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Algebraic Methods in Computer Aided Geometric Design: Theoretical and Practical Applications

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Algebraic Methods in Computer Aided Geometric Design: Theoretical and Practical Applications

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1 Introduction

The usefulness of CAD/CAM (Computer Aided Design/Computer Aided Modelling) systems as a means of increasing the efficiency of the design process is nowadays uncontested. Advantages such as

- reduction of lead times,
- quality improvements, and
- cost reduction by saving time spent implementing engineering changes in the design process

are often cited as the major benefits resulting from the introduction of specialized software for CAD/CAM. From a mathematical point of view almost all the CAD/CAM problems are

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related to the manipulation of geometric objects into the two or three dimensional space, mainly curves and surfaces and combinations of both. Since these geometric entities are usually presented through polynomials or rational functions via their implicit or their parametric representation, it is clear that the intersection between Computer Algebra, Algebraic Geometry and Computer Aided Geometric Design must be nonempty.

In this paper we plan to briefly survey some relevant results in the symbolic and symbolic–numeric manipulation of curves and surfaces and their applications in Computer Aided Geometric Design (CAGD). Since, some chapters of this volume (see [48], [62]) are devoted to symbolic algorithms for curves and surfaces, we will focus more on the approximate version of the problems as well as their applications. More precisely, we will focus on the following levels

- Implicititation and Parametrization Problems when approximate objects are under consideration.
- Applications in CAGD such as computing with implicit curves, offsetting and blending.
- Practical performance of algebraic techniques in CAGD.

2 Implicititation and Parametrization Problems: Exact versus Approximate

Algebraic curves and surfaces are the main geometric objects in CAGD, specially rational curves and surfaces. In many practical applications parametric representation of varieties are used (see e.g. [5], [6], [41], [46]) but other situations require the availability of the implicit equation or even produce as output of a geometric operation the implicit equation since this new geometric object does not have a parametric representation. These facts have motivated the emergence of a research area devoted to the construction of conversion algorithms for algebraic varieties, namely parametrization and implicitization algorithms (see e.g. [6], [42], [48], [62]).

This section is devoted to analyze the problem of how to adapt the symbolic algorithms of implicitization and parameterization which require exact coefficients to the more real case where the involved coefficients are floating–point real numbers.

2.1 Implicitization

One of the main problems arising in the manipulation of curves and surfaces in CAGD consists in finding efficient algorithms for computing the implicit equations of curves and surfaces parametrized by rational functions (see for example [14], [15], or the chapters 5 and 7 in [41]). This is due to the fact that, for example, while for tracing the considered curve or surface the parametric representation is the most convenient, for deciding in an efficient way the position of a point with respect to the considered curve or surface, the implicit equation is desired.

The implicitization problem for hypersurfaces (in real applications, curves in the real plane or surfaces in the three dimensional real space) parametrized in a rational way can be stated in the following terms: let $\mathcal{V}$ be a hypersurface in $\mathbb{R}^n$ (in real applications $n = 2$ or $n = 3$) parametrized by ($i \in \{1, \ldots, n\}$):

$$x_i = \frac{f_i(t_1, \ldots, t_{n-1})}{g_i(t_1, \ldots, t_{n-1})}$$

where f_i and g_i belong to $\mathbb{Z}[t_1, \ldots, t_{n-1}]$ with $\gcd(f_i, g_i) = 1$. The implicitization problem for $\mathcal{V}$ consists in finding a non zero element $\mathcal{R}_{\mathcal{V}}(x_1, \ldots, x_n)$ in $\mathbb{Z}[x_1, \ldots, x_n]$ with the smallest posible total degree and such that:

$$\mathcal{R}_{\mathcal{V}}\left(\frac{f_1(t_1, \ldots, t_{n-1})}{g_1(t_1, \ldots, t_{n-1})}, \ldots, \frac{f_n(t_1, \ldots, t_{n-1})}{g_n(t_1, \ldots, t_{n-1})}\right) = 0.$$

More general formulations of the implicitization problem for arbitrary parametric varieties can be found in [1], [9], [11], [30], [33] or [47].

The implicitization problem can be seen as a problem in elimination theory and therefore it can be approach by elimination techniques as Gröbner Basis. Nevertheless, alternative methods can be applied. Next example, extracted from [33], shows a non standard way of computing the implicit equation of a rational surface avoiding the use of Gröbner Bases or resultants (ie determinants of polynomial matrices).

Example 2.1.
The parametric equations of the bicubic surface $\mathcal{B}$ are:

$$\begin{aligned} x(u,v) &= 3v(v-1)^2 + (u-1)^3 + 3u \\ y(u,v) &= 3u(u-1)^2 + v^3 + 3v \\ z(u,v) &= -3u(u^2-5u+5)v^3 - 3(u^3+6u^2-9u+1)v^2 \\ &\qquad + v(6u^3+9u^2-18u+3) - 3u(u-1). \end{aligned}$$

We look for an implicit equation $\mathcal{H}_{\mathcal{B}}$ for $\mathcal{B}$. The first two equations, denoted by H_1 and H_2, (the third one will be denoted by H_3) do not have the desired structure in order to apply the ad–hoc technique presented in [33] but an easy linear combination of them

$$F_1 = \frac{-H_1 + 3H_2}{8} = u^3 + \frac{3v^2}{4} - \frac{15u^2}{8} + \frac{3v}{4} - \frac{3y}{8} + \frac{x}{8} + \frac{3u}{8} + \frac{1}{8}$$

$$F_2 = \frac{3H_1 - H_2}{8} = v^3 - \frac{9v^2}{4} - \frac{3u^2}{8} + \frac{3v}{4} + \frac{y}{8} - \frac{3x}{8} + \frac{15u}{8} - \frac{3}{8}$$

gives the good shape for the polynomial system to deal with. In this particular case the equation $\mathcal{H}_{\mathcal{B}}$ has the following structure:

$$\mathcal{H}_{\mathcal{B}}(x,y,z) = z^9 + \sum_{i=1}^{9} r_i(x,y) z^{9-i} = \prod_{F_1(\Delta)=0, F_2(\Delta)=0} (z - H_3(\Delta)) \tag{1}$$

The computation of the $r_i(x,y)$'s is performed by computing the Newton Sums of order 9, $\mathbf{S}_k(x,y)$ $(k \in \{1,\ldots,9\})$ for the equation (1):

$$\mathbf{S}_k(x,y) = \sum_{F_1(\Delta)=0, F_2(\Delta)=0} (H_3(\Delta))^k.$$

For that, first it is computed the Jacobian Determinant of F_1 and F_2:

$$\mathbf{Jac}(u,v) = 9v^2u^2 - \frac{27}{2}vu^2 + \frac{9}{4}u^2 - \frac{45}{4}v^2u + 18vu - \frac{9}{4}u + \frac{9}{8}v^2 - \frac{9}{2}v - \frac{9}{8}$$

Denoting by $\ell(\mathbf{Jac}\, u^i v^j)$ $(0 \le i,j \le 2)$ the coefficient of u^2v^2 into the normal form of $\mathbf{Jac}\, u^i v^j$ with respect to F_1 and F_2, then every $\mathbf{S}_k(x,y)$ is determined by the following expression:

$$\mathbf{S}_k(x,y) = \sum_{i,j=0}^{2} c_{ij}^{(k)} \ell(\mathbf{Jac} u^i v^j)$$

where the $c_{ij}^{(k)}$'s are the coefficients of the normal form of H_3^k with respect to F_1 and F_2:

$$(H_3(s,t))^k = \sum_{i,j=0}^{2} c_{ij}^{(k)} u^i v^j \qquad \bmod \ \langle F_1, F_2 \rangle.$$

Finally the desired result is obtained by using the classical Newton Identities of the univariate case. The first two coefficients in $\mathcal{H}_{\mathcal{B}}(x,y,z)$ are:

$$\begin{aligned}
\mathbf{r}_1(x,y) =& -\frac{233469x}{2048} + \frac{188595y}{2048} - \frac{112832595}{262144} - \frac{81x^2}{64} + \frac{135xy}{32} - \frac{81y^2}{64}\\
\mathbf{r}_2(x,y) =& -\frac{20972672709381x}{536870912} + \frac{17975329363179y}{536870912} - \frac{729y^4}{8192} - \frac{729x^4}{8192} + \frac{1215x^3y}{2048}\\
&+\frac{1215xy^3}{2048} - \frac{4105971x^3}{65536} + \frac{3129597y^3}{65536} + \frac{14456151x^2y}{65536} - \frac{13181049xy^2}{65536}\\
&+\frac{48101467761xy}{8388608} - \frac{38812918311y^2}{16777216} - \frac{226569919823911171}{137438953472} - \frac{1}{2}\Big(\frac{233469x}{2048}\\
&+\frac{112832595}{262144} + \frac{81x^2}{64} - \frac{188595y}{2048} - \frac{135xy}{32} + \frac{81y^2}{64}\Big)\Big(-\frac{233469x}{2048} + \frac{188595y}{2048}\\
&+\frac{135xy}{32} - \frac{81y^2}{64} - \frac{112832595}{262144} - \frac{81x^2}{64}\Big) - \frac{4779x^2y^2}{4096} - \frac{54187594407x^2}{16777216} + \cdots\cdots\cdots\cdots
\end{aligned}$$

The computing time was less than 15 seconds by using `Maple 9` on a Power PC at 1MHz (the implicitization time for the previous bicubic spline was 1500 seconds). The size of the file containing the full implicit equation of $\mathcal{B}$ is around 600 kbytes and it is available upon request. It is important to mention that no Computer Algebra was able to substitute the parametric equations of $\mathcal{B}$ inside its implicit equation and obtaining 0 as result. Therefore we have verified that several hundreds of points randomly generated on $\mathcal{B}$ verify the obtained implicit equation. □

Another problems with a similar formulation than the implicitization problem described before, and where the solution is obtained by eliminating some variables from the initial equations, are (see [41]):

- Computation of offset curves and surfaces.
- Computation of constant–radius blending surfaces.
- Computation of the convolution of two plane curves or surfaces.
- Computation of the convolution of two plane curves.
- Computation of the common tangent of two plane curves.
- Computation of the inversion formula for parametric surfaces.

These geometrical operations are often used when generating the boundary of a configuration space obstacles, in order to construct collision free motion paths for translating objects.

Two main difficulties are encountered when trying to use the usual elimination techniques offered by Computer Algebra (resultants, Gröbner bases, etc) to deal with the variable elimination problems mentioned before. For example the implicitation of a rational surface defined by

$$x = \frac{X(u,v)}{W(u,v)},\ y = \frac{Y(u,v)}{W(u,v)},\ z = \frac{Z(u,v)}{W(u,v)}$$

appearing into a real–world problem is difficult to achieve by applying directly resultants or Gröbner bases because, first, it is usually a very costly algebraic operation and, second, the coefficients of the polynomials in the parametrization are usually floating–point real numbers.

These difficulties can be currently overcome in two different ways:

- By using multivariate resultants (see [9]), the implicit equation is described as a non evaluated determinant. Then any question about the considered surface, requiring the implicit equation, is reduced to a Numerical Linear Algebra question over such matrix (usually an eigenvalue problem).
- By taking into account that, in general, a concrete object to be modelled is made by several hundreds (or thousands) of small patches, all of them sharing the same algebraic structure. For such an object a database is constructed containing the implicit equation of every class of patch appearing in its definition. This database must also contains the inversion formulae (providing the parameters in terms of the cartesian coordinates) and must be pruned to avoid specialization problems. Moreover the database for a specific object is kept into a bigger and general database for a further use (see [20]).

For this reason, a first option for the study of numeric implicitization algorithms the precomputation of the implicit representations of the algebraic models of the patches, defined in a generic way by means of parameters. This approach, called "*generic implicitation*", generates data bases which allow a quickly problem solving. For instance, the implicit equation of the parametric surface

$$x = a_1 v^2 + a_2 v, \quad y = b_1 u^2 + b_2, \quad z = c_1 uv + c_2 u + c_3 v \tag{2}$$

is, for almost all the values of the parameters a_i, b_i and c_i,

$$\begin{aligned} &c_1^4 x^2 y^2 - 2b_1 a_1 c_1^2 xyz^2 + b_1^2 a_1^2 z^4 + (-2c_2^2 a_1 c_1^2 + 2c_2 a_2 c_1^3) xy^2 \\ &+(-2c_3^2 b_1 c_1^2 - 2b_2 c_1^4) x^2 y + (-8c_2 c_3 b_1 a_1 c_1 + 2c_3 a_2 b_1 c_1^2) xyz + \ldots = 0. \end{aligned} \tag{3}$$

The data base may contain also the conditions which imply that the previous representation provides after specialization a bad implicit equation: for instance, if a_1 is considered to be equal to 0 in the previous representation, then the resulting implicit equation after specialization is

$$(c_1^2 y - c_3^2 b_1 - c_1^2 b_2) \cdot (c_1^2 x^2 y - (b_2 c_1^2 + b_1 c_3^2) x^2 + \ldots - a_2^2 c_2^2 b_2) = 0,$$

which contains not only the implicit equation of the surface in (2) with $a_1 = 0$, but also the extraneous factor corresponding (if $c_1 \neq 0$) to the equation of the plane $y = (c_3^2 b_1 + c_1^2 b_2)/c_1^2$. Even in this case this “bad” implicit equation can be useful since the availability of the parametric representation allows to discard in practice those points coming from the extraneous factor.

Moreover, the data base should contain the algebraic expressions which describe u and v in function of x, y and z. For the surface defined by (2), and whose implicit equation appears in (3), the value of v in function of x, y and z is given by the formula:

$$v = \frac{b_1 a_1 z^2 - c_1^2 xy + (b_1 c_3^2 + c_1^2 b_2) x - a_1 c_2^2 y + a_1 c_2^2 b_2}{2b_1 a_1 c_3 z + (2a_1 c_2 c_1 - c_1^2 a_2) y - 2a_1 c_2 c_1 b_2 + a_2 b_1 c_3^2 + c_1^2 a_2 b_2}.$$

The main drawback of the first approach is due to the existence of base points, i.e. solutions of the polynomial system

$$X(u, v) = 0,\ Y(u, v) = 0,\ Z(u, v) = 0,\ W(u, v) = 0,$$

since their existence implies the vanishing of the determinant defining the implicit equation. This problem is solved by looking for an appropriated submatrix of full rank in the resultant matrix defining the implicit equation as shown in [50].

The main drawback provided by the second approach is due to the fact that some algebraic structures arising in the data base construction are very complicated and the implicit equation can not be generated or even difficult to use due to its huge size. Namely:

$$x = \frac{X(u, v)}{W(u, v)}, \quad y = \frac{Y(u, v)}{W(u, v)}, \quad z = \frac{Z(u, v)}{W(u, v)}$$

with

$$W(u, v) = \sum_{j=0}^{3} (A_j v^2 + B_j v + C_j) u^j$$

and $(U \in \{X, Y, Z\}, i \in \{x, y, z\})$

$$U(u, v) = \sum_{j=0}^{3} (\alpha_j^{(i)} v^2 + \beta_j^{(i)} v + \gamma_j^{(i)}) u^j.$$

Thus this strategy has no practical applicability when considering cases such as the general bicubic patches, due to the fact that the generic implicit equation is too complex to be used in practice.

It is not also easy to deal in advance with specialization problems: up to this moment these are detected by substituting several points in the surface, uniformly generated by the parametrization, into the candidate to be the implicit equation.

A way of solving this problem is using techniques to reduce the degree of the considered surface. This degree reduction could affect the parametrization and the implicit equation as well. In the first case, by using a set of techniques already usual in Geometric Design, the considered surface is approximated by another one with parametrizations of lower degrees, which allows the use of the precomputed implicit equations in the data base. In the second case, an upper bound for the total degree of the implicit equation of the considered surface is determined and established from the beginning and, for each patch, the implicit equation coefficients are determined such that the error produced in the computations is smaller than the tolerance chosen to solve the problem.

It is important to mention that the algorithms sketched here involve techniques for computing singular values, formal expansion at infinity of rational functions constructed with the parametrization of the initial curve or surface, rewriting symmetric polynomials in terms of the solutions of certain equation systems like in our initial example, etc.

For instance, in the case of degree reducing the implicit equation, it is possible to reduce even the number of components/patches of the considered curve or surface (see [19] or for a similar formulation [13]). If the curve to be implicitized is defined by

$$\begin{pmatrix} x(u) \\ y(u) \end{pmatrix} = \begin{cases} (u, u^2) & \text{si } u > 0 \\ (u, -u^2) & \text{si } u \leq 0 \end{cases}$$

with $u \in [-1, 1]$, and the total degree of the implicit equation is decided to be bounded by 3, then by using the Bernstein bases and computing the singular values of the matrix generated by replacing the parametrizations in the implicit equation to be computed, the following result is obtained as the implicit equation:

$$\begin{aligned} &-0.14338002021536847y^3 + 0.50872225217688360xy^2 - 0.14431972674994731y \\ &\quad -0.69833143139600062x^2y + 0.017022876399572399x + 0.460281533430831x^3 = 0. \end{aligned}$$

In Figure 1 the inicial parametric curve and the curve associated to the implicit equation are displayed, being these two curves indistinguishable. In this case, using the Bernstein basis is essential (using the habitual power base does not produce the correct result with respect to the accuracy of the obtained result).

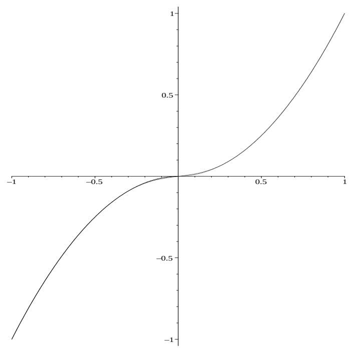

Figure 1: *Approximated implicitization.*

Another strategy consists in replacing the computations of resultants or Gröbner bases by developing evaluation outlines which supply directly the implicit equation. For instance, if the curve to be implicitized is defined by the polynomial parametrization

$$x = f(u), y = g(u),$$

then its implicit equation is given by the "formula"

$$\prod_{f(\alpha)-x=0} (y - g(\alpha)),$$

where the solutions of the equation $f(\alpha) - x = 0$ are considered in the algebraic closure of $\mathbb{K}(x)$ (where $\mathbb{K}$ is the base field). Using Newton Identities together with the Laurent expansion of the rational functions

$$\frac{f'(x)g(x)^k}{f(x)}$$

give the desired result, without computing any determinants (which could be a difficult task when the coefficients of $f(t)$ and $g(t)$ are given in an approximated form) or Gröbner bases. Another alternative to be considered is to compute the Puiseux expansion of the solutions of the equation $f(u) - x = 0$ (seen as equation in u) and to use the product to recover the implicit equation.

2.2 Parametrization

Although many authors have addressed the problem of globally and symbolically parametrizing algebraic curves and surfaces (see [62]), only few results have been achieved for the case of approximate algebraic varieties. Piecewise parametrizations are provided in [12], [29], [38] by means of combination of both algebraic and numerical techniques for solving differential equations and rational B-spline manipulations. In [7], the problem of finding a global approximate parametrization is studied for the case of approximate irreducible conics, rational cubics and quadrics. In [53], the results in [7] are generalized to the special case of curves parametrizable by lines.

The statement of the problem for the approximate case is slightly different to the classical symbolic parametrization question (see [62]). Intuitively speaking, one is given an irreducible affine algebraic plane real curve $\mathcal{C}$, that may or not be rational, and a tolerance $\epsilon > 0$, and the problem consists in computing a rational curve $\overline{\mathcal{C}}$, and its parametrization, such that almost all points of the rational curve $\overline{\mathcal{C}}$ are in the "*vicinity*" of $\mathcal{C}$. The notion of vicinity may be introduced as the offset region limited by the external and internal offset to $\mathcal{C}$ at distance ϵ, and therefore the problem consists in finding, if it is possible, a rational curve $\overline{\mathcal{C}}$ lying within the offset region of $\mathcal{C}$.

For instance, let us suppose that we are given a tolerance $\epsilon = 0.001$, and that we are given the quartic $\mathcal{C}$ defined by

$$x^4 + 2y^4 + 1.001x^3 + 3x^2y - y^2x - 3y^3 + 0.00001y^2 - 0.001x - 0.001y - 0.001.$$

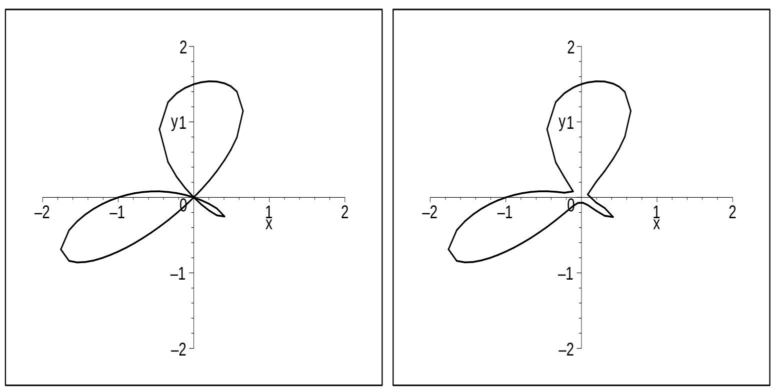

Figure 2: *Curve $\mathcal{C}$ (left), curve $\overline{\mathcal{C}}$ (right)*

Note that $\mathcal{C}$ has genus 3, and therefore the input curve is not rational. Thus, an answer to the problem is given by the quartic $\overline{\mathcal{C}}$ defined by

$$\begin{aligned} &x^4 + 2.y^4 + 1.001x^3 + 3.x^2y - y^2x - 3.y^3 + 10^{-6}y^2 - .6243761996 \cdot 10^{-13}x \\ &\quad -.6260915576 \cdot 10^{-13}y + .9744187291 \cdot 10^{-23} - .3522924910 \cdot 10^{-16}x^2 \\ &\quad +.9991263887 \cdot 10^{-6}xy \end{aligned}$$

that can be parametrized by $\overline{\mathcal{P}}(t) = (\overline{p}_1(t), \overline{p}_2(t))$, where

$$\overline{p}_1(t) = -.487671 \cdot \frac{2.0526 - 2.05055t^2 + 6.15167t + .512063 \cdot 10^{-6}t^4 - 6.15167t^3}{1 + 2t^4},$$

$$\overline{p}_2(t) = .487671 \cdot \frac{-2.05260t + 2.05055t^3 - 6.15167t^2 + 6.15167t^4 + .256287 \cdot 10^{-6}}{1 + 2t^4}.$$

In Figure 2 one may check that $\mathcal{C}$ and $\overline{\mathcal{C}}$ are close.

In the following, we briefly describe the ideas in [53] for parametrizing approximate curves by lines. More precisely, given a tolerance $\epsilon > 0$ and an algebraic plane real curve

$\mathcal{C}$ defined by an ϵ–irreducible polynomial $f(x,y) \in \mathbb{C}[x,y]$ of degree d, and having an ϵ-singularity of multiplicity $d-1$ (see below the notion of ϵ-singularity, and [12] for the concept of ϵ–irreducibility), the algorithm we describe computes a proper parametrization of a rational curve that is exactly parametrizable by lines (see [62] for the notion of properness). Furthermore, the error analysis shows that under certain initial conditions that ensures that points are projectively well defined, the output curve lies within the offset region of $\mathcal{C}$ at distance at most

$$2\sqrt{2}\epsilon^{1/(2d)}e^2.$$

We start with the notion of ϵ–singularity. We say that $\overline{P} \in \mathbb{C}^2$ is an ϵ*–affine singularity* of multiplicity r of an algebraic plane curve defined by a polynomial $f(x,y) \in \mathbb{R}[x,y]$ if, for $0 \leq i+j \leq r-1$, it holds that

$$\left|\frac{\partial^{i+j} f}{\partial^i x \partial^j y}(\overline{P})\right| / \|f\| < \epsilon,$$

where $\|f\|$ denotes the infinity norm of f. Now, let $\mathcal{C}$ be an ϵ–irreducible (over $\mathbb{C}$) real algebraic curve of degree d having an ϵ–singularity $\overline{P} = (\overline{a}, \overline{b})$ of multiplicity $d-1$ (for checking the existence and actual computation of ϵ–singularities see [53]), and let $f(x,y)$ its defining polynomials. Then the following theorems hold.

Theorem 2.2.
Let $\overline{p}_1(t)$ be the root in $\mathbb{R}(t)$ of the quotient of $f(x, tx + \overline{b} - \overline{a}t)$ and $(x-\overline{a})^{d-1}$, and let $\overline{p}_2(t) = t\overline{p}_1(t) + \overline{b} - t\overline{a}$. Then the implicit equation of the rational curve $\overline{\mathcal{C}}$ defined by the parametrization $\overline{P}(t) = (\overline{p}_1(t), \overline{p}_2(t))$ is

$$\overline{f}(x,y) = f(x,y) - T(x,y),$$

where $T(x,y)$ is the Taylor expansion up to order $d-1$ of $f(x,y)$ at $\overline{P}$. □

Theorem 2.3.
$\mathcal{C}$ is contained in the offset region of $\overline{\mathcal{C}}$ at distance $2\sqrt{2}\epsilon^{\frac{1}{2d}}e^2$. □

The following example illustrate the results stated in the previous theorems.

Example 2.4. (See [53])
We consider $\epsilon = 0.001$ and the curve $\mathcal{C}$ of degree 6 defined by the polynomial

$$f(x,y) = y^6 + x^6 + 2.yx^4 - 2.y^4x + 10^{-3}x + .10^{-3}y + 2 \cdot 10^{-3} + 10^{-3}x^4.$$

The point $\overline{P} = (.1875000000 \cdot 10^{-5}, -.50000002 \cdot 10^{-3})$ is an ϵ–singularity of multiplicity 5, and therefore $\mathcal{C} \in \mathcal{L}^6_{0.001}$. Applying Theorem 2.2 one gets the curve $\overline{\mathcal{C}}$ defined by

$$\begin{aligned}\overline{f}(x,y) = & -.1250000464 \cdot 10^{-12}x + .1125000100 \cdot 10^{-14}y + .9999999873 \cdot 10^{-3}x^4 + 2.yx^4 \\ & -2.y^4x - .1000000173 \cdot 10^{-8}yx + y^6 + x^6 - .7500000036 \cdot 10^{-8}x^3 \\ & +.2499999700 \cdot 10^{-8}y^3 + .2109375029 \cdot 10^{-13}x^2 - .3000000180 \cdot 10^{-12}y^4 \\ & +.2812500000 \cdot 10^{-11}y^2 - .1500000000 \cdot 10^{-4}x^3y - .4000000160 \cdot 10^{-2}xy^3 \\ & -.3000000240 \cdot 10^{-5}y^2x + .4218750000 \cdot 10^{-10}yx^2 + .1562500311 \cdot 10^{-18},\end{aligned}$$

and its parametrization $\overline{\mathcal{P}}(t) = (\overline{p}_1(t), \overline{p}_2(t))$ where

$$\overline{p}_1 = \frac{-2t + .3000000120 \cdot 10^{-2} t^5 + .1875000000 \cdot 10^{-5} t^6 + 2.t^4 - .9375000000 \cdot 10^{-5}}{1 + t^6},$$

$$\overline{p}_2 = \frac{-.4887500200 \cdot 10^{-3} - 2.t^4 - .3000000120 \cdot 10^{-2} t^5 + 2t - .5000000200 \cdot 10^{-3} t^6}{1 + t^6}.$$

See Figure 3 to compare the input curve and the rational output curve. □

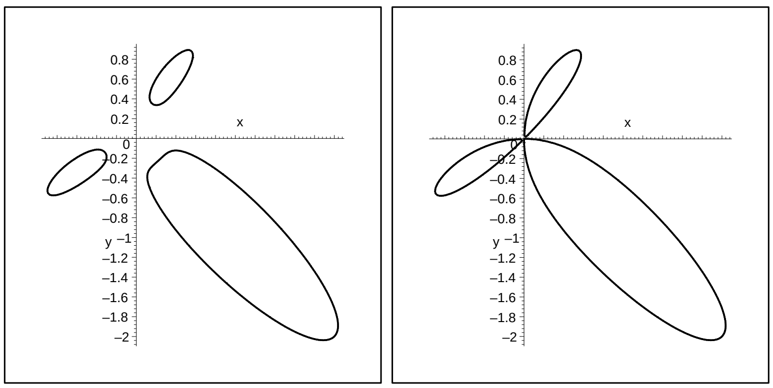

Figure 3: *Curve $\mathcal{C}$ (left), curve $\overline{\mathcal{C}}$ (right)*

3 Applications in CAGD

As we have mentioned, CAGD is a natural frame for applications of algebraic curves and surfaces. In this section four of these applications are analyzed, namely computing with real plane curves defined implicitly, dealing with offsets curves and surfaces including the consideration of topological problems and blending several surfaces.

3.1 Implicit Real Curves Plotting

A very common problem in CAD systems is the resolution of topological questions when dealing with geometric entities defined by algebraic objects. Probably the most simple one is the determination of the topology of an algebraic curve defined by its implicit equation. For instance, for the polynomial

$$\begin{aligned} f(x,y) \quad = \quad & 279756.0x - 559692.0xy^2 + 279936.0xy^4 + 15588.0y^2x^3 + 217.0x^5 \\ & -745286.4y^2 15583.0x^3 + 26043.6x^2 - 2303.9x^4 + 35.9x^6 + 370656.0y^4 \\ & -72774.0y^2x^2 + 2589.4y^2x^4 + 1296.0y^6 + 46728.0y^4x^2 + 373334.3900 \end{aligned}$$

its real drawing appears to the left of Figure 4. This picture (only quantitative) does not allow to determine which is the real configuration of the considered curve, while the graph appearing to the right of Figure 4 gives the right qualitative information looked for. This

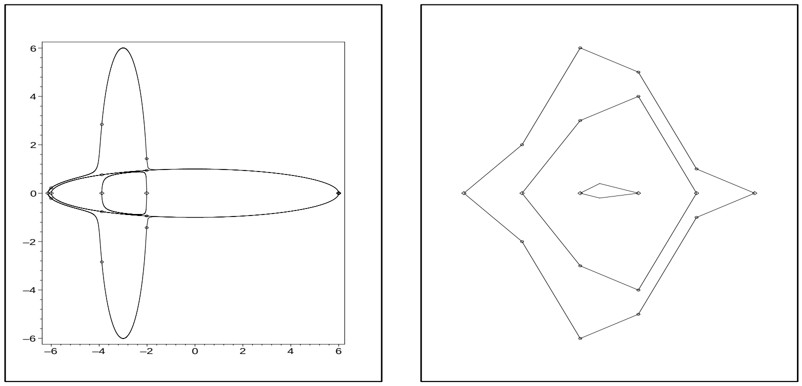

Figure 4: *Topological resolution of implicitly defined algebraic curves.* $f(x, y) = 0$.

graph can be computed in a very fast way (a few seconds) by using the algorithms in [34] and [36] and can be used to resolve topological problems (see Subsection 3.3).

The problem of computing the graph (even topologically) of a planar algebraic curve defined implicitly has received a special attention from Computer Algebra, since it has been responsible of many advances regarding subresultants, real root counting, infinitesimal computations, etc. From the seminal papers [2], [31] and [57], the interested reader can see in [10], [17], [26], [34], [36] and [45], how the theoretical and practical complexities of the algorithms dealing with this problem have been dramatically improved.

The usual strategy to compute the graph (even topologically) of a planar algebraic curve defined implicitly by a polynomial $f(x, y) \in \mathbb{R}[x, y]$ proceeds in the following way:

- Step I: Computation of the discriminant of f with respect to y, $R(x)$, and characterization of the real roots of $R(x)$, $\alpha_1 < \ldots < \alpha_r$.
- Step II: For every α_i, computation of the real roots of $f(\alpha_i, y)$, $\beta_{i,1} < \ldots < \beta_{i,s_i}$.
- Step III: For every α_i and $\beta_{i,j}$ computation of the number of half–branches to the right and to the left of the point $(\alpha_i, \beta_{i,j})$.

Following [36] and in order to avoid the numerical problems arising from the computation of the roots of $R(x)$ and of every $f(\alpha_i, y)$ which has always multiple roots, before starting the computations, a generic linear change of variables is performed in order to have the following condition for every $\alpha \in \mathbb{R}$:

$$\#\{\beta \in \mathbb{R} : f(\alpha, \beta) = 0, \frac{\partial f}{\partial y}(\alpha, \beta) = 0\} \leq 1.$$

This assures that for every α_i real root of $R(x)$, there is only one critical point of the curve in the vertical line $x = \alpha_i$, whose y-coordinate can be rationally described in terms of α_i. Moreover this allows to symbolically construct, from every $f(\alpha_i, y)$, a squarefree polynomial $g_i(\alpha_i, y)$ whose real roots need to be computed in order to finish with the so called Step II. Step III is thus accomplished by merely computing the number of real roots of the squarefree polynomials $f(\gamma_i, y)$ ($i \in \{0, 1, \ldots, r+1\}$) with $\gamma_0 = -\infty$, $\gamma_{r+1} = \infty$ and γ_i any real number in the open interval (α_i, α_{i+1}). These computations provide a graph of the considered curve which is very helpful when the curve is going to be traced numerically, since we know exactly how to proceed when coming closer to a complicated point.

A more complicated example is given by the squarefree polynomial

$$\begin{aligned} g(x,y) \;=\;& y^8 + y^7 + (-7x-8)y^6 + (21x^2-7)y^5 + (-35x^3+35x+20)y^4 + \\ & (35x^4 - 70x^2 + 14)y^3 + (-21x^5 + 70x^3 - 42x - 16)y^2 + \\ & (7x^6 - 35x^4 + 42x^2 - 7)y - x^7 + 7x^5 - 14x^3 + 7x + 2. \end{aligned}$$

Next, both, the topological structure and the true drawing of the real algebraic plane curve defined by f are displayed. The real drawing of the curve is obtained by using the information contained into the graph providing the topological structure.

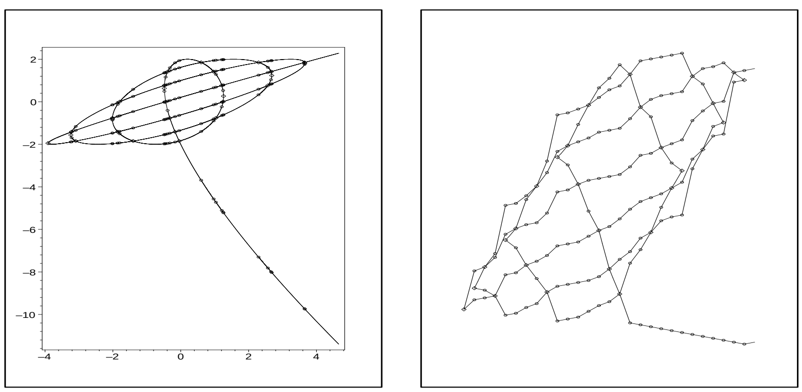

Figure 5: *Topological resolution of implicitly defined algebraic curves.* $g(x,y) = 0$.

3.2 Offsetting

Given an algebraic variety, in practice a curve or a surface, in some computer aided geometric design applications one needs to compute its offsets. That is, one considers a geometric manipulation of the original variety that generates a new algebraic variety. This offsetting construction essentially consists in computing the envelope $\mathcal{O}_d(\mathcal{V})$ of a system of hyperspheres with fixed, but probably undetermined, distance d and centered at the points of the original algebraic set $\mathcal{V}$ (for a formal definition of offsets see e.g. [3]). Alternatively, one may see the offset $\mathcal{O}_d(\mathcal{V})$ to a hypersurface $\mathcal{V}$, at distance d, as the Zariski closure of the constructible set consisting of the intersection points of the hyperspheres of radius d centered at each point $P \in \mathcal{V}_0$ and the normal line to $\mathcal{V}$ at P; where $\mathcal{V}_0 \subset \mathcal{V}$ is the set of regular points of $\mathcal{V}$, where the non-zero normal vectors to $\mathcal{V}$ are not isotropic. In Figure 6 we illustrate the offsetting of a parabola $y = x^2$ at distance 1.

Some interesting problems concerning offsets, and related to algebraic geometry, have been addressed by many authors. In particular, implicitization problems (see [40], [41], [64]),

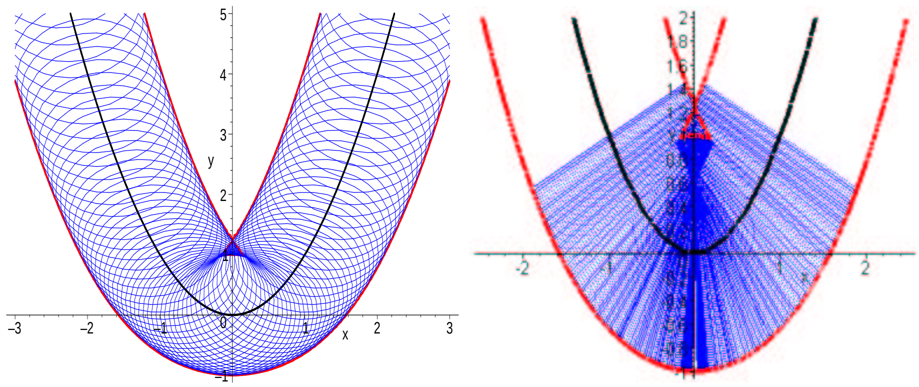

Figure 6: *Offsetting of $y = x^2$ at $d = 1$.*

parametrization problems (see [3], [49], [54], [55], [61]), analysis of topological, algebraic, and geometric properties of the offset in terms of the corresponding properties of the initial variety (see [4], [24], [25], [60]), etc.

In this paper we will report on the characterization of the unirationality of offsets and on the direct parametrization algorithm for offsets to rational surfaces following the ideas in [3] and [61]; a similar treatment can be done for the case of plane curves (see [3]).

Unirationality of offsets to surfaces can be characterized by means of the notion of *Rational Pythagorean Hodograph* and, more effectively, by means of the concept of *reparametrizing hypersurface* (see [3], [55]). These concepts does not depend on the distance. They depend only on the initial surface.

More precisely, let $\mathcal{P}(\bar{t}) = (P_1(\bar{t}), P_2(\bar{t}), P_3(\bar{t}))$ be a rational parametrization of a surface $\mathcal{V}$. Then, we say that $\mathcal{P}(\bar{t})$ is RPH (*Rational Pythagorean Hodograph*) if the normal vector $\mathcal{N}(\bar{t}) = (N_1(\bar{t}), N_2(\bar{t}), N_3(\bar{t}))$ associated with $\mathcal{P}(\bar{t})$ satisfies $N_1(\bar{t})^2 + N_2(\bar{t})^2 + N_3(\bar{t})^2 = m(\bar{t})^2$, with $m(\bar{t})$ is a rational function.

On the other hand, let $\mathcal{N}(\bar{t}) = (N_1(\bar{t}), N_2(\bar{t}), N_3(\bar{t}))$ be the normal vector of $\mathcal{V}$ associated with $\mathcal{P}(\bar{t})$ (w.l.o.g we assume that $N_2(\bar{t})$ in not identically zero). Then, we define the *reparametrizing surface of the offset $\mathcal{O}_d(\mathcal{V})$ to $\mathcal{V}$ associated with $\mathcal{P}(\bar{t})$* as the surface defined by the primitive part, w.r.t x_3, of the numerator of the irreducible expression of the rational function:

$$x_3^2 \sum_{i=2}^{3} N_i^2(x_1, x_2) - N_2^2(x_1, x_2) - 2\, x_3\, N_1(x_1, x_2)\, N_2(x_1, x_2)$$

We denote by $\mathcal{G}_{\mathcal{P}}(\mathcal{V})$ the reparametrizing surface of $\mathcal{O}_d(\mathcal{V})$ associated with $\mathcal{P}(\bar{t})$.

In this situation, the results presented in [3], one deduces the following characterization of the rationality.

Theorem 3.1.
The following statements are equivalent:

(1) $\mathcal{V}$ is rational and there exists an RPH *parametrization of $\mathcal{V}$.*

(2) All the components of $\mathcal{O}_d(\mathcal{V})$ are rational.

(3) There exists a proper parametrization $\mathcal{P}$ of $\mathcal{V}$ such that $\mathcal{G}_{\mathcal{P}}(\mathcal{V})$ has, at least, one rational component. Furthermore, if $\varphi(t_1,t_2) = (\varphi_1(t_1,t_2), \varphi_2(t_1,t_2), \varphi_3(t_1,t_2))$ is a rational parametrization of one component of $\mathcal{G}_{\mathcal{P}}(\mathcal{V})$ then $\mathcal{P}(\varphi_1(t_1,t_2), \varphi_2(t_1,t_2))$ is RPH.

(4) For all proper parametrization $\mathcal{P}$ of $\mathcal{V}$, $\mathcal{G}_{\mathcal{P}}(\mathcal{V})$ has, at least, one rational component. □

In addition, from the analysis of unirationality presented in [3], one deduces that offsets of rational surfaces have the following behavior: they are reducible with two rational components, or they are rational, or irreducible and non-rational. Furthermore, we can derive criteria to distinguish among these cases.

Theorem 3.2. *(Criterion of Double Rationality)*
Let $\mathcal{V}$ be rational, then the following statements are equivalent:

(1) There exists an RPH *proper parametrization of $\mathcal{V}$.*

(2) All proper parametrization of $\mathcal{V}$ are RPH.

(3) $\mathcal{O}_d(\mathcal{V})$ is reducible.

(4) There exists a proper parametrization $\mathcal{P}$ of $\mathcal{V}$ such that $\mathcal{G}_{\mathcal{P}}(\mathcal{V})$ is reducible.

(5) For all proper parametrization $\mathcal{P}$ of $\mathcal{V}$, $\mathcal{G}_{\mathcal{P}}(\mathcal{V})$ is reducible.

(6) There exists a proper parametrization $\mathcal{P}$ of $\mathcal{V}$ such that $\mathcal{G}_{\mathcal{P}}(\mathcal{V})$ has two rational components.

(7) For all proper parametrization $\mathcal{P}$ of $\mathcal{V}$, $\mathcal{G}_{\mathcal{P}}(\mathcal{V})$ has two rational components. □

Theorem 3.3. *(Criterion of Rationality)*
Let $\mathcal{V}$ be rational, then the following statements are equivalent:

(1) There exists an RPH *rational parametrization of $\mathcal{V}$ but there does not exist a proper* RPH *rational parametrization of $\mathcal{V}$.*

(2) $\mathcal{O}_d(\mathcal{V})$ is rational (and therefore irreducible).

(3) There exists a proper parametrization $\mathcal{P}$ of $\mathcal{V}$ such that $\mathcal{G}_{\mathcal{P}}(\mathcal{V})$ is rational.

(4) For all proper parametrization $\mathcal{P}$ of $\mathcal{V}$, $\mathcal{G}_{\mathcal{P}}(\mathcal{V})$ is rational. □

From these results one may derive an algorithm that deduces the rationality of the components of the offset to a rational surface given parametrically and that, in the affirmative case, obtains a rational parametrization. We illustrate these ideas by the following example.

Example 3.4. (See [61])
Let $\mathcal{V}$ be the surface in $\mathbb{C}^3$ defined by:

$$F(y_1, y_2, y_3) = 16\, y_1^4 - 3\, y_2^2\, y_3^4 - y_3^6 - y_2^6 - 3\, y_3^2\, y_2^4.$$

$\mathcal{V}$ can be properly parametrized as:

$$\mathcal{P}(t_1, t_2) = \left(\frac{t_2^3}{2}, \frac{(-1 + t_1^2)\, t_2^2}{t_1^2 + 1}, \frac{2\, t_1\, t_2^2}{t_1^2 + 1} \right).$$

First, we compute the normal vector of $\mathcal{V}$ associated with $\mathcal{P}$:

$$\mathcal{N}(t_1, t_2) = \left(\frac{4\, t_2^3}{t_1^2 + 1}, \frac{-3\, t_2^4\, (-1 + t_1^2)}{(t_1^2 + 1)^2}, \frac{-6\, t_1\, t_2^4}{(t_1^2 + 1)^2} \right)$$

and we check that

$$\|\mathcal{N}(t_1, t_2)\| = \frac{t_2^3\, \sqrt{16 + 9\, t_2^2}}{(t_1^2 + 1)} \notin \mathbb{C}(t_1, t_2).$$

Therefore, $\mathcal{P}$ is not RPH. As a consequence, $\mathcal{O}_d(\mathcal{V})$ is irreducible. In order to study whether $\mathcal{O}_d(\mathcal{V})$ is rational, we compute the reparametrizing surface:

$$H(x_1, x_2, x_3) = -3\, x_2 + 6\, x_2\, x_1^2 + 3\, x_2\, x_3 + 3\, x_2\, x_3\, x_1^4 + 6\, x_2\, x_3\, x_1^2 - 3\, x_2\, x_1^4 + 8\, x_3\, x_1^4 - 8\, x_3$$

which is a rational surface that can be parametrized by

$$\mathcal{R} = (R_1, R_2, R_3) = \left(t_1, \frac{-3\, t_2\, (t_1^4 - 1)}{8\, (t_2 + 2\, t_2\, t_1^2 + t_2\, t_1^4 - 1 + 2\, t_1^2 - t_1^4)}, t_2 \right).$$

In this situation, we conclude that $\mathcal{O}_d(\mathcal{V})$ is rational and that it can be parametrized as:

$$\mathcal{S}(t_1, t_2) = \mathcal{P}(R_1, R_2) + d\, \frac{R_3\, \mathcal{N}(R_1, R_2)}{M_1(R_1, R_2)\, R_3 + M_2(R_1, R_2)}$$

where $M_1(t_1, t_2)$ and $M_2(t_1, t_2)$ are the numerators of the first and second component of the normal vector, $\mathcal{N}(t_1, t_2)$. □

3.3 Topological problems

The resolution of topological problems in CAGD constitutes the main source of qualitative information, which guides the most part of the computation processes. In this section we present one problem in which the determination of the topology represents an indispensable step before starting the purely numeric resolution.

The problem is connected with the determination of the situations when changes of topology (between the topology of the considered curve or surface and the topology of the offset) appear. In the particular case of the parabola $y = x^2$, it can be proved that the topological change of the distance d offset curve appears when $d = 1/2$:

The implicit equation of the distance $d > 0$ offset of the parabola $y = x^2$ is

$$\begin{aligned}16x^6 + 16x^4y^2 - 40x^4y - 32x^2y^3 + (1 - 48d^2)x^4 + (-32d^2 + 32)x^2y^2 + 16y^4\\ +(8d^2 - 2)x^2y + (-32d^2 - 8)y^3 + (-20d^2 + 48d^4)x^2 + (-8d^2 + 1 + 16d^4)y^2\\ +(8d^2 + 32d^4)y - 8d^4 - 16d^6 - d^2\end{aligned}$$

and its discriminant with respect to the variable y is

$$x(64x^6 + (48 - 192d^2)x^4 + (192d^4 + 336d^2 + 12)x^2 - 64d^6 - 12d^2 + 48d^4 + 1).$$

Then, the offset of the parabola is not topologically a parabola when the polynomial

$$64x^3 + (48 - 192d^2)x^2 + (192d^4 + 336d^2 + 12)x - 64d^6 - 12d^2 + 48d^4 + 1 \tag{4}$$

has a real positive root. By using Sturm–Habicht sequence together with the techniques developed in [32], we conclude that:

- The number of real roots of the polynomial (4) is determined by the behavior of the polynomials
$$[1, 1, -d^2, -d^4(4d^2 + 1)^2].$$
In this particular case, this number is always equal to 1 (for any d).

- The number of real positive roots of the polynomial (4) is determined by the behavior of the polynomials
$$[1, (2d - 1)(2d + 1), -d^2(20d^2 + 1)(4d^2 + 5), -d^4(4d^2 + 1)^2(2d - 1)^3(2d + 1)^3].$$
In this particular case, this number is always equal to 0, for any $d \in (0, 1/2)$.

In Figure 7 it is presented the topological variation of the offsets of the parabola for different values of d. The isolated point appearing for $d = 0$ coincides with the parabola focal point and appears in the cases corresponding to $d \in (0, 1/2)$ from the complex part: if

$$u = \pm\sqrt{1 + 4d^2}$$

then the parametrization of the offset

$$x(u) = u \pm \frac{2du}{\sqrt{1 + 4u^2}}, \qquad y(u) = u^2 \mp \frac{d}{\sqrt{1 + 4u^2}}$$

gives the point $(0, d^2 + 1/4)$. From our point of view, the point $(0, d^2 + 1/4)$ (when $d \in (0, 1/2)$) does not belong to the offset of the parabola, because is not generated from a real point of the curve considered initially.

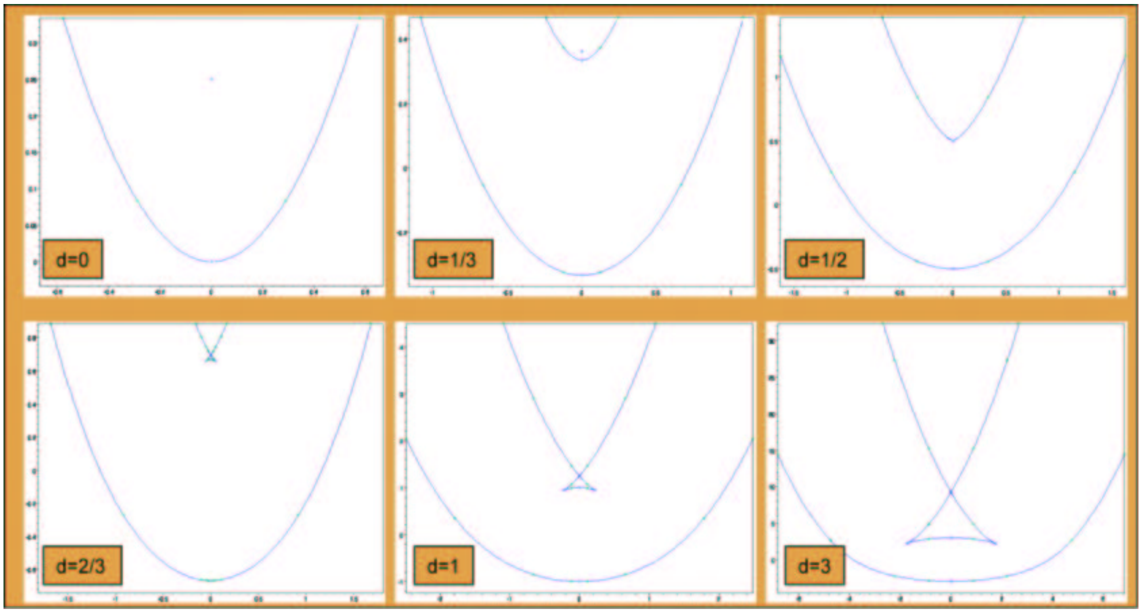

Figure 7: *Topological variation for the offsets of the parabola* $y = x^2$.

3.4 Blending

Computing blending and modeling surfaces is one of the central problems in CAGD (see [41], [46]). In many applications, objects are modeled as a collection of several surfaces whose pieces join smoothly. This situation leads directly to the blending problem in the sense that a blending surface is a surface that provides a smooth transition between distinct geometric features of an object (see [27], [43], [44], [65]).

More precisely, if one is given a collection of surfaces to be blended $V_1, \ldots, V_n$ (*primary surfaces*), and a collection of auxiliary surfaces $U_1, \ldots, U_n$ (*clipping surfaces*), then the blending problem deals with the computation of a surface V (*blending surface*) containing the space curves $C_i = U_i \cap V_i$ (*clipping curves*), and such that V meets each V_i at C_i with "certain" smooth conditions (for the notion of G^k–continuity see [18]).

Intuitively speaking the G^k–continuity consists in requiring that the Taylor expansions at C_i of the different pieces of the object agree till certain order with the corresponding Taylor expansion of the blending surface. In particular, let $C \subset V_1 \cap V_2$ be an irreducible curve such that V_1, V_2 are smooth at all but finitely many points on C. Then, we say that V_1 meets V_2 with *G^k–continuity* if there exists parametrizations $\mathcal{P}_1$, $\mathcal{P}_2$ of V_1, V_2 respectively such that all partial derivatives of $\mathcal{P}_1$, and $\mathcal{P}_2$ up to order k agree along C. If the surfaces V_1, V_2 are not rational, the G^k–continuity of V_1, V_2 along a irreducible curve $C \subset V_1 \cap V_2$ can be introduced by requiring that there exists two polynomials $A(x_1, x_2, x_3), B(x_1, x_2, x_3)$, not identically zero along C, such that all derivatives of $AF_1 - BF_2$ up to order k vanish along C, where F_1, and F_2 are the implicit equation of V_1, and V_2 respectively (see [66]).

In Figure 8, we illustrate an example of a blending where the primary surfaces are a cylinder a cone and an sphere, and the clipping surfaces are planes parallel to the floor.

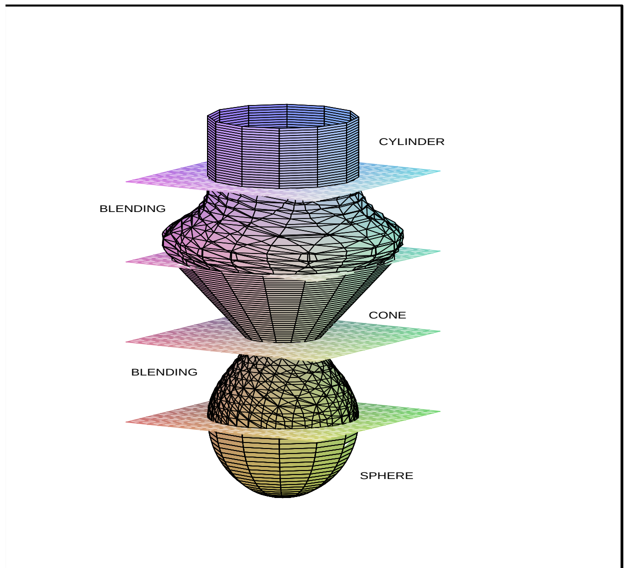

Figure 8: *Primary Surfaces (Cylinder, Cone, Sphere), Clipping Surfaces (planes parallel to the floor), and Blending Surface.*

The blending problem may be approached from two different points of view, namely, implicitly (see [44], [66]), where an implicit expression of the solution is computed, or parametrically (see [28], [39], [51], [52], [56], [63]) where parametric outputs are reached. Furthermore, a second consideration, depending on whether either symbolic or numerical techniques are used, can be made (see [8], [27], [37], [46] for numerical techniques, and [46], [63] for symbolic techniques).

For the implicit blending problem, Hoffmann and Hopcroft proved that using the potential method (see [44]) one may compute all possible implicit solutions for the case of two quadrics with G^1–continuity. Afterwards, Warren (see [65]) extended this results to the general case, stating that all solutions are in the intersection of some polynomial ideals generated by the implicit equations of V_i, and powers of the equations of U_i. This result (that we will refer as Hoffmann–Warren's Theorem) gives a description of the space of solutions for the surface blending problem.

For the symbolic parametric version of the problem, one considers that surfaces and curves are rational and that they are given by parametrizations. More precisely, in this case, one is given $k \in \mathbb{N}$ and a pair $\mathcal{S} = (\overline{P}, \overline{s})$, where:

- $\overline{\mathcal{P}} = (\mathcal{P}_1(t,h), \ldots, \mathcal{P}_n(t,h)) \in (\mathbb{K}(t,h)^3)^n$, $\mathbb{K}$ is an algebraically close field, and $\mathcal{P}_i(t,h)$ is a regular parametrization of the primary surface V_i in C_i (that is, for almost all point $P_i \in C_i$ such that there exists $(t_0,h_0) \in \mathbb{K}^2$ with $P_i = \mathcal{P}_i(t_0,h_0)$, it holds that the vectors $\{\partial\mathcal{P}_i(t_0,h_0)/\partial h,\, \partial\mathcal{P}_i(t_0,h_0)/\partial t\}$ are linearly independent),

- $\overline{s} = (s_0, \ldots, s_{n-1}) \in \mathbb{K}^n$ is a vector of n different elements such that $\mathcal{Q}_i(t) = \mathcal{P}_i(t, s_{i-1})$ parametrizes the clipping curve C_i,

and one looks for parametric solutions, $\mathcal{T}(t,h)$, for $\mathcal{S}$ with G^k–continuity; i.e, a regular parametrization $\mathcal{T}(t,h)$ in C_i such that for $i = 1, \ldots, n$, it holds that

$$\frac{\partial^j \mathcal{T}}{\partial^j h}(t, s_{i-1}) = \frac{\partial^j \mathcal{P}_i}{\partial^j h}(t, s_{i-1}), \;\; j = 0, \ldots, k,$$

(see [18], [52]). A pair $\mathcal{S}$ as above is called a *blending data.*

In this situation, the set of parametric solutions of the blending is also algebraically well structured, and therefore there exists a "parametric version" of Hoffmann–Warren's Theorem. In [52], it is shown that for a given blending data $\mathcal{S}$, the set all parametric solutions can be directly related to a free module of rank 3. More precisely, one has the following theorem.

Theorem 3.5.
Let $\mathcal{T}_p(t,h)$ be a particular parametric solution of the parametric blending problem. Then, the set of all the parametric solutions for $\mathcal{S}$ with G^k–continuity can be expressed as

$$\mathcal{T}_p(t,h) + \prod_{i=0}^{n-1}(h - s_i)^{k+1} \cdot \left(\frac{N_1}{M_1}, \frac{N_2}{M_2}, \frac{N_3}{M_3}\right),$$

where $N_i, M_i \in \mathbb{K}[t,h]$ and $\gcd\left(\prod_{i=0}^{n-1}(h - s_i), M_i\right) = 1.$ □

Therefore, taking into account this result, the problem of computing all rational G^k blendings for several surfaces is reduced to the determination of a particular parametric solution. There are several methods that approach this problem (see [28], [39], [51]). The following two theorems shows how to compute particular solutions for the blending data $\mathcal{S}$ with G^k–continuity.

Theorem 3.6.
Let $u_1, \ldots, u_n \in \mathbb{K} \setminus \{0, 1\}$ and for $i = 1, \ldots, n$ let

$$f_i(h) = \frac{u_i \prod_{j=1}^{i-2}(h - s_{j-1})^{k+1} \prod_{j=i-1, j\neq i}^{n} (s_{j-1} - h)^{k+1}}{(1 - u_i)(h - s_{i-1})^{k+1} + u_i \prod_{j=1}^{i-2}(h - s_{j-1})^{k+1} \prod_{j=i-1, j\neq i}^{n} (s_{j-1} - h)^{k+1}}.$$

Then, a parametric solution for $\mathcal{S}$ with G^k–continuity is given by

$$\mathcal{T}_p(t,h) = f_1(h)\mathcal{P}_1(t,h) + \cdots + f_n(h)\mathcal{P}_n(t,h). \quad \square$$

Theorem 3.7.
A parametric solution for $\mathcal{S}$ with G^k–continuity is given by

$$\mathcal{T}_p(t,h) = \sum_{i=1}^{n}\sum_{\ell=0}^{k}\frac{1}{\ell!}\left[\frac{\partial^\ell}{\partial^\ell h}\left(\frac{(h-s_{i-1})^{k+1}}{\prod_{i=1}^{n}(h-s_{i-1})^{k+1}}\right)\right]_{s_{i-1}}\frac{\prod_{i=1}^{n}(h-s_{i-1})^{k+1}}{(h-s_{i-1})^{k+1-\ell}}\mathcal{Q}_i(t)+$$

$$\sum_{i=1}^{n}\sum_{j=1}^{k}\frac{\partial^j\mathcal{P}_i}{\partial^j h}(t,s_{i-1})\sum_{\ell=0}^{k-j}\frac{1}{\ell!j!}\left[\frac{\partial^\ell}{\partial^\ell h}\left(\frac{(h-s_{i-1})^{k+1}}{\prod_{i=1}^{n}(h-s_{i-1})^{k+1}}\right)\right]_{s_{i-1}}\frac{\prod_{i=1}^{n}(h-s_{i-1})^{k+1}}{(h-s_{i-1})^{k+1-j-\ell}}.$$

□

Combining the above results one may derive an algorithm to compute all parametric solutions for a blending data $\mathcal{S}$. In the following example we illustrate these ideas.

Example 3.8.
Let V_i, $i = 1,\dots,4$ the primary surfaces parametrized by

$$\mathcal{P}_1(t,h) = \left(\frac{t^2-1}{2(t^2+1)}, h+4, \frac{t+6+6t^2}{t^2+1}\right),\quad \mathcal{P}_2(t,h) = \left(\frac{t^2-1}{t^2+1}, h+2, \frac{2t}{t^2+1}\right),$$

$$\mathcal{P}_3(t,h) = \left(\frac{t^2-1}{t^2+1}, 2-\frac{3(h-2)(5t^2+5-6t)}{4(t^2+1)}, \frac{2t}{t^2+1}\right),$$

$$\mathcal{P}_4(t,h) = \left(\frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1}, 2+\frac{4(h-3)(-t+t^2+1)}{t^2+1}\right),$$

and the clipping curves C_i, $i = 1,\dots,4$ defined by

$$\mathcal{Q}_1(t) = \mathcal{P}_1(t,0),\quad \mathcal{Q}_2(t) = \mathcal{P}_2(t,1),\quad \mathcal{Q}_3(t) = \mathcal{P}_3(t,2),\quad \mathcal{Q}_4(t) = \mathcal{P}_4(t,3).$$

Thus, we consider the problem of blending with G^1–continuity four surfaces. By applying Theorem 3.7. to the rational blending data $\mathcal{S} = ((\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3,\mathcal{P}_4), (0,1,2,3))$, one gets the following blending surface for $\mathcal{S}$ with G^1–continuity (see Figure 9):

$$\begin{aligned}
&\mathcal{T}_p(t,h) = \\
&\quad(-1/216(t^2-1)(-129h^6+11h^7+602h^5-1410h^4+1691h^3-873h^2-108)/(t^2+1),\\
&\quad-1/432(-1728-432h+4044th^6-398th^7+28248th^4-15596th^5+h^7+7224h^4\\
&\qquad-432ht^2-10823h^3+210h^6-2066h^5+7596th^2-23894th^3+6318h^2t^2\\
&\qquad-10823h^3t^2+6318h^2+h^7t^2-1728t^2+7224h^4t^2-2066h^5t^2+210h^6t^2)/(t^2+1),\\
&\quad1/108(648+108t+33th^6-th^7+822th^4-256th^5+56h^7-7872h^4+9674h^3-678h^6\\
&\qquad+3266h^5+729th^2-1219th^3-5094h^2t^2+9674h^3t^2-5094h^2+56h^7t^2+648t^2\\
&\qquad-7872h^4t^2+3266h^5t^2-678h^6t^2)/(t^2+1)).
\end{aligned}$$

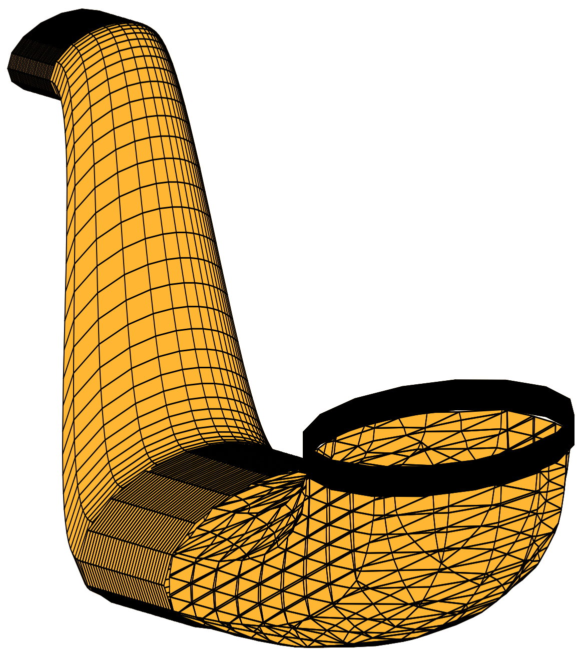

Figure 9: *Primary Surfaces and Blending Surface with G^1–continuity.*

Thus, by Theorem 3.5 all the parametric solutions for $\mathcal{S}$ with G^1–continuity are

$$\mathcal{T}_p(t,h) + h^2(h-1)^2(h-2)^2(h-3)^2\left(\frac{N_1}{M_1}, \frac{N_2}{M_2}, \frac{N_3}{M_3}\right),$$

where $N_i, M_i \in \mathbb{K}[t,h]$ and $\gcd(h(h-1)(h-2)(h-3), M_i) = 1$. □

Another interesting problem in this context is the computation and characterization of existence of polynomial parametric solutions because they avoid the unstable numerical behavior of the denominators when tracing the surface. In this situation, one may state the analogous result to Theorem 3.5 for the polynomial case (see [52]).

Theorem 3.9.
Let $\mathcal{T}_p^{\mathrm{Pol}}(t,h)$ be a particular polynomial solution of the parametric blending problem. Then, the set of all the parametric polynomial solutions for $\mathcal{S}$ with G^k–continuity can be expressed as

$$\mathcal{T}_p^{\mathrm{Pol}}(t,h) + \prod_{i=0}^{n-1}(h-s_i)^{k+1}\cdot(R_1,R_2,R_3) \quad \textit{where} \quad R_i \in \mathbb{K}[t,h]. \quad \square$$

Moreover, a criterion to decide whether there exists parametric polynomial solutions is stated in [52]. In addition, it also holds that Theorem 3.7 always reaches a polynomial parametrization if there exists any. To be more precisely, one has the following theorem.

Theorem 3.10.
There exists parametric polynomial solutions for $\mathcal{S}$ *with* G^k*–continuity if and only if the rational functions*

$$\frac{\partial^j \mathcal{P}_i}{\partial^j h}(t, s_{i-1}) \quad for \quad j = 0, \ldots, k, \;\; i = 1, \ldots, n,$$

are polynomials. Furthermore, Theorem 3.7 outputs a parametric polynomial solution for $\mathcal{S}$*, if there exists any.* □

4 Practical performance of algebraic techniques in CAGD

This last section contains two examples where some of the techniques already presented and using algebraic techniques are being currently applied in practice to solve two real problems in the company CANDEMAT devoted to construct bids for the automotive industry.

4.1 Sectioning B–spline surfaces

A successfully application of the generic implicitation procedures described in Subsection 2.1 has been the sectioning a B–spline surface: i.e. the intersection of the considered surface with a plane.

When the user needs, for instance, to section a surface, its type is determined (in our case, the type of the surface defined by (2) is polynomial of degrees

$$x : [u \to 0, v \to 2], \quad y : [u \to 2, v \to 0], \quad z : [u \to 1, v \to 1]$$

and then the data base is accessed, in order to obtain the generic algebraic expression (in our case, the equation in (3)). By evaluating this expression taking into account the concrete values of the parameters for the considered surface, the implicit equations are obtained.

Consider all the patches (implicitly represented) defining the B–spline surface to be sectioned by the plane $x = k$. For each patch, and with the equation $x(u, v) = k$, we compute the intersection of this curve (into the $u - -v$ domain) with the boundary of the definition domain (i.e. starting with $u = 0$, then $u = 1$, $v = 0$ and $v = 1$, usually two points are determined at most). By evaluating these points in the parametrization we obtain the extremes of the section on the B–spline surface. With each point computed before, and by using the implicit equation, every component of the section is discretized (always inside the plane $x = k$). The points computed before are interpolated by using a cubic spline curve representing the section of the considered patch. The previous steps are repeated for every patch of the considered surface (see [21] and [22]).

When topological problems appear, the algorithms described in Subsection 3.1 are applied in order to resolve the configuration problems (for example the appearing of closed components).

Figure 10 shows how the sectioning looks like, by using the generic implicitation, of a concrete object in the CAD/CAM environment CSIS of the company CANDEMAT.

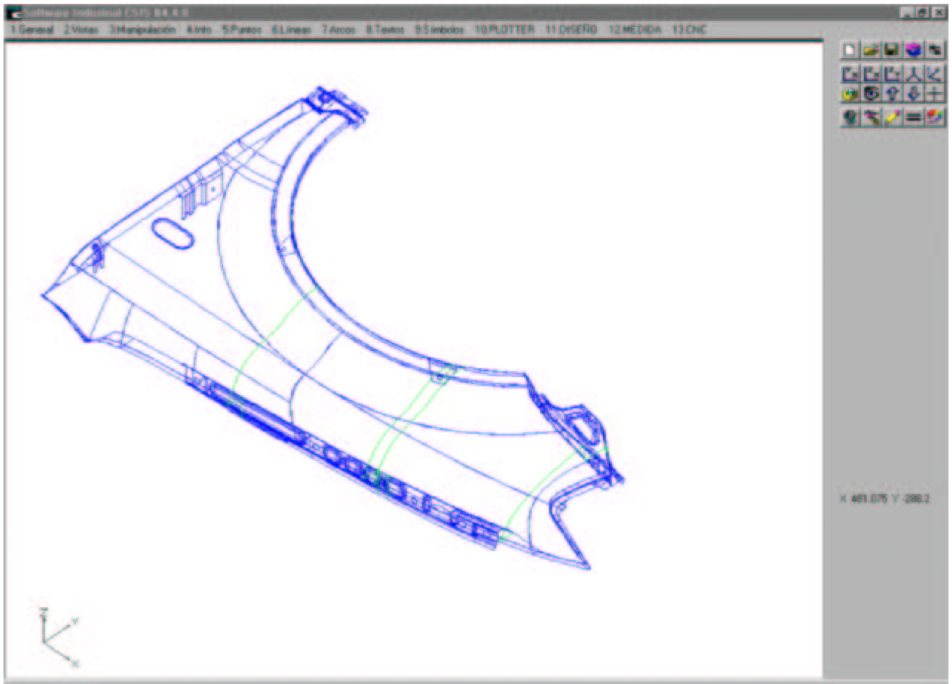

Figure 10: *Sectioning an implicit B–spline surface*

4.2 Shape Error Control

The availability of shape error measurement tools is an obvious necessity in the field of quality control in industry, where parts are made according to a theoretical mathematical model. When dealing with surfaces the objective is to establish whether or not the shape of a specific area is correct in terms of the theoretical definition of the mathematical entity irrespective of its position in 3D space.

In our case a theoretical model and an actual one made from the first as reference are available. The measuring of errors is based on corresponding points in the theoretical surface model and the actual surface. The points are previously chosen for the detection of faults in regions of the actual surface. The points are given in two different positions in 3D space and one tries to find an euclidean transformation between them allowing the evaluation of errors in such a way that it is guaranteed that the tolerances specified by the standards are verified.

The method (see [23] for more details) finds the rigid motion moving the first set of points as close as possible to the second one. This is made by introducing a non linear least–squares problem where the unknowns to be determined are the parameters of the rigid motion (the translation and the three angles of the rotation). The structure of this non linear least–squares problem allows its resolution in closed form by using several symbolic methods (and the Computer Algebra System Maple). Thus the error is computed by applying this optimal rigid motion to the first set of points and then making the differences.

Figure 11 shows how the algorithm sketched before works in practice, into the software CSIS of the company CANDEMAT.

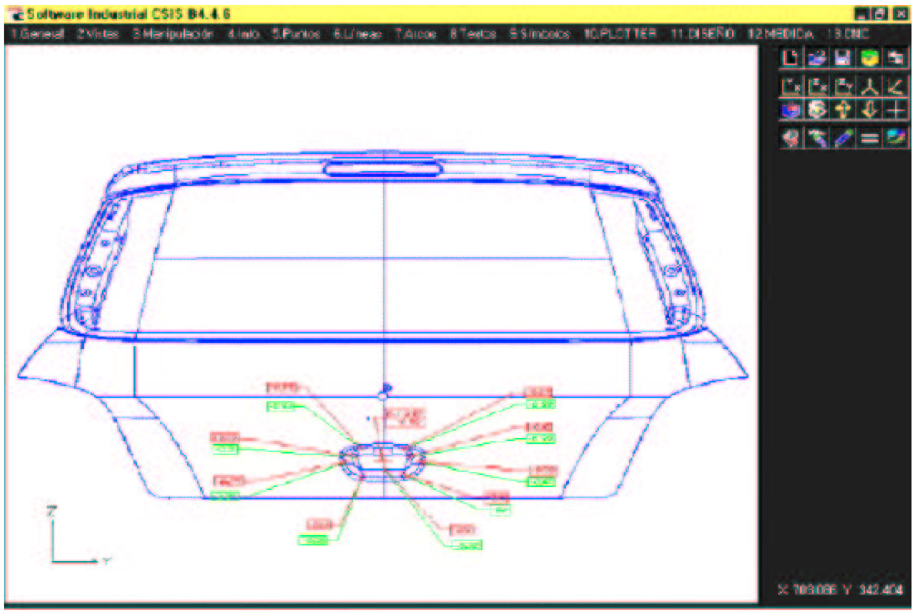

Figure 11: *Shape error computation*

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