that all the chosen points belong to the same connected component of the auxiliary curve. One of the ways to assure the latter condition is to use rational auxiliary curves (see [7] for a detailed discussion of the related questions).

The following statement is a corollary of Proposition 5.3.

Proposition 5.5 *For any nonsingular curve* A *of degree* m *in* $\mathbb{R}P^2$ *and any positive integer* d*, the sum of the depths of any* $3d-1$ *disjoint nests in* $\mathbb{R}A$ *does not exceed* $md/2$ *if no oval of one nest contains inside all the ovals of the other* $3d-2$ *nests.*

Proof. Choose one point inside of the most interior oval of each nest (in such a way that the chosen $3d-1$ points are generic), and consider a real rational curve of degree d passing through the chosen points. The statement now follows from the Bézout theorem. $\square$

Proposition 5.5 was known before Proposition 5.3 (see [7]). The proof of Proposition 5.5 presented in [7] is based on the possibility to trace a connected real cubic (of genus 0) through 8 points in $\mathbb{R}P^2$ and a connected real quartic (of genus 3) through 13 points in $\mathbb{R}P^2$. The advantage of the proof of Proposition 5.5 via Welschinger's theorem and Proposition 5.3 consists in the fact that in this way Proposition 5.5 immediately extends to the case of pseudo-holomorphic curves (it is not clear whether a proof similar to that presented in [7] works in the symplectic category).

Exercises

1. Let r be a positive integer. Prove the following statement: if r points in $\mathbb{R}^2$ are sufficiently generic, then the rank of any nodal tropical curve passing through these points is at least r.

2. Let m be a positive integer. Prove the following statement: if $\mathcal{C}$ is a sufficiently generic collection of $\frac{m(m+3)}{2}$ points in $\mathbb{R}^2$, then there exists exactly one nonsingular plane projective tropical curve of degree m which passes through all the points of $\mathcal{C}$.

3. Calculate Welschinger's invariant in the case of rational cubics in the projective plane.

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