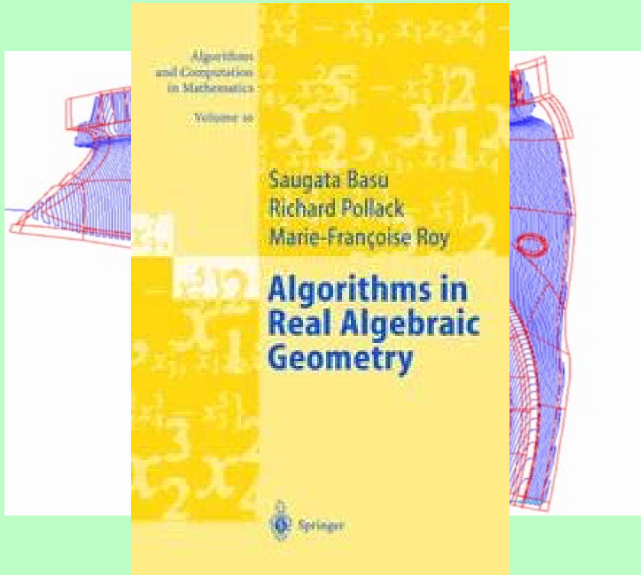
Applications of Real Algebraic Geometry to Computer Aided Geometric Design



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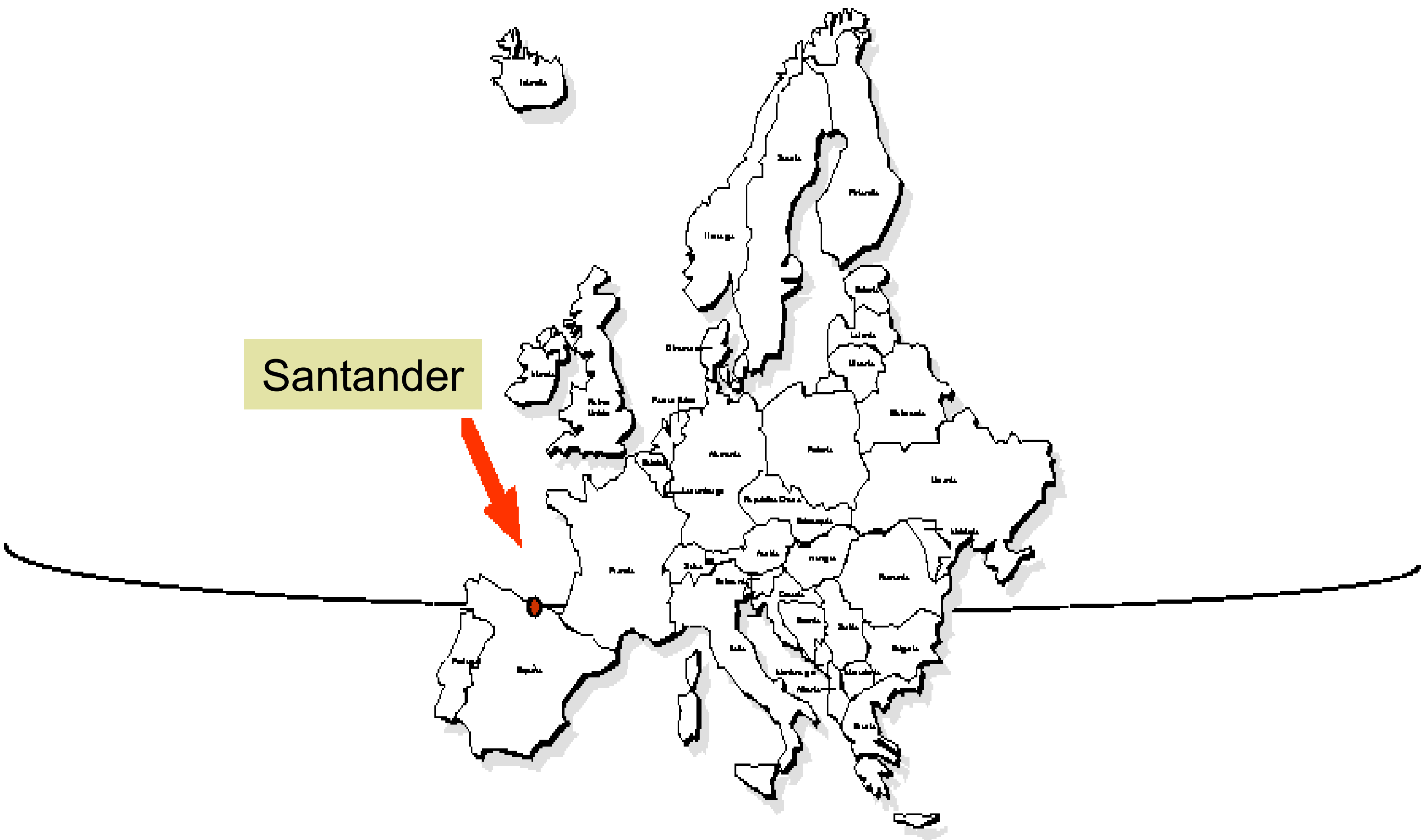
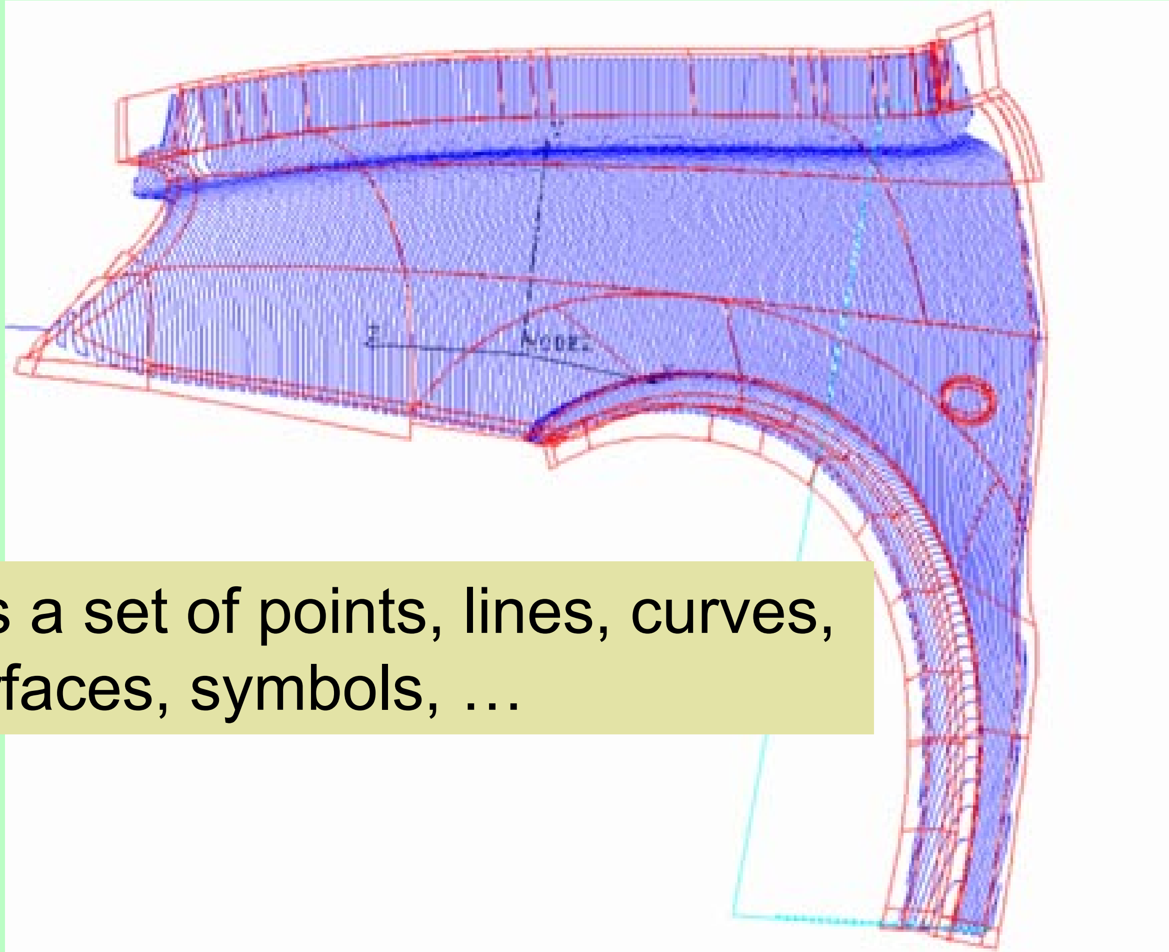
RAG&Applications, Trieste, August 2003


Santander

CAGD: One motivating example

Mould mathematical definition

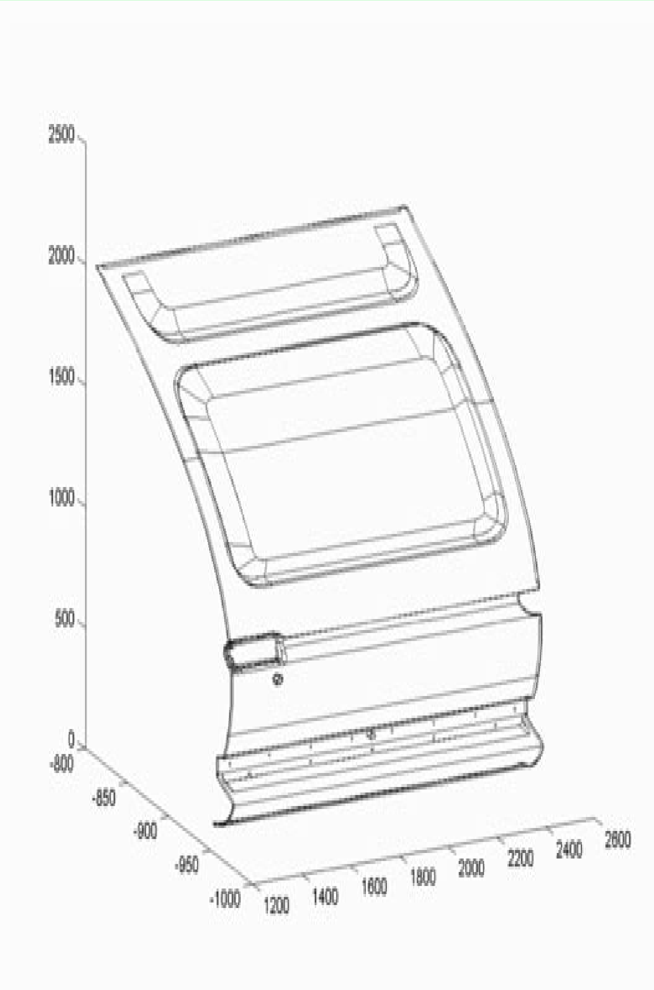


ASCII files

It is a set of points, lines, curves, surfaces, symbols, …

CAD formats:
- IGES
- VDA
- STEP
- ……………………

Semialgebraic Sets

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2.1 ,32,38,6,308,15,10Hpaso.felix,1.0D0,2,2HMM,32767,32.767D0,13H970226.G      3
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Piecewise defined and continuous polynomial/rational curves and surfaces are used to represent the considered objects

Definition

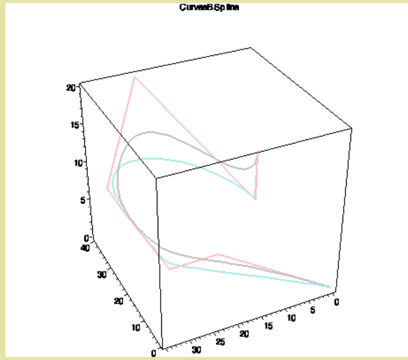
The N–**order B-spline basis functions** are defined recursively as:

$$B_{i,1}(u) = \begin{cases} 1 & u_i \leq u < u_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$B_{i,N}(u) = \frac{u-u_i}{u_{i+N-1}-u_i} B_{i,N-1}(u) + \frac{u_{i+N}-u}{u_{i+N}-u_{i+1}} B_{i+1,N-1}(u)$$

Definition

A N–order **rational B-spline curve** $R_N(u)$ is defined by:



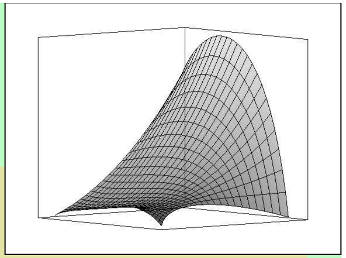
$$\frac{\displaystyle\sum_{i=0}^{n-1} B_{i,N}(u) w_i P_i}{\displaystyle\sum_{i=0}^{n-1} B_{i,N}(u) w_i}, \qquad 0 \leq u \leq 1$$

where $\{P_i\}$ are the *control points*, $\{w_i\}$ are the *weights* and $\{B_{i,N}\}$ are the N*–order B-spline basis functions* defined on the nonperiodic *knot vector*

$$U = \{u_0, u_1, \ldots, u_{n+N-2}, u_{n+N-1}\}$$

with $u_0 = u_1 = \ldots = u_{N-1} = 0$ and $u_n = u_{n+1} = \ldots = u_{n+N-1} = 1$.

If $w_i = c \neq 0)$ (for all i) the B–spline curve is said **integral**.



Definition

A N_1, N_2–order **rational B-spline surface** $R_{N_1,N_2}(u,v)$ is defined by

$$R_{N_1,N_2}(u,v) = \frac{\displaystyle\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1} B_{i,N_1}(u) N_{j,N_2}(v) w_{i,j} P_{i,j}}{\displaystyle\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1} B_{i,N_1}(u) B_{j,N_2}(v) w_{i,j}}, \qquad 0 \le u \le 1, 0 \le v \le 1$$

where $\{P_{i,j}\}$ are the *control points*, $\{w_{i,j}\}$ are the *weights*, $\{N_{i,N_1}\}$ and $\{N_{j,N_2}\}$ are the *B–spline basis functions* defined on the nonperiodic *knot vectors*

$$U = \{u_0, u_1, \dots, u_{n_1+N_1-1}\} \qquad \text{and} \qquad V = \{v_0, v_1, \dots, v_{n_2+N_2-1}\}$$

respectively, with $u_0 = u_1 = \dots = u_{N_1-1} = 0$, $u_{n_1} = u_{n_1+1} = \dots = u_{n_1+N_1-1} = 1$, $v_0 = v_1 = \dots = v_{N_2-1} = 0$ and $v_{n_2} = v_{n_2+1} = \dots = v_{n_2+N_2-1} = 1$.
If $w_{i,j} = c \neq 0)$ (for all i and j) then the B–spline surface is said **integral** .

Mould realization

From the mathematical definition a first real model is constructed

This model is improved by performed operations such as:

- sectioning
- offsetting
- blending

These operations are initially defined on the theoretical model mathematically

Control quality is performed in order to accomplish the initial requirements for the mould

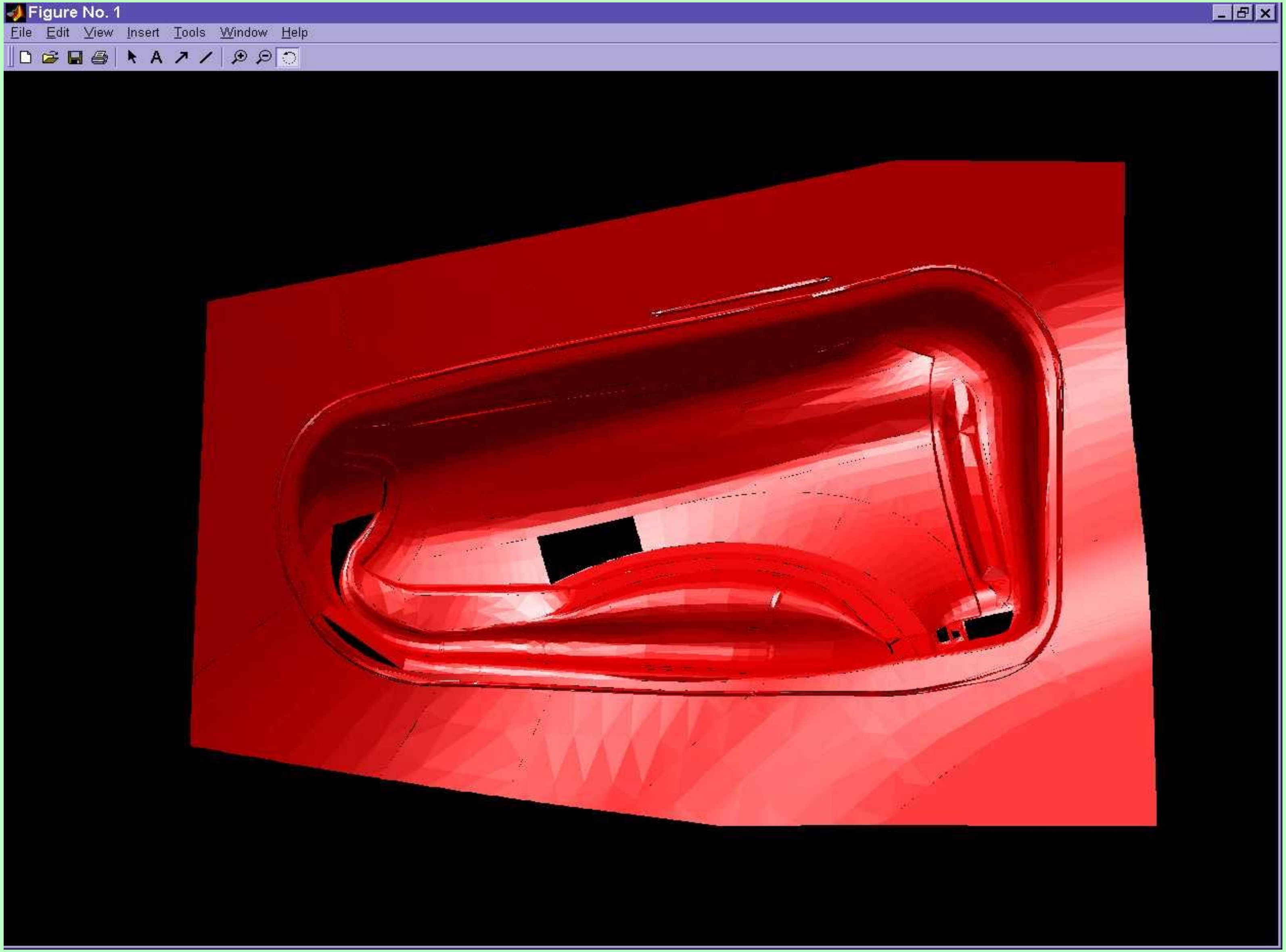


Figure No. 1
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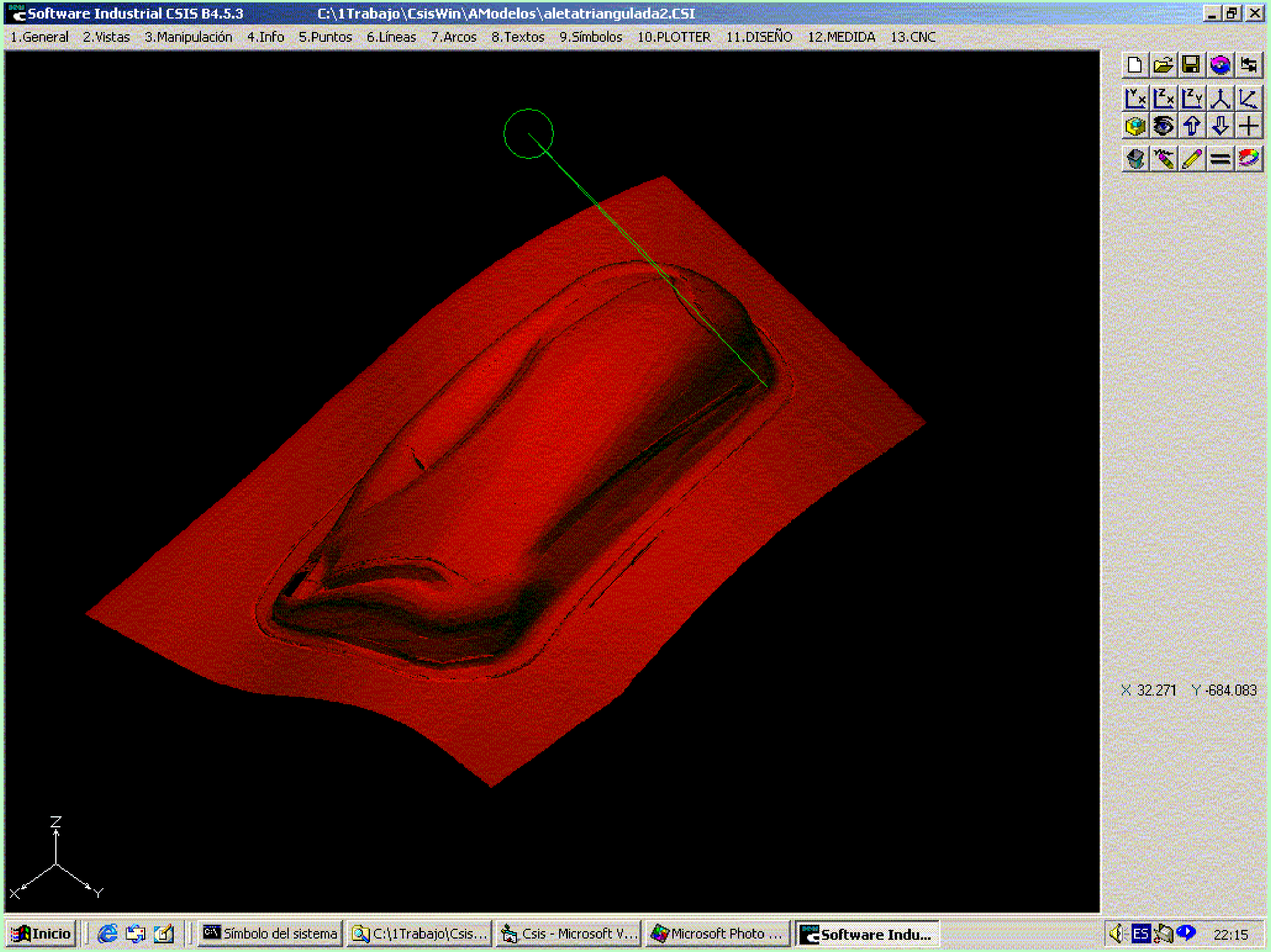
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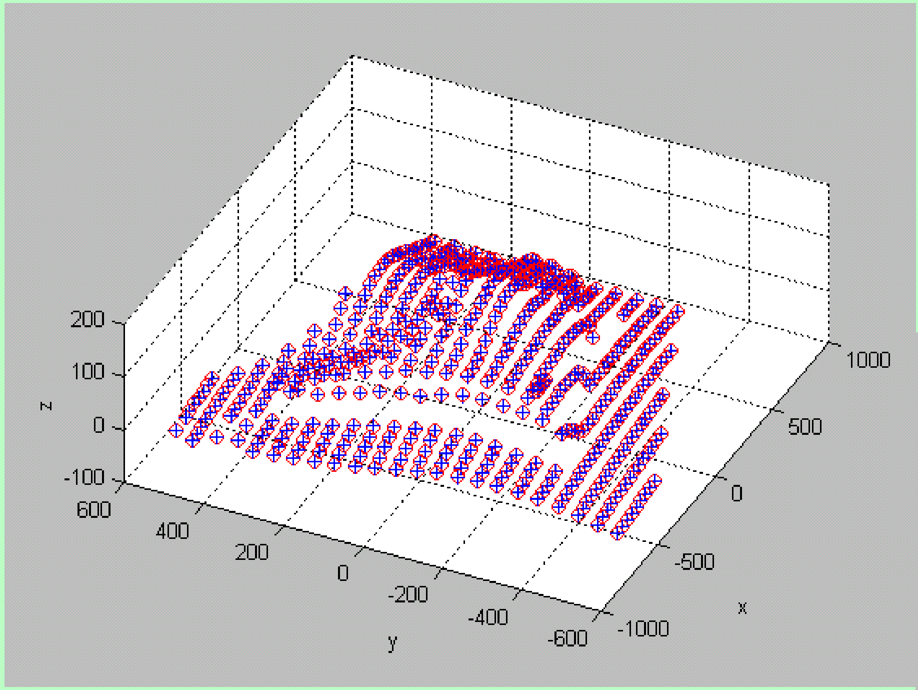

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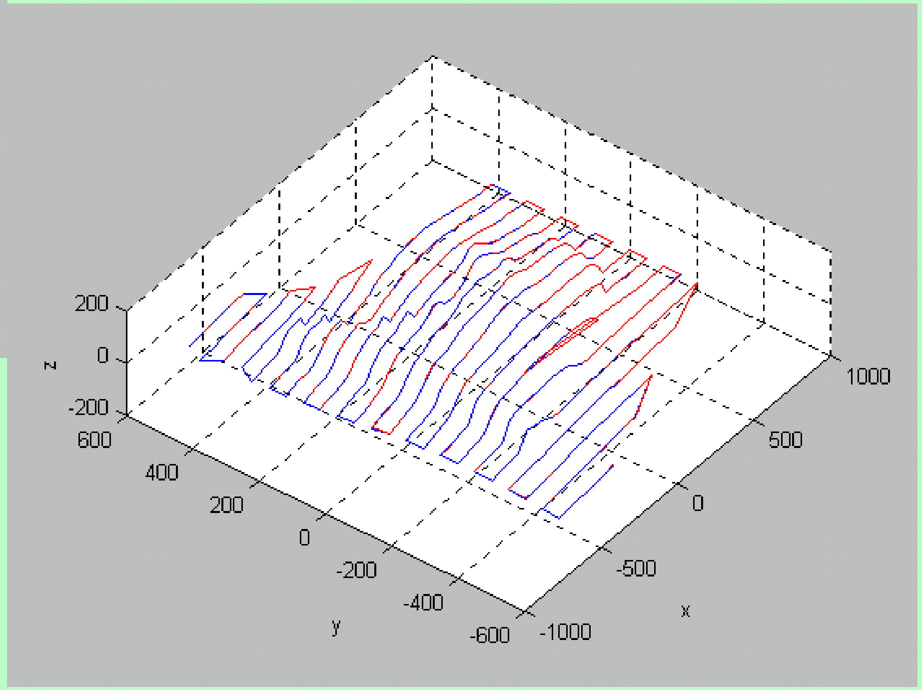

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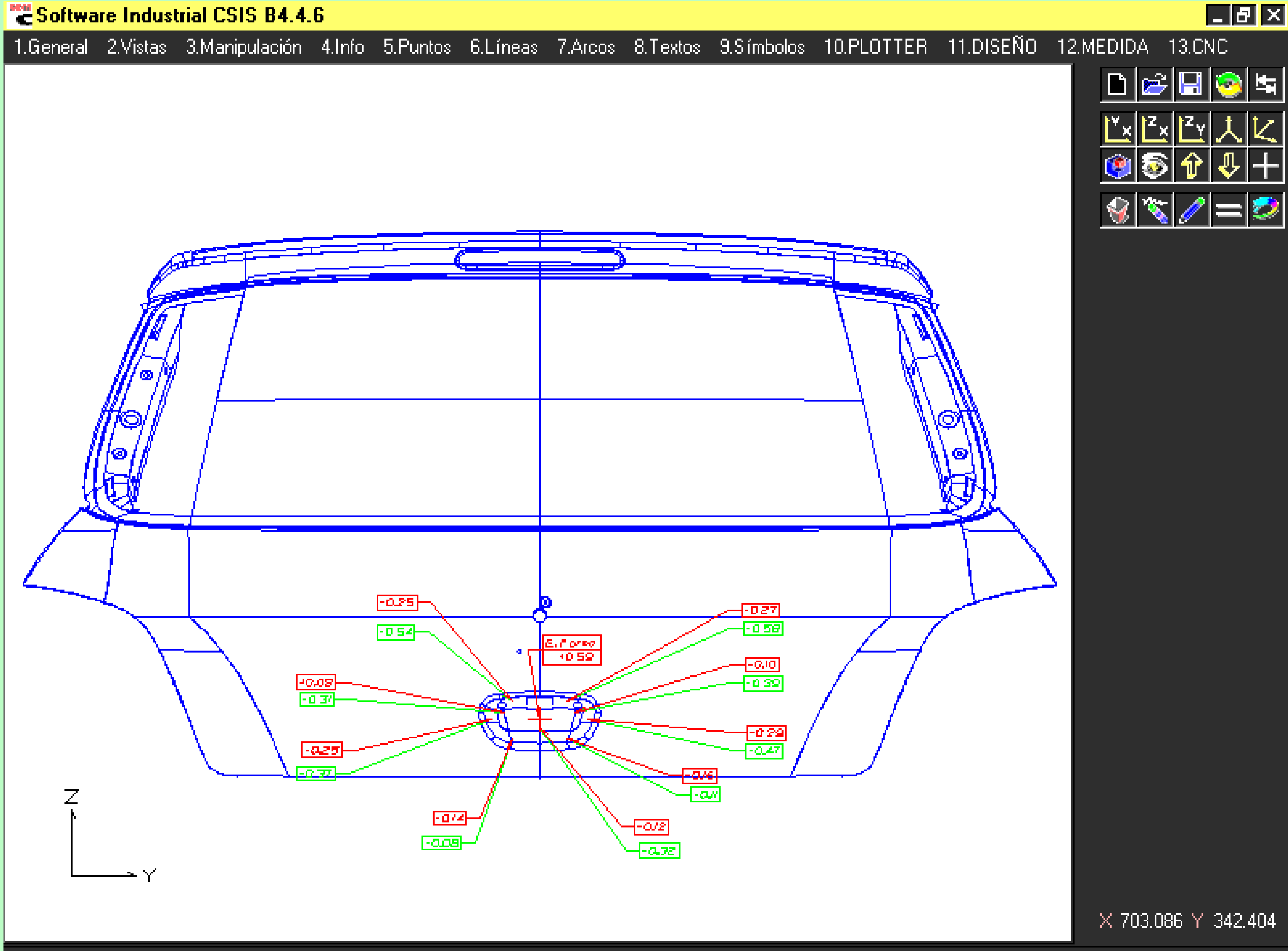
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11.DISEÑO
12.MEDIDA
13.CNC
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X 703.086 Y 342.404

- **CAGD Introduction and Motivation (0.5)**
- **Implicitation (1)**
- **Offsets Manipulation (0.5)**
- **Efficient Topology Computation: Real Plane Curves (1.5)**
- **Drawing Implicit Surfaces (0.5)**

GOAL:

to use tools from Computational Real Algebraic Geometry to deal with curves and surfaces in CAGD

Implicitation

Let $\mathcal{S}$ be an elementary bicubic B–spline surface defined by:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (n_0(u)\ n_1(u)\ n_2(u)\ n_3(u)) \cdot \begin{pmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{pmatrix} \cdot \begin{pmatrix} n_0(v) \\ n_1(v) \\ n_2(v) \\ n_3(v) \end{pmatrix}$$

with $0 \leq u, v \leq 1$, $P_{ij} \in \mathbb{R}^3$, and

$$n_0(s) = \frac{(1-s)^3}{6} \qquad n_1(s) = \frac{3s^3-6s^2+4}{6}$$

$$n_2(s) = \frac{-3s^3+3s^2+3s+1}{6} \qquad n_3(s) = \frac{s^3}{6}$$

The problem is to compute a polynomial H(x,y,z) with the smallest posible degree such that S is contained in

$$\{(\alpha,\beta,\gamma) \in \mathbb{R}^3 : H(\alpha,\beta,\gamma)=0\}$$

If available, the implicit equation is very useful for:

- Point-Surface Positioning
- Surface-Surface Intersections (sectioning)
- Surface-Curve Intersections
- Manipulating Trimmed Surfaces and Sculptured Solids
- ....

$$\mathcal{S}: \begin{cases} x = \dfrac{f_1(s,t)}{q(s,t)} \\ y = \dfrac{f_2(s,t)}{q(s,t)} \\ z = \dfrac{f_3(s,t)}{q(s,t)} \end{cases} \Longrightarrow \quad H(x,y,z) = 0$$

$$x = A \Longrightarrow \quad H(A,y,z) = 0$$

The Implicitation Problem is a classical question in Algebraic Geometry (Elimination Theory):

- through multivariate resultants
  - Manocha&Canny, Sederberg, Bajaj, GALAAD, ...
- through Gröbner Basis
  - Buchberger, Cox et al, Hoffmann, Kalkbrenner, Gao&Chou, Fing et al, ...
- through moving curves and surfaces
  - Cox, Sederberg, Goldman, Zhang, Chen, Du, ...
- through deformation and/or "ad-hoc" techniques
  - Canny, Gonzalez-Vega, ...

X=f(t), Y=g(t)

Sylvester Resultant[eliminates t]: gives P(X,Y)

X=f(s,t), Y=g(s,t), Z=h(s,t)

Multivariate Resultant[eliminates s,t]: gives P(X,Y,Z)

Examples

A base point is a real solution of the overdetermined system:

$$\begin{cases} q(s,t) &= 0 \\ f_1(s,t) &= 0 \\ f_2(s,t) &= 0 \\ f_3(s,t) &= 0 \end{cases}$$

Why the presence of base points makes the resultant to vanish identically?

X=f(t), Y=g(t)

Lex Grobner Basis with t > XY [eliminates t]:

- gives P(X,Y)

X=f(s,t), Y=g(s,t), Z=h(s,t)

Lex Grobner Basis with st > XYZ [eliminates s,t]:

- gives P(X,Y,Z)

What the presence of base points makes to the Grobner Basis?

Implicitation is not used or does not appear into CAD/CAM systems due mainly to:

It is a very costly algebraic operation.

The coefficients of the polynomials involved in the parametrization are usually floating point real numbers.

Generic Implicitization

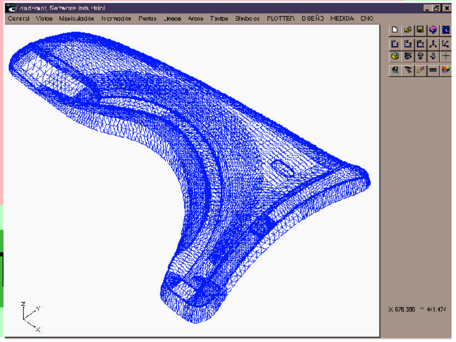
$$x = x_{00}\frac{t_2 - t}{t_2 - t_1} + x_{01}\frac{t - t_1}{t_2 - t_1}$$

$$y = y_{00}\frac{s_2 - s}{s_2 - s_1} + y_{11}\frac{s - s_1}{s_2 - s_1}$$

$$z = \left[z_{00}\frac{s_2 - s}{s_2 - s_1} + z_{10}\frac{s - s_1}{s_2 - s_1}\right]\frac{t_2 - t}{t_2 - t_1} + \left[z_{10}\frac{s_2 - s}{s_2 - s_1} + z_{11}\frac{s - s_1}{s_2 - s_1}\right]\frac{t - t_1}{t_2 - t_1}$$

$$\Downarrow \Downarrow \Downarrow \Downarrow \Downarrow \Downarrow \Downarrow \Downarrow \Downarrow \Downarrow$$

$$(z_{00} - 2z_{10} + z_{11})xy + (y_{00}z_{10} + y_{11}z_{10} - y_{00}z_{11} - y_{11}z_{00})x +$$
$$+(x_{00}z_{10} + x_{01}z_{10} - x_{00}z_{11} - x_{01}z_{00})y +$$
$$+(x_{00}y_{11} + x_{01}y_{00} - x_{00}y_{00} - x_{01}y_{11})z +$$
$$+x_{01}y_{11}z_{00} + x_{00}y_{00}z_{11} - x_{00}y_{11}z_{10} - x_{01}y_{00}z_{10}$$



How to overcome these difficulties (I):

Floating-point real numbers as coefficients

A concrete object to model (and then to construct) is made by several hundreds (or thousands) of small patches, **all of them sharing the same algebraic structure**

For such object, a data base is constructed containing the implicit equation of every kind of patch appearing in its definition.

The data base contains also the inversion formulae

u=U(x,y,z) v=V(x,y,z)

The data base is pruned “through evaluation” to avoid specialization problems.

The implicitation process is accomplished by using Sylvester Resultants, Gröbner Basis computations and ad-hoc techniques for specific cases.

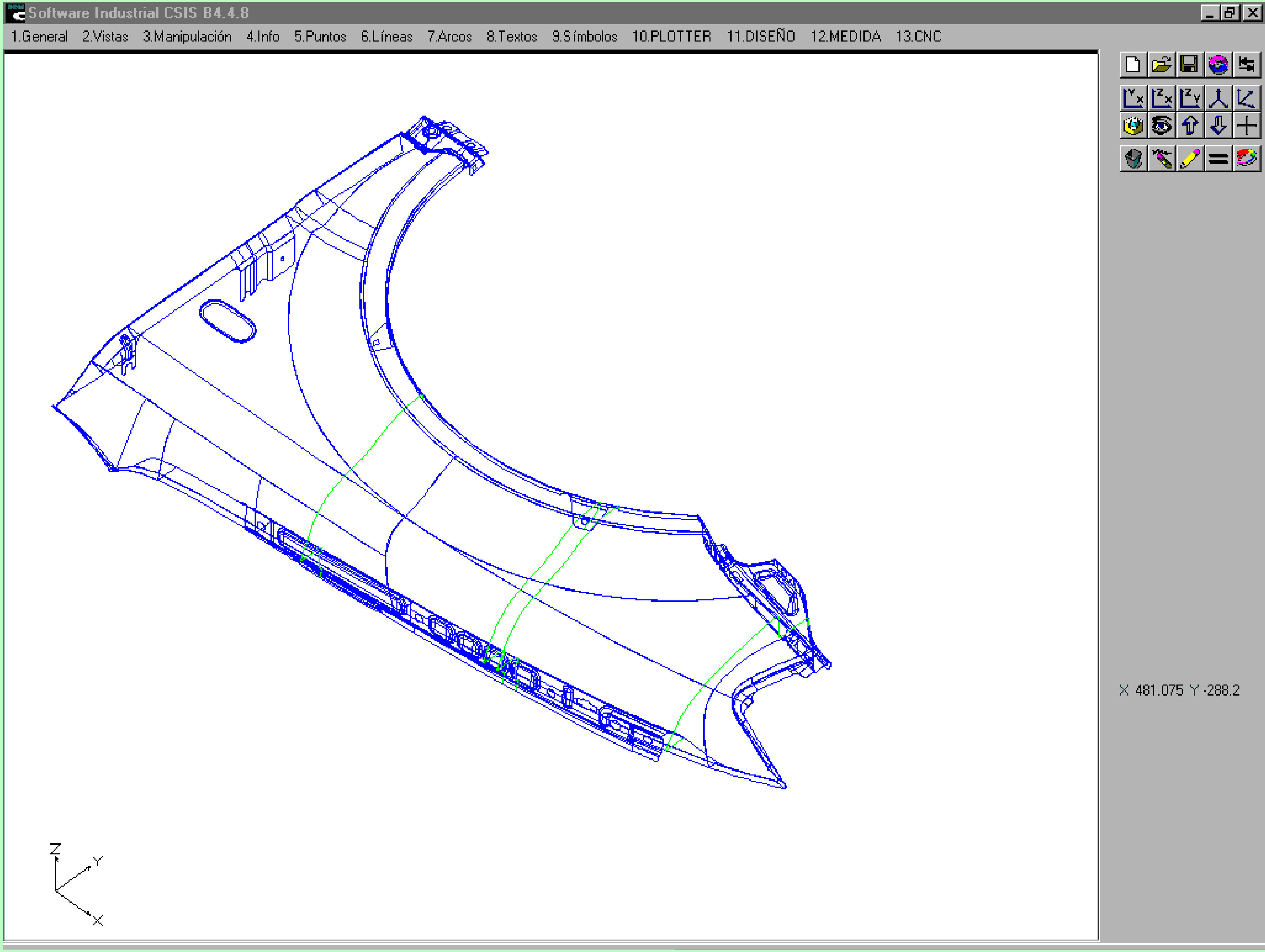
How to overcome these difficulties (II):

Computing times

Since the "generic" implicitation procedure is a preprocessing step, no computing time other than the one used

- in evaluating the generic implicit equation, and
- in its verification

is spent when the CAD/CAM user is working.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FURTHER WORK (I)

Some algebraic structures are not easy to generically implicitize:

$$x = \frac{f_1(s,t)}{q(s,t)}, \quad y = \frac{f_2(s,t)}{q(s,t)}, \quad z = \frac{f_3(s,t)}{q(s,t)}$$

with:

$$\begin{aligned} q(s,t) =& s^3(A_3t^2 + B_3t + C_3) + s^2(A_2t^2 + B_2t + C_2) \\ &+ s(A_1t^2 + B_1t + C_1) + (A_0t^2 + B_0t + C_0) \end{aligned}$$

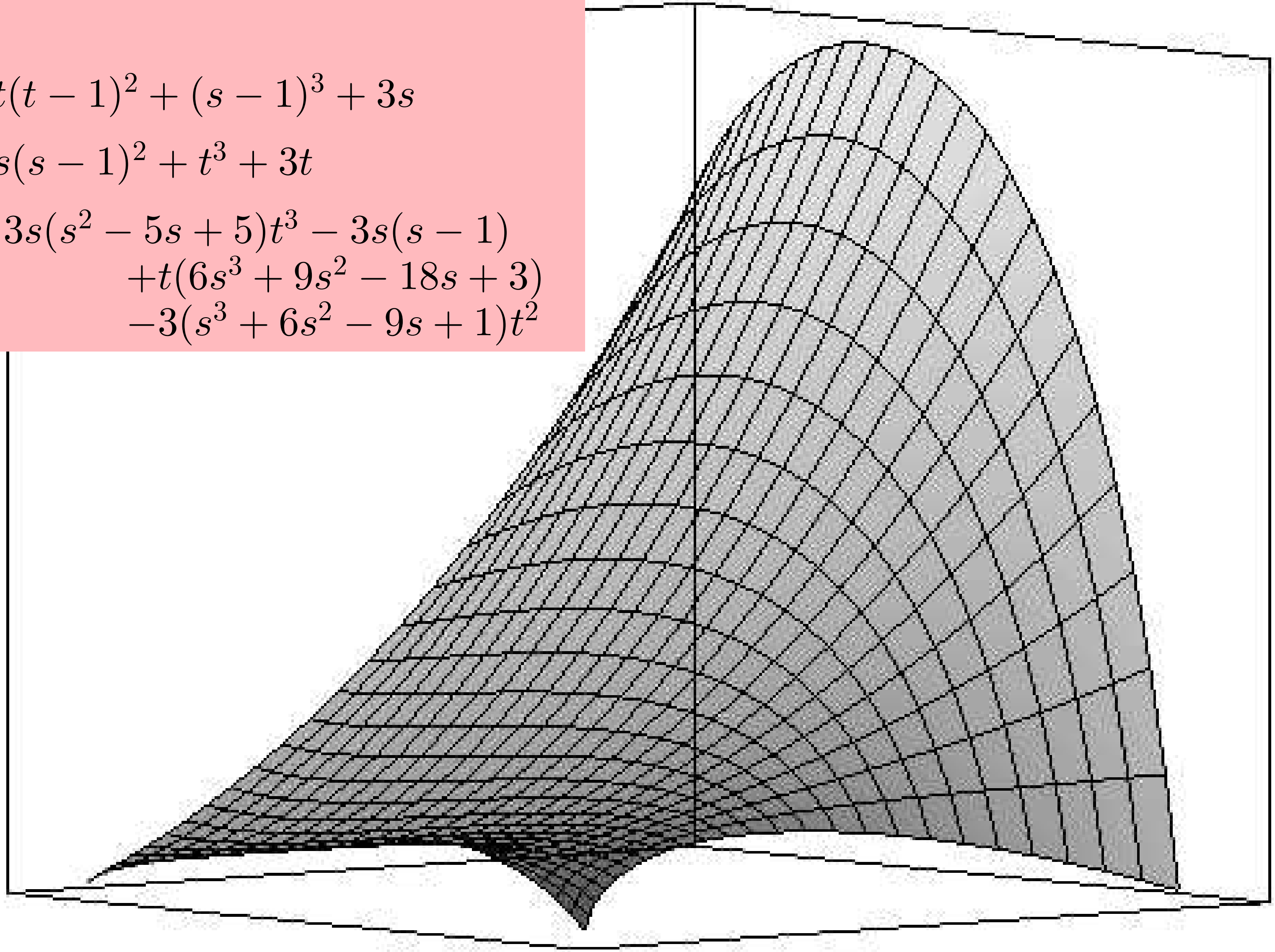
$$\begin{aligned} f_i(s,t) = & s^3(\alpha_{3i}t^2 + \beta_{3i}t + \gamma_{3i}) + s^2(\alpha_{2i}t^2 + \beta_{2i}t + \gamma_{2i}) \\ &+ s(\alpha_{1i}t^2 + \beta_{1i}t + \gamma_{1i}) + (\alpha_{0i}t^2 + \beta_{0i}t + \gamma_{0i}) \end{aligned}$$

The implicit equation of the bicubic surface $\mathcal{B}$

$$x = 3t(t-1)^2 + (s-1)^3 + 3s$$

$$y = 3s(s-1)^2 + t^3 + 3t$$

$$\begin{aligned} z = & -3s(s^2-5s+5)t^3 - 3s(s-1) \\ & +t(6s^3+9s^2-18s+3) \\ & -3(s^3+6s^2-9s+1)t^2 \end{aligned}$$



$$F_1 = \frac{-H_1 + 3H_2}{8} = s^3 + \frac{3t^2}{4} - \frac{15s^2}{8} + \frac{3t}{4} - \frac{3y}{8} + \frac{x}{8} + \frac{3s}{8} + \frac{1}{8}$$

$$F_2 = \frac{3H_1 - H_2}{8} = t^3 - \frac{9t^2}{4} - \frac{3s^2}{8} + \frac{3t}{4} + \frac{y}{8} - \frac{3x}{8} + \frac{15s}{8} - \frac{3}{8}$$

The computing time was 4.5 seconds by using Maple 8 on a Power PC G4 (1,24 MHz) versus the 10^6 seconds in Hoffmann's solution.

The size of the file containing the implicit equation (in non expanded form) is around 600 Kbytes.

$$- \frac{54187594407x^2}{16777216} + \frac{48101467761xy}{8388608} - \frac{38812918311y^2}{16777216}$$
$$- \frac{2265699198239117 1}{137438953472} - \frac{1}{2}\Big(\frac{233469x}{2048} - \frac{188595y}{2048}$$
$$+ \frac{112832595}{262144} + \frac{81x^2}{64} - \frac{135xy}{32} + \frac{81y^2}{64}\Big)\Big(-\frac{233469x}{2048}$$
$$+ \frac{188595y}{2048} - \frac{112832595}{262144} - \frac{81x^2}{64} + \frac{135xy}{32} - \frac{81y^2}{64}\Big)$$

Approximating expansions at infinity of rational functions

$$f(T) = a_0T^m + a_1T^{m-1} + \ldots + a_{m-1}T + a_m = a_0\prod_{j=1}^{s}(T - \alpha_j)^{e_j}$$

Newton Sums ($k \geq 0$):

$$\mathbf{S}_k = e_1\alpha_1^k + \ldots + e_s\alpha_s^k$$

Newton Identities ($j \geq 1$):

$$ja_j = -\mathbf{S}_j a_0 - \mathbf{S}_{j-1}a_1 - \ldots - \mathbf{S}_1 a_{j-1}$$

(if $i > m$ then $a_i = 0$)

$$\frac{f'(T)}{f(T)} = \frac{\mathbf{S}_0(\{\alpha_j\}_j)}{T} + \frac{\mathbf{S}_1(\{\alpha_j\}_j)}{T^2} + \frac{\mathbf{S}_2(\{\alpha_j\}_j)}{T^3} + \cdots\cdots\cdots$$

$$\frac{f'(T)g(T)}{f(T)} = \cdots\cdots\cdots\cdots\cdots + \frac{\mathbf{S}_1(\{g(\alpha_j)\}_j)}{T} + \cdots\cdots\cdots\cdots\cdots$$

x=f(t), y=g(t)

$n = \deg(g)$.

$$\mathcal{H}_{\mathcal{C}}(x,y) = x^n + r_1(y)x^{n-1} + \ldots + r_n(y)$$

$k \in \{1, \ldots, n\}$:

$$k \cdot r_k(y) = \mathbf{S}_k(y) - \mathbf{S}_{k-1}(y) \cdot r_1(y) - \ldots - \mathbf{S}_1(y) \cdot r_{k-1}(y)$$

Every $\mathbf{S}_j(y)$ is the coefficient of t^{-1} in the expansion in negative powers of t of:

$$\frac{(f(t))^j g'(t)}{g(t) - y}$$

$$x = f(t) = t^4 - t + 1 \qquad y = g(t) = t^3 + t + 1$$

$$\mathcal{H}_{\mathcal{C}}(x, y) = x^3 + r_1(y)x^2 + r_2(y)x + r_3(y)$$

$$\frac{3t^2+1}{t^3+t+1-y} = \frac{3}{t} - \frac{2}{t^3} - \frac{-3+3y}{t^4} + \frac{2}{t^5} + \frac{5-5y}{t^6} + \frac{1-6y+3y^2}{t^7} - \frac{-7+7y}{t^8} - \frac{-6+16y-8y^2}{t^9} + \frac{6-9y^2+3y^3}{t^{10}} + \frac{13-30y+15y^2}{t^{11}} - \frac{-22y+33y^2-11y^3}{t^{12}} - \frac{-19+36y-6y^2-12y^3+3y^4}{t^{13}} - \frac{-13+65y-78y^2+26y^3}{t^{14}} + \ldots\ldots$$

$$\mathbf{S}_1(y) = 5$$
$$\mathbf{S}_2(y) = -11 + 26y - 8y^2$$
$$\mathbf{S}_3(y) = -76 + 93y + 6y^2 - 21y^3 + 3y^4$$

$$\mathcal{H}_{\mathcal{C}}(x, y) = x^3 - 5x^2 + (18 - 13y + 4y^2)x - 23 + 34y - 22y^2 + 7y^3 - y^4$$

Example

Short Summary From Lecture I

Computer Aided Geometric Design:

Computations with curves and surfaces over the reals.

In practice:

- ❖ Parametric representations.
- ❖ The coefficients are floating-point real numbers.

Implicit Equations:

Very useful when available.

Extraneous factors:

- ❖ Algebraic.
- ❖ Geometric.

Difficult to compute even in the exact-coefficient case:

- ❖ Generic Implicitation.
- ❖ Newton Sums.

Recent (Semi)-Numerical Approaches:

- through Calculus of Variations and as an eigenvalue problem
  - Corless, Giesbrecht, Kotsireas & Watt (2000)

- through a SVD problem over the composite matrix and monoids
  - Dokken (1997)
  - Sederberg, Zheng, Klimaszewski & Dokken (1999)

- through Newton's method
  - Hartmann (1998)

The approximate implicitization factorization

- [Dokken 1997, 2001] Approximate Implicitization
- Assume that the surface $\mathbf{p}(s,t)$ has degree (n_1,n_2)
- Assume that q has total degree m and that $\mathbf{b}$ is a vector containing the unknown coefficients of q
- The combination $q(\mathbf{p}(s,t))$ is a polynomial of degrees (mn_1, mn_2)
- Collect basis functions of degree (mn_1,mn_2) in $\boldsymbol{\alpha}(s,t)$

Then $q(\mathbf{p}(s,t))$ can be factorized

$$q(\mathbf{p}(s,t)) = (\mathbf{Db})^T \alpha(s,t)$$

The factorization

$$q(\mathbf{p}(s,t)) = (\mathbf{Db})^T \alpha(s,t)$$

- The matrix **D** is built from products of the coefficients of **p**(*s,t*).
- An element in **D** is the product of a maximum of *m* such coefficients, where *m* the total degree of *q*.
- If **p**(*s,t*) was described in a Bernstein basis of degree (n_1,n_2) then $\boldsymbol{\alpha}(s,t)$ contains the Bernstein basis of degree (mn_1,mn_2).
- The first step of moving curves and surfaces use the same factorization.

Properties of the factorization

$$q(\mathbf{p}(s,t)) = (\mathbf{Db})^T \alpha(s,t)$$

- If **Db**=0 and **b**≠**0** then $q(\mathbf{p}(s,t))=0$ and **b** describes an implicitization q of $\mathbf{p}(s,t)$).
- If $\boldsymbol{\alpha}(s,t)$ describes a Bernstein basis then $\| \alpha(s,t) \|_2 \leq 1$

$$| q(\mathbf{p}(s,\mathrm{t})) | = \left| (\mathbf{Db})^{\mathrm{T}} \alpha(s,t) \right| \leq \| \mathbf{Db} \|_2$$

THE SINGULAR VALUE DECOMPOSITION

Let $\mathbf{A}$ be an arbitrary m–by–n matrix with $m \geq n$. Then there exist orthogonal matrices $\mathbf{U}$ and $\mathbf{V}$ and $\mathbf{\Sigma} = \text{diagonal}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \ldots \geq \sigma_n \geq 0$ such that

$$\mathbf{A} = \mathbf{U\Sigma V}.$$

If

$$\sigma_1 \geq \ldots \geq \sigma_r > \sigma_{r+1} = \ldots = \sigma_n = 0$$

then the rank of $\mathbf{A}$ is r and the nullspace of $\mathbf{A}$ is generated by the last $n - r$ columns of $\mathbf{V}$.

Properties of the inequality

$$| q(\mathbf{p}(s,\mathrm{t})) | = \left| (\mathbf{Db})^{\mathrm{T}} \alpha(s,t) \right| \leq \| \mathbf{Db} \|_2$$

Let σ_1 be the smallest singular value of **D** then

$$\min_{\| \mathbf{b} \|_2 = 1} \max_{(s,t) \in \Omega} | q(\mathbf{p}(s,t)) | \leq \sigma_1$$

Singular value decomposition (SVD) can be used to find approximate solutions of the implicitization problem.

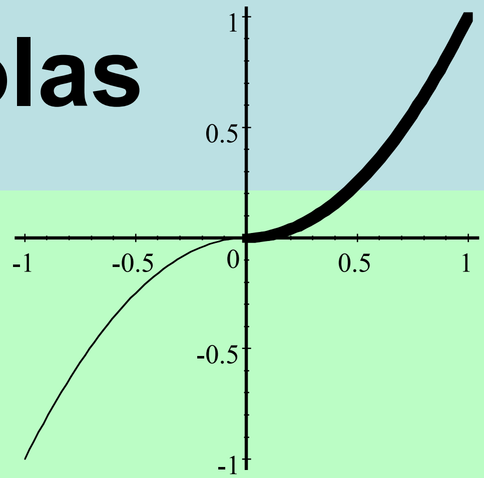
Piecewise polynomials can be approximated

- Approximation of multiple manifolds

$$\sum_{i=1}^{r}\left(q(\mathbf{p}_i(s_i,t_i))\right)^2 \leq \left\|\begin{pmatrix}\mathbf{D}_1\\ \vdots \\ \mathbf{D}_r\end{pmatrix}\mathbf{b}\right\|_2^2$$

- D_1 can be based on a curve, D_2 on a surface,...

Example two parabolas



$$\mathbf{p}_1(s) = (-1,-1)(1-s)^2 + \left(-\tfrac{1}{2},0\right)2(1-s)s + (0,0)s^2$$
$$\mathbf{p}_2(s) = (0,0)(1-s)^2 + \left(\tfrac{1}{2},0\right)2(1-s)s + (1,1)s^2$$

- We want to approximate both curve segments at the same time with one algebraic curve of degree 3.
- Thus we will make
  - $q(\mathbf{p}_1(s)) = (\mathbf{D}_1\mathbf{b})\boldsymbol{\alpha}_1(s)$
  - $q(\mathbf{p}_2(s)) = (\mathbf{D}_2\mathbf{b})\boldsymbol{\alpha}_2(s)$

  and combine the matrices: $\mathbf{D} = \begin{pmatrix} \mathbf{D}_1 \\ \mathbf{D}_2 \end{pmatrix}$

The combined matrix

Contribution from $\mathbf{D}_1$

Contribution from $\mathbf{D}_2$

$$\begin{pmatrix} \mathbf{D}_1 \\ \hline \mathbf{D}_2 \end{pmatrix} = \left(\begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 1 \\
0 & 0 & 0 & 0 & \frac{1}{15} & 0 & 0 & \frac{1}{3} & \frac{1}{15} & 1 \\
\frac{1}{20} & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{20} & 0 & \frac{1}{2} & \frac{1}{5} & 1 \\
\frac{1}{5} & \frac{1}{15} & 0 & 0 & \frac{2}{5} & \frac{1}{5} & \frac{1}{15} & \frac{2}{3} & \frac{2}{5} & 1 \\
\frac{1}{2} & \frac{1}{3} & \frac{1}{6} & 0 & \frac{2}{3} & \frac{1}{2} & \frac{1}{3} & \frac{5}{6} & \frac{2}{3} & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
-1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\
-\frac{1}{2} & -\frac{1}{3} & -\frac{1}{6} & 0 & \frac{2}{3} & \frac{1}{2} & \frac{1}{3} & -\frac{5}{6} & -\frac{2}{3} & 1 \\
-\frac{1}{5} & -\frac{1}{15} & 0 & 0 & \frac{2}{5} & \frac{1}{5} & \frac{1}{15} & -\frac{2}{3} & -\frac{2}{5} & 1 \\
-\frac{1}{20} & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{20} & 0 & -\frac{1}{2} & -\frac{1}{5} & 1 \\
0 & 0 & 0 & 0 & \frac{1}{15} & 0 & 0 & -\frac{1}{3} & -\frac{1}{15} & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{6} & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)$$

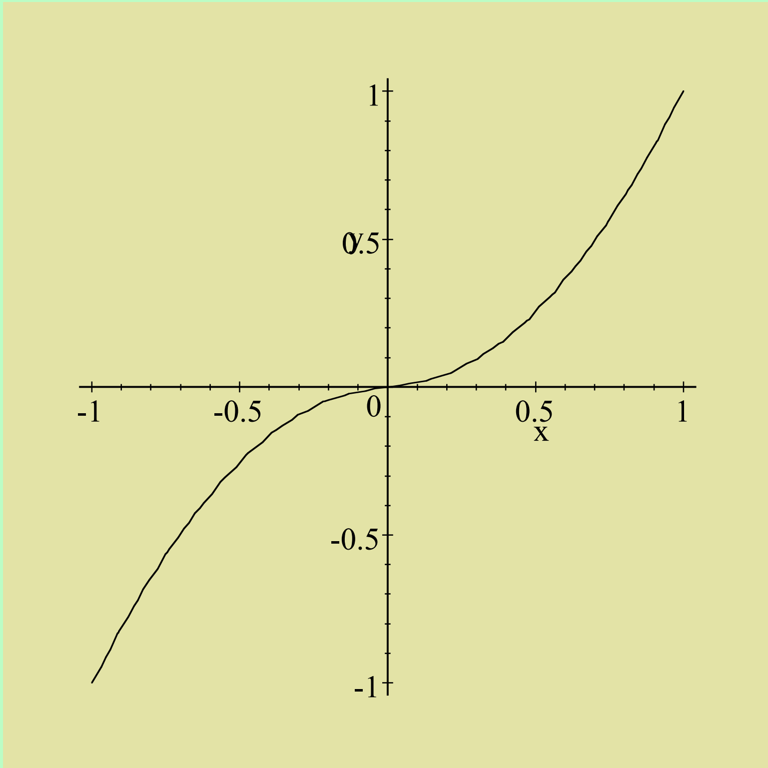
Implicit from combined D

Singular Values:

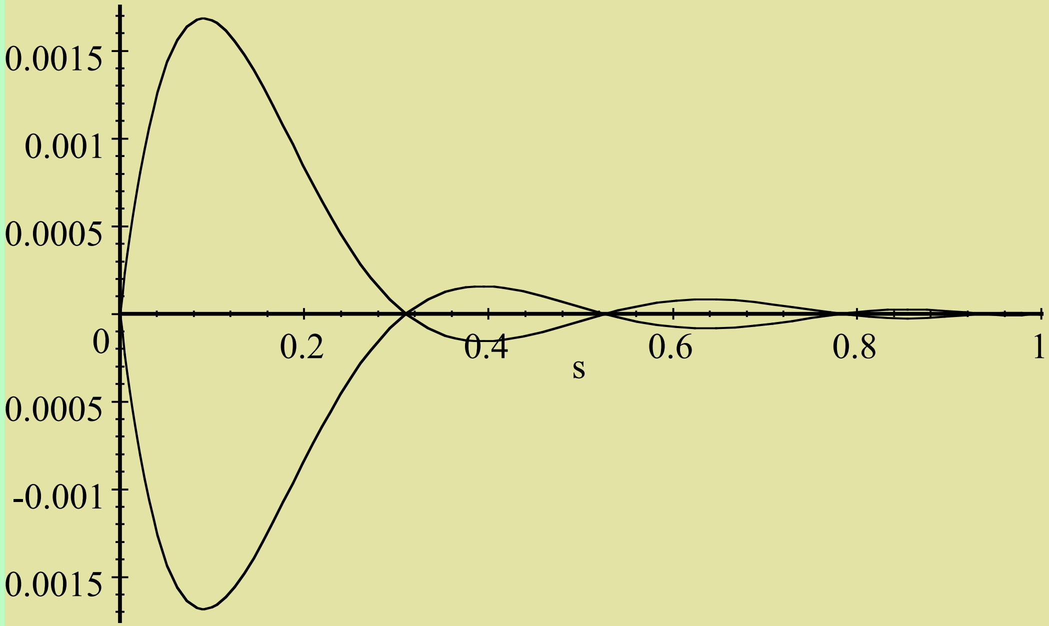
4.25, 3.91, 1.98, 1.31, 0.38, 0.37, 0.11, 0.05, 0.03, 0.007937

Combining eigenvector with basis functions and plot implicit:

$$\begin{pmatrix} x^3 \\ x^2 y \\ x y^2 \\ y^3 \\ x^2 \\ xy \\ y^2 \\ x \\ y \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -.4602815334 \\ .6983314313 \\ -.5087222523 \\ .1433800204 \\ 0 \\ 0 \\ 0 \\ -.0170228764 \\ .1443197267 \\ 0 \end{pmatrix} = 0$$



Error of $\mathbf{p}_1(s)$ and $\mathbf{p}_2(s)$



FURTHER WORK (II)

To detect in advance (before specialization) if specialization problems are going to appear.

To check in advance the existence of base points in the parameter domain

A base point is a real solution of the overdetermined system:

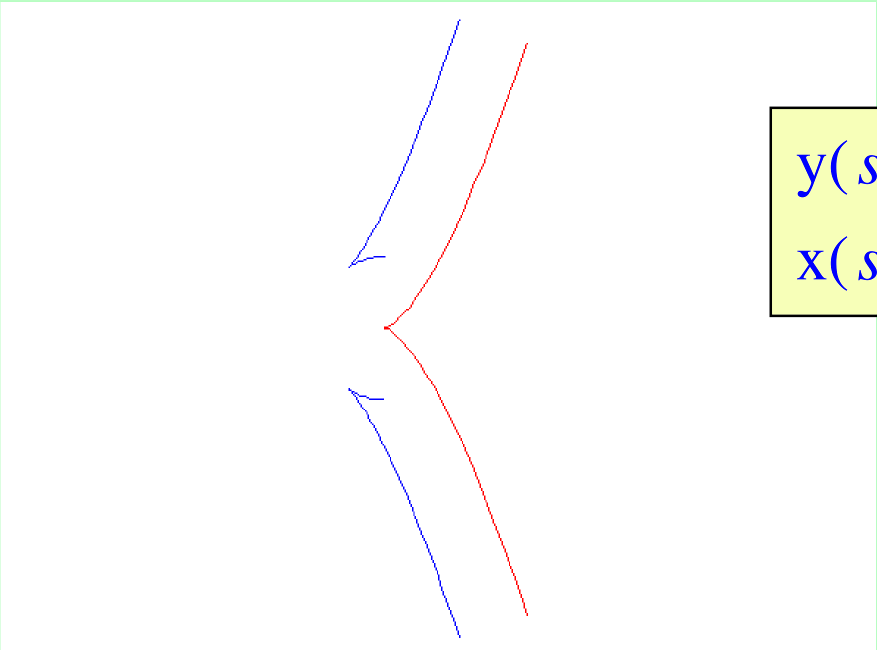
$$\begin{cases} q(s,t) & = & 0 \\ f_1(s,t) & = & 0 \\ f_2(s,t) & = & 0 \\ f_3(s,t) & = & 0 \end{cases}$$

OFFSETS

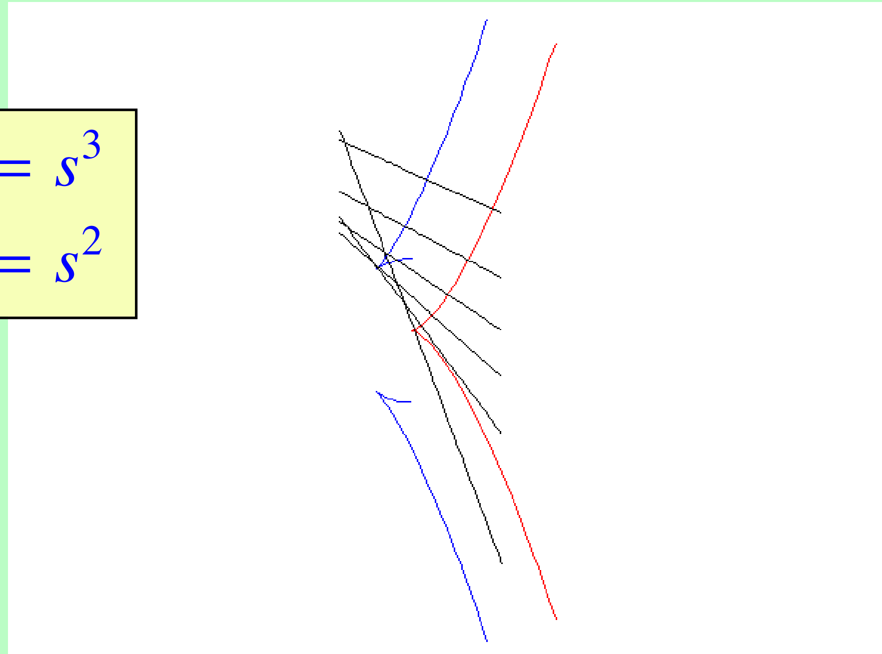
Very often used in CAD/CAM in order to define trajectories or safety regions

Geometric definition for the offset of a curve to distance *d*

Set of points whose euclidean distance from a corresponding point on the curve is *d*



$$y(s) = s^3$$
$$x(s) = s^2$$



$$x(t)=u(t)+\frac{d\left(\frac{\partial}{\partial t}v(t)\right)}{\sqrt{\left(\frac{\partial}{\partial t}u(t)\right)^2+\left(\frac{\partial}{\partial t}v(t)\right)^2}}$$

$$y(t)=v(t)-\frac{d\left(\frac{\partial}{\partial t}u(t)\right)}{\sqrt{\left(\frac{\partial}{\partial t}u(t)\right)^2+\left(\frac{\partial}{\partial t}v(t)\right)^2}}$$

$$x(t)=u(t)-\frac{d\left(\frac{\partial}{\partial t}v(t)\right)}{\sqrt{\left(\frac{\partial}{\partial t}u(t)\right)^2+\left(\frac{\partial}{\partial t}v(t)\right)^2}}$$

$$y(t)=v(t)+\frac{d\left(\frac{\partial}{\partial t}u(t)\right)}{\sqrt{\left(\frac{\partial}{\partial t}u(t)\right)^2+\left(\frac{\partial}{\partial t}v(t)\right)^2}}$$

Offsets of plane curves: algebraic definitions

$$f(u, v) := 0 \quad \rightarrow \quad \begin{aligned} & f(u, v) = 0 \\ & (x - u)^2 + (y - v)^2 - d^2 = 0 \\ & (x - u)\left(\frac{\partial}{\partial v} f\right)(u, v) - (y - v)\left(\frac{\partial}{\partial u} f\right)(u, v) = 0 \end{aligned}$$

$$\begin{aligned} u &= \mathrm{u}(t) \\ v &= \mathrm{v}(t) \end{aligned} \quad \rightarrow \quad \begin{aligned} & (\mathrm{u}(t) - x)^2 + (\mathrm{v}(t) - y)^2 - d^2 = 0 \\ & (x - \mathrm{u}(t))\left(\frac{\partial}{\partial t} u\right)(t) + (y - \mathrm{v}(t))\left(\frac{\partial}{\partial t} v\right)(t) = 0 \end{aligned}$$

Examples

$$y(s) = s^3$$
$$x(s) = s^2$$

$$(x^2 + y^2 - R^2)(729 R^8 + 16 R^4 - 216 R^6 + (-1944 R^6 + 288 R^4) x$$
$$+ (-32 R^2 + 1188 R^4 - 729 R^6) y^2 + (-2916 R^6 + 1728 R^4) x^2$$
$$+ (-504 R^2 + 6318 R^4) y^2 x + (4104 R^4 - 32 R^2) x^3 + (-1701 R^2 + 16) y^4$$
$$+ (-2484 R^2 + 2187 R^4) y^2 x^2 + 729 y^6 + (4374 R^4 - 504 R^2) x^4 - 2376 x^5 R^2 + 729 x^8$$
$$+ 216 x^7 + (-2916 R^2 + 16) x^6 + (216 - 4374 R^2) y^4 x + (-4860 R^2 - 32) y^2 x^3$$
$$+ 729 y^2 x^6 - 1458 y^2 x^5 + (-432 - 2187 R^2) y^2 x^4 - 1458 y^4 x^3 + 729 y^4 x^2)$$

Extraneous factors

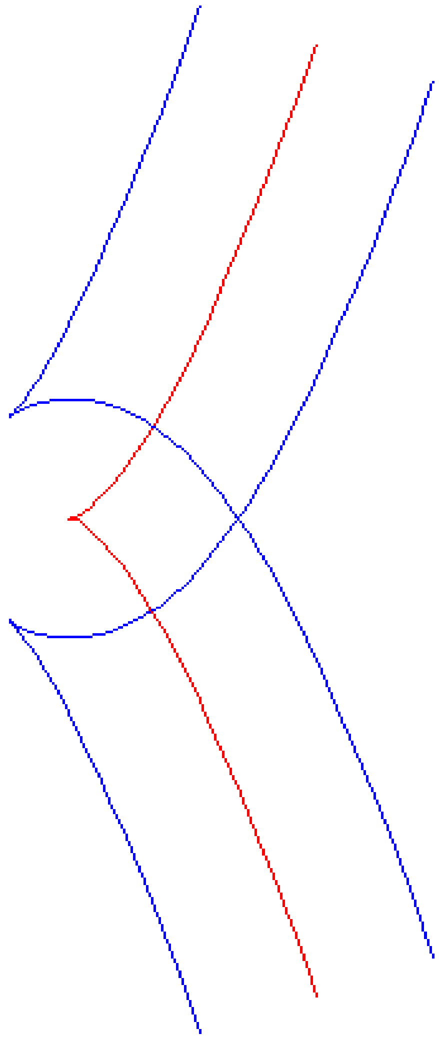
In the polynomial case, they are avoided through the division of the second equation by

$$\gcd\left(\frac{\partial}{\partial s} X(s), \frac{\partial}{\partial s} Y(s)\right)$$

Examples

$$729x^8 + 729x^6y^2 + 216x^7 - 1458x^5y^2 - 1458x^3y^4 - 11648x^6 - 9180x^4y^2 + 729x^2y^4 + 729y^6 - 9504x^5 - 19472x^3y^2 - 17280xy^4 + 67968x^4 + 25056x^2y^2 - 6788y^4 + 65536x^3 + 99072xy^2 - 158976x^2 - 27776y^2 - 119808x + 173056$$

$$y(s) = s^3$$
$$x(s) = s^2$$



$R = 2$

Non expanded curve implicitation

The implicit equation of the d-offset for the curve $\mathcal{C}$ defined by

$$x = u(t) \qquad\qquad y = v(t)$$

is a divisor of

$$\mathcal{H}_{\mathcal{C}}(x, y) = \mathrm{Resultant}_t \left[\begin{array}{c} (u(t)-x)^2 + (v(t)-y)^2 - d^2 \\ (u(t)-x)\frac{\partial u}{\partial t}(t) + (v(t)-y)\frac{\partial v}{\partial t}(t)) \end{array} \right] =$$

$$= \prod_{\{\tilde{t} : G(x,y;\tilde{t})=0\}} ((u(\tilde{t})-x)\frac{\partial u}{\partial t}(\tilde{t}) + (v(\tilde{t})-y)\frac{\partial v}{\partial t}(\tilde{t})))$$

$$G(x, y; t) = (u(t)-x)^2 + (v(t)-y)^2 - d^2 = 0 = 0$$

Offsets of surfaces: algebraic definitions

$$u = \mathrm{u}(s,t),\ v = \mathrm{v}(s,t),\ w = \mathrm{w}(s,t)$$

$$(x - \mathrm{u}(s,t))^2 + (y - \mathrm{v}(s,t))^2 + (z - \mathrm{w}(s,t))^2 - R^2 = 0$$

$$\left[\begin{array}{l} \left(\frac{\partial}{\partial s} u\right)(s,t)\,(x - \mathrm{u}(s,t)) + \left(\frac{\partial}{\partial s} v\right)(s,t)\,(y - \mathrm{v}(s,t)) + \left(\frac{\partial}{\partial s} w\right)(s,t)\,(z - \mathrm{w}(s,t)) = 0 \\ \left(\frac{\partial}{\partial t} u\right)(s,t)\,(x - \mathrm{u}(s,t)) + \left(\frac{\partial}{\partial t} v\right)(s,t)\,(y - \mathrm{v}(s,t)) + \left(\frac{\partial}{\partial t} w\right)(s,t)\,(z - \mathrm{w}(s,t)) = 0 \end{array}\right]$$

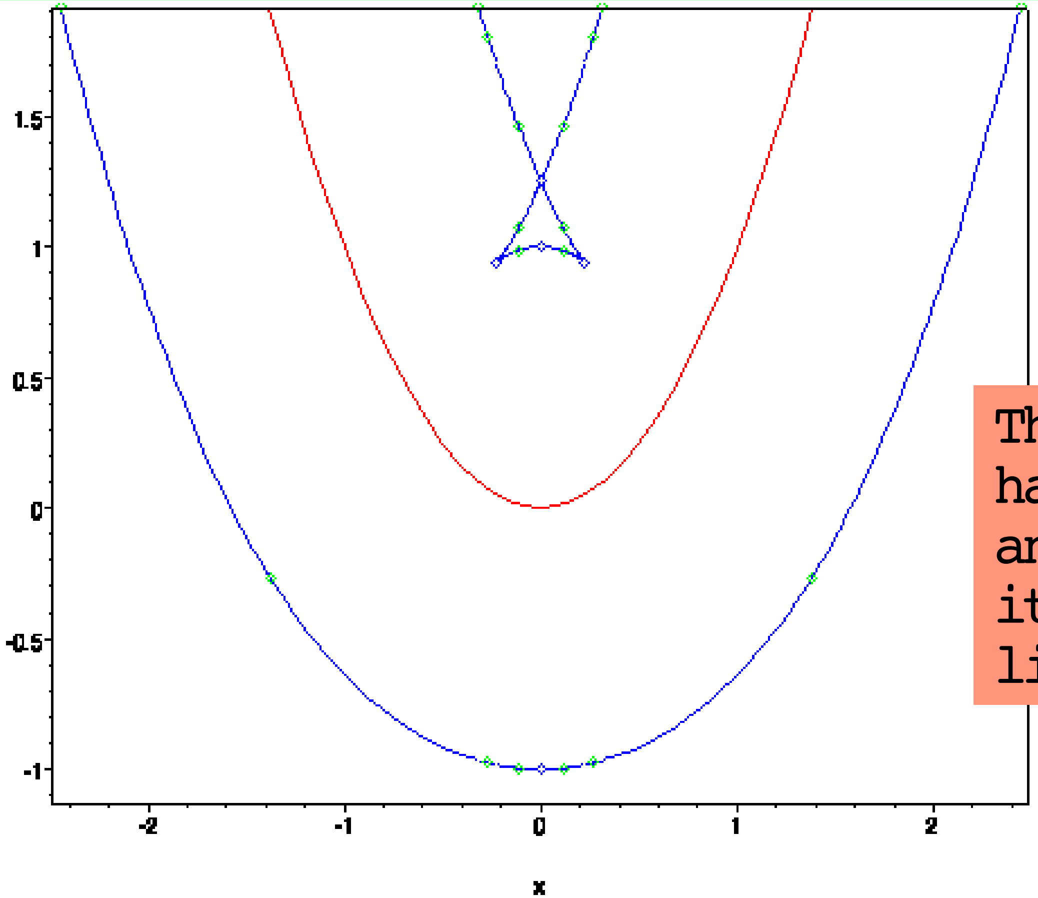
Flaws and Drawbacks

- Offsets are used very often in practice in CAD/CAM.
- Offsets of rational curves/surfaces are not in general rational curves/surfaces.
- Offset implicit equations are usually huge and difficul to manipulate.
- Offset computation produces extraneous factors in many cases [they could be eliminated in advance but this will complicate even more the elimination process].

and moreover ..

$y = x^2$ $R = 1$

$$16x^6 + 16x^4y^2 - 40x^4y - 32x^2y^3 - 47x^4 + 16y^4 + 6x^2y - 40y^3 + 28x^2 + 9y^2 + 40y - 25$$



The offset of a parabola has real singular points and topologically one of its two components is not like a parabola.

Offsets Topology Control

When the d-offset

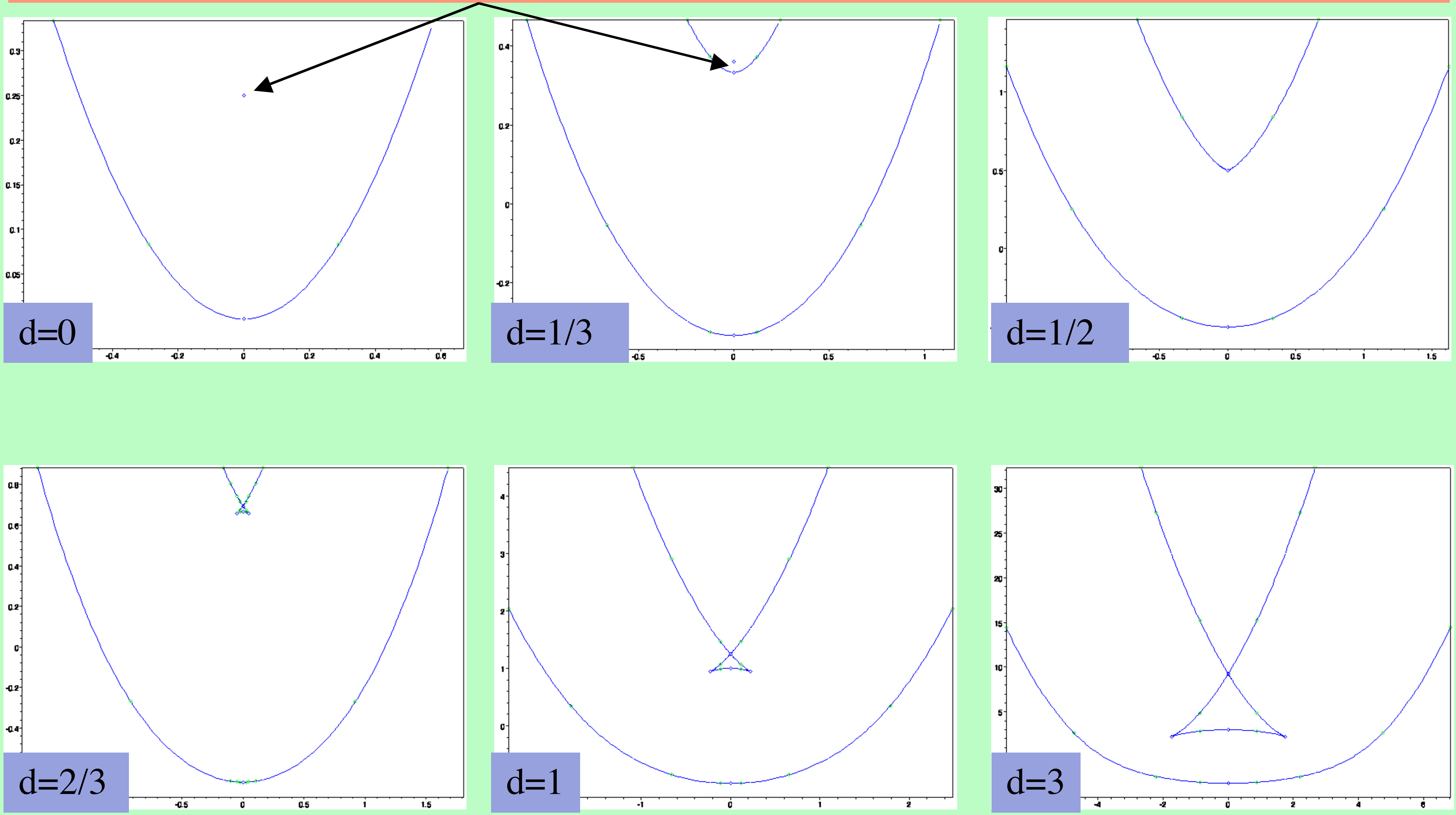
$$16x^6 + 16x^4y^2 - 40x^4y - 32x^2y^3 + (-48d^2 + 1)x^4 + (-32d^2 + 32)x^2y^2 \\ + 16y^4 + (-2 + 8d^2)x^2y + (-8 - 32d^2)y^3 + (-20d^2 + 48d^4)x^2 \\ + (16d^4 - 8d^2 + 1)y^2 + (32d^4 + 8d^2)y - 16d^6 - 8d^4 - d^2$$

of the parabola

$$\mathrm{u}(t) = t, \mathrm{v}(t) = t^2$$

is not topologically (like) a parabola?

The isolated point comes from the complex part ……………………………



Short Summary From Lectures I and II

Computer Aided Geometric Design:

Computations with curves and surfaces over the reals.

In practice:

- ❖ Parametric representations.
- ❖ The coefficients are floating-point real numbers.

Implicit Equations:

Very useful when available.

Extraneous factors: Algebraic and Geometric.

Difficult to compute even in the exact-coefficient case:

- ❖ Generic Implicitation.
- ❖ Newton Sums.

Approximate Implicitation through composition and the SVD.

Offsets:

Used in CAD/CAM to define trajectories for NC machine-tools or safety regions.

Offsets are algebraic sets but, in general, not rational.

Extraneous factors: Algebraic (easy to remove) and Geometric (difficult to remove) .

Offsets implicit equations are difficult to compute (in practice they are interpolated).

Offsets introduces new singularities (to analyze through Quantifier Elimination).

An easy to compute Quantifier Elimination

The offset is not a topologically a parabola when the discriminant with respect to y:

$$x^2 d^2 (64 x^6 + (48 - 192 d^2) x^4 + (192 d^4 + 336 d^2 + 12) x^2 - 64 d^6 - 12 d^2 + 48 d^4 + 1)^3$$

has more than 1 real root. Or, equivalently, if the polynomial

$$64 x^3 + (48 - 192 d^2) x^2 + (192 d^4 + 336 d^2 + 12) x - 64 d^6 - 12 d^2 + 48 d^4 + 1$$

has one positive real root.

$$P := a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \qquad Q := b_3 x^3 + b_2 x^2 + b_1 x + b_0$$

M. Coste's Lectures
M.-F. Roy's Lectures

$$M_0 := \begin{bmatrix} a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ 0 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & a_4 & a_3 & a_2 & a_1 & a_0 \\ b_3 & b_2 & b_1 & b_0 & 0 & 0 & 0 \\ 0 & b_3 & b_2 & b_1 & b_0 & 0 & 0 \\ 0 & 0 & b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & 0 & 0 & b_3 & b_2 & b_1 & b_0 \end{bmatrix}$$

Resultant:=det(M_0)=$Sres_0$(P,Q)

$$M_1 := \begin{bmatrix} a_4 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & a_4 & a_3 & a_2 & a_1 & a_0 \\ b_3 & b_2 & b_1 & b_0 & 0 & 0 \\ 0 & b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & 0 & b_3 & b_2 & b_1 & b_0 \end{bmatrix}$$

$Sres_1$(P,Q):=det(M_{11})x+det(M_{10})

$$M_2 := \begin{bmatrix} a_4 & a_3 & a_2 & a_1 & a_0 \\ b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & b_3 & b_2 & b_1 & b_0 \end{bmatrix}$$

$Sres_2$(P,Q):= det(M_{22})x^2+ det(M_{21})x+ det(M_{20})

Coming back to the initial problem:

For any $d > 0$ the number of real roots of

$$64\,x^3 + (48 - 192\,d^2)\,x^2 + (192\,d^4 + 336\,d^2 + 12)\,x - 64\,d^6 - 12\,d^2 + 48\,d^4 + 1$$

is always equal to *1*:

$$[\,1, 1, -d^2, -d^4\,(4\,d^2 + 1)^2\,]$$

For any $d > 1/2$ the number of real roots of

$$64\,x^3 + (48 - 192\,d^2)\,x^2 + (192\,d^4 + 336\,d^2 + 12)\,x - 64\,d^6 - 12\,d^2 + 48\,d^4 + 1$$

with $x > 0$ is always equal to *1*:

$$[\,1, (2\,d - 1)\,(2\,d + 1), -d^2\,(20\,d^2 + 1)\,(4\,d^2 + 5), -d^4\,(4\,d^2 + 1)^2\,(2\,d - 1)^3\,(2\,d + 1)^3\,]$$

$$P_n(\underline{a}, x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

$$\forall x \quad P_n(\underline{a}, x) > 0 \iff \mathbf{C}(\mathbf{stha}_n(P_n, 1), \ldots, \mathbf{stha}_0(P_n, 1)) = 0$$

$$\exists x \quad P_n(\underline{a}, x) = 0 \iff \mathbf{C}(\mathbf{stha}_n(P_n, 1), \ldots, \mathbf{stha}_0(P_n, 1)) > 0$$

Laureano González-Vega
A Combinatorial Algorithm solving some Quantifier Elimination Problems
Libro: Quantifier Elimination and Cylindrical Algebraic Decomposition, Texts and Monographs in Symbolic Computation, 365-375, Springer-Verlag, 1998

COMPUTING $\mathbb{H}_4$:

$$
\begin{array}{rl}
 & \{(a_3,a_2,a_1,a_0)\in\mathbb{R}^4 : \mathbf{S}_2>0,\mathbf{S}_1<0,\mathbf{S}_0>0\}\cup \\
\cup & \{(a_3,a_2,a_1,a_0)\in\mathbb{R}^4 : \mathbf{S}_2<0,\mathbf{S}_1>0,\mathbf{S}_0>0\}\cup \\
\cup & \{(a_3,a_2,a_1,a_0)\in\mathbb{R}^4 : \mathbf{S}_2<0,\mathbf{S}_1<0,\mathbf{S}_0>0\}\cup \\
\cup & \{(a_3,a_2,a_1,a_0)\in\mathbb{R}^4 : \mathbf{S}_2<0,\mathbf{S}_1=0,\mathbf{S}_0>0\}\cup \\
\cup & \{(a_3,a_2,a_1,a_0)\in\mathbb{R}^4 : \mathbf{S}_2<0,\mathbf{S}_1=0,\mathbf{S}_0<0\}\cup \\
\cup & \{(a_3,a_2,a_1,a_0)\in\mathbb{R}^4 : \mathbf{S}_2=0,\mathbf{S}_1>0,\mathbf{S}_0<0\}\cup \\
\cup & \{(a_3,a_2,a_1,a_0)\in\mathbb{R}^4 : \mathbf{S}_2=0,\mathbf{S}_1<0,\mathbf{S}_0>0\}\cup \\
\cup & \{(a_3,a_2,a_1,a_0)\in\mathbb{R}^4 : \mathbf{S}_2=0,\mathbf{S}_1=0,\mathbf{S}_0>0\}\cup \\
\cup & \{(a_3,a_2,a_1,a_0)\in\mathbb{R}^4 : \mathbf{S}_2<0,\mathbf{S}_1=0,\mathbf{S}_0=0\}
\end{array}
$$

$$
\begin{array}{rl}
\mathbf{S}_2 = & 3a_3^2-8a_2 \\
\mathbf{S}_1 = & 2a_2^2a_3^2-8a_2^3+32a_2a_0+a_1a_2a_3-12a_3^2a_0-6a_1a_3^3-36a_1^2 \\
\mathbf{S}_0 = & -27a_1^4-4a_3^3a_1^3+18a_2a_3a_1^3-6a_3^2a_0a_1^2+144a_2a_0a_1^2+a_2^2a_3^2a_1^2-4a_2^3a_1^2- \\
 & 192a_3a_0^2a_1+18a_0a_2a_3^3a_1-80a_0a_2^2a_3a_1+256a_0^3-27a_3^4a_0^2+144a_2a_3^2a_0^2- \\
 & 128a_2^2a_0^2-4a_2^3a_3^2a_0+16a_2^4a_0
\end{array}
$$

REDUCTION:

$$\begin{array}{rl} & \{(a_3,a_2,a_1,a_0)\in\mathbb{R}^4 : \mathbf{S}_2<0, \mathbf{S}_1\neq 0, \mathbf{S}_0>0\}\cup \\ \cup & \{(a_3,a_2,a_1,a_0)\in\mathbb{R}^4 : \mathbf{S}_2=0, \mathbf{S}_1\leq 0, \mathbf{S}_0>0\}\cup \\ \cup & \{(a_3,a_2,a_1,a_0)\in\mathbb{R}^4 : \mathbf{S}_2>0, \mathbf{S}_1< 0, \mathbf{S}_0>0\}\cup \\ \cup & \{(a_3,a_2,a_1,a_0)\in\mathbb{R}^4 : \mathbf{S}_2=0, \mathbf{S}_1> 0, \mathbf{S}_0<0\}\cup \\ \cup & \{(a_3,a_2,a_1,a_0)\in\mathbb{R}^4 : \mathbf{S}_2<0, \mathbf{S}_1=0\} = \end{array}$$

$$\begin{array}{rl} = & \{(a_3,a_2,a_1,a_0)\in\mathbb{R}^4 : \mathbf{S}_2<0, \mathbf{S}_1\neq 0, \mathbf{S}_0>0\}\cup \\ \cup & \{(a_3,a_2,a_1,a_0)\in\mathbb{R}^4 : \mathbf{S}_2=0, \mathbf{S}_1\leq 0, \mathbf{S}_0>0\}\cup \\ \cup & \{(a_3,a_2,a_1,a_0)\in\mathbb{R}^4 : \mathbf{S}_2>0, \mathbf{S}_1< 0, \mathbf{S}_0>0\}\cup \\ \cup & \{(a_3,a_2,a_1,a_0)\in\mathbb{R}^4 : \mathbf{S}_2<0, \mathbf{S}_1=0\} \end{array}$$

$$\mathbf{S}_2=0 \quad\Longrightarrow\quad a_2=\frac{3a_3^2}{8} \quad\Longrightarrow\quad \mathbf{S}_1=-(16a_1+a_3^3)^2\leq 0$$

Multiple Modes in densities with normal conditionals

$$f_{X,Y}(x,y) — \exp(-(\alpha x^2y^2+x^2+y^2+\beta xy+\gamma x+\delta y))$$

MODES

$$x = -\frac{\beta y + \gamma}{2(\alpha y^2 + 1)}$$

$$y = -\frac{\beta x + \gamma}{2(\alpha x^2 + 1)}$$

Conditions: giving the modes according to the real roots of

$$4\alpha^2 y^5 + 2\alpha^2 \delta y^4 + 8\alpha y^3 + \alpha(4\delta + \beta\gamma)y^2 + \ldots$$

	signo(S_3)	signo(S_2)	signo(S_1)	signo(S_0)
1	$\bullet$	< 0	$= 0$	$\bullet$
2	$\bullet$	< 0	$\neq 0$	> 0
3	$\bullet$	$= 0$	$= 0$	> 0
4	$= 0$	$= 0$	$\neq 0$	> 0
5	< 0	≥ 0	> 0	> 0
6	> 0	≥ 0	< 0	> 0
7	$= 0$	$= 0$	$= 0$	$= 0$

Existence of three real roots

N. Ioakimidis: *Beams on tensionless elastic foundation*.
Internat. J. Numer. Methods Engrg. 39 (1996), no. 4, 663-686.

R. Liska and B. Wendroff: *Analysis and computation with stratified fluid models*
J. Comput. Phys. 137 (1997), 2, 161-187.

H. Hong, R. Liska and S. Steinberg: *Testing stability by quantifier elimination*
J. Symbolic Comput. 24 (1997), 1-2, 225-242.

N. Ioakimidis: *QE in applied mechanics problems with CAD*.
Internat. J. Solids Structures 34 (1997), no. 30, 4037-4070.

H. Hong and J. Dalibor:*Testing positiveness of polynomials*.
J. Automat. Reason. 21 (1998), 1, 23-38.

N. Ioakimidis:
Fracture initiation at an elastic crack tip: A computational implementation of the T-criterion.
Internat. J. of Fracture. 98 (1999), 3/4, 293-311.

P. Pau and J. Schicho:
Quantifier elimination for trigonometric polynomials by cylindrical trigonometric decomposition.
J. Symbolic Comput. 29 (2000), 6, 971-983.

Pytaghorean Hodograph Curves

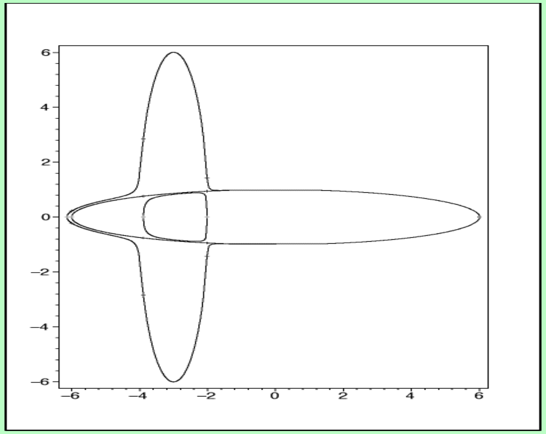
$x = f(t)$
$y = g(t)$

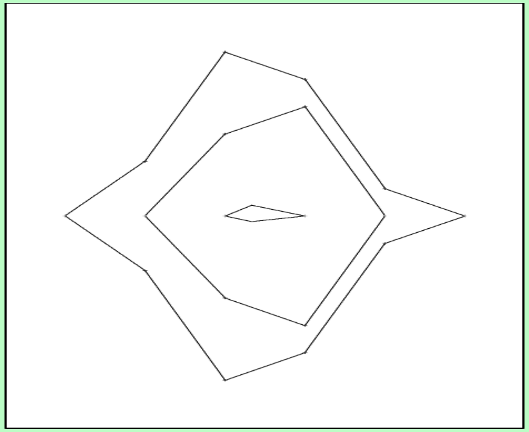
$$x'(t)^2 + y'(t)^2 = h(t)^2$$

The determination of these curves requires to characterize the existence of real roots for a degree 2k (4) univariate polynomial whose coefficients depend on several parameters

B. Juttler: Hermite Interpolation by PHC of degree seven. Math. of Comp. , 2000

A motivating example:
Efficient Topology Computation for Implicit Plane Curves





$$g(x,y) = \quad 279756x + 279936xy^4 - 559692y^2x - 15583x^3 + 217x^5 + \frac{130218}{5}x^2 - \frac{23039}{10}x^4 + \frac{359}{10}x^6 + 370656y^4$$

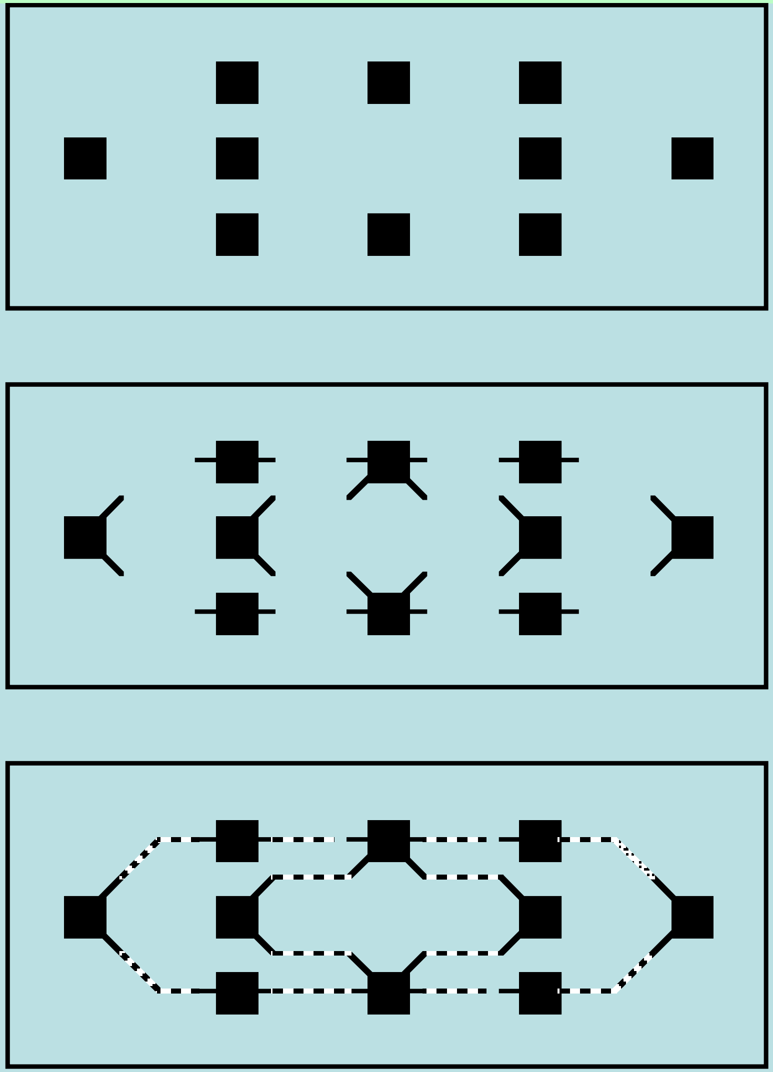
$$-\frac{3726432}{5}y^2 - 72774y^2x^2 + \frac{12947}{5}y^2x^4 + 1296y^6 + 46728y^4x^2 + 15588y^2x^3 + \frac{37333439}{100}$$

M. El Kahoui, I. Necula, L. Gonzalez-Vega

RAG&Applications, Trieste, August 2003

$$P(x,y) := 2x^4 - 3x^2y + y^4 - 2y^3 + y^2$$

$$R(x) = x(2048x^6 - 4608x^4 + 37x^2 + 12)$$



A curve defined by a squarefree polynomial $f(x,y) \in \mathbb{R}[x,y]$ is in general position

- if its y–leading coefficient has no real roots, and
- if the number of critical points over any $\alpha \in \mathbb{R}$ is 0 or 1.

The algorithm (the curve in general position)

$$f(x,y) = \sum_{j=0}^{d} a_j(x)y^j$$

Step I

- $j \in 0, 1, 2, \ldots, d$:

$$H_j(x,y) = h_j(x)y^j + h_{j,j-1}(x)y^{j-1} + h_{j,1}(x)y + h_{j,0}(x)$$

  the subresultant sequence of P and P_y (as polynomials in y).

- $R(x)$ the squarefree part of $h_0(x)$.
- $R(x) = \Gamma_0(x) \cdot \Gamma_1(x) \cdot \ldots \cdot \Gamma_d(x)$, the decomposition of $R(x)$ with respect to the polynomials $h_j(x)$.

The roots of $\Gamma_i(x)$ are the roots of $R(x)$ such that

$$h_0(x) = \ldots = h_{i-1}(x) = 0, \quad h_i(x) \neq 0$$

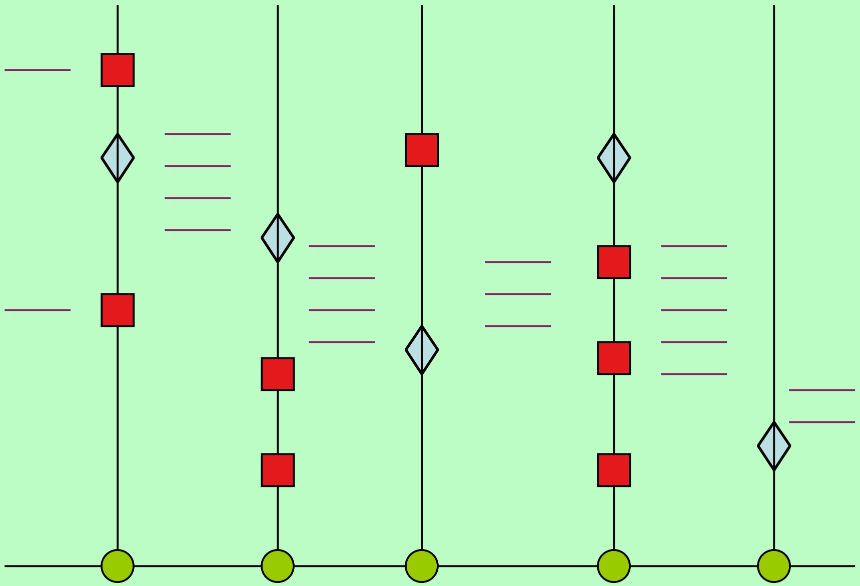
and moreover

$$\Gamma_i(\alpha) = 0 \quad \Longleftrightarrow \quad \gcd(P(\alpha, y), P_y(\alpha, y)) = H_i(\alpha, y)$$

Since the curve is in general position:

$$\Gamma_i(\alpha) = 0 \quad \Longrightarrow \quad H_i(\alpha, y) \quad \text{has only one root} \quad (\beta) \quad \Longrightarrow$$

$$\Longrightarrow \quad \beta = -\frac{1}{i+1} \frac{h_{i+1,i}(\alpha)}{h_i(\alpha)}$$



$$f(x, y) = 0$$

Generic Position

$$D(x) = \text{resultant}_y\left(f, \frac{\partial f}{\partial y}\right)$$

$$\alpha_1 < \alpha_2 < \ldots < \alpha_m$$

$$H_j(x, y) = \mathbf{StHa}_j(f) = h_j(x)y^j + h_{j,j-1}(x)y^{j-1} + \ldots + h_{j,0}(x)$$

$$r_k(x) = -\frac{1}{k}\frac{h_{k,k-1}(x)}{h_k(x)}$$

$$\beta_j^{(k)} = r_k(\alpha_j^{(k)})$$

$$F_k(\alpha_j^{(k)}, \beta_j^{(k)}, y) = \frac{f(\alpha_j^{(k)}, y)}{(y - \beta_j^{(k)})^k}$$

If for every $k \in \{1, \ldots, t\}$ and $j \in \{1, \ldots, s_k\}$ the following condition is verified:

$$H_k(\alpha_j^{(k)}, y) = h_k(\alpha_j^{(k)})(y - \beta_j^{(k)})^k$$

then the curve $\mathcal{C}(f)$ is in generic position.

$$f\left(\frac{\alpha_k + \alpha_{k+1}}{2}, y\right) = 0$$

Examples

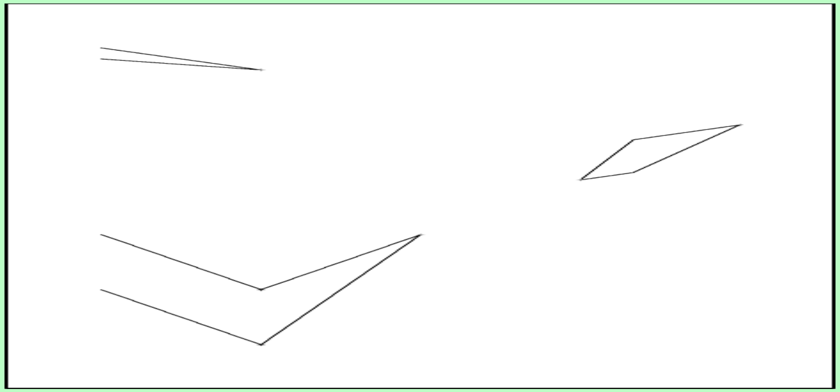
	Time	d	r	GP	Precision
Pol_1	2.481	8	28	y	20
Pol_2	0.130	5	4	y	15
Pol_3	0.269	5	5	y	15
Pol_4	0.171	4	4	n	15
Pol_5	0.061	4	3	n	10
Pol_6	1.170	8	8	n	20
Pol_7	0.120	4	2	n	10
Pol_8	0.360	6	6	y	15
Pol_9	0.869	8	13	n	15
Pol_{10}	0.351	6	5	n	10
Pol_{11}	2.409	6	4	y	15
Pol_{12}	116.959	8	4	y	40
Pol_{13}	0.110	8	3	y	10
Pol_{14}	3.569	6	8	n	30
Pol_{15}	0.180	4	6	n	10
Pol_{16}	0.230	6	7	n	10

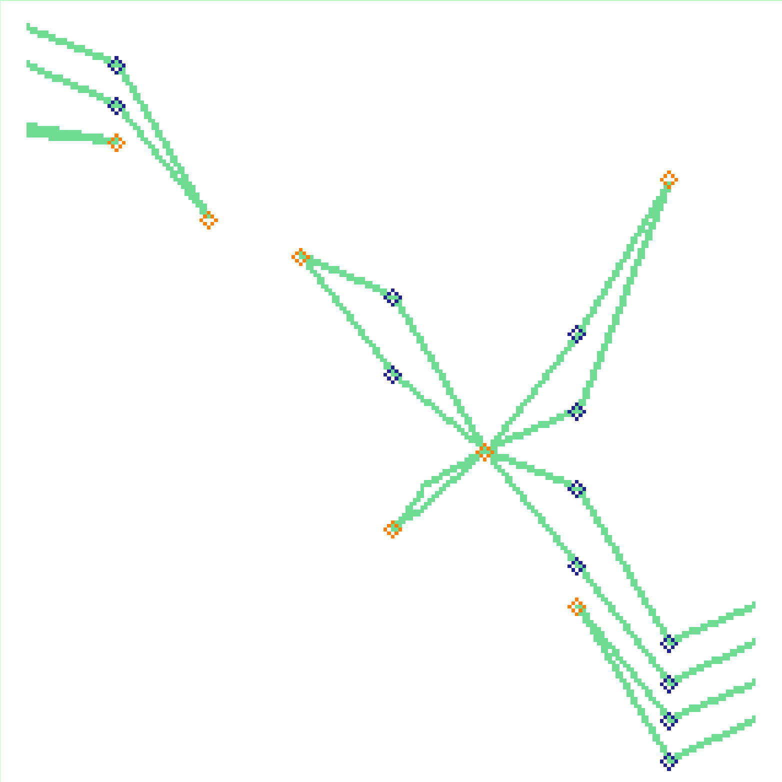
Table 1: Experimental results

$$\mathrm{Pol}_{12} = y^8 + 25y^4x^4 + 61y^6 + 62x^5y^4 + x^6 - 5x^2y^5 - 11xy + \frac{1}{10000000}y$$

Digits	α_2 and α_3
10	$0.9090909091 \cdot 10^{-8}$ $0.9090909091 \cdot 10^{-8}$
20	$0.90909090909090909091 \cdot 10^{-8}$ $0.90909090909090909091 \cdot 10^{-8}$
30	$0.909090909090909090909090909091 \cdot 10^{-8}$ $0.909090909090909090909090909091 \cdot 10^{-8}$
40	$0.9090909090909090909090909090909073334909 \cdot 10^{-8}$ $0.9090909090909090909090909090909108483272 \cdot 10^{-8}$

Table 2: Two real roots of the discriminant very close





```
(4637850728449*x^10)/281474976710656000000000000
+ (107121545878451*x^11)/70368744177664000000000000
+ (6360959449597*x^12)/17592186044416000000000000
- (239972680189053*x^9*y)/70368744177664000000000000
- (140653252266267*x^10*y)/70368744177664000000000000
+ (287674673451211*x^11*y)/281474976710656000000000000
- (442652693575939*x^8*y^2)/281474976710656000000000000
- (4009380758206559*x^9*y^2)/1407374883553280000000000000
- (1082242807229*x^10*y^2)/17592186044416000000000000
- (293971090400737*x^7*y^3)/70368744177664000000000000
- (3905296774251887*x^8*y^3)/1407374883553280000000000000
- (479026579521471*x^9*y^3)/175921860444160000000000000
+ (5471731524766511*x^6*y^4)/2814749767106560000000000000
+ (1222121019127831*x^7*y^4)/1407374883553280000000000000
- (980602721653537*x^8*y^4)/1407374883553280000000000000
+ (66805941205967*x^5*y^5)/87960930222080000000000000
+ (413471726083331*x^6*y^5)/281474976710656000000000000
- (13130211656779*x^7*y^5)/28147497671065600000000000
+ (3319914912300829*x^4*y^6)/1407374883553280000000000000
+ (2633963543959379*x^5*y^6)/1407374883553280000000000000
+ (52707229014351*x^6*y^6)/703687441776640000000000000
+ (3605872821703211*x^3*y^7)/1407374883553280000000000000
+ (322723471360697 7*x^4*y^7)/7036874417766400000000000000
+ (6275505371940339*x^5*y^7)/2814749767106560000000000000
+ (156971899888277*x^2*y^8)/175921860444160000000000000
+ (223637971466669*x^3*y^8)/703687441776640000000000000
+ (2070981496486811*x^4*y^8)/7036874417766400000000000000
+ (159358201507923*x*y^9)/28147497671065600000000000
+ (2942715212482791*x^2*y^9)/2814749767106560000000000000
+ (1353042667527611*x^3*y^9)/5629499534213120000000000000
+ (212364788238237*y^10)/70368744177664000000000000
+ (1419551874856703*x*y^10)/1407374883553280000000000000
+ (86404948006809*x^2*y^10)/703687441776640000000000000
+ (1712063301607367*y^11)/28147497671065600000000000
+ (782320953287937*x*y^11)/70368744177664000000000000
+ (405475175411651*y^12)/70368744177664000000000000
```

Becuwe, Stefan and Cuyt
On the Froissart phenomenon in multivariate homogeneous Pade approximation.
Adv. Comput. Math., 11(1), 21--40, 1999

- Gnuplot
- Maple
- Mathematica
- GraphEq

Short Summary From Lectures I, II and II

Computer Aided Geometric Design:

Computations with curves and surfaces over the reals.

In practice:

- ❖ Parametric representations.
- ❖ The coefficients are floating-point real numbers.

Implicit Equations:

Very useful when available.

Extraneous factors: Algebraic and Geometric.

Difficult to compute even in the exact-coefficient case:

- ❖ Generic Implicitation.
- ❖ Newton Sums.

Approximate Implicitation through composition and the Singular Value Decomposition.

Offsets:

Used in CAD/CAM to define trajectories for NC machine-tools or safety regions.

Offsets are algebraic sets but, in general, not rational.

Extraneous factors: Algebraic (easy to remove) and Geometric (difficult to remove).

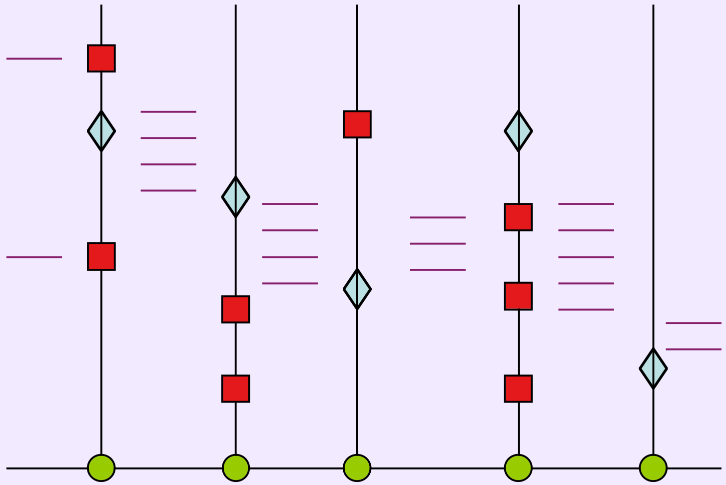
Offsets implicit equations are difficult to compute (in practice they are interpolated).

Offsets introduces new singularities (to analyze through Quantifier Elimination).

PHC curves are surfaces are designed by using Quantifier Elimination.

Efficient Topology Computation for Real Algebraic Implicit Curves:

A very common subtask in CAD/CAM when dealing with curves and surfaces.
The easiest and non trivial case of the Cylindrical Algebraic Decomposition.
Generic position reduces the complexity of the involved computations.
Generic position allows to perform numerically most of the involved computations.



$$f(x, y) = 0$$

Generic Position

$$D(x) = \mathrm{resultant}_y\left(f, \frac{\partial f}{\partial y}\right)$$

$$\alpha_1 < \alpha_2 < \ldots < \alpha_m$$

$$H_j(x, y) = \mathbf{StHa}_j(f) = h_j(x)y^j + h_{j,j-1}(x)y^{j-1} + \ldots + h_{j,0}(x)$$

$$r_k(x) = -\frac{1}{k}\frac{h_{k,k-1}(x)}{h_k(x)}$$

$$\beta_j^{(k)} = r_k(\alpha_j^{(k)})$$

$$F_k(\alpha_j^{(k)}, \beta_j^{(k)}, y) = \frac{f(\alpha_j^{(k)}, y)}{(y - \beta_j^{(k)})^k}$$

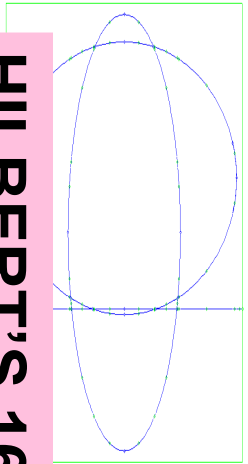
If for every $k \in \{1, \ldots, t\}$ and $j \in \{1, \ldots, s_k\}$ the following condition is verified:

$$H_k(\alpha_j^{(k)}, y) = h_k(\alpha_j^{(k)})(y - \beta_j^{(k)})^k$$

then the curve $\mathcal{C}(f)$ is in generic position.

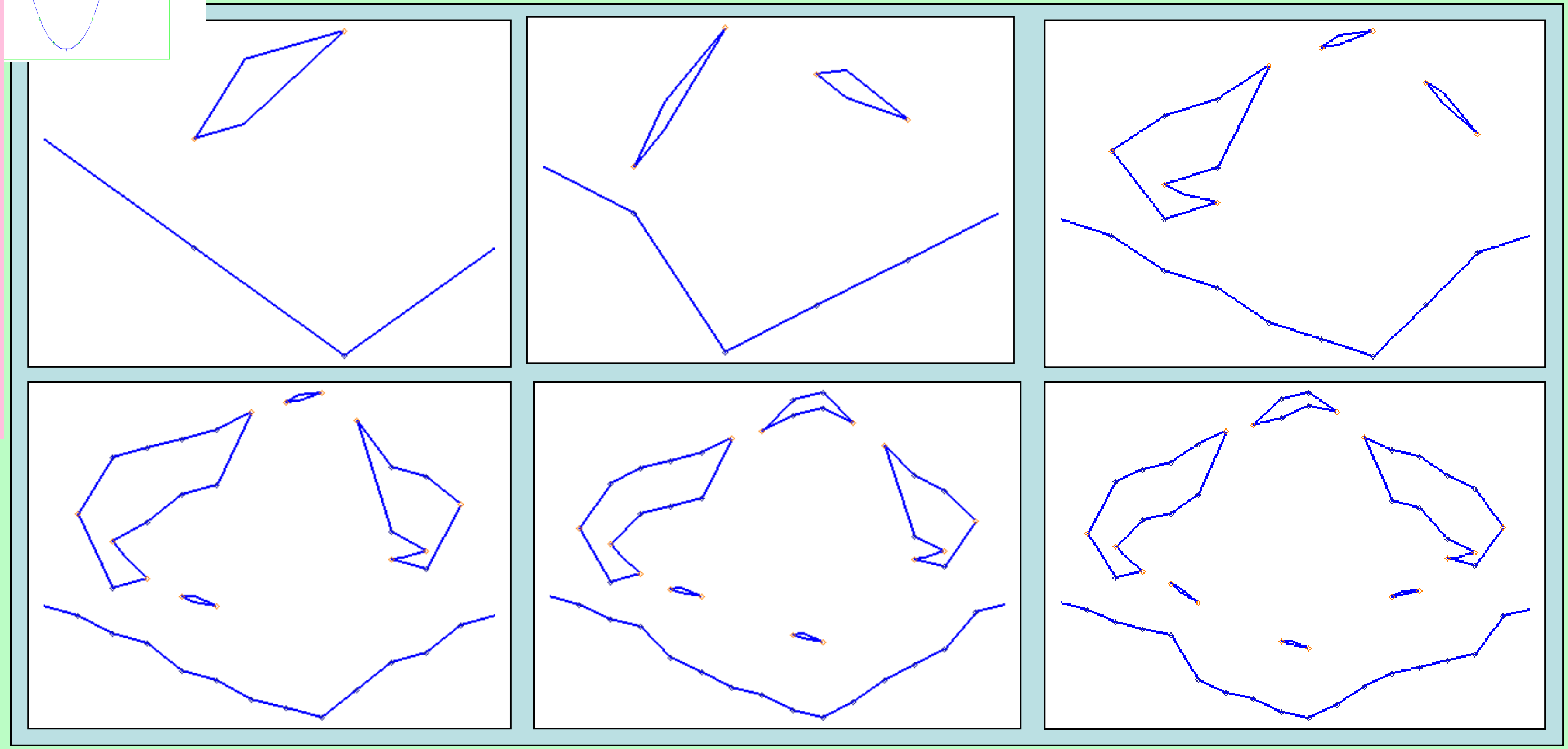
$$f\left(\frac{\alpha_k + \alpha_{k+1}}{2}, y\right) = 0$$

HILBERT'S 16th PROBLEM



$$P := (4x^2+y^2-16)\left(16(y-1)^2+\frac{25}{4}x^2-100\right)(5y+7)$$

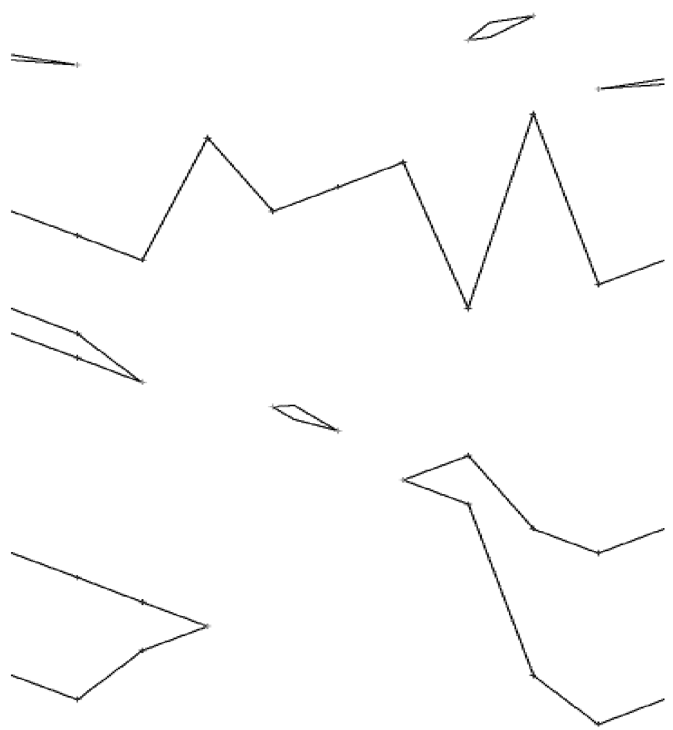
$$Q := t \rightarrow P + t(x+7)(x+8)(x+9)(x+5)(x+6)$$



$t = 1,\ 1/10,\ 1/100,\ 1/1000,\ 5/100000,\ 1/1000000$

THE GENERAL CASE

$$y^{15} + 7\,y - 5\,xy^4 + 5\,y^6 + 2\,x^5y^3 + 7\,x^4y^4 - y^8 - x^7y^4 - 2\,x^7y^5 + 8\,x^3y^9 + 8\,x^2y^{10}$$
$$+8\,x^8y^5 + x^5y^8 + 10\,x^4y^9 + 4\,x^8y^6 + y^{10}x^4 - 4\,x^3y^{11} - 10\,xy^{13}$$



Precision=50, Time= 248 s.

Exact case solving weakness:
- Discriminant *D(x)* computation

Tools:
- Bezout Matrix (and different bases)
- Generalized Eigenvalue Problem:
  - ❖ for computing the real roots of *D(x)*
- SVD [kernel computations]:
  - ❖ for computing β

Robustness Issues:
- Matrix Perturbation for GEP and SVD
- Geometrical clustering

Going into a Generalized Eigenvalue Problem

$$A(x) = \begin{bmatrix} a_{11}(x) & \dots & a_{1n}(x) \\ \vdots & & \vdots \\ a_{n1}(x) & \dots & a_{nn}(x) \end{bmatrix} = \sum_{i=0}^{m} A_i x^i$$

$$C_0 = \begin{bmatrix} \mathbf{0} & I & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & I \\ -A_0 & -A_1 & -A_2 & \dots & -A_{m-2} & -A_{m-1} \end{bmatrix} \qquad C_1 = \begin{bmatrix} I & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & A_m \end{bmatrix}$$

$$\det(C_0 - xC_1) = \det(A(x))$$

Discarding infinite generalized eigenvalues

THE SINGULAR VALUE DECOMPOSITION

Let $\mathbf{A}$ be an arbitrary m–by–n matrix with $m \geq n$. Then there exist orthogonal matrices $\mathbf{U}$ and $\mathbf{V}$ and $\boldsymbol{\Sigma} = \text{diagonal}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \ldots \geq \sigma_n \geq 0$ such that

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}.$$

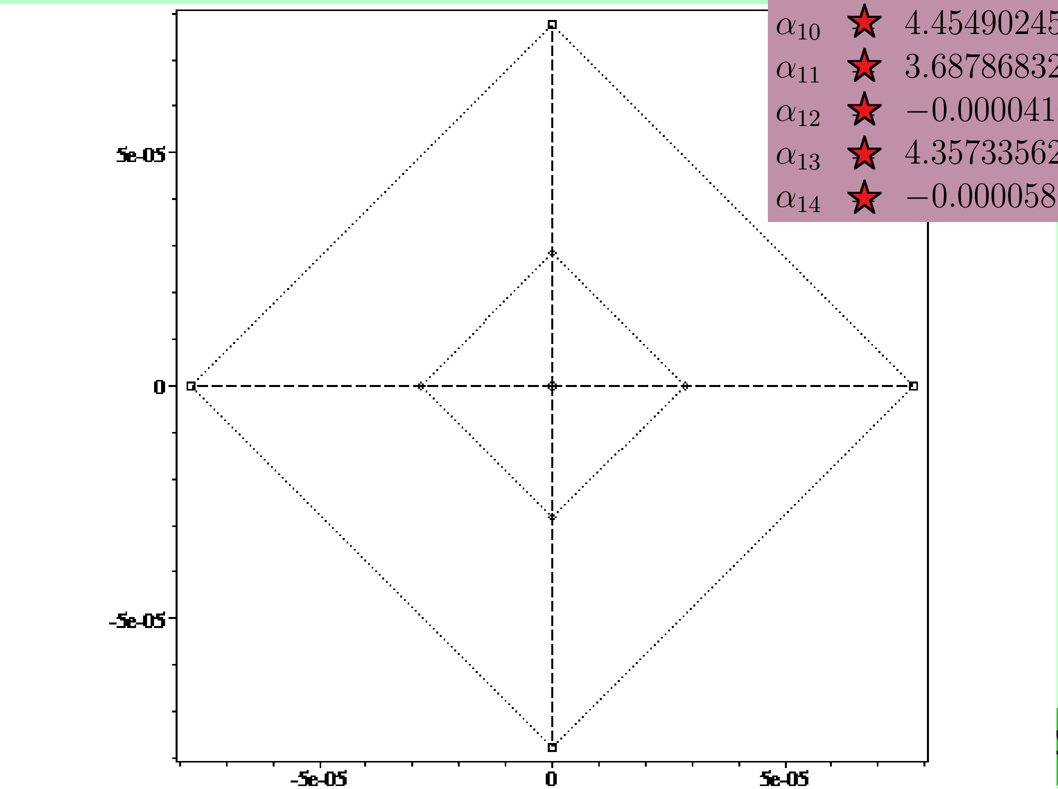
If

$$\sigma_1 \geq \ldots \geq \sigma_r > \sigma_{r+1} = \ldots = \sigma_n = 0$$

then the rank of $\mathbf{A}$ is r and the nullspace of $\mathbf{A}$ is generated by the last $n - r$ columns of $\mathbf{V}$.

$$f := .07922069123\,x^6 - .3656339596\,x^5\,y + .5498572239\,x^4\,y^2 - .6101111042\,x^3\,y^3 + .2960769667\,x^2\,y^4 - .1060122131\,x\,y^5 + .01236809153\,y^6 - .2030818995\,x^2\,y^2 + .1476959269\,x^3\,y - .02769298629\,x^4 + .1476959269\,x\,y^3 - .02769298629\,y^4$$

$$\begin{aligned}
\alpha_1 &\star\ 0.0 + 0.0i \\
\alpha_2 &\star\ 0.0 + 0.0i \\
\alpha_3 &\star\ 0.0 + 0.0i \\
\alpha_4 &= 0.60864768859999958 - 2.7132641149999998 \times 10^{-16}i \\
\alpha_5 &= -0.60864768859999958 + 9.6018646160000019 \times 10^{-18}i \\
\alpha_6 &\star\ 0.000058059093789999983 - 0.000058059117369999979i \\
\alpha_7 &\star\ 0.000058059112620000033 + 0.000058059117369999979i \\
\alpha_8 &\star\ 0.000041004277989999997 - 0.000000000144029060199999987i \\
\alpha_9 &\star\ -0.000058059112620000033 + 0.000058059117450000033i \\
\alpha_{10} &\star\ 4.4549024529999993 \times 10^{-13} + 0.000041004421769999974i \\
\alpha_{11} &\star\ 3.6878683239999987 \times 10^{-17} + 6.5344718359999984 \times 10^{-16}i \\
\alpha_{12} &\star\ -0.000041004278880000011 - 0.000000000144038914500000004i \\
\alpha_{13} &\star\ 4.3573356269999982 \times 10^{-13} - 0.000041004133689999976i \\
\alpha_{14} &\star\ -0.000058059093809999980 - 0.000058059117420000004i
\end{aligned}$$



Geometric Clustering

A very useful strategy: Numerical Linear Algebra and Clustering

f(x,y)=0

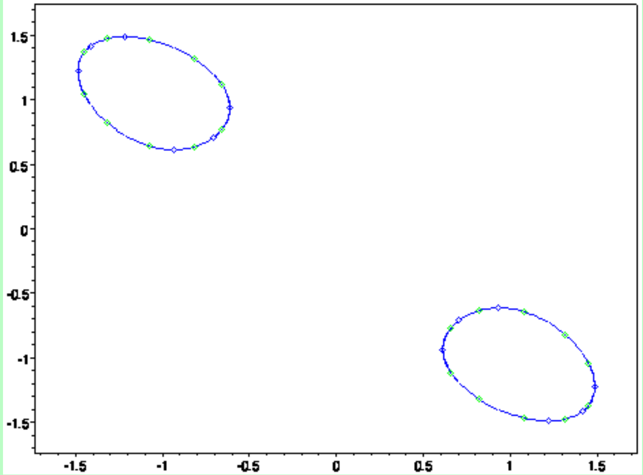
1. Compute the Bezoutian with respect to y: $\mathbf{B}(x) = \mathrm{Bez}(f(x,y), f_y(x,y))$.
2. Compute the companion matrix pencil (C_0, C_1) for $B(x)$: $\det(\mathbf{B}(x)) = \det(C_0 - xC_1)$.
3. Find all generalized eigenvalues for (C_0, C_1) and discard non–real and non finite eigenvalues.
4. For every α, real eigenvalue of the pencil (C_0, C_1):
   - Compute the nullspace $\mathbf{A}$ of $\mathbf{B}(\alpha)$ by performing SVD. Let k be the dimension of the nullspace.
   - Compute
     $$\beta = \frac{\sum_{j=1}^{k} \mathbf{A}_{k+1,j} X_j}{k!},$$
     where β is the unique multiple root (of multiplicity $k+1$) of $f(\alpha, y)$, by solving the linear system
     $$\mathbf{A}X = Y$$
     where $\mathbf{A}_{i,j}$ is the i–th element of the j–th column vector of the nullspace and
     $$Y_i = \begin{cases} 0 & \text{if } 1 \le i \le k-1 \\ (k-1)! & \text{if } i = k. \end{cases}$$
   - Compute the real roots of every $f(\alpha, y)$ knowing in advance the only multiple root β of $f(\alpha, y)$ (together with its multiplicity).
   - Choose x–values c strictly between each two consecutive α's, and solve $f(c, y) = 0$.
5. Deduce the topology of $f(x,y) = 0$.

R. Corless, A. Shakoori, I. Necula, L. Gonzalez-Vega

An application: Plane Section of Implicit Surfaces

$$(x^2 + y^2 + z^2 + 2)^2 - 9\,x^2 - 9\,y^2 = 0$$

$$x + y + z = 0$$



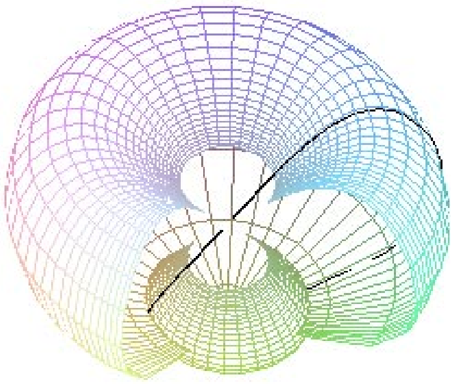
The algebraic/topological analysis concludes that:

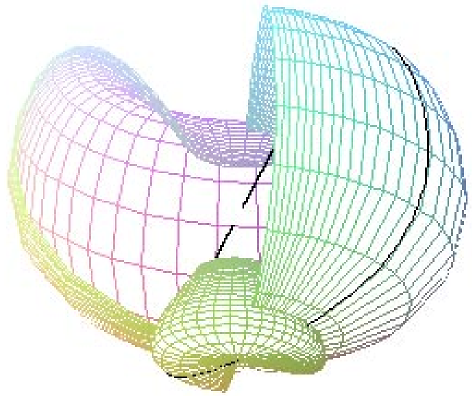
- The curve section has no singular points
- The curve section has two closed components with starting points:

[.6126044921, -.9332990412, .3206945491]

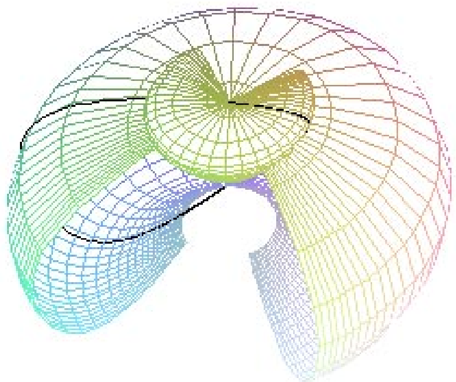
[-1.486514595, 1.218778958, .267735637]

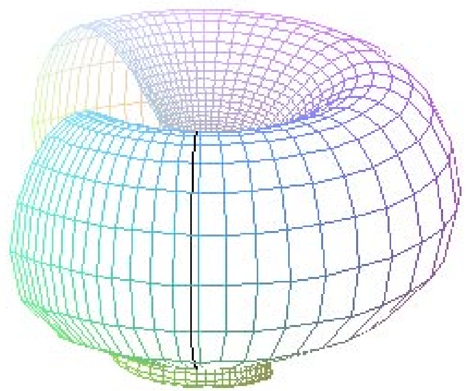
$$\frac{\partial}{\partial s}\mathrm{x}(s) = \frac{\frac{\partial}{\partial y} f}{\sqrt{\left(\frac{\partial}{\partial x} f\right)^2 + \left(\frac{\partial}{\partial y} f\right)^2}}, \quad \frac{\partial}{\partial s}\mathrm{y}(s) = -\frac{\frac{\partial}{\partial x} f}{\sqrt{\left(\frac{\partial}{\partial x} f\right)^2 + \left(\frac{\partial}{\partial y} f\right)^2}}$$





Putting altogether !



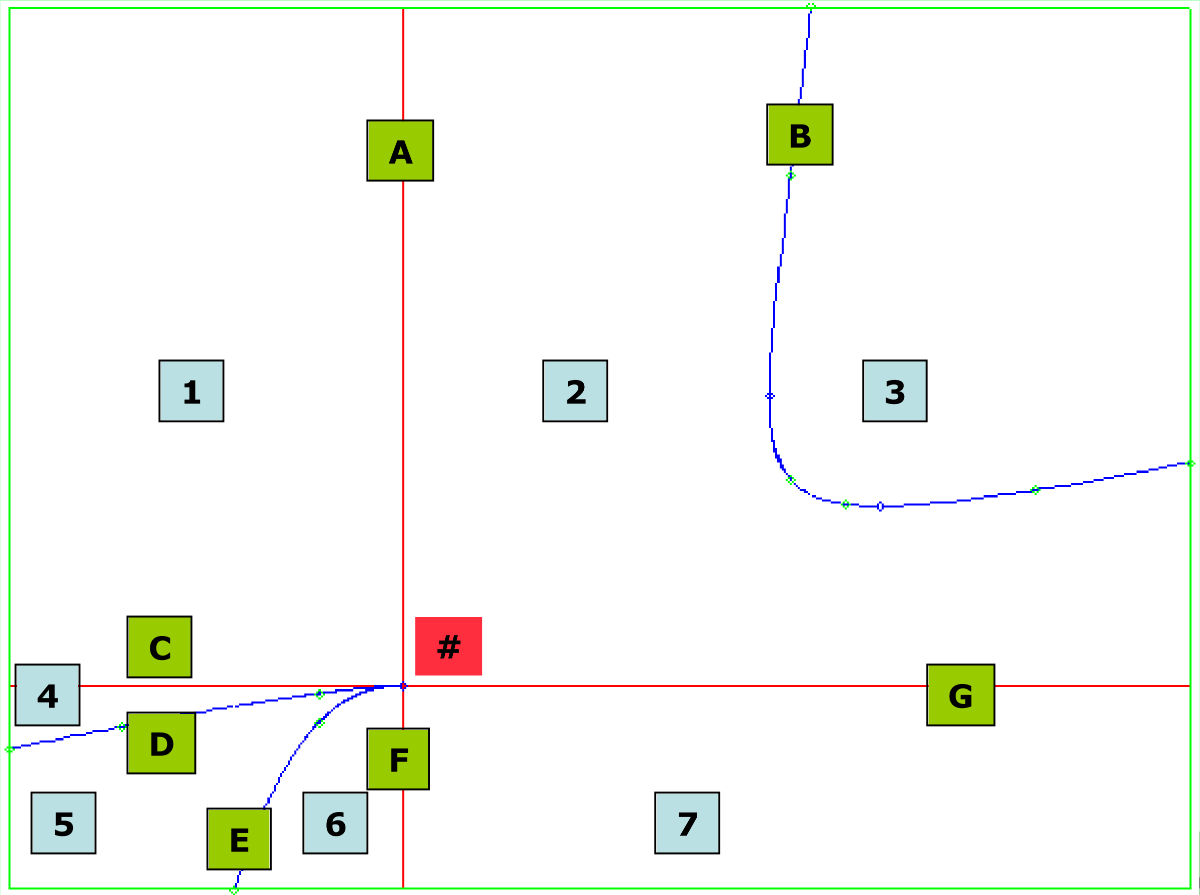


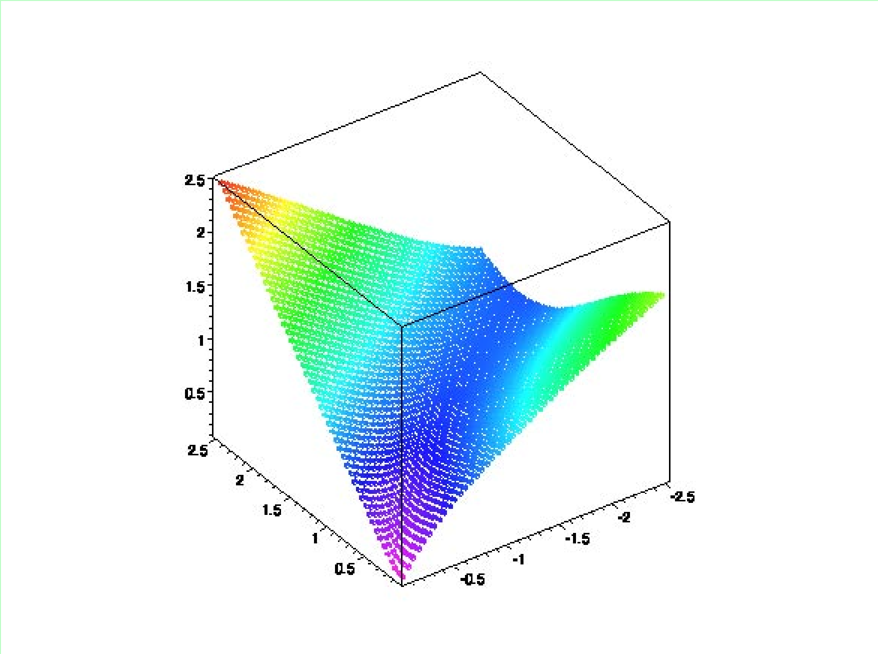
Another motivating example: Algebraic drawing of implicitely defined Surfaces

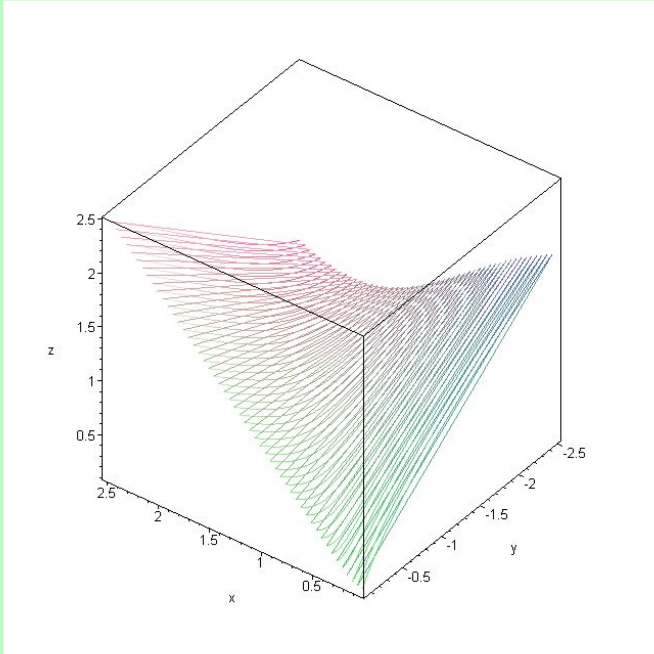
$$H(x,y,z)=0$$

$$H(x,y,z) = x^3 + x^2y - z^3 + xyz + y^2$$

$$\begin{aligned} P_0(x,y) = \ & 4x^3y^3 - 27x^6 - 54x^5y - 54x^3y^2 \\ & -27x^4y^2 - 54x^2y^3 - 27y^4 \\ P_1(x,y) = \ & xy \end{aligned}$$




2.5
2
1.5
1
0.5
2.5
2
1.5
1
0.5
-2.5
-2
-1.5
-1
-0.5


z
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2
1.5
1
0.5
x
2.5
2
1.5
1
0.5
y
-2.5
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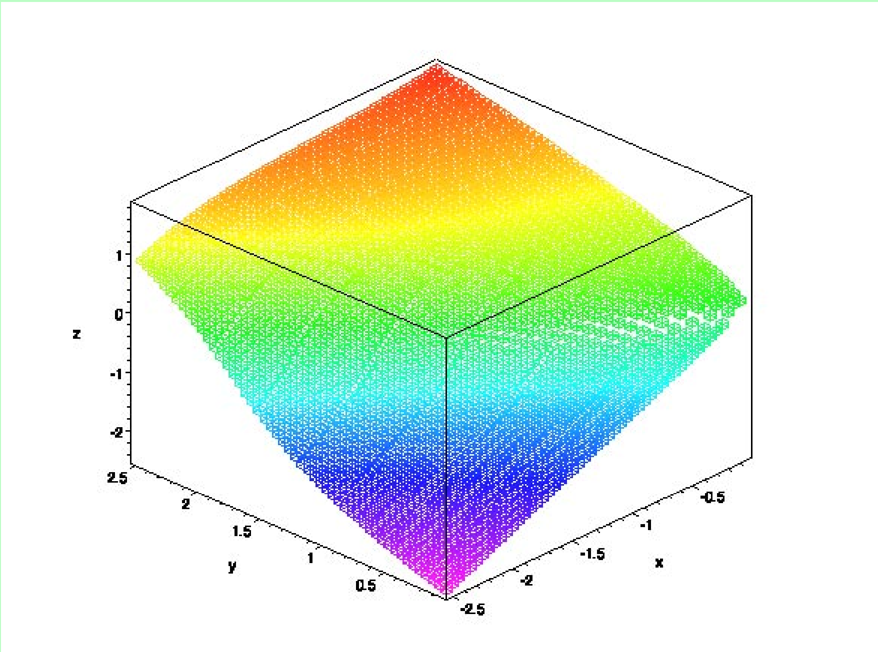
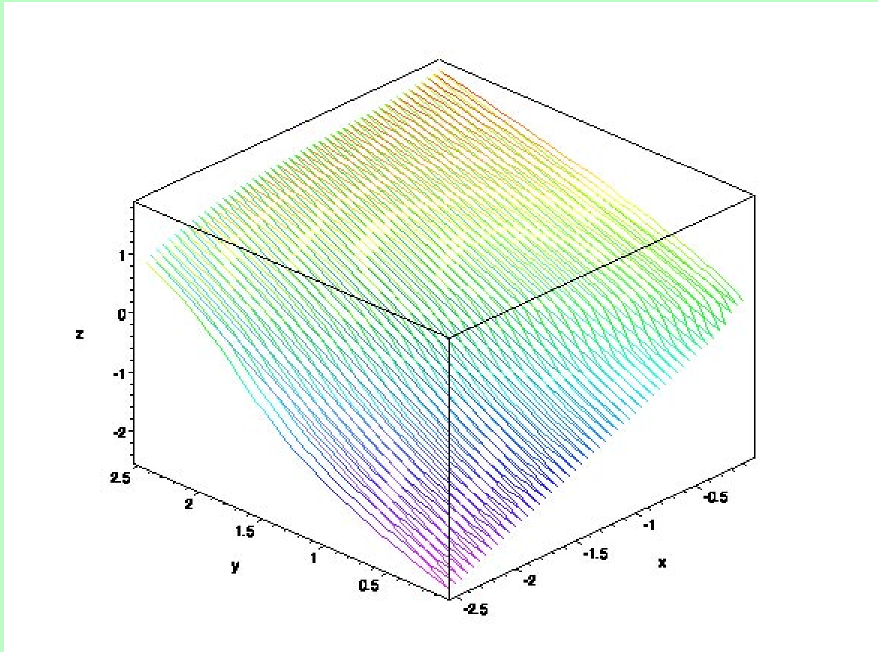
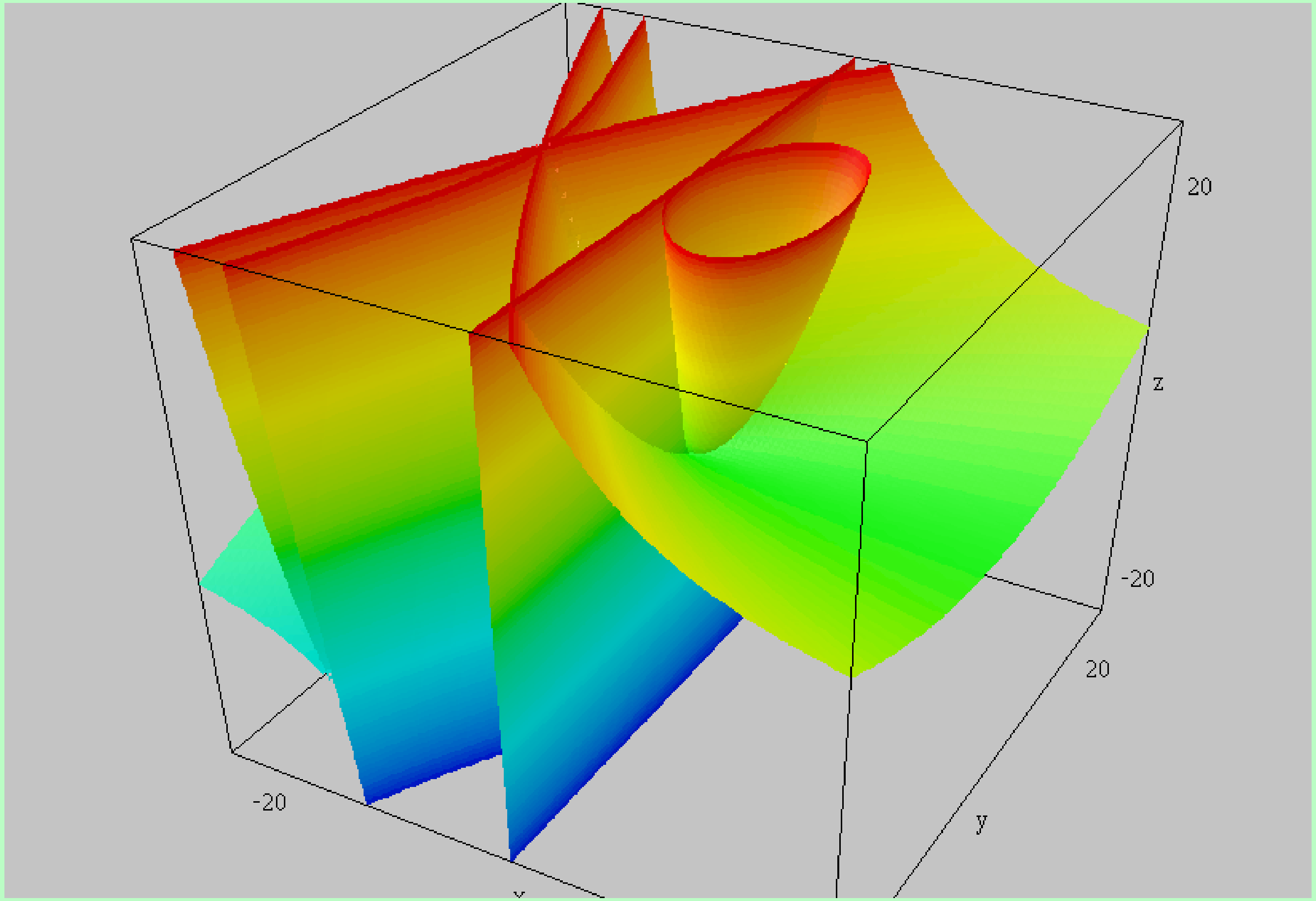
-1.5
-1
-0.5


z
1
0
-1
-2
y
2.5
2
1.5
1
0.5
x
-0.5
-1
-1.5
-2
-2.5


z
1
0
-1
-2
y
2.5
2
1.5
1
0.5
x
-0.5
-1
-1.5
-2
-2.5


20
z
-20
20
y
-20
x

A
Symbolic / Numeric
Problem Solving Environment
for
Computer Aided Geometric Design

Some capabilities

GENERATION OF POLYNOMIAL AND RATIONAL B-SPLINE CURVES AND SURFACES:

- Reading IGES/VDA format
- Plotting IGES/VDA format

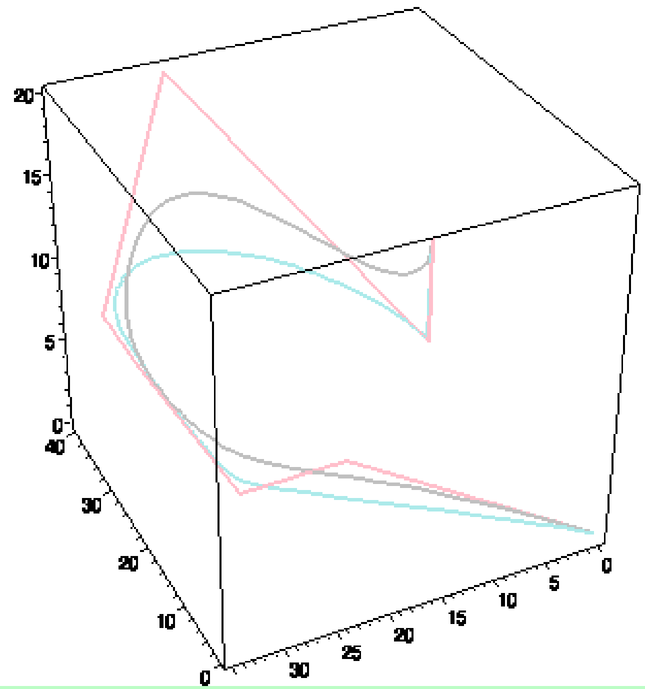
POLYNOMIAL AND RATIONAL B-SPLINE CURVE/SURFACE MANIPULATION

- Numerical generation of polynomial and rational B-spline curves and surfaces.
- Computation of the B-spline curve/surfaces derivatives, using:
  - basis function method
  - control point method.
- Knot vector refinement.
- Bezier decomposition.
- Approximate implicitation.
- Offset computations
- Approximation of rational B-spline curves/surfaces with polynomial ones, degree raising, ...

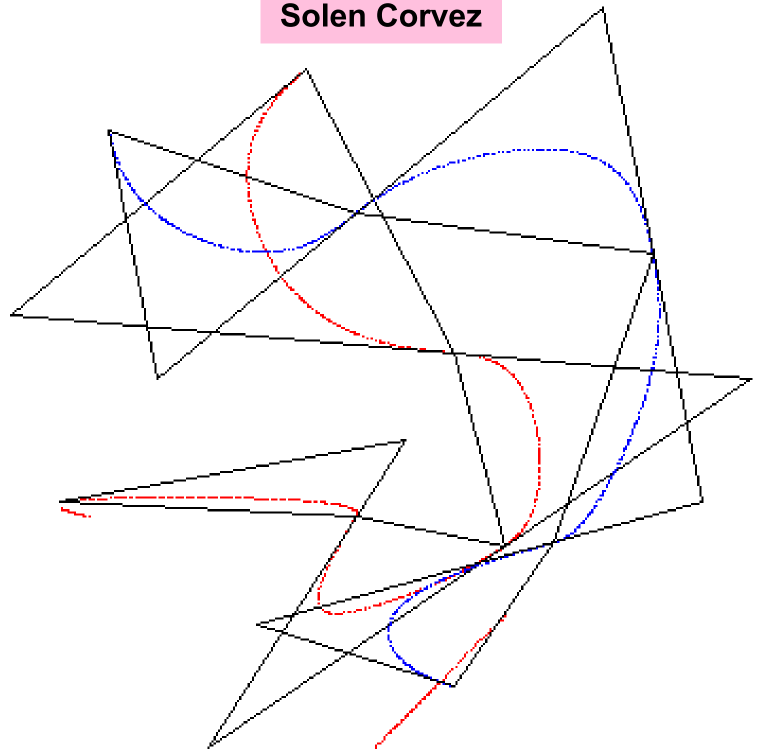
A CAGD SYSTEM WITH EXACT ARITHMETIC

- Generic implicititation.
- A-splines.
- Topologically reliable computation of $f(x,y)=0$.
- Symbolic/Numerical Polynomial System Solving.

Ioana Necula



Solen Corvez



Example 6 *We consider the B-spline curves (polynomial and rational) defined by the following parameters:*

```
> read "simbC.txt";

Curve order: 6
Knot vector: [0,0,0,0,0,0,.1210225739,1.,1.,1.,1.,1.,1.]

Control points: [[1813.213542, 3155.072848, 7292.017654,
10852.09613, 11970.06777, 13138.01982, 5307.115079],
[15411.10969, 6888.316017, 7878.233333, 4268.225166,
9045.040000, 6034.080537, 9315.020747], [22834.01291,
13631.11796, 26325.11671, 23448.19923, 5654.282051,
18408.02733, 22853.31278]]

Weights: [.1883026555, .1081886648e-1, .5431078818e-1,
.5421364074, .1062056055, .6872239468, .2581224202e-1]
```

$$Aspline_1 := \left[\left[\left[[2,10],[3,6],\left[7,\frac{26}{3}\right]\right],\left[1,1,\frac{1}{2},\frac{1}{3},0,1\right]\right],\left[\left[\left[7,\frac{26}{3}\right],[12,12],[13,8]\right],\left[1,2,1,\frac{1}{3},\frac{1}{2},1\right]\right],\left[\left[[13,8],[14,4],\left[11,\frac{10}{3}\right]\right],[2,1,1,1,-1,1]\right],\left[\left[\left[11,\frac{10}{3}\right],[5,2],[9,1]\right],\left[3,1,\frac{1}{2},\frac{1}{3},0,\frac{1}{3}\right]\right]\right]$$

$$Aspline_2 := \left[\left[\left[[6,11],[0,7],\left[9,\frac{32}{5}\right]\right],[2,3,1,1,0,4]\right],\left[\left[\left[9,\frac{32}{5}\right],[15,6],\left[10,\frac{36}{11}\right]\right],[1,1,1,3,-1,2]\right],\left[\left[\left[10,\frac{36}{11}\right],[4,0],\left[7,\frac{15}{4}\right]\right],\left[1,3,2,1,\frac{1}{2},\frac{1}{5}\right]\right],\left[\left[\left[7,\frac{15}{4}\right],[8,5],[1,4]\right],\left[\frac{1}{2},\frac{1}{6},1,3,0,2\right]\right]\right]$$

MORE PROBLEMS

- Self-intersection detection for curves and surfaces
- A-splines
- Implicit surfaces topology computations
- Space curve topology computations
- Computing parameterized families of curves and surfaces with fixed topology
- And much more ……………………………………………………

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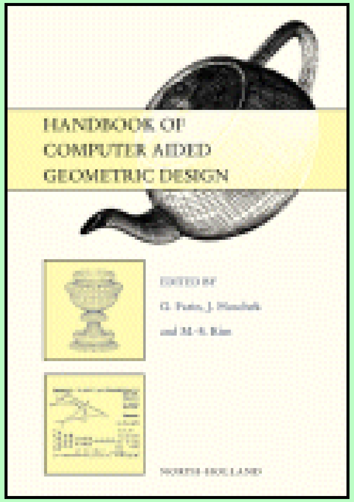
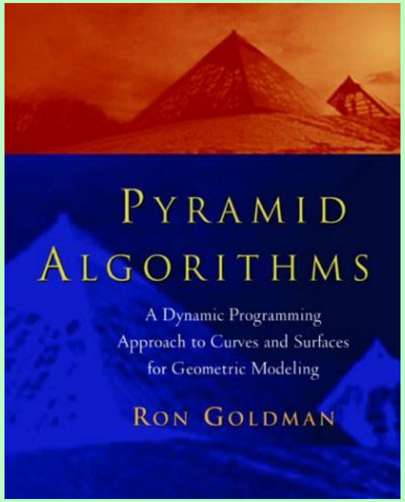
THE RAAG NETWORK

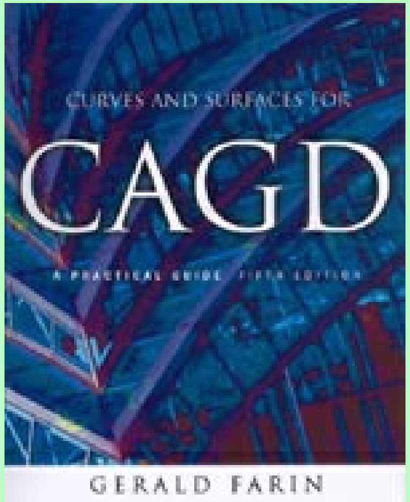
THE GAIA II PROJECT

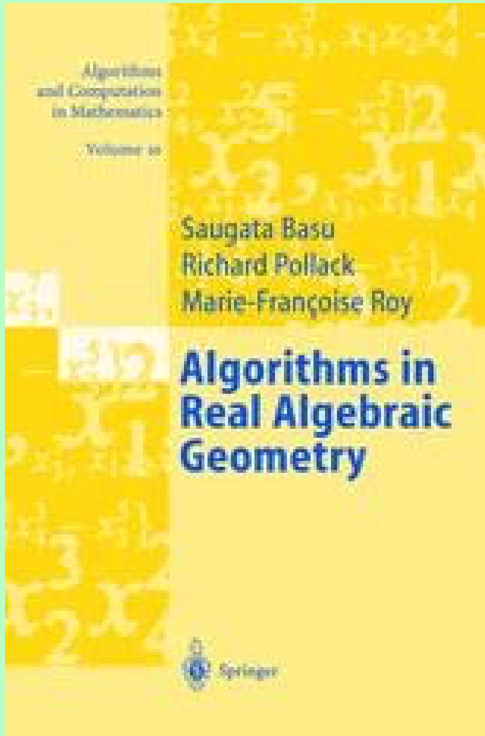
CANDEMAT



http://www.elsevier.nl/







C. Hoffmann
Geometric and Solid Modeling: An Introduction
Morgan Kaufmann, San Mateo, CA, 1989.
http://www.cs.purdue.edu/homes/cmh/distribution/books/geo.html

REAL ALGEBRAIC GEOMETRY

AND

COMPUTER AIDED GEOMETRIC DESIGN

Some Easy Examples

by

Laureano Gonzalez-Vega

IMPLICITATION OF CURVES AND SURFACES

A first easy example

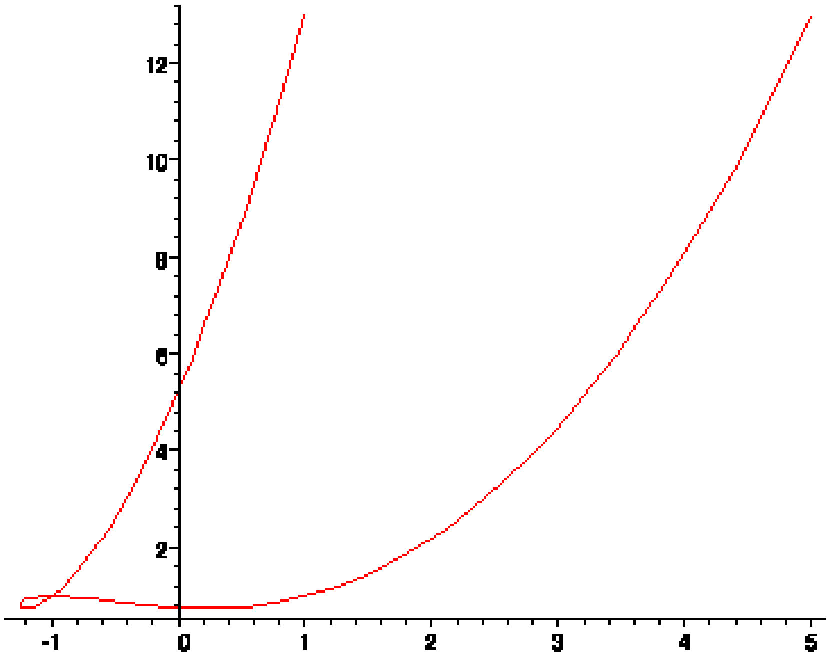
```
> with(plots):with(algcurves):
Warning, the name changecoords has been redefined
```

```
> f:=t->t^2+t-1;g:=t->t^4-t^2+1;
```

$$f := t \to t^2 + t - 1$$

$$g := t \to t^4 - t^2 + 1$$

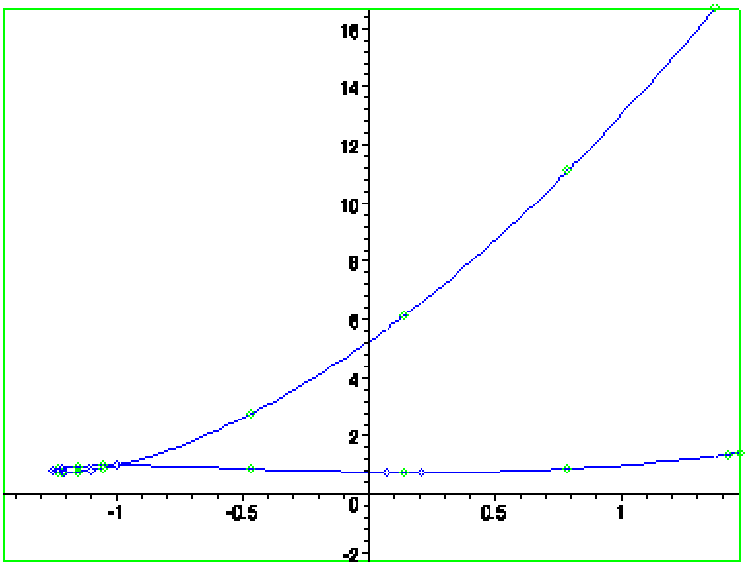
```
> plot([f(t),g(t),t=-2..2]);D1:=%:
```



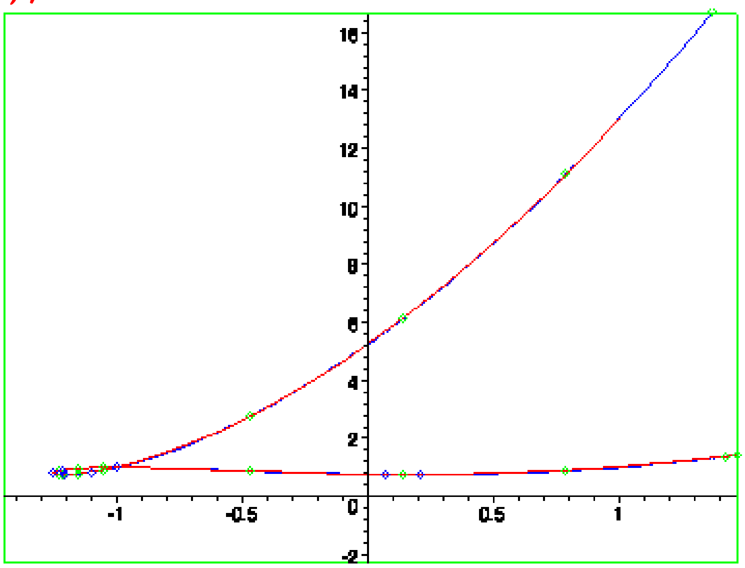
```
> Eq:=resultant(f(t)-x,g(t)-y,t);
```

$$Eq := 4-6\,y+4\,x+2\,x^2+y^2-6\,y\,x-2\,y\,x^2+2\,x^3+x^4$$

```
> plot_real_curve(Eq,x,y);D2:=%:
```



```
> display([D1,D2]);
```



A first difficulty (but easy to solve under some additional hypothesis)

```
> f:=t->t^4+2*t^2+4;g:=t->t^6+3*t^4+4*t^2+1;
```

$$f := t \to t^4+2\,t^2+4$$

$$g := t \to t^6+3\,t^4+4\,t^2+1$$

```
> Eq:=resultant(f(t)-x,g(t)-y,t);
```

$$Eq := (13+2\,y-16\,x+y^2+7\,x^2-x^3)^2$$

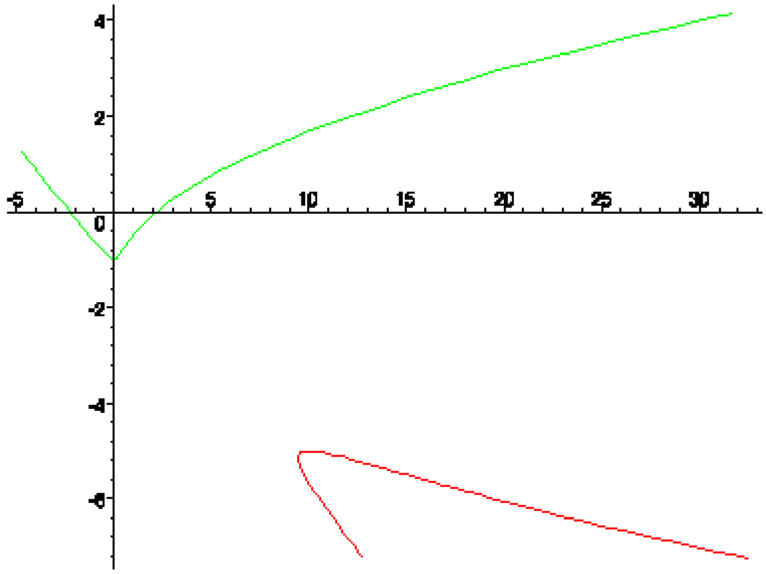
A more complicated situation

```
> f:=t->(t^3+t)/(t+1);g:=t->(t^2-t-1)/(t+1);
```

$$f := t \to \frac{t^3+t}{t+1}$$

$$g := t \to \frac{t^2-t-1}{t+1}$$

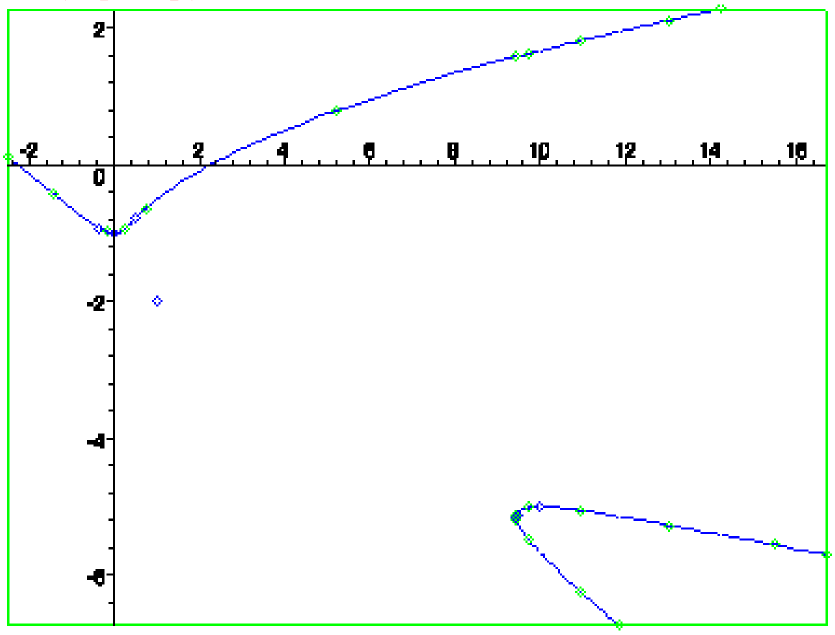
```
> display([plot([f(t),g(t),t=-.75..6],colour=green),
          plot([f(t),g(t),t=-5..-1.25])]);D1:=%:
```



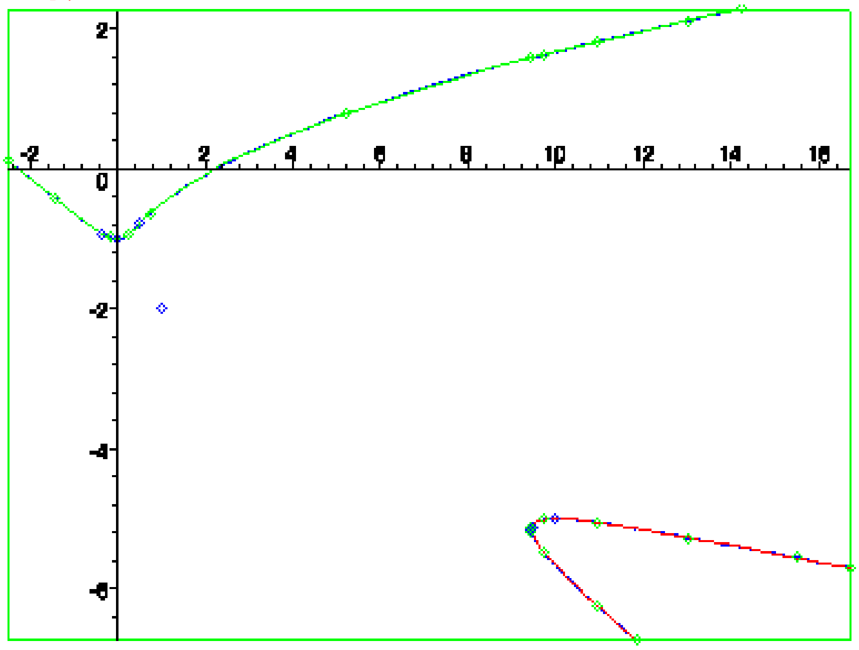
```
> Eq:=resultant(numer(f(t))-denom(f(t))*x,numer(g(t))-denom(g(t))*y,t);
```

$$Eq := -5-11\,y-8\,y^2-y\,x-2\,y^3+x^2-x\,y^2$$

```
> plot_real_curve(Eq,x,y);D2:=%:
```



```
> display([D1,D2]);
```



Where this point comes from ?

```
> subs(x=1,y=-2,Eq);
```

$$0$$

```
> factor(Eq);
```

$$-5-11\,y-8\,y^2-y\,x-2\,y^3+x^2-x\,y^2$$

```
> solve(f(t)-1,t);solve(g(t)+2,t);
```

$$1,-\frac{1}{2}+\frac{1}{2}I\sqrt{3},-\frac{1}{2}-\frac{1}{2}I\sqrt{3}$$

$$-\frac{1}{2}+\frac{1}{2}I\sqrt{3},-\frac{1}{2}-\frac{1}{2}I\sqrt{3}$$

GENERIC IMPLICITATION OF REVOLUTION SURFACES

One example of computing generic implicit equations for revolution surfaces

```
> restart;
read "Macintosh HD:Desktop Folder:RAAGRennes03:ImpliSupRevol.txt";
> c1:=(c12*t**2+c11*t+c10);
c2:=(c22*t**2+c21*t+c20);
c3:=(c32*t**2+c31*t+c30);
```

$$c1 := c12\,t^2+c11\,t+c10$$

$$c2 := c22\,t^2+c21\,t+c20$$

$$c3 := c32\,t^2+c31\,t+c30$$

```
> e2X:=ImplicitSuperfRevolX(c1,c2,c3,x,y,z,t);
```

$$
\begin{aligned}
e2X := {} & 6\,(y^2+z^2)\,c12^2\,c11\,c32\,c31\,x+6\,(y^2+z^2)\,c12^2\,c11\,c22\,c21\,x \\
& -6\,(y^2+z^2)\,c12^2\,c11\,c32\,c31\,c10-4\,(y^2+z^2)\,c12\,c11^2\,c22^2\,x \\
& +4\,(y^2+z^2)\,c12\,c11^2\,c32^2\,c10-2\,(y^2+z^2)\,c12^2\,c11^2\,c32\,c30 \\
& -4\,(y^2+z^2)\,c12^3\,c22\,c20\,x+2\,(y^2+z^2)\,c12\,c11^3\,c32\,c31 \\
& +4\,(y^2+z^2)\,c12^2\,c22^2\,c10\,x+4\,(y^2+z^2)\,c12\,c11^2\,c22^2\,c10 \\
& +2\,(y^2+z^2)\,c12^3\,c11\,c21\,c20-2\,(y^2+z^2)\,c12^3\,c31^2\,x+(y^2+z^2)^2\,c12^4 \\
& -2\,(y^2+z^2)\,c12^2\,c22^2\,c10^2-2\,(y^2+z^2)\,c12^2\,c22^2\,x^2 \\
& -(y^2+z^2)\,c12^2\,c11^2\,c21^2-2\,(y^2+z^2)\,c12^2\,c32^2\,x^2 \\
& -2\,(y^2+z^2)\,c12^3\,c21^2\,x+2\,(y^2+z^2)\,c12^3\,c21^2\,c10 \\
& -(y^2+z^2)\,c12^2\,c11^2\,c31^2-2\,(y^2+z^2)\,c12^2\,c32^2\,c10^2 \\
& +2\,(y^2+z^2)\,c12^3\,c31^2\,c10+4\,(y^2+z^2)\,c12^2\,c32^2\,c10\,x \\
& +2\,(y^2+z^2)\,c12^3\,c11\,c31\,c30-2\,(y^2+z^2)\,c12^2\,c11^2\,c22\,c20 \\
& +4\,(y^2+z^2)\,c12^3\,c32\,c30\,c10-4\,(y^2+z^2)\,c12^3\,c32\,c30\,x \\
& -6\,(y^2+z^2)\,c12^2\,c11\,c22\,c21\,c10+4\,(y^2+z^2)\,c12^3\,c22\,c20\,c10 \\
& +2\,(y^2+z^2)\,c12\,c11^3\,c22\,c21-(y^2+z^2)\,c11^4\,c32^2-2\,(y^2+z^2)\,c12^4\,c20^2 \\
& -2\,(y^2+z^2)\,c12^4\,c30^2-2\,c31^2\,c30^2\,c12^3\,x+2\,c21^2\,c20^2\,c12^3\,c10 \\
& -2\,c21^2\,c20^2\,c12^3\,x+2\,c31^2\,c30^2\,c12^3\,c10+c12^4\,c30^4+c12^4\,c20^4 \\
& -2\,c21^2\,c12^3\,c10\,c30^2+6\,c12^2\,c22^2\,c10^2\,c20^2+2\,c12^2\,c22^2\,c10^2\,c30^2 \\
& +6\,c12^2\,c22^2\,x^2\,c20^2+2\,c12^2\,c22^2\,x^2\,c30^2+c12^2\,c11^2\,c21^2\,c20^2 \\
& +c12^2\,c11^2\,c21^2\,c30^2+2\,c12^2\,c11^2\,c22\,c20^3+2\,c12^2\,c11^2\,c32\,c30\,c20^2 \\
& +4\,c12\,c11^2\,c22^2\,x\,c20^2+4\,c12\,c11^2\,c22^2\,x\,c30^2 \\
& -4\,c12\,c11^2\,c32^2\,c10\,c20^2-4\,c12\,c11^2\,c32^2\,c10\,c30^2 \\
& +4\,c12\,c11^2\,c32^2\,x\,c20^2+4\,c12\,c11^2\,c32^2\,x\,c30^2 \\
& +6\,c12^2\,c11\,c22\,c21\,c10\,c30^2-6\,c12^2\,c11\,c22\,c21\,x\,c30^2
\end{aligned}
$$

$+6\, c12^2\, c11\, c32\, c31\, c10\, c20^2 - 6\, c12^2\, c11\, c32\, c31\, x\, c20^2$

$-4\, c22\, c21^2\, c12^2\, x^2\, c20 - 8\, c22\, c21\, c12^2\, x^2\, c31\, c30 + c12^2\, c11^2\, c31^2\, c20^2$

$+c12^2\, c11^2\, c31^2\, c30^2 + 2\, c12^2\, c11^2\, c32\, c30^3 + 2\, c12^2\, c32^2\, c10^2\, c20^2$

$+6\, c12^2\, c32^2\, c10^2\, c30^2 + 2\, c12^2\, c32^2\, x^2\, c20^2 + 6\, c12^2\, c32^2\, x^2\, c30^2$

$+4\, c22\, c20^3\, c12^3\, x - 4\, c32\, c30^3\, c12^3\, c10 + 4\, c32\, c30^3\, c12^3\, x$

$-4\, c22\, c20^3\, c12^3\, c10 - 2\, c11^3\, c22^2\, c10\, c21\, c20 - 2\, c11^3\, c22^2\, c10\, c31\, c30$

$+2\, c11^3\, c22^2\, x\, c21\, c20 + 2\, c11^3\, c22^2\, x\, c31\, c30 - 2\, c11^3\, c32^2\, c10\, c21\, c20$

$-2\, c11^3\, c32^2\, c10\, c31\, c30 - 4\, c22\, c21^2\, c12^2\, c10^2\, c20$

$-8\, c22\, c21\, c12^2\, c10^2\, c31\, c30 - 8\, c32\, c31\, c12^2\, c10^2\, c21\, c20$

$-4\, c32\, c31^2\, c12^2\, c10^2\, c30 - 8\, c32\, c31\, c12^2\, x^2\, c21\, c20$

$-4\, c32\, c31^2\, c12^2\, x^2\, c30 - 2\, c12^2\, c11\, c31^2\, c10\, c21\, c20$

$-2\, c12^2\, c11\, c31^3\, c10\, c30 + 2\, c12^2\, c11\, c31^2\, x\, c21\, c20$

$+2\, c12^2\, c11\, c31^3\, x\, c30 + 8\, c21\, c20\, c12^3\, c10\, c31\, c30$

$-4\, c12\, c11^2\, c32\, c31^2\, x\, c30 + 8\, c22\, c21^2\, c12^2\, c10\, x\, c20$

$+16\, c22\, c21\, c12^2\, c10\, x\, c31\, c30 - 4\, c12\, c11\, c22^2\, c10\, x\, c21\, c20$

$-12\, c12\, c11\, c22^2\, c10\, x\, c31\, c30 - 12\, c12\, c11\, c32^2\, c10\, x\, c21\, c20$

$-4\, c12\, c11\, c32^2\, c10\, x\, c31\, c30 + 4\, c12\, c11^2\, c22\, c21^2\, c10\, c20$

$+4\, c12\, c11^2\, c22\, c21\, c10\, c31\, c30 - 4\, c12\, c11^2\, c22\, c21^2\, x\, c20$

$-4\, c12\, c11^2\, c22\, c21\, x\, c31\, c30 - 4\, c12^2\, c11\, c32\, c30\, c10\, c21\, c20$

$+2\, c12^2\, c11\, c32\, c30^2\, c10\, c31 + 4\, c12^2\, c11\, c32\, c30\, x\, c21\, c20$

$-2\, c12^2\, c11\, c32\, c30^2\, x\, c31 + 2\, c11^3\, c32^2\, x\, c21\, c20 + 2\, c11^3\, c32^2\, x\, c31\, c30$

$-8\, c31\, c30\, c12^3\, x\, c21\, c20 + 2\, c12^2\, c11\, c21^2\, x\, c31\, c30$

$-4\, c12\, c11^2\, c32\, c31\, x\, c21\, c20 + 2\, c12^2\, c11\, c21^3\, x\, c20 + 2\, c12^4\, c20^2\, c30^2$

$+c11^4\, c22^2\, c20^2 + c11^4\, c22^2\, c30^2 + c11^4\, c32^2\, c20^2 + c11^4\, c32^2\, c30^2$

$+2\, c12\, c11\, c22^2\, c10^2\, c21\, c20 + 6\, c12\, c11\, c22^2\, c10^2\, c31\, c30$

(two more pages have been removed !)

$-8\,c32^2\,x^3\,c22^2\,c10-6\,c11\,c22^3\,c10\,x^2\,c21+6\,c11\,c22^2\,c10^2\,x\,c32\,c31$

$-2\,c22^2\,c21^2\,c12\,x^3+2\,c12\,c22^2\,x^3\,c31^2+4\,c12\,c22^3\,x^3\,c20$

$+2\,c22^2\,c21^2\,c12\,c10^3+4\,c12\,c32^3\,x^3\,c30+2\,c12\,c32^2\,x^3\,c21^2$

$+2\,c31^2\,c12^2\,c10^2\,c21^2-4\,c32^4\,c10^3\,x-4\,c12\,c22^2\,c10^3\,c32\,c30$

$+4\,c12\,c22^2\,x^3\,c32\,c30+6\,c12\,c32^2\,c10^2\,x\,c21^2$

$-12\,c12\,c32^2\,c10\,x^2\,c22\,c20+6\,c12\,c22^2\,c10^2\,x\,c31^2$

$+12\,c12\,c32^2\,c10^2\,x\,c22\,c20-4\,(y^2+z^2)\,c12\,c11^2\,c32^2\,x$

$+c11^2\,c32^2\,c31^2\,x^2+6\,c22^2\,c21^2\,c12\,c10\,x^2-2\,c11\,c12^3\,c20^2\,c31\,c30$

$-2\,c11\,c12^3\,c30^2\,c21\,c20-2\,c12\,c11\,c21^3\,x^2\,c22$

$-4\,c12\,c11\,c32\,c30\,x^2\,c22\,c21-4\,c12\,c11\,c22\,c20\,c10^2\,c32\,c31$

$+c11^2\,c22^2\,x^2\,c31^2+2\,c11^2\,c22^3\,x^2\,c20+2\,c11^2\,c32^3\,c10^2\,c30$

$-2\,c11^2\,c32^2\,c31^2\,x\,c10+6\,c11\,c32^2\,c10^2\,x\,c22\,c21$

$-6\,c11\,c32^2\,c10\,x^2\,c22\,c21-6\,c11\,c32^3\,c10\,x^2\,c31+4\,c12\,c32^2\,x^3\,c22\,c20$

$+2\,c11^2\,c32^2\,c10^2\,c22\,c20-2\,c11\,c32^2\,c10^3\,c22\,c21$

$-2\,c12\,c11\,c31^3\,x^2\,c32-4\,c12\,c11\,c22\,c20\,x^2\,c32\,c31$

$+8\,c22\,c21\,c12\,c10^3\,c32\,c31-8\,c32\,c31\,c12\,x^3\,c22\,c21$

$+2\,c11^2\,c32^2\,x^2\,c22\,c20-2\,c12\,c11\,c21^2\,x^2\,c32\,c31$

$-4\,c12\,c32^2\,c10^3\,c22\,c20+c11^2\,c32^2\,c10^2\,c21^2$

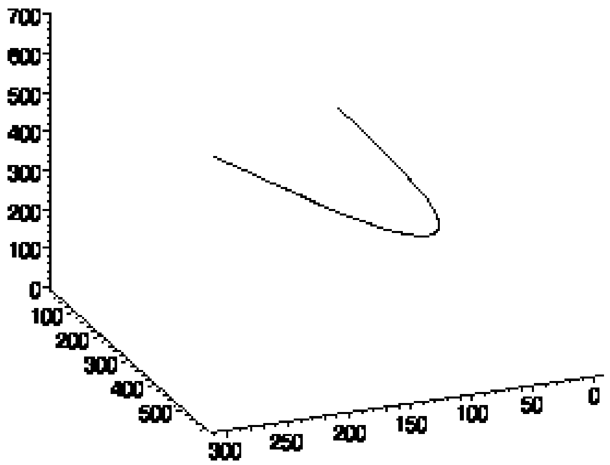
```
> c12:=rand(1..50)():c11:=rand(1..50)():c10:=rand(1..50)():
c22:=rand(1..50)():c21:=rand(1..50)():c20:=rand(1..50)():
c32:=rand(1..50)():c31:=rand(1..50)():c30:=rand(1..50)():
> f1:= t -> c1;
f2:= t -> c2;
f3:= t -> c3;
```

$$f1 := t \to c1$$

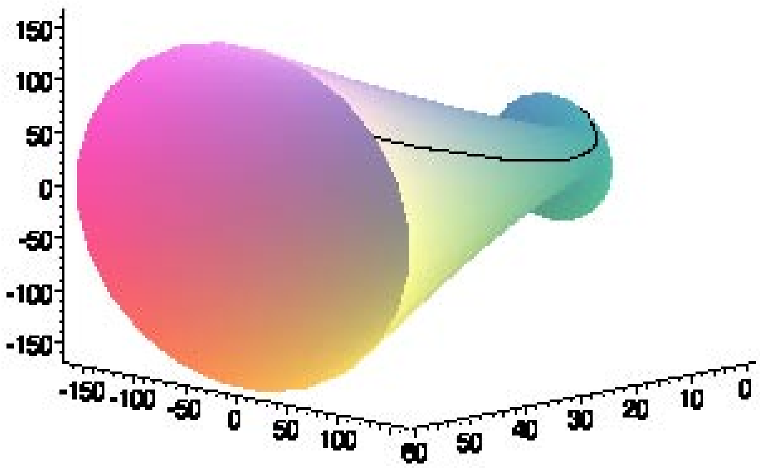
$$f2 := t \to c2$$

$$f3 := t \to c3$$

```
> plot3d([f1(t),f2(t),f3(t)],t=-6..6,u=0..1,
axes=FRAME,orientation=[45,75]);
```




```
> plots[display]([
plot3d([f1(t),f2(t),f3(t)],t=-
2..2,u=0..1,axes=FRAME,orientation=[45,75]),
plot3d([f1(t),f2(t)*cos(u)+f3(t)*sin(u),f3(t)*cos(u)-f2(t)*sin(u)],
t=-2..2,u=0..2*Pi,axes=FRAME,orientation=[45,75],style=PATCHNOGRID)]);
```



CURVE APPROXIMATE IMPLICITIZATION

Newton Sums to Elementary Symmetric Functions

```
> symmetrize:=proc(newton,m)
 local sol,j;
   sol:=array(1..m);
   sol[1]:=-newton[1];
   for j from 2 to m do
     sol[j]:=-(1/j)*(newton[j]+sum('newton[k]*sol[j-k]','k'=1..j-1))
od;       sol;
 end:
```

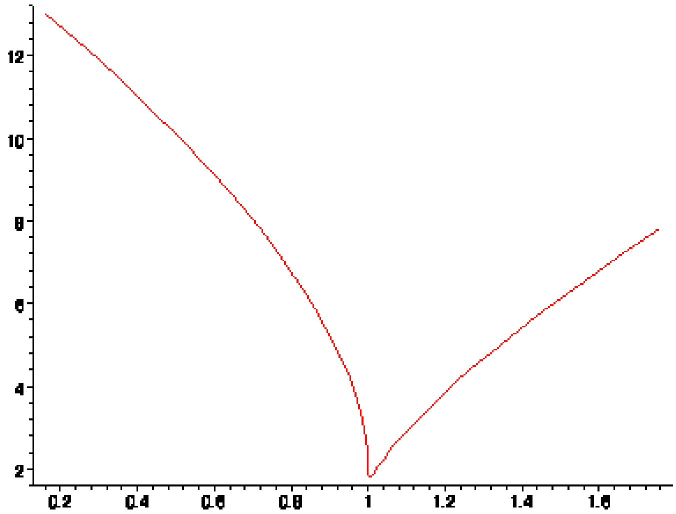
Curve whose implicit equation is to be computed

```
> with(plots):
> f:=t->0.1*t^3-0.01*t^2+1.001;g:=t->2.1*t^2-1.3*t+2.01;
```

$$f := t \to .1\, t^3 - .01\, t^2 + 1.001$$

$$g := t \to 2.1\, t^2 - 1.3\, t + 2.01$$

```
> plot([f(t),g(t),t=-2..2]);
```



Implicitization

Rational function to be expanded

```
> frat:=diff(g(t),t)/(g(t)-y);
```

$$frat := \frac{4.2\, t - 1.3}{2.1\, t^2 - 1.3\, t + 2.01 - y}$$

Expansion at infinity

```
>
ser0:=convert(series(frat,t=infinity,degree(f(t),t)*degree(g(t),t)+2),
polynom);
```

$$ser0 := 2.000000000\,\frac{1}{t} + \frac{.6190476190}{t^2} + \frac{-1.531065760 + .9523809524\,y}{t^3}$$

$$+ \frac{-1.540319620 + .8843537414\,y}{t^4} + ($$

$$.4761904762\,(1.531065760 - .9523809524\,y)\,(2.01 - 1.\,y) - .9535311934$$

$$+ .5474570781\,y)/t^5 + ($$

$$.4761904762\,(1.540319620 - .8843537414\,y)\,(2.01 - 1.\,y) + .3169012864$$

$$- .6767344881\,y + .2807472195\,y^2) \Big/ t^6 + (.4761904762$$

$$(-.5119174626 + 1.093186481\,y - .4535147392\,y^2)\,(2.01 - 1.\,y)$$

$$+ 1.108842557 - 1.396987978\,y + .4344897444\,y^2) \Big/ t^7$$

Computing the Newton Sums

```
> ser:=array(1..degree(g(t),t)):ser[1]:=ser0*f(t):
for j from 2 to degree(g(t),t) do ser[j]:=ser[j-1]*f(t) od:
newton_suma:=array(1..degree(g(t),t)):
for j from 1 to degree(g(t),t) do newton_suma[j]:=coeff(ser[j],t**(-1)) od;
```

$$newton\_suma_1 := 1.863278696 + .07891156462\,y$$

$$newton\_suma_2 :=$$

$$-.0009523809524\,(1.540319620 - .8843537414\,y)\,(2.01 - 1.\,y)$$

$$+ 1.736641218 + .1454192872\,y + .003783403005\,y^2 + .004761904762$$

$$(-.5119174626 + 1.093186481\,y - .4535147392\,y^2)\,(2.01 - 1.\,y)$$

$$+ .00004761904762\,(1.531065760 - .9523809524\,y)\,(2.01 - 1.\,y)$$

Computing the Elementary Symmetric Functions

```
> sim_elem:=evalm(symmetrize(newton_suma,degree(g(t),t)));
```

$$sim\_elem := \Big[-1.863278696 - .07891156462\,y,$$

$$.0004761904762\,(1.540319620 - .8843537414\,y)\,(2.01 - 1.\,y)$$

$$- .8683206090 - .07270964360\,y - .001891701502\,y^2 - .002380952381$$

$$(-.5119174626 + 1.093186481\,y - .4535147392\,y^2)\,(2.01 - 1.\,y)$$

$$- .00002380952381\,(1.531065760 - .9523809524\,y)\,(2.01 - 1.\,y)$$

$$-\frac{1}{2}(1.863278696 + .07891156462\,y)\,(-1.863278696 - .07891156462\,y) \Big]$$

Implicit Equation

```
> Ecuacion:=x**degree(g(t),t)+sum(sim_elem[degree(g(t),t)-
k]*x**k,'k'=0..degree(g(t),t)-1);
```

$$Ecuacion := x^2$$

$$+.0004761904762\,(1.540319620-.8843537414\,y)\,(2.01-1.\,y)$$

$$-.8683206090-.07270964360\,y-.001891701502\,y^2-.002380952381$$

$$(-.5119174626+1.093186481\,y-.4535147392\,y^2)\,(2.01-1.\,y)$$

$$-.00002380952381\,(1.531065760-.9523809524\,y)\,(2.01-1.\,y)$$

$$-\frac{1}{2}(1.863278696+.07891156462\,y)\,(-1.863278696-.07891156462\,y)$$

$$+(-1.863278696-.07891156462\,y)\,x$$

```
> expand(%);
```

$$x^2+.8714340652+.06637615810\,y+.006393478028\,y^2\ \ -.001079796998\,y^3-1.863278696\,x-.07891156462\,x\,y$$

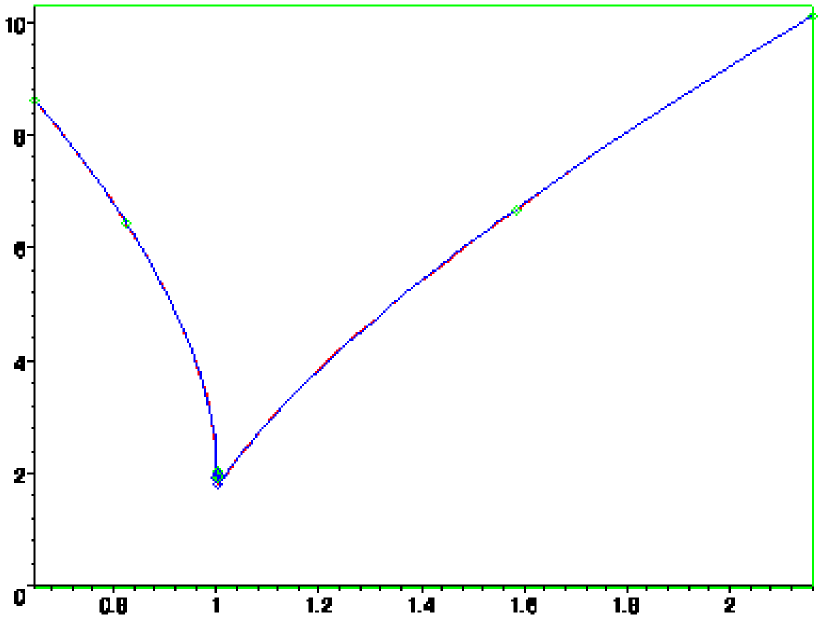
Verification

```
> expand(subs(x=f(t),y=g(t),Ecuacion));
```

$$-.4\,10^{-11}\,t^3+.158\,10^{-9}\,t^2+.1\,10^{-10}\,t^4-.1\,10^{-9}\,t$$

Final Verification

```
> with(algcurves):
> display([plot_real_curve(Ecuacion,x,y),plot([f(t),g(t),t=-2..2])]);
```



COMPUTING WITH OFFSETS

```
> restart:with(plots):with(algcurves):
```

The Offset of the Parabola (I): Visualizing and Computing the Implicit Equation

Parametric Equation of the Parabola

```
> u:=t->t;v:=t->t^2;
```

$$u := t \to t$$

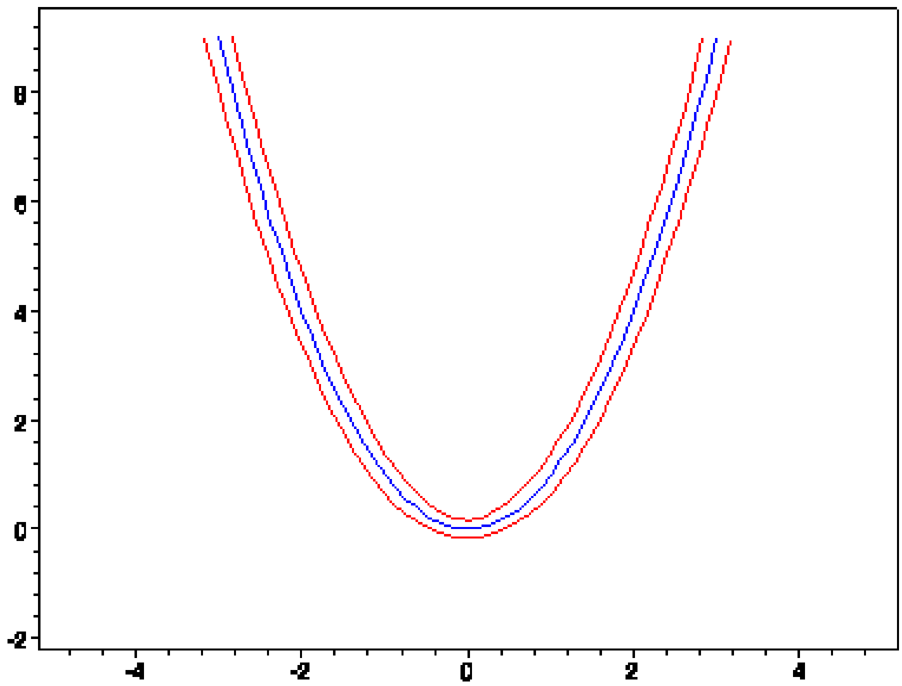
$$v := t \to t^2$$

The first picture from the parametric (and not rational) equations of the offset

```
> H[1]:=animate(
[u(t)+(2*t*d)/sqrt(1+4*t^2),v(t)-d/sqrt(1+4*t^2),t=-3..3],
                                              d=1/6..2,frames=25):
H[2]:=animate(
[u(t)-(2*t*d)/sqrt(1+4*t^2),v(t)+d/sqrt(1+4*t^2),t=-3..3],
                                              d=1/6..2,frames=25):
```

$x(t)=u(t)+\dfrac{d\left(\frac{\partial}{\partial t}v(t)\right)}{\sqrt{\left(\frac{\partial}{\partial t}u(t)\right)^2+\left(\frac{\partial}{\partial t}v(t)\right)^2}}$	$y(t)=v(t)-\dfrac{d\left(\frac{\partial}{\partial t}u(t)\right)}{\sqrt{\left(\frac{\partial}{\partial t}u(t)\right)^2+\left(\frac{\partial}{\partial t}v(t)\right)^2}}$
$x(t)=u(t)-\dfrac{d\left(\frac{\partial}{\partial t}v(t)\right)}{\sqrt{\left(\frac{\partial}{\partial t}u(t)\right)^2+\left(\frac{\partial}{\partial t}v(t)\right)^2}}$	$y(t)=v(t)+\dfrac{d\left(\frac{\partial}{\partial t}u(t)\right)}{\sqrt{\left(\frac{\partial}{\partial t}u(t)\right)^2+\left(\frac{\partial}{\partial t}v(t)\right)^2}}$

```
> display([H[1],H[2],plot([t,t^2,t=-3..3],colour=blue)],axes=BOXED);
```



Computing the Implicit Equation of the Offset

```
> pol[1]:=(u(t)-x)^2+(v(t)-y)^2-d^2;
pol[2]:=(x-u(t))*diff(u(t),t)+(y-v(t))*diff(v(t),t);
```

$$pol_1 := (t-x)^2+(t^2-y)^2-d^2$$

$$pol_2 := x-t+2\,(y-t^2)\,t$$

```
> Eq_Offset:=collect(resultant(pol[1],pol[2],t),[x,y],distributed);
```

$$Eq\_Offset := -8\,d^4-d^2-16\,d^6+(16\,d^4+1-8\,d^2)\,y^2+(48\,d^4-20\,d^2)\,x^2$$
$$-40\,x^4\,y-32\,x^2\,y^3+16\,x^4\,y^2+(-32\,d^2+32)\,x^2\,y^2+(8\,d^2-2)\,y\,x^2$$
$$+16\,x^6+(-8-32\,d^2)\,y^3+(-48\,d^2+1)\,x^4+16\,y^4+(32\,d^4+8\,d^2)\,y$$

The Offset of the Parabola (II): Analyzing the Implicit Equation

Computing the discriminant of the implicit equation

```
> factor(discrim(Eq_Offset,y));
```

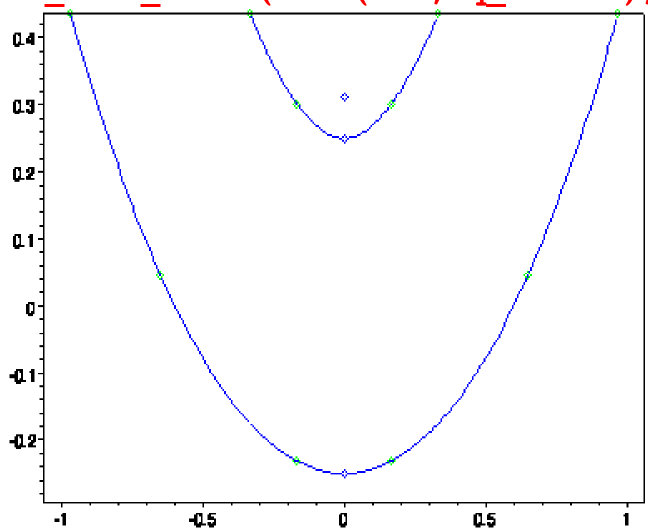
$$-4096\,d^2\,x^2\,(64\,x^6-192\,d^2\,x^4+48\,x^4+192\,d^4\,x^2+12\,x^2+336\,d^2\,x^2$$
$$+48\,d^4+1-12\,d^2-64\,d^6)^3$$

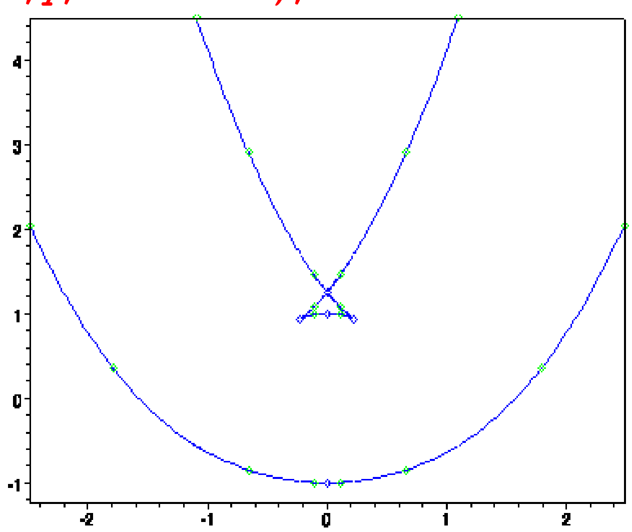
```
> Disc:=collect(64*x^6-
192*d^2*x^4+48*x^4+336*d^2*x^2+192*d^4*x^2+12*x^2-64*d^6+48*d^4-
12*d^2+1,x);
```

$$Disc := 64\,x^6+(-192\,d^2+48)\,x^4+(192\,d^4+12+336\,d^2)\,x^2+48\,d^4+1\quad-12\,d^2-64\,d^6$$

Studying some pictures from the implicit equation

```
> plot_real_curve(subs(d=1/4,Eq_Offset),x,y,axes=BOXED);
plot_real_curve(subs(d=1,Eq_Offset),x,y,axes=BOXED);
```





The Offset of the Parabola (III): Where the isolated point comes from?

It is always in the vertical line $y = 0$

```
> factor(subs(x=0,Eq_Offset));
```

$$-(d-y)\,(d+y)\,(1+4\,d^2-4\,y)^2$$

```
> solve(-1+4*y-4*d^2,y);
```

$$\frac{1}{4}+d^2$$

```
> solve(u(t)+(2*t*d)/sqrt(1+4*t^2),t);
solve(v(t)-d/sqrt(1+4*t^2)-1/4-d^2,t);
```

$$0, \frac{1}{2}\sqrt{-1+4\,d^2}, -\frac{1}{2}\sqrt{-1+4\,d^2}$$

$$\frac{1}{2}\sqrt{1+2\,d^2+2\,\sqrt{2\,d^2+d^4}}, -\frac{1}{2}\sqrt{1+2\,d^2+2\,\sqrt{2\,d^2+d^4}},$$
$$\frac{1}{2}\sqrt{1+2\,d^2-2\,\sqrt{2\,d^2+d^4}}, -\frac{1}{2}\sqrt{1+2\,d^2-2\,\sqrt{2\,d^2+d^4}}, \frac{1}{2}\sqrt{-1+4\,d^2},$$
$$-\frac{1}{2}\sqrt{-1+4\,d^2}$$

And for $d < 1/2$ the isolated point comes from the complex part ....

The Offset of the Cuspidal Curve: Removing algebraic extraneous factors

Parametric Equation of the Curve

```
> u:=t->t^2;v:=t->t^3;
```

$$u := t \to t^2$$

$$v := t \to t^3$$

Computing the Implicit Equation of the Offset

```
> pol[1]:=(u(t)-x)^2+(v(t)-y)^2-d^2;
pol[2]:=(x-u(t))*diff(u(t),t)+(y-v(t))*diff(v(t),t);
```

$$pol_1 := (t^2-x)^2+(t^3-y)^2-d^2$$

$$pol_2 := 2\,(x-t^2)\,t+3\,(y-t^3)\,t^2$$

```
> Eq_Offset:=resultant(pol[1],pol[2],t);
```

$$Eq\_Offset := (x^2+y^2-d^2)\,(-32\,x^3\,d^2+288\,x\,d^4+1728\,d^4\,x^2-504\,d^2\,x^4$$
$$-432\,x^4\,y^2-2916\,d^6\,x^2+4374\,d^4\,x^4-2916\,d^2\,x^6-2376\,x^5\,d^2$$
$$+4104\,x^3\,d^4-1944\,x\,d^6+16\,x^6-216\,d^6+16\,d^4+16\,y^4+1188\,y^2\,d^4$$
$$+6318\,y^2\,x\,d^4-729\,y^2\,d^6-4860\,x^3\,y^2\,d^2-2187\,x^4\,y^2\,d^2+729\,y^2\,x^6$$

$-1458\, y^2 x^5 - 2484\, y^2 x^2 d^2 - 504\, y^2 x\, d^2 + 216\, y^4 x - 32\, y^2 x^3 - 32\, y^2 d^2$

$+729\, x^2 y^4 + 729\, y^6 - 1701\, y^4 d^2 - 1458\, y^4 x^3 - 4374\, y^4 x\, d^2 \quad + 2187\, y^2 x^2 d^4 + 216\, x^7 + 729\, x^8 + 729\, d^8)$

Cleaning the equations

```
> pol[3]:=simplify(pol[2]/gcd(diff(u(t),t),diff(v(t),t)));
```

$$pol_3 := 2\, x - 2\, t^2 + 3\, t\, y - 3\, t^4$$

and computing again the Implicit Equation of the Offset

```
> Eq_Offset:=resultant(pol[1],pol[3],t);
```

$Eq\_Offset := -32\, x^3 d^2 + 288\, x\, d^4 + 1728\, d^4 x^2 - 504\, d^2 x^4 - 432\, x^4 y^2$

$-2916\, d^6 x^2 + 4374\, d^4 x^4 - 2916\, d^2 x^6 - 2376\, x^5 d^2 + 4104\, x^3 d^4$

$-1944\, x\, d^6 + 16\, x^6 - 216\, d^6 + 16\, d^4 + 16\, y^4 + 1188\, y^2 d^4 + 6318\, y^2 x\, d^4$

$-729\, y^2 d^6 - 4860\, x^3 y^2 d^2 - 2187\, x^4 y^2 d^2 + 729\, y^2 x^6 - 1458\, y^2 x^5$

$-2484\, y^2 x^2 d^2 - 504\, y^2 x\, d^2 + 216\, y^4 x - 32\, y^2 x^3 - 32\, y^2 d^2 + 729\, x^2 y^4$

$+729\, y^6 - 1701\, y^4 d^2 - 1458\, y^4 x^3 - 4374\, y^4 x\, d^2 + 2187\, y^2 x^2 d^4 \quad + 216\, x^7 + 729\, x^8 + 729\, d^8$

which is free of (algebraic) extraneous factors.

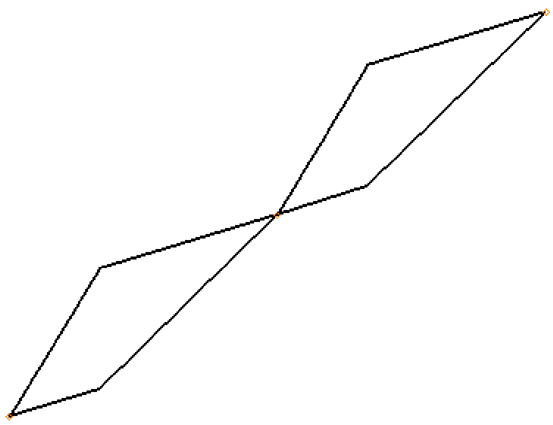
CURVE TOPOLOGY COMPUTATIONS (The exact case)

```
> restart;
> read "Macintosh HD:Desktop Folder:RAAGRennes03:topcur.txt":
```

Computing the topology of some curves. Try your favourite example !

```
> poli:=y**8-x*y+x**2;
tiempo:=time():principal(poli,x,y,10,'black');time()-tiempo;nrCambio;
```

$$poli := y^8 - x\, y + x^2$$

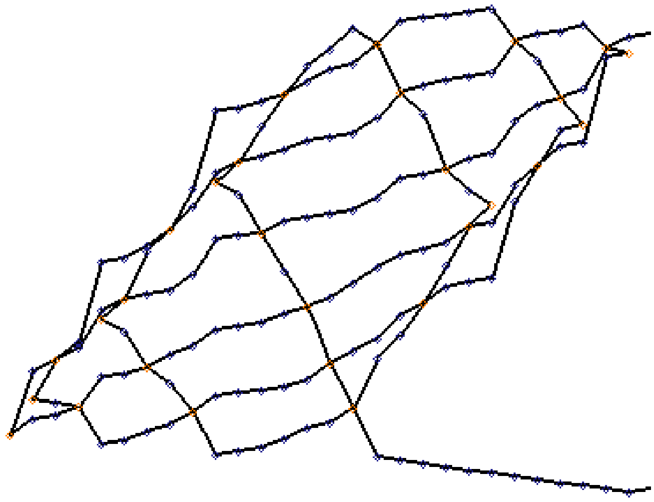


1.950

0

```
> poli:=-(-2-7*x+14*x**3-7*x**5+x**7+(7-42*x**2+35*x**4-7*x**6)*y)-
        ((16+42*x-70*x**3+21*x**5)*y**2+(-14+70*x**2-35*x**4)*y**3)-
        ((-20-35*x+35*x**3)*y**4+(7-21*x**2)*y**5+(8+7*x)*y**6-y**7-
y**8);
tiempo:=time():principal(poli,x,y,15,'black');time()-tiempo;nrCambio;
```

$$poli := 2+7x-14x^3+7x^5-x^7-(7-42x^2+35x^4-7x^6)\,y\;-(16+42x-70x^3+21x^5)\,y^2-(-14+70x^2-35x^4)\,y^3$$
$$-(-20-35x+35x^3)\,y^4-(7-21x^2)\,y^5-(8+7x)\,y^6+y^7+y^8$$

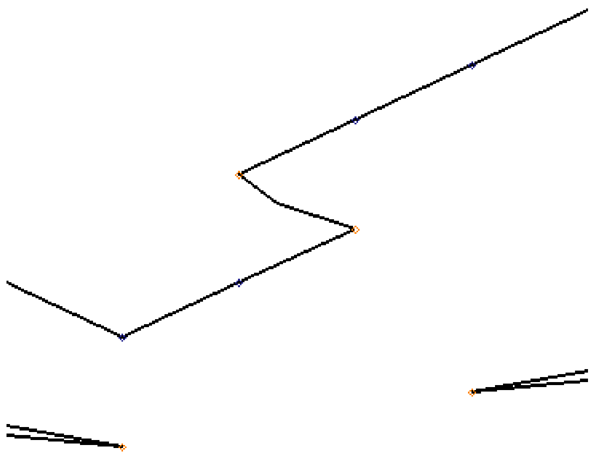


12.317

0

```
> poli:=y**5+(-x-1)*y**4+(-2*x+1)*y**3+(2*x+1)*y**2+(2*x-1)*y-2*x-1;
tiempo:=time():principal(poli,x,y,10,'black');time()-tiempo;nrCambio;
```

$$poli := y^5+(-x-1)\,y^4+(-2x+1)\,y^3+(2x+1)\,y^2+(2x-1)\,y-2x-1$$

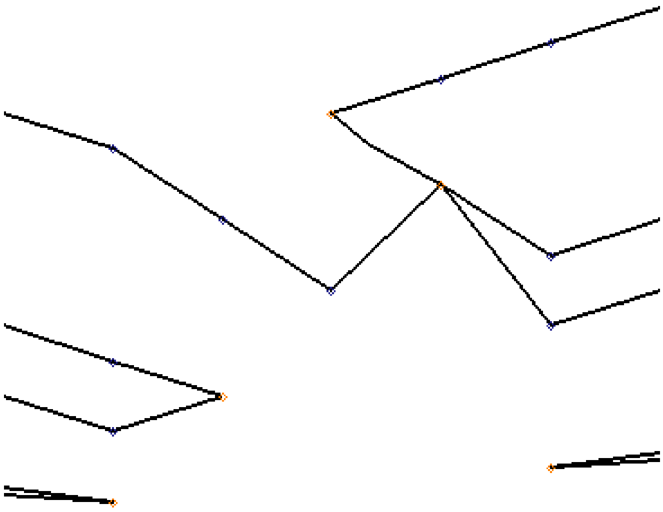


.450

-1

```
> poli:=y**5+(-x-1)*y**4+(-2*x+1)*y**3+(2*x+1)*y**2+(2*x-1)*y-x-1;
tiempo:=time():principal(poli,x,y,15,'black');time()-tiempo;nrCambio;
```

$$poli := y^5+(-x-1)\,y^4+(-2\,x+1)\,y^3+(2\,x+1)\,y^2+(2\,x-1)\,y-x-1$$

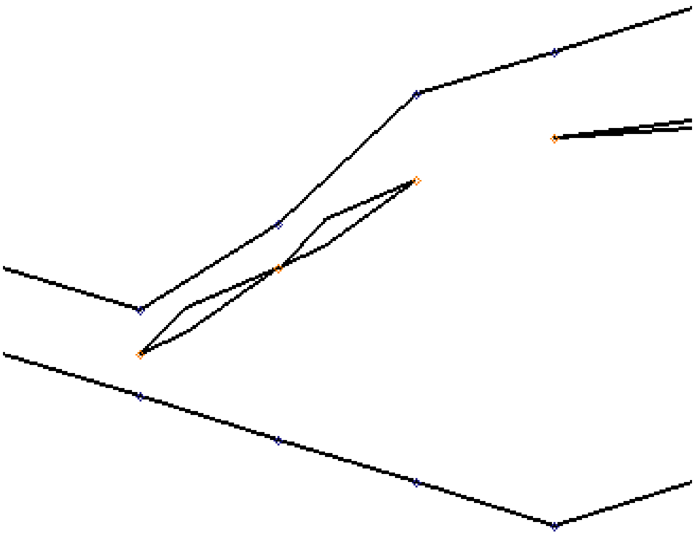


.483

0

```
> poli:=y**4-6*x*y**2+x**2-4*x**2*y**2+24*x**3;
tiempo:=time():principal(poli,x,y,10,'black');time()-tiempo;nrCambio;
```

$$poli := y^4-6\,y^2\,x+x^2-4\,y^2\,x^2+24\,x^3$$



1.084

-1

SECTIONING REVOLUTION SURFACES

The curve C into the plane Y=0 defined by x=C_1(t), z=C_2(t) is going to be rotated around the axiz OZ

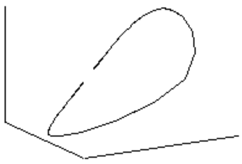
```
> restart; with(plots):with(algcurves):
> plots[setoptions3d](scaling=CONSTRAINED,axes=FRAMED);
plots[setoptions](scaling=CONSTRAINED,axes=FRAMED);
> C1:=(2*t-1)/(1+t**2);C2:=0;C3:=(1-t+t**2)/(2+t+t**2);
```

$$C1 := \frac{2\,t-1}{1+t^2}$$

$$C2 := 0$$

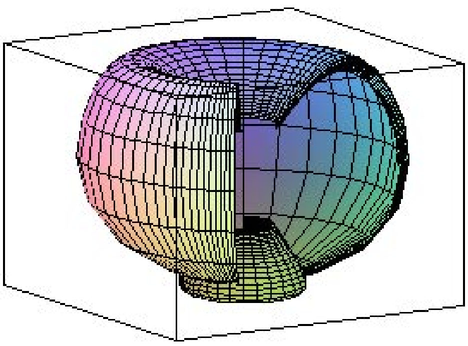
$$C3 := \frac{1-t+t^2}{2+t+t^2}$$

```
> plot3d([C1,C2,C3],t=-25..25,s=-1..1,
orientation=[60,75],grid=[175,2],tickmarks=[0,0,0]);
```



Picture of the Revolution Surface generated by the curve C

```
> T1:=C1*(2*s)/(1+s**2)-C2*(1-s**2)/(1+s**2):
T2:=C2*(2*s)/(1+s**2)+C1*(1-s**2)/(1+s**2):T3:=C3:
P1:=(a,b)->plot3d([T1,T2,T3],s=a..b,t=-
6..6,grid=[50,50],axes=BOXED,scaling=UNCONSTRAINED,projection=1,tickma
rks=[0,0,0],orientation=[59,75]):P1(-3,3);
```



Implicit Equation of the Revolution Surface

```
> read "Macintosh HD:Desktop Folder:RAAGRennes03:ImpliSupRevol.txt";
> toro:=ImplicitSuperfRevolZ(C1,C2,C3,x,y,z,t);
```

$$toro := 121\,z^4+4\,(x^2+y^2)^2\,z^4+19\,(x^2+y^2)\,z^4+8\,(x^2+y^2)^2\,z^3-308\,z^3$$
$$-72\,(x^2+y^2)\,z^3+8\,(x^2+y^2)^2\,z^2+262\,z^2-28\,(x^2+y^2)\,z^2+4\,(x^2+y^2)^2\,z$$
$$-84\,z+20\,(x^2+y^2)\,z+9+(x^2+y^2)^2-3\,x^2-3\,y^2$$

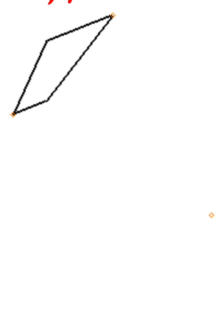
Section of the revolution surface by the plane y=1:

```
> Seccion1:=subs(y=1,toro);
```

$$Seccion1 := 121\,z^4+4\,(x^2+1)^2\,z^4+19\,(x^2+1)\,z^4+8\,(x^2+1)^2\,z^3-308\,z^3$$
$$-72\,(x^2+1)\,z^3+8\,(x^2+1)^2\,z^2+262\,z^2-28\,(x^2+1)\,z^2+4\,(x^2+1)^2\,z-84\,z$$
$$+20\,(x^2+1)\,z+6+(x^2+1)^2-3\,x^2$$

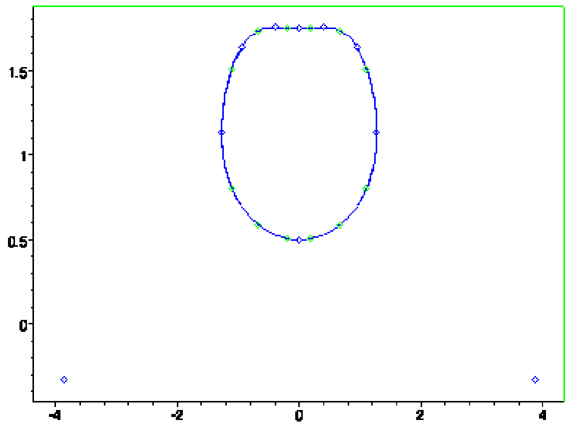
Topological Graph of this section (it has two isolated points !):

```
> read "Macintosh HD:Desktop Folder:RAAGRennes03:topcur.txt":
> principal(Seccion1,x,z,10,'black');
```

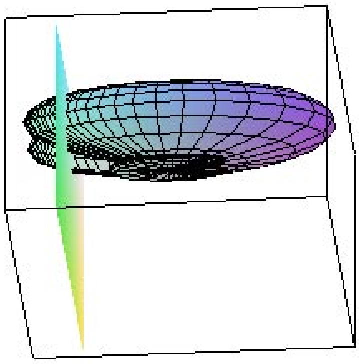


Graph of the section (it has two isolated points !) and real picture of the section:

```
> plot_real_curve(Seccion1,x,z,scaling=UNCONSTRAINED);
```



```
> P2:=y->plot3d([s,y,t],s=-4..4,t=-2..2,style=PATCHNOGRID):
> plots[display](P1(-4,4),P2(1),grid=[50,50],axes=BOXED,scaling=UNCONSTRAINED,
projection=1,tickmarks=[0,0,0],orientation=[-175,128]);
```



Section of the revolution surface by the plane y=1/2:

```
> Seccion2:=subs(y=1/2,toro);
```

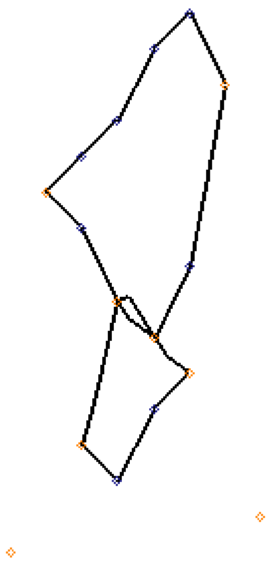
$$Seccion2 := 121\,z^4+4\left(x^2+\frac{1}{4}\right)^2 z^4+19\left(x^2+\frac{1}{4}\right)z^4+8\left(x^2+\frac{1}{4}\right)^2 z^3-308\,z^3$$

$$-72\left(x^2+\frac{1}{4}\right)z^3+8\left(x^2+\frac{1}{4}\right)^2 z^2+262\,z^2-28\left(x^2+\frac{1}{4}\right)z^2+4\left(x^2+\frac{1}{4}\right)^2 z-84\,z$$

$$+20\left(x^2+\frac{1}{4}\right)z+\frac{33}{4}+\left(x^2+\frac{1}{4}\right)^2-3\,x^2$$

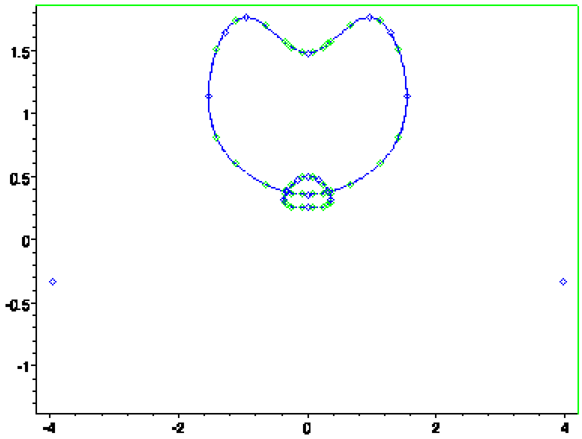
Topological Graph of this section:

```
> principal(Seccion2,x,z,20,'black');
```

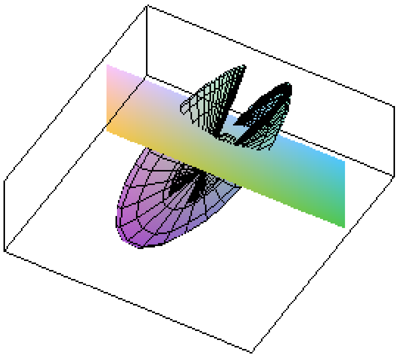


Graph of the section and real picture of the section:

```
> plot_real_curve(Seccion2,x,z,scaling=UNCONSTRAINED);
```




```
> plots[display](P1(-4,4),P2(1/2),grid=[50,50],axes=BOXED,
scaling=UNCONSTRAINED,projection=1,tickmarks=[0,0,0],orientation=[60,1
61]);
```



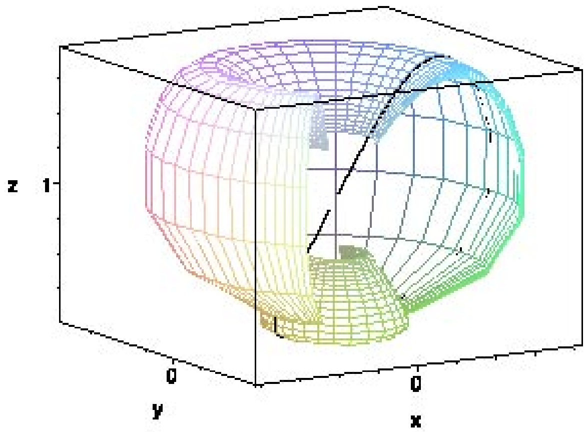
Checking that everything is correct (Where the isolated component comes from?)
The circle of radius 4 with center (0,0,-1/3) in the plane z=-1/3 is inside the considered surface (when presented by its implicit equation).

```
> factor(subs(z=-1/3,toro));
```

$$\frac{25}{81}(x^2+y^2-16)^2$$

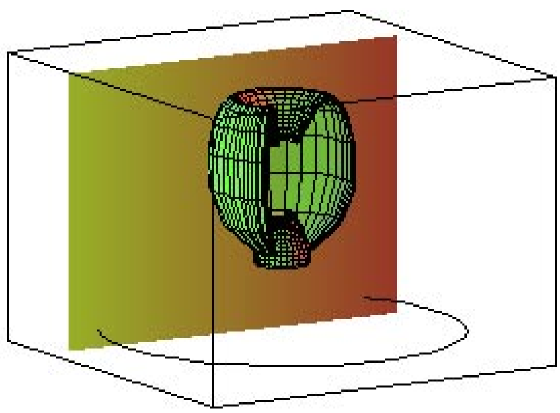
Plotting the revolution surface together with the initial curve:

```
> Initial:=spacecurve([C1,C2,C3],t=-50..50,
                    colour=black,numpoints=3000,labels=[x,y,z]):
Q:=(a,b)->plot3d([T1,T2,T3],s=a..b,t=-6..6,grid=[40,40],
projection=1,tickmarks=[0,0,0],orientation=[59,75],style=HIDDEN):
> display3d([Initial,Q(-
2.5,2.5)],axes=BOXED,tickmarks=[2,2,2],scaling=UNCONSTRAINED);
```



Plotting the revolution surface:
```
> P:=(a,b)->plot3d([T1,T2,T3],s=a..b,t=-6..6,grid=[40,40],
      projection=1,tickmarks=[0,0,0],orientation=[59,75],style=PATCH):
```
Plotting the extraneous geometric component:
```
> T:=spacecurve([4*cos(t),4*sin(t),-1/3],t=0..2*Pi,colour=black):
```
Plotting the different planes:
```
> H:=animate3d([s,y,t],s=-4.2..4.2,t=-1/2..2,y=-2..2.5,

style=PATCHNOGRID,scaling=UNCONSTRAINED,frames=12):
```
Plotting altogether !
```
> display3d([H,T,P(-2.5,2.5)],
    scaling=UNCONSTRAINED,axes=BOXED,lightmodel=light1,shading=XY);
```



DRAWING IMPLICIT SURFACES

```
> restart;
> with(plots):with(algcurves):
```

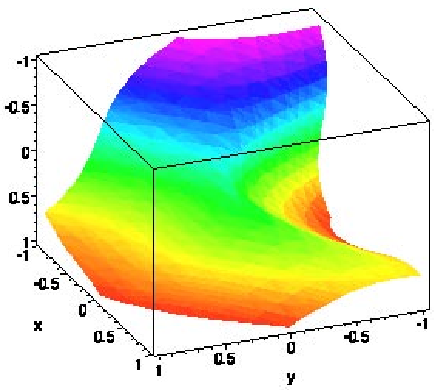
Warning, the name changecoords has been redefined

```
> read "Macintosh HD:Desktop Folder:RAAGRennes03:topcur.txt":
> P:=x^3+x^2*y-z^3+x*y*z+y^2;
```

$$P := x^3+x^2\,y-z^3+x\,y\,z+y^2$$

A first approximation to $P(x,y,z)=0$

```
> implicitplot3d(P,x=-1..1,y=-1..1,z=-1..1,grid=[13,13,13],style=PATCHNOGRID,shading=ZHUE);
```




```
> D0:=implicitplot3d(P,x=-1..1,y=-1..1,z=-1..1,grid=[13,13,13],style=PATCHNOGRID,shading=ZHUE):
```

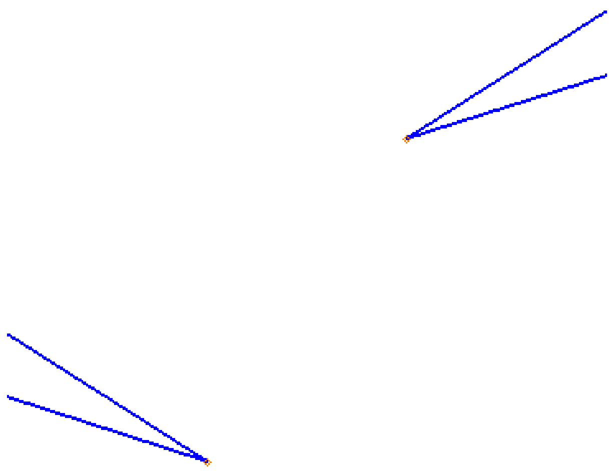
Computing the right subresultants

```
> Pol_StHA:=[StHa(P,1,z)];
```

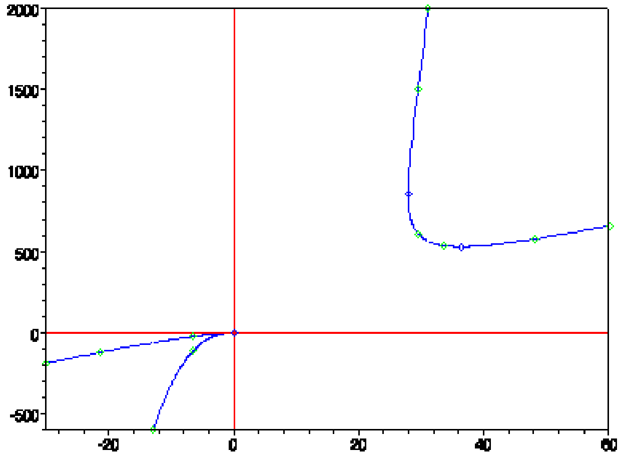
$$Pol\_StHA := [x^3+x^2\,y-z^3+x\,y\,z+y^2, -3\,z^2+x\,y, -9\,x^3-9\,x^2\,y-6\,x\,y\,z-9\,y^2,$$
$$-4\,x^3\,y^3+27\,x^6+54\,x^5\,y+54\,x^3\,y^2+27\,x^4\,y^2+54\,x^2\,y^3+27\,y^4]$$

Plotting the projection on the xy-plane

```
> principal(Pol_StHA[4],x,y,30,blue);
```



```
> D1:=plot_real_curve(Pol_StHA[4],x,y,view=[-30..60,-600..2000]):
> D2:=plot([t,0,t=-30..60]): D3:=plot([0,t,t=-600..2000]):
> display({D1,D2,D3},axes=BOXED);
```



What happens on some regions .... ?

```
> fsolve(subs(x=-3,y=2,P));fsolve(subs(x=1,y=-5,P));
  fsolve(subs(x=-6,y=-45,P));
```

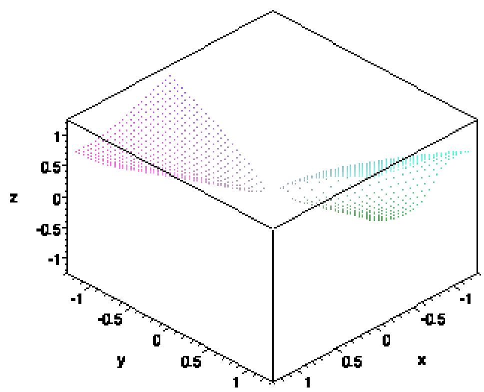
-.7601324178

2.166424046

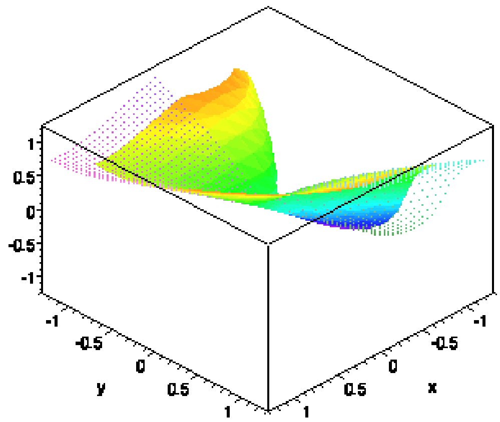
-16.06981070, -.7012773375, 16.77108804

See the surface over $x > 0$ and $y < 0$ (one slice) *or* $x < 0$ and $y > 0$ (one slice)

```
> Lista1:=[]:
for j from 1 to 25 do for k from 1 to 25 do
Lista1:=[op(Lista1),[j/20,-k/20,fsolve(subs(x=j/20,y=-k/20,P))]]od:
od:
> D1:=pointplot3d(Lista1,connect=false,labels=[x,y,z]):
> Lista2:=[]:
for j from 1 to 25 do for k from 1 to 25 do
Lista2:=[op(Lista2),[-j/20,k/20,fsolve(subs(x=-j/20,y=k/20,P))]]od:
od:
> D2:=pointplot3d(Lista2,connect=false,labels=[x,y,z]):
> display({D1,D2});
```

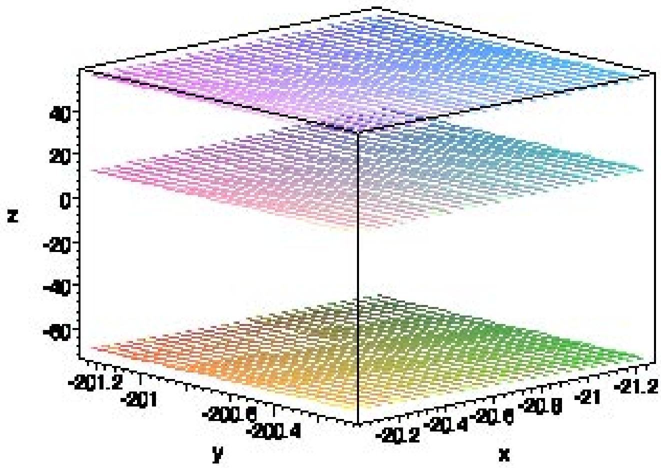


```
> display({D0,D1,D2});
```



See the surface over $x < 0$ and $y < 0$ (three slices)

```
> Lista31:=[]:Lista32:=[]:Lista33:=[]:
for j from 1 to 25 do for k from 1 to 25 do
  lista_aux:=[fsolve(subs(x=-20-j/20,y=-200-k/20,P))];
  Lista31:=[op(Lista31),[-20-j/20,-200-k/20,lista_aux[1]]];
  Lista32:=[op(Lista32),[-20-j/20,-200-k/20,lista_aux[2]]];
  Lista33:=[op(Lista33),[-20-j/20,-200-k/20,lista_aux[3]]];  od: od:
> D31:=pointplot3d(Lista31,connect=true,labels=[x,y,z]):
D32:=pointplot3d(Lista32,connect=true,labels=[x,y,z]):
D33:=pointplot3d(Lista33,connect=true,labels=[x,y,z]):
> display([D31,D32,D33],scaling=UNCONSTRAINED);
```



CURVE TOPOLOGY COMPUTATIONS: The non exact case

Loading the required programs.

```
> restart;
read "Macintosh HD:Desktop Folder:code Folder
Rob:Rob:CompanionMatrixPencil.mpl";
read "Macintosh HD:Desktop Folder:code Folder
Rob:Rob:BezoutLagrange.mpl";
read "Macintosh HD:Desktop Folder:code Folder Rob:Rob:aux.mpl";
> with(plots):with(LinearAlgebra):
```

Defining the polynomial and computing the Bezoutian.

```
> f:=-.6949946950+.8972437112e-2*y^4+.8397759899e-1*y^3-.1580181124e-
1*x^3+.4515930970e-1*x^2*y^2+.1059610946e-
1*x^4+.2972723975*x^2+.2735499120*y^2-.3297741797e-1*x^3*y-
.4625884690*x*y-.3034580357e-1*x*y^3+.2130279206*x^2*y-
.2878723627*x*y^2;
```

$$f := -.6949946950 + .008972437112\,y^4 + .08397759899\,y^3 - .01580181124\,x^3 + .04515930970\,x^2\,y^2 + .01059610946\,x^4 + .2972723975\,x^2 + .2735499120\,y^2 - .03297741797\,x^3\,y - .4625884690\,x\,y - .03034580357\,x\,y^3 + .2130279206\,x^2\,y - .2878723627\,x\,y^2$$

```
> g:=diff(f,y): A:=convert(nmat(f,g,y),Matrix):map(degree,A,x);
```

$$\begin{bmatrix} 6 & 5 & 4 & 3 \\ 5 & 4 & 3 & 2 \\ 4 & 3 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}$$

Solving the Generalized Eigenvalue Problem.

```
> C,B:=CompanionMatrixPencil(A,x,datatype=complex[8]);
```

$$C,B := \begin{bmatrix} \text{24 x 24 Matrix} \\ \text{Data Type: complex[8]} \\ \text{Storage: sparse} \\ \text{Order: Fortran\_order} \end{bmatrix}, \begin{bmatrix} \text{24 x 24 Matrix} \\ \text{Data Type: complex[8]} \\ \text{Storage: sparse} \\ \text{Order: Fortran\_order} \end{bmatrix}$$

```
> disc_roots:=Eigenvalues(C,B);
```

$$disc\_roots := \begin{bmatrix} \text{24 Element Column Vector} \\ \text{Data Type: complex[8]} \\ \text{Storage: rectangular} \\ \text{Order: Fortran\_order} \end{bmatrix}$$

```
> for j from 1 to 24 do print([disc_roots[j]]) od;
```

$$[\mathrm{Float}(\infty)+0.\ I]$$

$$[\mathrm{Float}(\infty)+0.\ I]$$

$$[\mathrm{Float}(\infty)+0.\ I]$$

$$[\mathrm{Float}(\infty)+0.\ I]$$

$$[\mathrm{Float}(\infty)+0.\ I]$$

$$[\mathrm{Float}(\infty)+0.\ I]$$

$$[\mathrm{Float}(\infty)+0.\ I]$$

$$[\mathrm{Float}(\infty)+0.\ I]$$

$$[\mathrm{Float}(\infty)+0.\ I]$$

$$[\mathrm{Float}(\infty)+0.\ I]$$

$$[-.750417638799999952\ 10^{8}-.492614880999999986\ 10^{-8}\ I]$$

$$[-.215177780599999987\ 10^{7}-.521716401799999971\ 10^{-6}\ I]$$

$$[.107565337999999989\ 10^{7}+.182532955600000010\ 10^{7}\ I]$$

$$[.107565337999999989\ 10^{7}-.182532955600000010\ 10^{7}\ I]$$

$$[-5.95578912900000023+.164056575800000007\ 10^{-11}\ I]$$

$$[-5.96825786600000008+.00721405826399999957\ I]$$

$$[-5.96825786500000000-.00721405826300000001\ I]$$

$$[5.00424524000000037+.00732660497699999958\ I]$$

$$[4.99150984100000006+.529911512700000010\ 10^{-11}\ I]$$

$$[5.00424523800000021-.00732660498100000044\ I]$$

$$[.967112885000000033-.207212310699999991\ 10^{-14}\ I]$$

$$[.962597384399999978+.00261442308699999992\ I]$$

$$[.962597384399999978-.00261442308799999992\ I]$$

$$[.934956535999999949+0.\ I]$$

Removing: non finite generalized eigenvalues and non real generalized eigenvalues

```
> BB:=
  max(max(op(map(VectorNorm,[Row(C,21..24)],infinity))),
  max(op(map(VectorNorm,[Row(B,21..24)],infinity)))):
m:=degree(f,x):n:=degree(f,y):
cota:=1+evalf((m+1)^(2*n-1)*norm(diff(f,y),2)^n);
cota2:=((2*m+1)*BB)^n*sqrt(n);
```

$$cota := 73882.15551$$

$$cota2 := 336.3921578$$

```
> real_disc_roots:=[]: for j from 1 to 24 do if
Re(disc_roots[j])<>Float(infinity) and abs(disc_roots[j])<cota2 then
real_disc_roots:=[op(real_disc_roots),disc_roots[j]] fi; od;
> real_disc_roots;
```

$[-5.955789129000000023+.1640565758000000007\ 10^{-11}\ I,$

$-5.968257866000000008+.007214058263999999957\ I,$

$-5.968257865000000000-.007214058263000000001\ I,$

$5.004245240000000037+.007326604976999999958\ I,$

$4.991509841000000006+.5299115127000000010\ 10^{-11}\ I,$

$5.004245238000000021-.007326604981000000044\ I,$

$.9671128850000000033-.2072123106999999991\ 10^{-14}\ I,$

$.9625973843999999978+.002614423086999999992\ I,$

$.9625973843999999978-.002614423087999999992\ I,$

$.9349565359999999949+0.\ I]$

Clustering:

```
> Lista:=[]: for j from 1 to 10 do Lista:=[op(Lista),
  [Re(real_disc_roots[j]),Im(real_disc_roots[j])]] od: Lista;
```

$[[-5.955789129000000023, .1640565758000000007\ 10^{-11}],$

$[-5.968257866000000008, .007214058263999999957],$

$[-5.968257865000000000, -.007214058263000000001],$

$[5.004245240000000037, .007326604976999999958],$

$[4.991509841000000006, .5299115127000000010\ 10^{-11}],$

$[5.004245238000000021, -.007326604981000000044],$

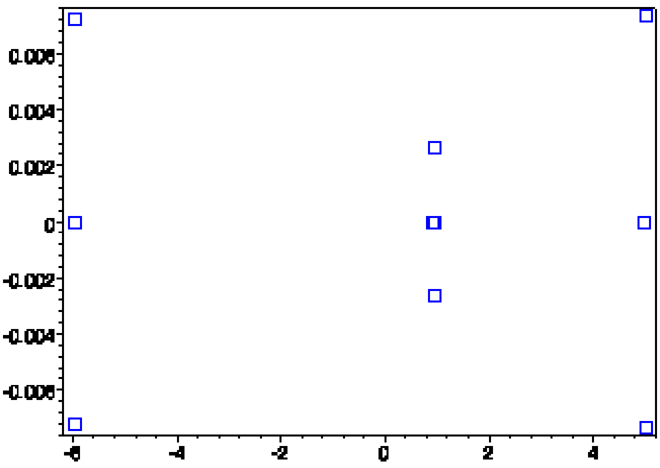
$[.9671128850000000033, -.2072123106999999991\ 10^{-14}],$

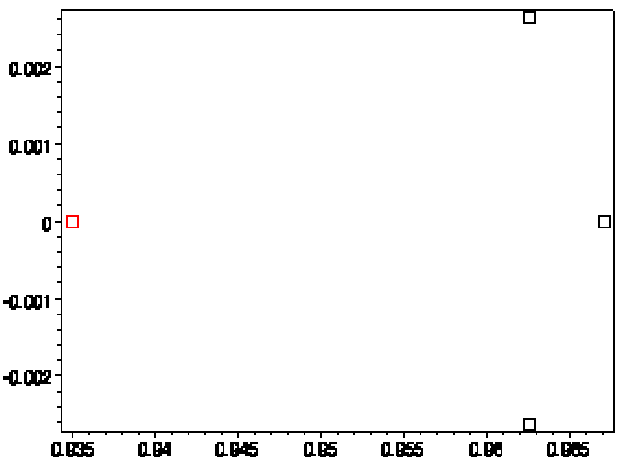
$[.9625973843999999978, .002614423086999999992],$

$[.9625973843999999978, -.002614423087999999992],$

$[.9349565359999999949, 0.]]$

```
> pointplot({op(Lista)},color=blue,axes=BOXED,symbol=BOX,symbolsize=14);
display([pointplot({seq(Lista[j],j=7..9)},color=black,axes=BOXED),
         pointplot({seq(Lista[j],j=10)},color=red,axes=BOXED)],
symbol=BOX,symbolsize=14,scaling=UNCONSTRAINED);
```






```
> cluster1:=(
real_disc_roots[1]+real_disc_roots[2]+real_disc_roots[3])/3;
```

$$cluster1 := -5.964101619 + .2\ 10^{-11}\ I$$

```
> cluster2:=(
real_disc_roots[4]+real_disc_roots[5]+real_disc_roots[6])/3;
```

$$cluster2 := 5.000000106 + .1\ 10^{-11}\ I$$

```
> cluster3:=(
real_disc_roots[7]+real_disc_roots[8]+real_disc_roots[9])/3;
```

$$cluster3 := .9641025512 - .4\ 10^{-12}\ I$$

```
> cluster4:=real_disc_roots[10];
```

$$cluster4 := .934956535999999949 + 0.\ I$$

```
> disc_real_roots:=map(Re,[cluster1,cluster4,cluster3,cluster2]);
```

$disc\_real\_roots :=$ [-5.964101619, .934956535999999949, .9641025512, 5.000000106]

Computing the critical points through the SVD.

Over the first real root on the projection

```
> SingularValues(subs(x=disc_real_roots[1],A), output='list');
```

[124.510294785310989, .160141592434733443, .00131609296767612764,

$.671612325813015415\ 10^{-10}$]

```
> UU[1]:=NullSpace(subs(x=disc_real_roots[1],A));
```

$$UU_1 := \left\{ \begin{bmatrix} .0202274565403114427 \\ -.0732640242882630355 \\ .265363020569301055 \\ -.961148115568737582 \end{bmatrix} \right\}$$

```
> V[1]:=[seq(UU[1][1][j]/UU[1][1][1],j=1..4)];
```

$$V_1 := [1.000000000, -3.622008736, 13.11895146, -47.51700312]$$

Over the second real root on the projection

```
> SingularValues(subs(x=disc_real_roots[2],A), output='list');
```

[.151062799786309348, .0210334065944654397, .702221996529291054 10^{-5},

.115997216353959397 10^{-11}]

```
> UU[2]:=NullSpace(subs(x=disc_real_roots[2],A));
```

$$UU_2 := \left\{ \begin{bmatrix} -.0462667727204963616 \\ .125801915460097224 \\ -.342060248400026023 \\ .930068841684911551 \end{bmatrix} \right\}$$

```
> V[2]:=[seq(UU[2][1][j]/UU[2][1][1],j=1..4)];
```

$$V_2 := [1.000000000, -2.719055342, 7.393216088, -20.10230641]$$

Over the third real root on the projection

```
> SingularValues(subs(x=disc_real_roots[3],A), output='list');
```

[.145496796656071430, .0206935091659732479, .000108433922884654830,

.697066892041253995 10^{-11}]

```
> UU[3]:=NullSpace(subs(x=disc_real_roots[3],A));
```

$$UU_3 := \left\{ \begin{bmatrix} -.0334675453395313905 \\ .101897472189956400 \\ -.310243620128823949 \\ .944587595059137808 \end{bmatrix} \right\}$$

```
> V[3]:=[seq(UU[3][1][j]/UU[3][1][1],j=1..4)];
```

$$V_3 := [1.000000000, -3.044665247, 9.269984307, -28.22398791]$$

Over the forth real root on the projection

```
> SingularValues(subs(x=disc_real_roots[4],A), output='list');
```

[3.47364829024996391, 1.95956569997015517, .0138333616541434709,

.305271812477011899 10^{-11}]

```
> UU[4]:=NullSpace(subs(x=disc_real_roots[4],A));
```

$$UU_4 := \left\{ \begin{bmatrix} -.00333681551105791206 \\ -.0222454370678382071 \\ -.148302916672250595 \\ -.988686123650175053 \end{bmatrix} \right\}$$

```
> V[4]:=[seq(UU[4][1][j]/UU[4][1][1],j=1..4)];
```

$$V_4 := [1.000000000, 6.666666766, 44.44444597, 296.2963102]$$

Checking the obtained results:

```
> subs(x=disc_real_roots[1],y=V[1][2],f);
subs(x=disc_real_roots[2],y=V[2][2],f);
subs(x=disc_real_roots[3],y=V[3][2],f);
subs(x=disc_real_roots[4],y=V[4][2],f);
```

$$-.4\ 10^{-7}$$

$$-.1830\ 10^{-6}$$

$$-.2\ 10^{-9}$$

$$-.34\ 10^{-7}$$

Some final remarks !

```
> plot_real_curve(f,x,y);
```

$\text{plot\_real\_curve}(-.6949946950+.008972437112\ y^4+.08397759899\ y^3$

$-.01580181124\ x^3+.04515930970\ x^2\ y^2+.01059610946\ x^4+.2972723975\ x^2$

$+.2735499120\ y^2-.03297741797\ x^3\ y-.4625884690\ x\ y-.03034580357\ x\ y^3$

$+.2130279206\ x^2\ y-.2878723627\ x\ y^2, x, y)$

```
> discrim(f,y);fsolve(%,x);
```

$.0001751645062\ x^3-.0002510092344\ x^2-.00003561347558\ x^4$

$-.00001309613391\ x^5+.4536326194\ 10^{-5}\ x^6-.1538900449\ 10^{-8}\ x^9$

$+.2839270485\ 10^{-6}\ x^7-.1518392238\ 10^{-6}\ x^8+.1645953659\ 10^{-8}\ x^{10}$

$+.5795526828\ 10^{-17}\ x^{11}-.5641705256\ 10^{-18}\ x^{12}-.00003656991138$

$+.0001564556219\ x$

-54008.99044, -5.960527884, .9340107318, .9729668332, 4.985004749,

54018.32814

```
> disc_real_roots;
```

[-5.964101619, .934956535999999949, .9641025512, 5.000000106]

```
> norm(f,2);
```

1.000000000