Be Careful : This worksheet need the GbRS package by J.C. Faugere and F. Rouillier available at http://fgbrs.lip6.fr

```
Gb/Maple interface package Version 0.47

JC Faugere (jcf@calfor.lip6.fr)

Type ?Gb for documentation

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Type ?Gb for documentation
```

Set "fast" options for Gb

```
> advance("fast");
```

1 A simple example

1.1 equations

```
> sys:=[-1*x4+1*x2*x1-1*x3*x1+1,-1*x1+1*x3*x2-1*x4*x2+1,-1*x2-1*x3*x1+1
> *x4*x3+1,-1*x3+1*x4*x1-1*x4*x2+1]:
> vars := [x1, x2, x3, x4]:
```

1.2 Groebner basis

A lexicographic Groebner basis

```
> gb_lex:=gbasis(sys,Lex(vars)):
```

Conversion to a standard Maple list of polynomials

```
> lgb_lex:=convert(gb_lex,gbasis);
```

$$\begin{aligned}&lgb\_lex := [\\ &25\,x1 - 25\,x3 - 12\,x4^9 - 21\,x4^8 - 12\,x4^7 + 4\,x4^6 + 12\,x4^4 + 46\,x4^3 - 13\,x4^2 - 4\,x4\,,\\ &25\,x2 + 13\,x4^9 + 14\,x4^8 + 13\,x4^7 - 11\,x4^6 - 38\,x4^4 - 14\,x4^3 - 13\,x4^2 + 11\,x4\,,\\ &25\,x3^2 - 25\,x3 - 9\,x4^9 - 17\,x4^8 - 9\,x4^7 + 8\,x4^6 + 9\,x4^4 + 42\,x4^3 + 9\,x4^2 - 33\,x4\,,\\ &5\,x4\,x3 - 5\,x3 - 3\,x4^9 - 10\,x4^8 - 13\,x4^7 - 10\,x4^6 - 5\,x4^5 - 2\,x4^4 + 15\,x4^3 + 13\,x4^2\\ &\;+ 10\,x4 + 5,\\ &x4^{10} + 2\,x4^9 + 3\,x4^8 + 2\,x4^7 + 2\,x4^6 - x4^5 - 2\,x4^4 - 3\,x4^3 - 2\,x4^2 - 2\,x4]\end{aligned}$$

The smallest polynomial

```
> lgb_lex[nops(lgb_lex)];
```

$$x4^{10} + 2\,x4^9 + 3\,x4^8 + 2\,x4^7 + 2\,x4^6 - x4^5 - 2\,x4^4 - 3\,x4^3 - 2\,x4^2 - 2\,x4$$

The number of polynomials in the system and in the G. Basis.

```
> nops(sys);nops(lgb_lex);
```

4

5

A Degree Reverse Lexicographic G. basis

```
> gb_drl:=gbasis(sys,DRL(vars)):
```
Conversion to a standard Maple list of polynomials
```
> lgb_drl:=convert(gb_drl,gbasis);
```

$$lgb\_drl := [$$
$$3\,x4^4 + 2\,x4^3 - 5\,x2^2 + x3^2 + 5\,x4\,x2 - 5\,x4\,x3 - x4^2 + x1 + 2\,x2 + 3\,x3 - x4 - 5,$$
$$x2^3 - x4^3 - x3^2 - x4^2 + x3 + x4,\; x3^3 - x4^3 + x2^2 + x4\,x2 - x4\,x3 - 1,$$
$$3\,x2^2\,x4 + x4^3 - x2^2 - x3^2 + x4\,x2 - x4\,x3 - 2\,x4^2 + 2\,x1 - 2\,x2 + x4 - 1, 3\,x3^2\,x4$$
$$+ x4^3 + 2\,x2^2 + 2\,x3^2 - 5\,x4\,x2 + 2\,x4\,x3 + x4^2 - 4\,x1 - 2\,x2 - 3\,x3 - 2\,x4 + 5,$$
$$3\,x2\,x4^2 + x4^3 - x2^2 + 2\,x3^2 + x4\,x2 - x4\,x3 + x4^2 - x1 + x2 - 5\,x4 - 1,$$
$$3\,x3\,x4^2 + x4^3 - x2^2 - x3^2 - 5\,x4\,x2 - x4\,x3 + x4^2 - x1 + x2 + x4 + 2,$$
$$x1^2 + x2^2 + x3^2 + x4^2 - x1 - x2 - x3 - x4,\; x2\,x1 - x4\,x3 + x2 - x4,$$
$$x3\,x1 - x4\,x3 + x2 - 1,\; -x1 + x3\,x2 - x4\,x2 + 1,\; -x3 + x4\,x1 - x4\,x2 + 1]$$

The number of polynomials in the system and in the G. Basis.
```
> nops(sys);nops(lgb_drl);
```
$$4$$
$$12$$

1.3 Has the system a finite number of complex roots ?

Print all the leading terms of the DRL G. Basis
```
> for i from 1 to nops(lgb_drl) do
> lm:=[LM(lgb_drl[i],DRL(vars))]:print(lm[1]*lm[2],indets(lgb_drl[i]))od
> :
```
$$3\,x4^4,\ \{x4,\ x2,\ x1,\ x3\}$$
$$x2^3,\ \{x4,\ x2,\ x3\}$$
$$x3^3,\ \{x4,\ x2,\ x3\}$$
$$3\,x2^2\,x4,\ \{x4,\ x2,\ x1,\ x3\}$$
$$3\,x3^2\,x4,\ \{x4,\ x2,\ x1,\ x3\}$$
$$3\,x2\,x4^2,\ \{x4,\ x2,\ x1,\ x3\}$$
$$3\,x3\,x4^2,\ \{x4,\ x2,\ x1,\ x3\}$$
$$x1^2,\ \{x4,\ x2,\ x1,\ x3\}$$
$$x2\,x1,\ \{x4,\ x2,\ x1,\ x3\}$$
$$x3\,x1,\ \{x4,\ x2,\ x1,\ x3\}$$
$$x3\,x2,\ \{x4,\ x2,\ x1,\ x3\}$$
$$x4\,x1,\ \{x4,\ x2,\ x1,\ x3\}$$

Print all the leading terms of the Lex G. Basis
```
> for i from 1 to nops(lgb_lex) do
> lm:=[LM(lgb_lex[i],Lex(vars))]:print(lm[1]*lm[2],indets(lgb_lex[i])):o
> d:
```
$$25\,x1,\ \{x4,\ x1,\ x3\}$$
$$25\,x2,\ \{x4,\ x2\}$$
$$25\,x3^2,\ \{x4,\ x3\}$$
$$5\,x4\,x3,\ \{x4,\ x3\}$$
$$x4^{10},\ \{x4\}$$

1.4 Can I solve the system ?

Solve symbolically the smallest polynomial of the Lexicographic G. Basis

```
> sols:=solve(lgb_lex[nops(lgb_lex)]);
```

$$sols := 0,\ 1,\ I,\ -I,\ -1+I,\ -1-I,\ -\frac{1}{4}+\frac{\sqrt{5}}{4}+\frac{1}{4}I\sqrt{2}\sqrt{5+\sqrt{5}},\ -\frac{1}{4}-\frac{\sqrt{5}}{4}+\frac{1}{4}I\sqrt{2}\sqrt{5-\sqrt{5}},$$
$$-\frac{1}{4}-\frac{\sqrt{5}}{4}-\frac{1}{4}I\sqrt{2}\sqrt{5-\sqrt{5}},\ -\frac{1}{4}+\frac{\sqrt{5}}{4}-\frac{1}{4}I\sqrt{2}\sqrt{5+\sqrt{5}}$$

put the solutions in a Maple list.

```
> vals_x4:=[sols]:
```

Substitute the first solution in the Grobner basis

```
> lgb_lex1_1:=subs(x4=vals_x4[1],lgb_lex);
```

$$lgb\_lex1\_1 := [25\,x1 - 25\,x3,\ 25\,x2,\ 25\,x3^2 - 25\,x3,\ 5 - 5\,x3,\ 0]$$

Substitute the second solution in the Grobner basis

```
> lgb_lex1_2:=subs(x4=vals_x4[2],lgb_lex);
```

$$lgb\_lex1\_2 := [25\,x1 - 25\,x3,\ 25\,x2 - 25,\ 25\,x3^2 - 25\,x3,\ 0,\ 0]$$

Substitute the third solution in the Grobner basis

```
> sub_x4:=subs(x4=vals_x4[3],lgb_lex);
```

$$sub\_x4 := [$$
$$25\,x1 - 25\,x3 - 50\,I,\ 25\,x2 + 25\,I,\ 25\,x3^2 - 25\,x3 - 25 - 75\,I,\ 5\,I\,x3 - 5\,x3 - 10,\ 0$$
$$]$$

Solve symbolically the fourth equation and substitute the solution it in this simple system

```
> subs(x3=solve(sub_x4[4]),sub_x4);
```

$$[25\,x1 + 25 - 25\,I,\ 25\,x2 + 25\,I,\ 0,\ 0,\ 0]$$

Now study the 7th solution of the smallest polynomial of the lexico. G. Basis

```
> lgb_lex1_7:=simplify(subs(x4=vals_x4[7],lgb_lex));
```

$$lgb\_lex1\_7 := [25\,x1 - 25\,x3 - \frac{25}{4}I\sqrt{5}\sqrt{2}\sqrt{5+\sqrt{5}} + \frac{25}{4}I\sqrt{2}\sqrt{5+\sqrt{5}},$$
$$25\,x2 - \frac{25\sqrt{5}}{4} + \frac{25}{4} + \frac{25}{4}I\sqrt{2}\sqrt{5+\sqrt{5}},$$
$$-25\,x3 + 25\,x3^2 - \frac{25\sqrt{5}}{2} - \frac{25}{8}I\sqrt{2}\sqrt{5+\sqrt{5}} - \frac{25}{8}I\sqrt{5}\sqrt{2}\sqrt{5+\sqrt{5}},$$
$$\frac{15}{8}I\sqrt{2}\sqrt{5+\sqrt{5}} + \frac{5\,x3\sqrt{5}}{4} + \frac{5}{4}I\,x3\sqrt{2}\sqrt{5+\sqrt{5}} - \frac{5}{8}I\sqrt{5}\sqrt{2}\sqrt{5+\sqrt{5}} - \frac{5\sqrt{5}}{2}$$
$$-\frac{25\,x3}{4},\ 0]$$

Solve symbolically the fourth equation and substitute the solution it in this simple system

```
> simp7:=simplify(subs(x3=solve(lgb_lex1_7[4]),lgb_lex1_7));
```

$$simp7 := [\frac{25\,(x1\,\sqrt{5} + x1\,\sqrt{2}\,\sqrt{5+\sqrt{5}}\,I - 5\,x1 - \sqrt{2}\,\sqrt{5+\sqrt{5}}\,I + \sqrt{5}\,\sqrt{2}\,\sqrt{5+\sqrt{5}}\,I)}{\sqrt{5} + \sqrt{2}\,\sqrt{5+\sqrt{5}}\,I - 5},$$

$$25\,x2 - \frac{25\,\sqrt{5}}{4} + \frac{25}{4} + \frac{25}{4}\,I\,\sqrt{2}\,\sqrt{5+\sqrt{5}},\ 0,\ 0,\ 0]$$

```
> simplify(map(solve,simp7));
```

$$[\frac{-I\,\sqrt{2}\,\sqrt{5+\sqrt{5}}\,(\sqrt{5}-1)}{\sqrt{5} + \sqrt{2}\,\sqrt{5+\sqrt{5}}\,I - 5},\ -\frac{1}{4} + \frac{\sqrt{5}}{4} - \frac{1}{4}\,I\,\sqrt{2}\,\sqrt{5+\sqrt{5}}]$$

1.5 Conclusion

On a simple exemple, one can hope being able to solve the system by hand from a lexicographic Groebner basis.

2 A less simple but simple example.

2.1 Equations

```
> sys:=[2*x^2+2*y^2+2*z^2+2*t^2+2*u^2+v^2-v,
> x*y+y*z+2*z*t+2*t*u+2*u*v-u,
> 2*x*z+2*y*t+2*z*u+u^2+2*t*v-t,
> 2*x*t+2*y*u+2*t*u+2*z*v-z,
> t^2+2*x*v+2*y*v+2*z*v-y,
> 2*x+2*y+2*z+2*t+2*u+v-1]:
> vars:= [x,y,z,t,u,v]:
```

2.2 Lexicographic Grobner basis

Compute the Lexicographic Grobner basis

```
> gb_lex:=gbasis(sys,Lex(vars),verbose):
```

Convert the Gb object to a Maple list of polynomials

```
> lgb_lex:=convert(gb_lex,gbasis);
```

Print all the leading terms (to test if the system is zero-dimensional)

```
> for i from 1 to nops(lgb_lex) do
> lm:=[LM(lgb_lex[i],Lex(vars))]:print(lm[2],indets(lgb_lex[i])):od:
```

$$x,\ \{v,\ x\}$$
$$y,\ \{v,\ y\}$$
$$z,\ \{v,\ z\}$$
$$t,\ \{v,\ t\}$$
$$u,\ \{u,\ v\}$$
$$v^{32},\ \{v\}$$

2.3 Can I solve the system ?

```
> lgb_lex[nops(lgb_lex)];
```

$$
\begin{aligned}
&-28162995118535968723144\,v + 113036676348417613323116877792\,v^6\\
&- 1525446186996942069762502640\,v^5 + 1629413965971924983647908 12\,v^4\\
&- 143162848032426450764883133632 0\,v^9\\
&+ 340497173589885376999566219074\,v^8\\
&- 6812615007072324241958563057 6\,v^7\\
&+ 4144701326719316454507672949767 6\,v^{12}\\
&- 1568463939268980668804042085288 0\,v^{11}\\
&+ 511821039005400738148182747915 2\,v^{10} + 76668167471886607679468 0\,v^2\\
&- 13234604967120210059321744\,v^3 + 8455392499839632798543052 8\,v^{32}\\
&+ 2307491742725230364750490562 56\,v^{27}\\
&- 796210550526414860608569344\,v^{31} + 378677057544005032536170496\,v^{30}\\
&+ 26833986333475831991202988032\,v^{29}\\
&- 152443439352118767800745154560\,v^{28}\\
&+ 1442617811629533832379318318592\,v^{26}\\
&- 11161666092537923547163549506560\,v^{25}\\
&+ 42814172956417214580926709440192\,v^{24}\\
&- 115687740411863158877715772402432\,v^{23}\\
&+ 242659489780862639208985746260736\,v^{22}\\
&- 413206786102694275595225480253376\,v^{21}\\
&+ 585961496263634599512370847468672\,v^{20}\\
&- 703251856550682185351644824075552\,v^{19}\\
&+ 722054622357603636976253056711864\,v^{18}\\
&- 638875118400692342463120613375288\,v^{17}\\
&+ 489493327739551778597686880326975\,v^{16}\\
&- 325713063015681683570833788021840\,v^{15}\\
&+ 188481466207694278337123922558128\,v^{14}\\
&- 94847024759930361924750310965968\,v^{13} + 4919368402321894925 67
\end{aligned}
$$

Try to solve symbolically the smallest polynomial (Maple is not able !) do not run :solve(lgb_lex[nops(lgb_lex)]);

Try to solve numerically the smallest polynomial with 10 Digits

```
> Digits:=10;sols_1:=[fsolve(lgb_lex[nops(lgb_lex)])];
```

$$Digits := 10$$

sols_1 := [0.1362473215, 0.2385949476, 0.2772103707, 0.2918655885, 0.4085801161, 0.4411464549, 0.4615870330, 0.5903351456, 0.6797708135, 0.7263011196, 0.7533578076, 1.]

Substitute the first numerical solution in the coordinate functions (Shape position)

```
> sols:=[sols_1[1],op(map(solve,subs(v=sols_1[1],lgb_lex)))];
```

sols := [0.1362473215, −312.3112725, 16.45935763, −206.1114662, −186.6961423, 116.8547211]

Substitute the "solutions" in the initial system

```
> subs(seq(vars[i]=sols[i],i=1..nops(vars)),sys);
```

[363831.4631, 21546.55694, 109491.9218, 197349.1344, −26317.29594, −1261.191832]

2.4 Conclusion 1

On an example where the Grobner basis computation is possible, even if the system is in shape position, computing a numerical approximation of the solutions is not easy.

2.5 RUR Computation

Compute a RUR from the lexicographic Grobner basis

```
> rr:=rur(gb_lex,verbose=1):
```

Convert the RS object to a standard Maple list of polynomials

```
> l_rr:=convert(rr,RUR);
```

$l\_rr := [-28162995118$
$-\ 15254461869969420$
$-\ 14316284803242645$
$+\ 34049717358988537$
$-\ 68126150070723242$
$+\ 41447013267193164$
$-\ 15684639392689806$
$+\ 51182103900540073$
$-\ 13234604967120210$
$+\ 23074917427252303$
$-\ 79621055052641486$
$+\ 26833986333475831$
$-\ 15244343935211876$
$+\ 14426178116295338$
$-\ 11161666092537923$
$+\ 42814172956417214$
$-\ 11568774041186315$
$+\ 24265948978086263$
$-\ 41320678610269427$
$+\ 58596149626363459$
$-\ 70325185655068218$
$+\ 72205462235760363$
$-\ 63887511840069234$
$+\ 48949332773955177$
$-\ 32571306301568168$
$+\ 18848146620769427$
$-\ 94847024759930361$
53667717230320625375
$+\ 23737702033167698$
$-\ 26695308272446486$
$+\ 17913736365189025$
$-\ 45096297130214331$
$+\ 95339208605167905$
$-\ 43155396265768314$
$+\ 17407745572221129$
$-\ 60385861661855755$
$-\ 13896335215476220$
$-\ 14939457056507639$
$+\ 94700395998203887$
$-\ 86388844732116012$
$+\ 39761091042120528$
$+\ 27236496128477969$
$+\ 21805796968753426$
$+\ 13127822085828757$
$-\ 97664578309706831$
$+\ 35963905283390460$
$-\ 93128631031549842$
$+\ 18684780713126423$
30370698778548029

Solve numerically the first polynomial of the RUR

```
> Digits:=10:sols_1:=fsolve(l_rr[1]);
```

$$sols\_1 := 0.1362473215, 0.2385949476, 0.2772103707, 0.2918655885, 0.4085801161, 0.4411464549, 0.4615870330, 0.5903351456, 0.6797708135, 0.7263011196, 0.7533578076, 1.$$

Plug the numerical values in the coordinates

```
> sols:=subs(v=sols_1[1],[seq(l_rr[i]/l_rr[2],i=3..nops(l_rr))]);
```

$$sols := [0.2099343558, 0.09600865841, 0.03954133249, 0.04159485276, 0.04262745215, 0.1358674840]$$

Substitute the solutions in the initial system

```
> subs(seq(vars[i]=sols[i],i=1..nops(vars)),sys);
```

$$[-0.0006058214, -0.00025665967, -0.00051478423, 0.00039918164, -0.00039834873, -0.0047192128]$$

The same computation with 15 digits

```
> Digits:=15:sols_1:=fsolve(l_rr[1]);sols:=subs(v=sols_1[1],[seq(l_rr[i
> ]/l_rr[2],i=3..nops(l_rr))]);
> subs(seq(vars[i]=sols[i],i=1..nops(vars)),sys);
> Digits:=10:
```

$$sols\_1 := 0.136247321545774, 0.238594947571154, 0.277210370739032, 0.291865588475904, 0.408580116118409, 0.441146454872566, 0.461587032954515, 0.590335145554309, 0.679770813538602, 0.726301119612819, 0.753357807614709, 1.00000000000000$$

$$sols := [0.210588110619363, 0.0963978678385328, 0.0403916135291411, 0.0417053486899400, 0.0427934035886771, 0.136247326079335]$$

$$[-0.1812388\,10^{-8}, 0.6048784\,10^{-9}, 0.9621696\,10^{-9}, -0.3201232\,10^{-9}, 0.43670407\,10^{-8}, 0.14610643\,10^{-7}]$$

2.6 Conclusion 2

The RUR seems to be well suited for computing numerical approximations of the solutions : it seems to be numerically more stable.

One can also veryfy that the coefficients in the RUR are smaller than those in the lexicographic Grobner basis for this system in shape position.

3 Another use of lexicographic Grobner bases.

3.1 Equations and variables

```
> sys:=[-2*x-1*z+2*y*x*t+1*z*y^2,-10*y*t+4*z*x+4*x^2-10*y^2+2*y^3*t-1*z
> *x^3+4*y*x^2*t+4*z*y^2*x+2,-1*x-2*z+2*z*y*t+1*x*t^2,-10*y*t+4*z*x-10*t
> ^2+4*z^2+2*y*t^3-1*z^3*x+4*z*x*t^2+4*z^2*y*t+2]:
```

```
> vars:=[t,x,y,z]:
```

3.2 The lexicographic Groebner basis

Compute a lexicographic Grobner basis and convert the result to a standard Maple list of polynomials.

```
> gb_lex:=gbasis(sys,Lex(vars)):
> lgb_lex:=convert(gb_lex,gbasis):
```

Check the system is zero-dimensional

```
> for i from 1 to nops(lgb_lex) do
> lm:=[LM(lgb_lex[i],Lex(vars))]:print(lm[2],indets(lgb_lex[i])):od:
```

$$t,\ \{t,\ z,\ y\}$$

$$x,\ \{x,\ z,\ y\}$$

$$y^8,\ \{z,\ y\}$$

$$y^6\, z,\ \{z,\ y\}$$

$$y^4\, z^3,\ \{z,\ y\}$$

$$y^2\, z^7,\ \{z,\ y\}$$

$$z^{17},\ \{z\}$$

3.3 the RUR

```
> rr:=rur(gb_lex,verbose=1):
> rr;
```

$$[-71538974319760970241763957553904366726869024250\,T^{18}$$
$$+\,2888801377810421617835177273139026555093260141875\,T^{16}$$
$$+\,229375899545663350517973668872891277856956181562500\,T^{14}$$
$$+\,2301022671857069687903388747063537760205795681250000\,T^{12}$$
$$-\,130404601576695797267881984269730053654212030039062500\,T^{10}$$
$$-\,3323259416124234683082999997424827913739669488525390625\,T^{8}$$
$$-\,9425380884758962448330437283505141598540083947753906250\,T^{6}$$
$$+\,320978252501526730473629248832499841410531967163085937500\,T^{4}$$
$$+\,1500223605239840720069101073651474780194569099426269531250\,T^{2}$$
$$+\,T^{56}-6226\,T^{54}+16430436\,T^{52}-24662608662\,T^{50}+23608811371681\,T^{48}$$
$$-\,15096905462934364\,T^{46}+6452159239278102192\,T^{44}$$
$$-\,1735074108381058460100\,T^{42}+229304834970591078707533\,T^{40}$$
$$+\,8246639716972793671366234\,T^{38}-4754942556742015396897760588\,T^{36}$$
$$-\,918741570902131728651426786210\,T^{34}$$
$$+\,361976104218704606205931330938093\,T^{32}$$
$$-\,5536434054784382753593642557773950884961018466949462890625$$
$$+\,7075140389118159368525322817995416\,T^{30}$$
$$-\,11512117926875707540346506316490909056\,T^{28}$$
$$+\,259588445837246406851216092221176474088\,T^{26}$$
$$-\,996696996901348069556415353407767687788926\,T^{22}$$
$$-\,1767923973954117355906236877071864628616820596\,T^{20}$$
$$+\,191364854797117316753523982725150155404179\,T^{24},56\,T^{31}-228012\,T^{29}$$
$$+\,325999824\,T^{27}-210036710036\,T^{25}+65468626349448\,T^{23}$$
$$-\,8227069662118252\,T^{21}+3172410996052608\,T^{19}$$
$$-\,77732733487075906740\,T^{17}+50036039615009287909064\,T^{15}$$
$$-\,4492529347102356418073060\,T^{13}-152296792174459433176326000\,T^{11}$$
$$+\,15931300936510839365736822500\,T^{9}$$
$$-\,115927250949372267644189125000\,T^{7}$$
$$-\,3951555332574713845494051562500\,T^{5}$$
$$+\,17232473112149181158755312500000\,T^{3}$$
$$+\,59613367781864670668446289062500\,T,340\,T^{30}-1171008\,T^{28}$$
$$+\,1333707852\,T^{26}-587333037960\,T^{24}+137318898077012\,T^{22}$$
$$-\,27252637450476176\,T^{20}+5540012246350863980\,T^{18}$$
$$-\,734170556207228426616\,T^{16}+28807800369683443036316\,T^{14}$$
$$+\,1037251105244351844980960\,T^{12}+551441775810865647216938500\,T^{10}$$
$$-\,24955795819560860336392875000\,T^{8}-3248290470983639600096562500\,T^{6}$$
$$+\,841014067983532698135093750000\,T^{4}$$
$$+\,49405632727488034831250976562500\,T^{2}$$
$$-\,90389630764659400100595703125000,1408\,T^{30}-4932736\,T^{28}$$
$$+\,6050367360\,T^{26}-3296295403136\,T^{24}+790408975157120\,T^{22}$$
$$-\,52670864384737920\,T^{20}-3539630872350611072\,T^{18}$$
$$-\,1417702802907961093248\,T^{16}+412486209385752284051584\,T^{14}$$
$$-\,19141249272664096667596160\,T^{12}-442680268512565341069808000\,T^{10}$$
$$+\,62273346196047290308130000000\,T^{8}$$
$$-\,1080803555330956530335590000000\,T^{6}$$
$$-\,2484398804069621204142750000000\,T^{4}$$
$$+\,9023045075172433199593125000000\,T^{2}$$
$$-\,197105464567863025108593750000000,20\,T^{30}-129832\,T^{28}$$
$$+\,462337788\,T^{26}-527566762240\,T^{24}+198706922899828\,T^{22}$$

3.4 Decomposition of the Lexicographic G. Base

Compute a decomposition into triangular sets from the lexicographic Grobner basis

```
> tri:=lextriangular(gb_lex):
```

Convert the Gb object to a standard Maple list of lists of polynomials

```
> l_tri:=convert(tri,gbasis);
```

$$l\_tri := [[t - y,\ 48\,x + z^5 - 16\,z^3 - 32\,z,\ y^2 - 1,\ z^6 - 12\,z^4 - 48\,z^2 + 64],\ \%1,\ \%1,\ \%1,$$
$$[24\,t - 5\,y^5 - 22\,y^3 + 3\,y,\ 16\,x + y^4\,z + 10\,z\,y^2 + 5\,z,\ y^6 + 5\,y^4 + 3\,y^2 - 9,\ z^2 - 4],$$
$$[24\,t - 5\,y^5 - 22\,y^3 + 3\,y,\ 16\,x + y^4\,z + 10\,z\,y^2 + 5\,z,\ y^6 + 5\,y^4 + 3\,y^2 - 9,\ z^2 - 4],$$
$$[24\,t - 5\,y^7 + 49\,y^5 + 5\,y^3 - 25\,y,\ x,\ y^8 - 10\,y^6 + 10\,y^2 - 1,\ z]]$$
$$\%1 := [t + 3\,y^3 + 2\,y,\ 4\,x - 3\,z\,y^2 + 3\,z,\ 9\,y^4 + 6\,y^2 + 1,\ 3\,z^2 + 4]$$

Compute a RUR of the biggest component

```
> rr:=rur(pretend(l_tri[1],Lex(vars)),verbose=1):
> rr;
```

$$[T^{12} - 3062\,T^{10} + 2206911\,T^8 + 21058700\,T^6 - 41236820625\,T^4 - 13389782093750\,T^2$$
$$+ 415573706640625, 12\,T^{11} - 30620\,T^9 + 17655288\,T^7 + 126352200\,T^5$$
$$- 164947282500\,T^3 - 26779564187500\,T, 60\,T^{10} - 94860\,T^8 + 93716440\,T^6$$
$$- 11825935000\,T^4 + 4045786787500\,T^2 + 57619969062500, 736\,T^{10}$$
$$- 1316256\,T^8 + 310075840\,T^6 + 25203032000\,T^4 - 1715249700000\,T^2$$
$$- 149793572500000, 60\,T^{10} - 94860\,T^8 + 93716440\,T^6 - 11825935000\,T^4$$
$$+ 4045786787500\,T^2 + 57619969062500, 544\,T^{10} - 715104\,T^8 - 151885760\,T^6$$
$$+ 42327272000\,T^4 + 14637423300000\,T^2 - 621924647500000]$$

3.5 Conclusion

Even if the RUR is the most friendly terminal expression, the lexicographic Grobner basis is usefull, when the system is not in shape position, to decompose (through triangular sets) the system.

When the system can not be put in shape position, the coefficients of the RUR are in general bigger than the coefficients in the lexicographic Grobner basis.

4 A more difficult example

4.1 Equations

```
> sys:=[1*z0+1*z1+1*z2+1*z3+1*z4+1*z5,1*z1*z0+1*z2*z1+1*z5*z0+1*z3*z2+1
> *z4*z3+1*z5*z4,1*z2*z1*z0+1*z5*z1*z0+1*z3*z2*z1+1*z5*z4*z0+1*z4*z3*z2+
> 1*z5*z4*z3,1*z3*z2*z1*z0+1*z5*z2*z1*z0+1*z5*z4*z1*z0+1*z4*z3*z2*z1+1*z
> 5*z4*z3*z0+1*z5*z4*z3*z2,1*z4*z3*z2*z1*z0+1*z5*z3*z2*z1*z0+1*z5*z4*z2*
> z1*z0+1*z5*z4*z3*z1*z0+1*z5*z4*z3*z2*z0+1*z5*z4*z3*z2*z1,1*z5*z4*z3*z2
> *z1*z0-1]:
> vars:= [z0,z1,z2,z3,z4,z5]:
```

4.2 Computing the Lexicographic Grobner base by change of ordering

On this example, the direct computation of the lexicographic Grobner basis can not be done quickly.

Compute a DRL Grobner basis.

```
> gb_drl:=gbasis(sys,DRL(vars),verbose):
> gb_drl:
```

The hilbert polynomial of I can easily be computed from a Grobner basis for a degree ordering. In particular it gives the dimension and degree (number of complex roots counted with multiplicities in the zero-dimensional case) :

```
> dimension(gb_drl);vdegree(gb_drl);
```

$$0$$

$$156$$

Deduce a Lexicographic Grobner basis from the DRL G. basis

```
> gb_lex:=fglm(gb_drl,verbose):
> gb_lex;
```

$[z0 + z1 + z2 + z3 + z4 + z5, -292382820431504$

$+ 23810276989699200\, z4^{3}\, z5^{5} + 2063557339107 2$

$+ 230166010900425600\, z4^{2} + 3968379498283200$

$- 159604974228968981505 3\, z5^{8} - 114881713301$

$+ 3968379498283200\, z1^{2} - 227189846763186549$

$- 3988812642545399\, z5^{44} + 10187423878429609$

$+ 1994739308439916238065\, z5^{26} + 15968400880$

$- 1993494118301162145413\, z5^{14} + 15943775234$

$+ 549961185828911895\, z4\, z5^{19} + 396837949828 3$

$- 15873517993132800\, z4^{4}\, z5^{4} - 72970598731668$

$+ 1863667496867205421\, z4\, z5^{37} + 291674853771$

$+ 3652859946911069217 45\, z4\, z5^{25} - 3650484040$

$+ 15873517993132800\, z5\, z1 + 158735179931328 0$

$71430830969097600\, z2\, z5 + 332776131113858709$

$- 174608697924460800\, z4^{3}\, z5^{5} - 2428648252949$

$- 2611193709870345600\, z4^{2} - 23810276989699 2$

$+ 23810276989699200\, z2^{2} + 183299447818672 4$

$+ 1302585310020204 20699\, z5^{2} + 258853406082 9$

$+ 23810276989699200\, z2\, z1 + 4581589762901032$

$- 1170137655821518912 07\, z5^{38} - 229099712396$

$- 16133250761305157265\, z5^{20} + 2289730585763 6$

$- 1831316684897086507418 7\, z5^{32} - 60727216075$

$- 55557312975964800\, z4^{5}\, z5^{3} + 17460869792446$

$+ 8305444561289527\, z4\, z5^{43} - 212120871519454$

$- 3319815883093451385381\, z4\, z5^{31} - 415769164$

$+ 4154986709036460221649\, z4\, z5^{13} - 238102769$

$- 95241107958796800\, z5\, z3, 7936758996566400\, z$

$- 7936758996566400\, z5\, z1 - 7936758996566400$

$- 7936758996566400\, z4^{5}\, z5^{3} + 23810276989699 2$

$- 23810276989699200\, z4^{3}\, z5^{5} - 337312257354 07$

$- 369059293340337600\, z4^{2} + 1176345388640471$

$- 3004383582891473073\, z4\, z5^{37} - 470203502707$

$- 5888581834026443480 85\, z4\, z5^{25} - 8569393086$

$+ 5884726742423405263 77\, z4\, z5^{13} + 4713132419$

$+ 3659742549078552381\, z5\, z4 + 64231705139569$

$- 16404772137036480803\, z5^{38} - 25674191652275$

$- 3211938090825682172335\, z5^{26} - 233049033269$

$+ 3210100109444754864587\, z5^{14} + 256985831539$

$+ 18326089487427735751\, z5^{2}, 1190513849484960$

$- 11905138494849600\, z5\, z1 - 3968379498283200$

$+ 15873517993132800\, z4^{4}\, z5^{4} - 27778656487982$

$- 208339923659868000\, z4^{2}\, z5^{6} - 2400869596461$

$+ 786029984751110\, z4\, z5^{43} - 2007519008182245$

$- 3141880629080738070 90\, z4\, z5^{31} - 3934236675$

$- 550329120654394950\, z4\, z5^{19} + 3931964087288$

$+ 3148923727991764957 30\, z4\, z5^{7} + 24093865151$

4.3 RUR or Triangular sets or both ?

Compute a RUR from de DRL G. Basis

```
> rr:=rur(gb_drl);
```

Compute triangular sets decomposition from the lexicographic Grobner basis

```
> tri:=lextriangular(gb_lex):
```

Converts the list of triangular sets to a standard Maple list of lists of polynomials

```
> l_tri:=convert(tri,gbasis);
```

$$
\begin{aligned}
&l\_tri := [[1387545279120\, z0 + 1387545279120\, z4 + 4321823003\, z5^{31}\\
&- 11037922310209\, z5^{25} - 1727506390124986\, z5^{19} - 2176188913464634\, z5^{13}\\
&- 1732620732685741\, z5^{7} - 13506088516033\, z5 , 1387545279120\, z1\\
&+ 1128983050\, z4\ z5^{30} - 2883434331830\, z4\ z5^{24} - 451234998755840\, z4\ z5^{18}\\
&- 562426491685760\, z4\ z5^{12} - 447129055314890\, z4\ z5^{6} + 165557857270\, z4\\
&- 3283058841\, z5^{31} + 8384938292463\, z5^{25} + 1312252817452422\, z5^{19}\\
&+ 1646579934064638\, z5^{13} + 1306372958656407\, z5^{7} + 4694680112151\, z5 ,\\
&1387545279120\, z2 + 778171189\, z5^{31} - 1987468196267\, z5^{25}\\
&- 310993556954378\, z5^{19} - 383262822316802\, z5^{13} - 300335488637543\, z5^{7}\\
&+ 5289595037041\, z5 , 1387545279120\, z3 - 1128983050\, z4\ z5^{30}\\
&+ 2883434331830\, z4\ z5^{24} + 451234998755840\, z4\ z5^{18}\\
&+ 562426491685760\, z4\ z5^{12} + 447129055314890\, z4\ z5^{6} - 165557857270\, z4\\
&- 1816935351\, z5^{31} + 4640452214013\, z5^{25} + 726247129626942\, z5^{19}\\
&+ 912871801716798\, z5^{13} + 726583262666877\, z5^{7} + 4909358645961\, z5 ,\\
&1387545279120\, z4^{2} + 4321823003\, z4\ z5^{31} - 11037922310209\, z4\ z5^{25}\\
&- 1727506390124986\, z4\ z5^{19} - 2176188913464634\, z4\ z5^{13}\\
&- 1732620732685741\, z4\ z5^{7} - 13506088516033\, z5\ z4 + 24177661775\, z5^{32}\\
&- 61749727185325\, z5^{26} - 9664082618092450\, z5^{20} - 12152237485813570\, z5^{14}\\
&- 9672870290826025\, z5^{8} - 68544102808525\, z5^{2},\\
&z5^{36} - 2554\, z5^{30} - 399709\, z5^{24} - 502276\, z5^{18} - 399709\, z5^{12} - 2554\, z5^{6} + 1],\\
&[z0 - z5,\ z1 - z5,\ z2 - z5,\ z3 + z4 + 4\, z5,\ z4^{2} + 4\, z5\ z4 + z5^{2},\ z5^{6} - 1],\\
&[z0 - z5,\ z1 - z5,\ z2 + z3 + 4\, z5,\ z3^{2} + 4\, z5\ z3 + z5^{2},\ z4 - z5,\ z5^{6} - 1],\\
&[z0 - z5,\ z1 + z2 + 4\, z5,\ z2^{2} + 4\, z2\ z5 + z5^{2},\ z3 - z5,\ z4 - z5,\ z5^{6} - 1],\\
&[z0 + z1 + 4\, z5,\ z1^{2} + 4\, z5\ z1 + z5^{2},\ z2 - z5,\ z3 - z5,\ z4 - z5,\ z5^{6} - 1], [\\
&z0 - z4^{5}\, z5^{2} + 3\, z4^{4}\, z5^{3} - 3\, z4^{3}\, z5^{4} + 4\, z4^{2}\, z5^{5} + 3\, z4 - 3\, z5,\\
&3\, z1 + 2\, z4^{5}\, z5^{2} - 5\, z4^{4}\, z5^{3} + 5\, z4^{3}\, z5^{4} - 10\, z4^{2}\, z5^{5} - 4\, z4 + 7\, z5,\ z2 + z5,\\
&3\, z3 + z4^{5}\, z5^{2} - 4\, z4^{4}\, z5^{3} + 4\, z4^{3}\, z5^{4} - 2\, z4^{2}\, z5^{5} - 2\, z4 + 2\, z5,\\
&z4^{6} - 3\, z4^{5}\, z5 + 3\, z4^{4}\, z5^{2} - 4\, z4^{3}\, z5^{3} + 3\, z4^{2}\, z5^{4} - 3\, z4\ z5^{5} - 1,\ z5^{6} + 1]]
\end{aligned}
$$

Print the first triangular set

```
> l_tri[1];
```

$$
\begin{aligned}
&[1387545279120\, z0 + 1387545279120\, z4 + 4321823003\, z5^{31} - 11037922310209\, z5^{25} \\
&- 1727506390124986\, z5^{19} - 2176188913464634\, z5^{13} - 1732620732685741\, z5^{7} \\
&- 13506088516033\, z5 ,\, 1387545279120\, z1 + 1128983050\, z4\, z5^{30} \\
&- 2883434331830\, z4\, z5^{24} - 451234998755840\, z4\, z5^{18} \\
&- 562426491685760\, z4\, z5^{12} - 447129055314890\, z4\, z5^{6} + 165557857270\, z4 \\
&- 3283058841\, z5^{31} + 8384938292463\, z5^{25} + 1312252817452422\, z5^{19} \\
&+ 1646579934064638\, z5^{13} + 1306372958656407\, z5^{7} + 4694680112151\, z5 , \\
&1387545279120\, z2 + 778171189\, z5^{31} - 1987468196267\, z5^{25} \\
&- 310993556954378\, z5^{19} - 383262822316802\, z5^{13} - 300335488637543\, z5^{7} \\
&+ 5289595037041\, z5 ,\, 1387545279120\, z3 - 1128983050\, z4\, z5^{30} \\
&+ 2883434331830\, z4\, z5^{24} + 451234998755840\, z4\, z5^{18} \\
&+ 562426491685760\, z4\, z5^{12} + 447129055314890\, z4\, z5^{6} - 165557857270\, z4 \\
&- 1816935351\, z5^{31} + 4640452214013\, z5^{25} + 726247129626942\, z5^{19} \\
&+ 912871801716798\, z5^{13} + 726583262666877\, z5^{7} + 4909358645961\, z5 , \\
&1387545279120\, z4^{2} + 4321823003\, z4\, z5^{31} - 11037922310209\, z4\, z5^{25} \\
&- 1727506390124986\, z4\, z5^{19} - 2176188913464634\, z4\, z5^{13} \\
&- 1732620732685741\, z4\, z5^{7} - 13506088516033\, z5\, z4 + 24177661775\, z5^{32} \\
&- 61749727185325\, z5^{26} - 9664082618092450\, z5^{20} - 12152237485813570\, z5^{14} \\
&- 9672870290826025\, z5^{8} - 68544102808525\, z5^{2}, \\
&z5^{36} - 2554\, z5^{30} - 399709\, z5^{24} - 502276\, z5^{18} - 399709\, z5^{12} - 2554\, z5^{6} + 1]
\end{aligned}
$$

Convert a standard Maple object to a lexicographic Grobner basis (be sure of what you are doing !)

```
> lex1:=pretend(l_tri[1],Lex(vars)):
```

Compute a RUR of this component

```
> rr1:=rur(lex1,verbose=1);
```

$rr1 := [T^{72} - 1759953663216\, T^{66} - 4281100524244784594860$

$+ 372781806863800793988957117095581056\, T^{54}$

$- 7516520229285316793562586032691168505209308969\, T^{48}$

$- 222261607104184258220988031781155320661542466478592$

$23620621422832042073911616607868382043449028729377867$

$T^{36} + 415546218333580118657496393710173581508116280295$

$180398802928761616T^{30} - 357650134823732516829801547180$

$771333284673690161635533551270134632794003569T^{24} - 652$

$011843081635914250295345084151148378066785248469284789$

$202761832291574631776T^{18} - 248839630160868500742200557$

$087489008891580337484231301965701808939175759979341462$

$98906T^{12} + 1444291562533472598625486868423263647093063$

$752933470826169457014602105772120702458452816712159888$

$141664667141491493779905343068666159186096658435890618$

$15382023724033733770929246153804195605125312211289, 72$

$- 116156941772256\, T^{65} - 2568660314546870756916120\, T^{59}$

$+ 20130217570645242875403684323161377024\, T^{53}$

$- 360792971005695206091004129569176088250046830512\, T^{47}$

$- 933498749837573884528149733480852346778478359210089$

$850342371221953514660818197883261753564165034257603231$

$T^{35} + 124663865500074035597248918113052074452434884088$

$75411964087862848480T^{29} - 8583603235769580403915237132$

$946511998832168563879252805230483231187056085656T^{23} -$

$991482131754694464565053162115147206708052021344724471$

$45819649712981248343371968T^{17} - 2986075561930422008906$

$405841049868106698964049810775623588421707270109119752$

$71573186872T^{11} + 8665749375200835591752921210539581882$

$949628517600824957016742087612634632724214750716900272$

$016T^{5}, -16023907944\, T^{66} - 146942233356125450657520\, T^{60}$

$+ 75946419876140530987896831701810808\, T^{54}$

$- 2458565804702999778205280505783066231552650736\, T^{48}$

$- 621726892545824172488980653363426741774870921266864$

$48668181183158419269305769816414217092421840714530647$

$T^{36} + 171397799418582390685884368138911473968013904615$

$266678024114094736T^{30} - 494706276585717869543940303601$

$197826979206823233436687796147244795824444000T^{24} - 259$

$785968308940490048105057533253328921098663894662325589$

$708289853557565402408T^{18} - 308065518541487857739336032$

$595859780044058815138468335089060614378427405690969859$

$0976T^{12} + 31961363202495967645928063149508383390722826$

$076386071301899395576782584251253707334013781985043186$

$497197347839979861186897181800207623887107180270647670$

$85714331677045418072819405573042386607752822990672, -9$

$- 8469166552735565660652\, T^{60}$

$+ 71405578836540208807951434776652276\, T^{54}$

$- 2233367655754297614920343527364135797714676540\, T^{48}$

$- 103167531406268767418535219970950874995374220538364$

4.4 The end-user step.

converts the RS object "RUR" to a standard Maple list of univariate polynomials

```
> r1:=convert(rr,RUR):
```

The number of distinct complex roots of the system equals the degree of the squarefree part of the first polynomial of the RUR

```
> degree(simplify(r1[1]/gcd(r1[1],diff(r1[1],op(indets(r1[1]))))));
```

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The number of distinct real roots of the system equals the number of real roots of the first polynomial of the RUR. In order not to have ambigous answer, you may use Sturm sequences or isolate the real roots symbolically. Here , we choose the second option˜:

```
> s1:=isole(r1[1]);
```

$$s1 := [[\frac{-30531796567025}{34359738368}, \frac{-1908237285439}{2147483648}], [\frac{-27046484365757}{34359738368}, \frac{-6761621091439}{8589934592}],$$
$$[\frac{-12213653530713}{34359738368}, \frac{-1526706691339}{4294967296}], [\frac{-1496950285045}{4294967296}, \frac{-11975602280359}{34359738368}],$$
$$[\frac{-11627071060233}{34359738368}, \frac{-1453383882529}{4294967296}], [\frac{-703102830529}{2147483648}, \frac{-11249645288463}{34359738368}],$$
$$[\frac{-1364114663647}{4294967296}, \frac{-10912917309175}{34359738368}], [\frac{-9867323648795}{34359738368}, \frac{-4933661824397}{17179869184}],$$
$$[\frac{-3862431197811}{17179869184}, \frac{-7724862395621}{34359738368}], [\frac{-4588081414481}{34359738368}, \frac{-286755088405}{2147483648}],$$
$$[\frac{-1839302345041}{34359738368}, \frac{-114956396565}{2147483648}], [\frac{-1757716926909}{34359738368}, \frac{-439429231727}{8589934592}],$$
$$[\frac{439429231727}{8589934592}, \frac{1757716926909}{34359738368}], [\frac{114956396565}{2147483648}, \frac{1839302345041}{34359738368}],$$
$$[\frac{286755088405}{2147483648}, \frac{4588081414481}{34359738368}], [\frac{7724862395621}{34359738368}, \frac{3862431197811}{17179869184}],$$
$$[\frac{4933661824397}{17179869184}, \frac{9867323648795}{34359738368}], [\frac{10912917309175}{34359738368}, \frac{1364114663647}{4294967296}],$$
$$[\frac{11249645288463}{34359738368}, \frac{703102830529}{2147483648}], [\frac{1453383882529}{4294967296}, \frac{11627071060233}{34359738368}],$$
$$[\frac{11975602280359}{34359738368}, \frac{1496950285045}{4294967296}], [\frac{1526706691339}{4294967296}, \frac{12213653530713}{34359738368}],$$
$$[\frac{6761621091439}{8589934592}, \frac{27046484365757}{34359738368}], [\frac{1908237285439}{2147483648}, \frac{30531796567025}{34359738368}]]$$

```
> nops(s1);
```

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The roots can also be approximated numerically (no information about the possible numerical errors)

```
> s2:=[fsolve(r1[1])];
```

$$s2 := [-888.5922308, -787.1562954, -355.4641016, -348.5358984, -338.3923048, -327.4077692, -317.6076952, -287.1769145, -224.8230855, -133.5307436, -53.53074361, -51.15629543, 51.15629543, 53.53074361, 133.5307436, 224.8230855, 287.1769145, 317.6076952, 327.4077692, 338.3923048, 348.5358984, 355.4641016, 787.1562954, 888.5922308]$$

```
> nops(s2);
```

To get an approximation of the system's roots, one may substitute the numerical solutions in the rational functions defined by the RUR.

```
> sols_num:=[seq(subs(T=s2[j],[seq(r1[i]/r1[2],i=3..nops(r1))]),j=1..no
> ps(s2))];
```

$sols\_num$:= [[1.000000016, 1.000000016, 1.000000016, 1.000000017, −0.2679491971, −3.732050868], [−0.2679491966, 1.000000021, 1.000000022, 1.000000021, 1.000000020, −3.732050886], [3.732040322, 0.2679480680, −0.9999975427, −0.9999978345, −0.9999975194, −0.9999979534], [0.2679505274, 3.732071147, −1.000005426, −1.000004780, −1.000005385, −1.000007639], [−0.9999976004, 3.732040310, 0.2679490185, −0.9999969829, −0.9999958083, −1.000001644], [1.000002401, 1.000003123, 1.000000226, 1.000001051, −3.732056276, −0.2679467163], [−0.9999994661, 0.2679484755, 3.732047602, −0.9999987061, −0.9999988570, −1.000000423], [−1.000000058, −1.000000089, 3.732051050, 0.2679491951, −1.000000060, −1.000000071], [−0.9999999869, −1.000000001, 0.2679491712, 3.732050768, −0.9999999863, −0.9999999687], [−1.000000002, −0.9999999982, −1.000000002, 3.732050814, 0.2679491922, −1.000000005], [1.000000006, 0.9999999884, 0.9999999902, −0.2679491891, −3.732050777, 0.9999999871], [3.732050792, −1.000000000, −0.9999999970, −0.9999999990, −0.9999999950, 0.2679491974], [−3.732050792, 1.000000000, 0.9999999970, 0.9999999990, 0.9999999950, −0.2679491974], [−1.000000006, −0.9999999884, −0.9999999902, 0.2679491891, 3.732050777, −0.9999999871], [1.000000002, 0.9999999982, 1.000000002, −3.732050814, −0.2679491922, 1.000000005], [0.9999999869, 1.000000001, −0.2679491712, −3.732050768, 0.9999999863, 0.9999999687], [1.000000058, 1.000000089, −3.732051050, −0.2679491951, 1.000000060, 1.000000071], [0.9999994661, −0.2679484755, −3.732047602, 0.9999987061, 0.9999988570, 1.000000423], [−1.000002401, −1.000003123, −1.000000226, −1.000001051, 3.732056276, 0.2679467163], [0.9999976004, −3.732040310, −0.2679490185, 0.9999969829, 0.9999958083, 1.000001644], [−0.2679505274, −3.732071147, 1.000005426, 1.000004780, 1.000005385, 1.000007639], [−3.732040322, −0.2679480680, 0.9999975427, 0.9999978345, 0.9999975194, 0.9999979534], [0.2679491966, −1.000000021, −1.000000022, −1.000000021, −1.000000020, 3.732050886], [−1.000000016, −1.000000016, −1.000000016, −1.000000017, 0.2679491971, 3.732050868]]

Plug the solutions in the original system. Note that the result seems to be correct.

```
> seq(subs(seq(vars[i]=sols_num[j][i],i=1..nops(vars)),sys),j=1..nops(s
> ols_num));
```

$[0., 0.1\,10^{-8}, 0.4\,10^{-8}, 0.67\,10^{-8}, 0.57\,10^{-8}, 0.99\,10^{-7}],$
$[0.1\,10^{-8}, 0.1\,10^{-8}, -0.6\,10^{-8}, -0.164\,10^{-7}, -0.204\,10^{-7}, 0.121\,10^{-6}], [-0.24608\,10^{-5},$
$-0.896\,10^{-6}, 0.2568\,10^{-5}, -0.53306\,10^{-5}, 0.72646\,10^{-5}, -0.0000161560],$
$[-0.1556\,10^{-5}, -0.1442\,10^{-6}, -0.4071\,10^{-5}, 0.6190\,10^{-5}, 0.3176\,10^{-5}, 0.000033662],$
$[-0.27073\,10^{-5}, 0.6461\,10^{-5}, 0.7429\,10^{-5}, -0.1105\,10^{-5}, -0.47883\,10^{-5},$
$-0.0000114262], [0.38087\,10^{-5}, -0.51834\,10^{-5}, -0.000011188, -0.000027228,$
$-0.000040896, -0.9748\,10^{-6}],$
$[-0.13750\,10^{-5}, 0.2092\,10^{-5}, -0.883\,10^{-6}, -0.5753\,10^{-5}, 0.76685\,10^{-5}, -0.60825\,10^{-5}]$
$, [-0.33\,10^{-7}, -0.111\,10^{-6}, 0.123\,10^{-6}, -0.180\,10^{-6}, 0.224\,10^{-6}, 0.353\,10^{-6}],$
$[-0.40\,10^{-8}, -0.80\,10^{-7}, 0.62\,10^{-7}, -0.106\,10^{-6}, 0.2415\,10^{-6}, -0.1468\,10^{-6}],$
$[-0.1\,10^{-8}, -0.65\,10^{-8}, -0.8\,10^{-8}, 0.10\,10^{-7}, 0.8\,10^{-8}, 0.8\,10^{-8}],$
$[0.51\,10^{-8}, 0.29\,10^{-7}, -0.33\,10^{-7}, -0.339\,10^{-7}, -0.215\,10^{-7}, -0.488\,10^{-7}],$
$[-0.16\,10^{-8}, 0.133\,10^{-7}, -0.34\,10^{-7}, 0.47\,10^{-7}, -0.79\,10^{-7}, 0.5\,10^{-8}],$
$[0.16\,10^{-8}, 0.133\,10^{-7}, 0.34\,10^{-7}, 0.47\,10^{-7}, 0.79\,10^{-7}, 0.5\,10^{-8}],$
$[-0.51\,10^{-8}, 0.29\,10^{-7}, 0.33\,10^{-7}, -0.339\,10^{-7}, 0.215\,10^{-7}, -0.488\,10^{-7}],$
$[0.1\,10^{-8}, -0.65\,10^{-8}, 0.8\,10^{-8}, 0.10\,10^{-7}, -0.8\,10^{-8}, 0.8\,10^{-8}],$
$[0.40\,10^{-8}, -0.80\,10^{-7}, -0.62\,10^{-7}, -0.106\,10^{-6}, -0.2415\,10^{-6}, -0.1468\,10^{-6}],$
$[0.33\,10^{-7}, -0.111\,10^{-6}, -0.123\,10^{-6}, -0.180\,10^{-6}, -0.224\,10^{-6}, 0.353\,10^{-6}],$
$[0.13750\,10^{-5}, 0.2092\,10^{-5}, 0.883\,10^{-6}, -0.5753\,10^{-5}, -0.76685\,10^{-5}, -0.60825\,10^{-5}],$
$[-0.38087\,10^{-5}, -0.51834\,10^{-5}, 0.000011188, -0.000027228, 0.000040896,$
$-0.9748\,10^{-6}], [0.27073\,10^{-5}, 0.6461\,10^{-5}, -0.7429\,10^{-5}, -0.1105\,10^{-5}, 0.47883\,10^{-5},$
$-0.0000114262],$
$[0.1556\,10^{-5}, -0.1442\,10^{-6}, 0.4071\,10^{-5}, 0.6190\,10^{-5}, -0.3176\,10^{-5}, 0.000033662], [$
$0.24608\,10^{-5}, -0.896\,10^{-6}, -0.2568\,10^{-5}, -0.53306\,10^{-5}, -0.72646\,10^{-5},$
$-0.0000161560], [-0.1\,10^{-8}, 0.1\,10^{-8}, 0.6\,10^{-8}, -0.164\,10^{-7}, 0.204\,10^{-7}, 0.121\,10^{-6}],$
$[0., 0.1\,10^{-8}, -0.4\,10^{-8}, 0.67\,10^{-8}, -0.57\,10^{-8}, 0.99\,10^{-7}]$

The roots of the system can be isolated symbolically from the RUR (the result is expressed using floating point numbers but the number of real roots is exact and the precision of the numerical approximation is the precision givent by the variable Digits of Maple). Not that Maple take some time to print the result : the reason is that the roots are sent to Maple by RS as products of intervals with rational coefficient that may be huge.

```
> iso:=isolecoords(rr,verbose=1);
```

$iso := \{\{z5 = -3.732050808,\ z4 = -0.2679491924,\ z3 = 1.000000000,\ z2 = 1.000000000,$
$z1 = 1.000000000,\ z0 = 1.000000000\}, \{z2 = -1.000000000,\ z0 = 3.732050808,$
$z4 = -1.000000000,\ z3 = -1.000000000,\ z1 = -1.000000000,\ z5 = 0.2679491924\},$
$\{z2 = 1.000000000,\ z1 = 1.000000000,\ z0 = 1.000000000,\ z4 = -3.732050808,$
$z3 = -0.2679491924,\ z5 = 1.000000000\}, \{z2 = -1.000000000,\ z5 = -1.000000000,$
$z0 = -1.000000000,\ z1 = -1.000000000,\ z3 = 3.732050808,\ z4 = 0.2679491924\}, \{$
$z5 = -1.000000000,\ z4 = -1.000000000,\ z2 = 0.2679491924,\ z0 = -1.000000000,$
$z1 = -1.000000000,\ z3 = 3.732050808\}, \{z5 = -1.000000000,\ z4 = -1.000000000,$
$z0 = -1.000000000,\ z2 = 3.732050808,\ z1 = -1.000000000,\ z3 = 0.2679491924\}, \{$
$z1 = 0.2679491924,\ z5 = -1.000000000,\ z4 = -1.000000000,\ z3 = -1.000000000,$
$z0 = -1.000000000,\ z2 = 3.732050808\}, \{z3 = 1.000000000,\ z2 = 1.000000000,$
$z1 = 1.000000000,\ z0 = 1.000000000,\ z4 = -3.732050808,\ z5 = -0.2679491924\}, \{$
$z1 = 3.732050808,\ z5 = -1.000000000,\ z4 = -1.000000000,\ z3 = -1.000000000,$
$z2 = 0.2679491924,\ z0 = -1.000000000\}, \{z2 = -1.000000000,\ z1 = 3.732050808,$
$z0 = 0.2679491924,\ z5 = -1.000000000,\ z4 = -1.000000000,\ z3 = -1.000000000\},$
$\{z2 = -1.000000000,\ z1 = 0.2679491924,\ z0 = 3.732050808,\ z5 = -1.000000000,$
$z4 = -1.000000000,\ z3 = -1.000000000\}, \{z4 = 1.000000000,\ z0 = -0.2679491924,$
$z5 = -3.732050808,\ z3 = 1.000000000,\ z2 = 1.000000000,\ z1 = 1.000000000\}, \{$
$z4 = 1.000000000,\ z3 = 1.000000000,\ z2 = 1.000000000,\ z1 = 1.000000000,$
$z5 = -0.2679491924,\ z0 = -3.732050808\}, \{z4 = 1.000000000,\ z3 = 1.000000000,$
$z0 = 1.000000000,\ z5 = 1.000000000,\ z2 = -3.732050808,\ z1 = -0.2679491924\}, \{$
$z4 = 1.000000000,\ z1 = 1.000000000,\ z0 = 1.000000000,\ z3 = -0.2679491924,$
$z5 = 1.000000000,\ z2 = -3.732050808\}, \{z4 = 1.000000000,\ z1 = 1.000000000,$
$z0 = 1.000000000,\ z5 = 1.000000000,\ z3 = -3.732050808,\ z2 = -0.2679491924\}, \{$
$z4 = -0.2679491924,\ z2 = 1.000000000,\ z1 = 1.000000000,\ z0 = 1.000000000,$
$z5 = 1.000000000,\ z3 = -3.732050808\}, \{z2 = -1.000000000,\ z5 = -1.000000000,$
$z0 = -1.000000000,\ z1 = -1.000000000,\ z3 = 0.2679491924,\ z4 = 3.732050808\}, \{$
$z4 = 1.000000000,\ z3 = 1.000000000,\ z0 = 1.000000000,\ z5 = 1.000000000,$
$z2 = -0.2679491924,\ z1 = -3.732050808\}, \{z2 = -1.000000000,$
$z3 = -1.000000000,\ z0 = -1.000000000,\ z1 = -1.000000000,\ z5 = 0.2679491924,$
$z4 = 3.732050808\}, \{z2 = -1.000000000,\ z0 = 0.2679491924,\ z4 = -1.000000000,$
$z3 = -1.000000000,\ z1 = -1.000000000,\ z5 = 3.732050808\}, \{z4 = 1.000000000,$
$z3 = 1.000000000,\ z2 = 1.000000000,\ z0 = -3.732050808,\ z5 = 1.000000000,$
$z1 = -0.2679491924\}, \{z4 = 1.000000000,\ z0 = -0.2679491924,\ z3 = 1.000000000,$
$z2 = 1.000000000,\ z5 = 1.000000000,\ z1 = -3.732050808\}, \{z2 = -1.000000000,$
$z3 = -1.000000000,\ z0 = -1.000000000,\ z1 = -1.000000000,\ z4 = 0.2679491924,$
$z5 = 3.732050808\}\}$

Note the precision of the evaluation of the system a the founded roots !

```
> seq(subs(iso[j],sys),j=1..nops(iso));
```

$[0., 0., 0., 0., 0., 0.], [0.4\,10^{-9}, -0.4\,10^{-9}, 0.4\,10^{-9}, -0.4\,10^{-9}, 0.4\,10^{-9}, 0.],$
$[0., 0., 0., 0., 0., 0.], [0., -0.4\,10^{-9}, 0., 0., 0., 0.], [0., 0., 0., 0., 0., 0.],$
$[0., 0., 0.4\,10^{-9}, 0., 0., 0.], [0., 0., 0., 0., 0., 0.], [-0.4\,10^{-9}, 0., 0., 0., 0., 0.],$
$[0., 0., 0., -0.4\,10^{-9}, 0., 0.], [0., 0., 0., 0., 0., 0.], [0., 0., 0., 0., 0.4\,10^{-9}, 0.],$
$[0., 0., 0., 0., 0., 0.], [-0.4\,10^{-9}, -0.4\,10^{-9}, -0.4\,10^{-9}, -0.4\,10^{-9}, -0.4\,10^{-9}, 0.],$
$[0., 0., 0., 0., 0., 0.], [0., 0., -0.4\,10^{-9}, 0., 0., 0.], [0., 0., 0., 0., 0., 0.],$
$[0., -0.4\,10^{-9}, 0., 0., 0., 0.], [0., 0., 0., 0., 0., 0.], [0., 0., 0., -0.4\,10^{-9}, 0., 0.],$
$[0.4\,10^{-9}, 0., 0., 0., 0., 0.], [0., 0., 0., 0., 0., 0.], [0., 0., 0., 0., -0.4\,10^{-9}, 0.],$
$[0., 0., 0., 0., 0., 0.], [0., 0., 0., 0., 0., 0.]$

5 Choosing the right strategy in practice.

The first thing to do is to use modular computations to get (probabilistic) informations about the system and its properties.

5.1 Equations

```
> sys:=[-6*x1+3*x2*x1+3*x3*x1+3*x4*x1+3*x5*x1+2*x1^2+4*x1^3-1*x2^2+1*x2
> ^3-1*x3^2+1*x3^3-1*x4^2+1*x4^3-1*x5^2+1*x5^3+3*x2^2*x1+3*x2*x1^2+3*x3^
> 2*x1+3*x3*x1^2+3*x4^2*x1+3*x4*x1^2+3*x5^2*x1+3*x5*x1^2,-6*x2+3*x2*x1+3
> *x3*x2+3*x4*x2+3*x5*x2-1*x1^2+1*x1^3+2*x2^2+4*x2^3-1*x3^2+1*x3^3-1*x4^
> 2+1*x4^3-1*x5^2+1*x5^3+3*x2^2*x1+3*x2*x1^2+3*x3^2*x2+3*x3*x2^2+3*x4^2*
> x2+3*x4*x2^2+3*x5^2*x2+3*x5*x2^2,-6*x3+3*x3*x1+3*x3*x2+3*x4*x3+3*x5*x3
> -1*x1^2+1*x1^3-1*x2^2+1*x2^3+2*x3^2+4*x3^3-1*x4^2+1*x4^3-1*x5^2+1*x5^3
> +3*x3^2*x1+3*x3*x1^2+3*x3^2*x2+3*x3*x2^2+3*x4^2*x3+3*x4*x3^2+3*x5^2*x3
> +3*x5*x3^2,-6*x4+3*x4*x1+3*x4*x2+3*x4*x3+3*x5*x4-1*x1^2+1*x1^3-1*x2^2+
> 1*x2^3-1*x3^2+1*x3^3+2*x4^2+4*x4^3-1*x5^2+1*x5^3+3*x4^2*x1+3*x4*x1^2+3
> *x4^2*x2+3*x4*x2^2+3*x4^2*x3+3*x4*x3^2+3*x5^2*x4+3*x5*x4^2,-6*x5+3*x5*
> x1+3*x5*x2+3*x5*x3+3*x5*x4-1*x1^2+1*x1^3-1*x2^2+1*x2^3-1*x3^2+1*x3^3-1
> *x4^2+1*x4^3+2*x5^2+4*x5^3+3*x5^2*x1+3*x5*x1^2+3*x5^2*x2+3*x5*x2^2+3*x
> 5^2*x3+3*x5*x3^2+3*x5^2*x4+3*x5*x4^2]:
```

```
> vars:= [x1,x2,x3,x4,x5]:
```

5.2 A Grobner basis modulo a prime number for the DRL ordering.

Modular computations are not subject to growth of coefficients so that the computations are fast.

By choosing a prime number less than 2ˆ16, the hardware arithmetic is used by Gb.

The prime number must be as big as possible to get information with a good probability.

Here , we choos 65521.

```
> gb_drl:=gbasis(sys,DRL(char=65521,vars)):
```

degree and dimension of the system :

```
> vdegree(gb_drl);
```

$$213$$

```
> dimension(gb_drl);
```

$$0$$

This computation shows that the system is zero-dimensional and that it has 213 solutions counted with multiplicities. We have no choice for the computations since it is to big for our version of FGLM (computation of a lexicographic grobner basis by change of ordering).

5.3 Resolution

So we solve the system using the RUR computation (quite long but less than 3 minute, you may skip)

```
> gb_drl:=gbasis(sys,DRL(vars),split,verbose):
> rr:=rur(gb_drl,verbose=1):
> sols:=isolecoords(rr);
```

$sols := \{\{x5 = -1.033072678,\ x4 = 0.5394707040,\ x3 = 0.5394707040,$
$x2 = 0.5394707040,\ x1 = 0.5394707042\}, \{x5 = -0.3466653777,$
$x4 = -0.3466653778,\ x3 = -0.3466653778,\ x2 = -0.3466653778,\ x1 = 1.800439994$
$\}, \{x2 = -0.6165912457,\ x1 = -0.6165912457,\ x5 = -0.6165912457,$
$x4 = -0.6165912457,\ x3 = -0.6165912457\}, \{x4 = 0.1313535790\,10^{-10},$
$x3 = 1.296630263,\ x2 = 0.2921715392\,10^{-10},\ x1 = 0.2271312335\,10^{-10},$
$x5 = -0.5645794553\}, \{x2 = 1.296630264,\ x1 = -0.3985374442\,10^{-9},$
$x5 = -0.5645794552,\ x4 = -0.4439379278\,10^{-9},\ x3 = -0.4831654125\,10^{-9}\}, \{$
$x5 = -0.3466653778,\ x4 = -0.3466653775,\ x3 = -0.3466653774,\ x2 = 1.800439993,$
$x1 = -0.3466653775\}, \{x5 = -0.5645794554,\ x4 = -0.2996970927\,10^{-10},$
$x3 = -0.4590359555\,10^{-10},\ x2 = -0.4214096115\,10^{-10},\ x1 = 1.296630263\}, \{$
$x1 = 0.4192805984,\ x5 = -0.3983217896,\ x4 = 0.4192805984,\ x3 = 0.4192805984,$
$x2 = 0.4192805984\}, \{x1 = 1.296630263,\ x4 = -0.6127414252\,10^{-10},$
$x3 = -0.5645794555,\ x2 = -0.6605817191\,10^{-10},\ x5 = -0.6465568327\,10^{-10}\}, \{$
$x2 = 1.296630263,\ x1 = -0.4451135808\,10^{-10},\ x5 = -0.4098877112\,10^{-10},$
$x4 = -0.5645794554,\ x3 = 0.1544291783\,10^{-10}\}, \{x5 = -0.3466653777,$
$x1 = -0.3466653777,\ x3 = 1.800439994,\ x2 = -0.3466653777,\ x4 = -0.3466653777$
$\}, \{x1 = 1.296630263,\ x3 = 0.1546008679\,10^{-11},\ x2 = 0.2243506723\,10^{-11},$
$x5 = 0.2023647794\,10^{-11},\ x4 = -0.5645794553\}, \{x5 = -0.5645794553,$
$x4 = 1.296630263,\ x3 = 0.5937061126\,10^{-10},\ x2 = 0.2204724944\,10^{-9},$
$x1 = -0.3883178704\,10^{-9}\}, \{x5 = 0.,\ x4 = 0.,\ x3 = 0.,\ x2 = 0.,\ x1 = 1.\}, \{$
$x3 = 1.296630262,\ x2 = 0.1004028464\,10^{-9},\ x1 = 0.3978648749\,10^{-9},$
$x4 = -0.5645794549,\ x5 = 0.9427773151\,10^{-10}\}, \{x2 = 1.296630263,$
$x5 = 0.1080910435\,10^{-9},\ x4 = 0.4960985635\,10^{-10},\ x3 = -0.5645794550,$
$x1 = 0.1113389399\,10^{-9}\}, \{x1 = 1.296630263,\ x2 = -0.5645794553,$
$x4 = -0.1268935732\,10^{-11},\ x3 = -0.1154856381\,10^{-11},\ x5 = -0.3228046719\,10^{-11}\},$
$\{x1 = 0.,\ x5 = 0.,\ x4 = 0.,\ x3 = 0.,\ x2 = 0.\}, \{x1 = -0.5645794584,$
$x5 = 0.1227348620\,10^{-9},\ x4 = 0.3994437186\,10^{-10},\ x3 = 0.1030347594\,10^{-9},$
$x2 = 1.296630265\}, \{x1 = 0.,\ x5 = 0.,\ x4 = 0.,\ x3 = 0.,\ x2 = 1.\},$
$\{x1 = 0.,\ x5 = 0.,\ x4 = 0.,\ x2 = 0.,\ x3 = 1.\}, \{x4 = -0.8684761570\,10^{-10},$
$x3 = 1.296630264,\ x2 = -0.5645794566,\ x1 = 0.2971659446\,10^{-10},$
$x5 = 0.1896696633\,10^{-9}\}, \{x2 = -0.3466653781,\ x1 = -0.3466653781,$
$x5 = -0.3466653782,\ x4 = 1.800439994,\ x3 = -0.3466653780\}, \{$
$x3 = 1.296630263,\ x2 = -0.1005980660\,10^{-10},\ x1 = -0.5645794554,$
$x5 = 0.2076882192\,10^{-10},\ x4 = 0.1952473034\,10^{-10}\}, \{x4 = -1.033072679,$
$x3 = 0.5394707036,\ x2 = 0.5394707047,\ x1 = 0.5394707047,\ x5 = 0.5394707048\}$
$, \{x5 = 0.3040912456,\ x4 = 0.3040912457,\ x3 = 0.3040912457,$
$x2 = 0.3040912457,\ x1 = 0.3040912457\}, \{x2 = -0.5645794553,$
$x4 = 1.296630263,\ x3 = 0.1596989870\,10^{-10},\ x1 = -0.8206639381\,10^{-11},$
$x5 = -0.1367894456\,10^{-9}\}, \{x4 = 1.296630263,\ x3 = -0.5645794559,$
$x2 = -0.1437542460\,10^{-10},\ x1 = -0.8606287869\,10^{-11},\ x5 = 0.1128839593\,10^{-9}\},$
$\{x1 = 0.,\ x5 = 0.,\ x3 = 0.,\ x2 = 0.,\ x4 = 1.\}, \{x3 = 0.4192805985,$
$x2 = 0.4192805985,\ x1 = 0.4192805985,\ x4 = -0.3983217899,\ x5 = 0.4192805985$
$\}, \{x4 = 0.4192805984,\ x5 = 0.4192805984,\ x3 = -0.3983217896,$
$x2 = 0.4192805983,\ x1 = 0.4192805983\}, \{x4 = 1.296630263,$

```
> nops(sols);
```

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5.4 Another system

```
> sys:=[62500*x1^2 + 62500*y1^2 + 62500*z1^2 -74529,
> 625*x2^2 + 625*y2^2 + 625*z2^2 -1250*x2 -2624,
> 12500*x3^2 + 12500*y3^2 + 12500*z3^2 + 2500*x3 -44975*y3 -10982,
> 400000*x1*x2 + 400000*y1*y2 + 400000*z1*z2 -400000*x2 + 178837,
> 1000000*x1*x3 + 1000000*y1*y3 + 1000000*z1*z3 + 100000*x3 -1799000*y3
> -805427,
> 2000000*x2*x3 + 2000000*y2*y3 + 2000000*z2*z3 -2000000*x2 + 200000*x3
> -3598000*y3 -1403,
> 113800000000000*x3*y2*z1 -113800000000000*x2*y3*z1
> -113800000000000*x3*y1*z2 + 113800000000000*x1*y3*z2 +
> 113800000000000*x2*y1*z3 -113800000000000*x1*y2*z3
> -206888400000000*x2*y1 + 206888400000000*x3*y1 + 206888400000000*x1*y2
> -206888400000000*x3*y2 -206888400000000*x1*y3 + 206888400000000*x2*y3
> -2014260000000*x2*z1 + 2014260000000*x3*z1 -61907200000000*y2*z1 +
> 61907200000000*y3*z1 + 2014260000000*x1*z2 -2014260000000*x3*z2 +
> 61907200000000*y1*z2 -61907200000000*y3*z2 -2014260000000*x1*z3 +
> 2014260000000*x2*z3 -61907200000000*y1*z3 + 61907200000000*y2*z3
> -362960716800000*x1 + 38025201600000*x2 + 292548849600000*x3 +
> 11809567440000*y1 + 1475978220000*y2 -825269402280000*y3
> -1212982689600000*z1 -151600474800000*z2 + 825859951200000*z3
> -19295432410527,
> -777600000000*x3*y2*z1 + 777600000000*x2*y3*z1 +
> 777600000000*x3*y1*z2 -777600000000*x1*y3*z2 -777600000000*x2*y1*z3 +
> 777600000000*x1*y2*z3 -1409011200000*x2*y1 + 1409011200000*x3*y1 +
> 1409011200000*x1*y2 -1409011200000*x3*y2 -1409011200000*x1*y3 +
> 1409011200000*x2*y3 -1065312000000*x2*z1 + 1065312000000*x3*z1
> -805593600000*y2*z1 + 805593600000*y3*z1 + 1065312000000*x1*z2
> -1065312000000*x3*z2 + 805593600000*y1*z2 -805593600000*y3*z2
> -1065312000000*x1*z3 + 1065312000000*x2*z3 -805593600000*y1*z3 +
> 805593600000*y2*z3 + 235685027200*x1 + 398417510400*x2 +
> 158626915200*x3 -311668424000*y1 -268090368000*y2 + 72704002800*y3 +
> 412221302400*z1 + 354583756800*z2 + 307085438400*z3 + 282499646407,
> 3200*x2 + 1271]:
> vars:=[x1, x2, x3, y1, y2, y3, z1, z2, z3]:
```

5.5 Groebner basis modulo p

```
> gb_drl:=gbasis(sys,DRL(char=65521,vars)):
> vdegree(gb_drl);
```

40

```
> dimension(gb_drl);
```

0

Are we in the shape lemma case ? (if so, it is better to use the RUR). We compute the lexicographic Grobner basis modulo 65521 :

```
> gb_lex:=fglm(gb_drl):
```

`Is the system in shape position (modulo p) ?`

```
> lgb_lex:=convert(gb_lex,gbasis):
> for i from 1 to nops(lgb_lex) do
> lm:=[LM(lgb_lex[i],Lex(vars))]:print(lm[2],indets(lgb_lex[i])):od:
```

$$x1, \{x1, z3\}$$

$$x2, \{x2\}$$

$$x3, \{x3, z3\}$$

$$y1, \{z3, y1\}$$

$$y2, \{y2, z3\}$$

$$y3, \{y3, z3\}$$

$$z1, \{z1, z3\}$$

$$z2, \{z2, z3\}$$

$$z3^{40}, \{z3\}$$

5.6 Resolution

Since the system is in shape position, we choose a RUR based resolution.

```
> gb_drl:=gbasis(sys,DRL(vars),split,verbose):
> rr:=rur(gb_drl,verbose=1):
> sols:=isolecoords(rr):
> sols;
```

$\{\{y1 = 0.5560438428,\ y2 = -1.693494642,\ y3 = 0.03292552609,\ x3 = -0.7063546081,$
$x1 = -0.8537298698,\ x2 = -0.3971875000,\ z1 = -0.3929688972,$
$z3 = -0.7989217996,\ z2 = 0.6150959166\}, \{x2 = -0.3971875000,$
$z1 = 0.7126347279,\ z3 = 0.4547635030,\ z2 = -1.769177274,\ y1 = -0.6906182390,$
$y3 = -0.09855964385,\ y2 = -0.3409973514,\ x1 = -0.4556997195,$
$x3 = -0.6633991607\}, \{x2 = -0.3971875000,\ z2 = -1.411231861,$
$z3 = 0.4548194149,\ y2 = 1.120130264,\ z1 = 0.5434761821,\ y3 = 0.1823292299,$
$y1 = -0.3771020886,\ x3 = -1.242136567,\ x1 = -0.8688450182\}, \{$
$x2 = -0.3971875000,\ z2 = 1.572064787,\ z3 = -0.4581603854,\ y2 = -0.8802723425,$
$z1 = -0.3770274062,\ y3 = -0.02457935045,\ y1 = 0.6451015212,$
$x3 = -0.8678596317,\ x1 = -0.7963406077\}, \{x2 = -0.3971875000,$
$z2 = -0.3414912217,\ z3 = 0.8471680467,\ y2 = -1.769081918,\ z1 = 0.3815573626,$
$y3 = 0.09809414888,\ y1 = 0.5910946833,\ x3 = -0.8170680466,$
$x1 = -0.8351557054\}, \{x2 = -0.3971875000,\ z3 = 1.149925390,$
$y3 = 0.1273202508,\ z2 = -0.3895102943,\ z1 = 0.8757123681,$
$x3 = -0.1901077837,\ y2 = 1.759132974,\ y1 = -0.4020403385,\ x1 = -0.5137659135$
$\}, \{x2 = -0.3971875000,\ z1 = 0.1596272060,\ z3 = 0.8046378631,$
$z2 = 1.257737671,\ y1 = 0.5182348410,\ y3 = 0.3523702341,\ y2 = -1.290101949,$
$x1 = 0.9478479861,\ x3 = 1.076767279\}, \{x2 = -0.3971875000,$
$z3 = 0.4472900343,\ y3 = -0.1603677435,\ z2 = -1.801560597,\ z1 = 0.6441761276,$
$x3 = -0.3928663012,\ y2 = -0.02542640965,\ y1 = 0.3307495088,$
$x1 = -0.8173774392\}, \{x2 = -0.3971875000,\ z3 = 0.4853252145,$
$y3 = 0.3739979523,\ z2 = 1.730246515,\ z1 = -0.1339437032,\ x3 = 1.263374342,$
$y2 = -0.5025077859,\ y1 = 0.4340157805,\ x1 = 0.9930525583\}, \{$
$x2 = -0.3971875000,\ z1 = 0.6639605740,\ z3 = 0.4048715059,\ z2 = -1.668257127,$
$y1 = -0.1145360362,\ y3 = -0.1865443108,\ y2 = 0.6805771427,$
$x1 = -0.8593613046,\ x3 = -0.2365791091\}, \{x2 = -0.3971875000,$
$z2 = -0.1696466337,\ z3 = 0.4059833887,\ y2 = 1.793735508,\ z1 = -0.09522157782,$
$y3 = 0.2104838725,\ y1 = -0.2449957765,\ x3 = 1.098647146,\ x1 = 1.059893332\},$
$\{x2 = -0.3971875000,\ x1 = 0.9807961793,\ z3 = 0.3419059279,$
$y3 = 0.3571399336,\ z2 = 1.801362993,\ z1 = -0.2606803143,\ x3 = 1.288920763,$
$y2 = 0.03685724449,\ y1 = 0.4031731998\}, \{x2 = -0.3971875000,$
$z1 = 0.3374247674,\ z3 = 0.3088039024,\ z2 = -1.800053287,\ y1 = -0.7291792106,$
$y3 = -0.1470497213,\ y2 = -0.07794388340,\ x1 = 0.7395310705,$
$x3 = 0.3924343933\}, \{x2 = -0.3971875000,\ z2 = -1.340708886,$
$z3 = 0.03663915324,\ y2 = 1.203647280,\ z1 = -0.1482621753,$
$y3 = 0.08768735709,\ y1 = -0.5613974274,\ x3 = 0.9931731844,$
$x1 = 0.9248325558\}, \{x2 = -0.3971875000,\ z3 = 0.2850897002,$
$y3 = -0.1538481001,\ z2 = -1.283347710,\ z1 = 0.3758575681,\ x3 = 0.3796552523,$
$y2 = 1.264628700,\ y1 = 0.03562912669,\ x1 = 1.024658799\}, \{$
$x2 = -0.3971875000,\ y1 = 0.6501307388,\ x3 = 0.9810877054,\ z1 = 0.3115492352,$
$z2 = -1.800556175,\ z3 = -0.01850506587,\ y2 = 0.06530352158,$
$y3 = 0.07973622270,\ x1 = 0.8202018633\}, \{x2 = -0.3971875000,$
$z1 = -0.8655274640,\ z3 = -0.4271350279,\ z2 = 0.8804584941,\ y1 = 0.2117996141,$
$y3 = 0.1063882127,\ y2 = -1.571960537,\ x1 = -0.6312425310,\ x3 = -1.138066475\}$
$, \{x2 = -0.3971875000,\ x3 = -0.4863955309,\ y2 = -0.02856942835,$
$y1 = -0.8082200173,\ z3 = -0.4466457766,\ y3 = -0.1442362295,$
$z1 = -0.5805959520,\ z2 = 1.801513496,\ x1 = -0.4496139960\}, \{$
$x2 = -0.3971875000,\ z1 = -0.7376556797,\ z3 = -0.4913068446,\ z2 = 1.629526733,$

```
> nops(sols);
```

$$24$$

5.7 Another system

```
> sys:=[1*z0+1*z1+1*z2+1*z3+1*z4+1*z5,1*z1*z0+1*z2*z1+1*z5*z0+1*z3*z2+1
> *z4*z3+1*z5*z4,1*z2*z1*z0+1*z5*z1*z0+1*z3*z2*z1+1*z5*z4*z0+1*z4*z3*z2+
> 1*z5*z4*z3,1*z3*z2*z1*z0+1*z5*z2*z1*z0+1*z5*z4*z1*z0+1*z4*z3*z2*z1+1*z
> 5*z4*z3*z0+1*z5*z4*z3*z2,1*z4*z3*z2*z1*z0+1*z5*z3*z2*z1*z0+1*z5*z4*z2*
> z1*z0+1*z5*z4*z3*z1*z0+1*z5*z4*z3*z2*z0+1*z5*z4*z3*z2*z1,1*z5*z4*z3*z2
> *z1*z0-1]:
> vars:= [z0,z1,z2,z3,z4,z5]:
```

5.8 modulo p

```
> gb_drl:=gbasis(sys,DRL(char=65521,vars)):
> gb_drl:
> vdegree(gb_drl);
```

$$156$$

```
> dimension(gb_drl);
```

$$0$$

```
> gb_lex:=fglm(gb_drl):
> lgb_lex:=convert(gb_lex,gbasis):
> for i from 1 to nops(lgb_lex) do
> lm:=[LM(lgb_lex[i],Lex(vars))]:print(lm[2],indets(lgb_lex[i])):od:
```

$$z0,\ \{z0,\ z4,\ z5,\ z1,\ z2,\ z3\}$$

$$z1^2,\ \{z4,\ z5,\ z1,\ z3\}$$

$$z2\ z1,\ \{z4,\ z5,\ z1,\ z2,\ z3\}$$

$$z3\ z1,\ \{z4,\ z5,\ z1,\ z3\}$$

$$z1\ z4,\ \{z4,\ z5,\ z1\}$$

$$z1\ z5^6,\ \{z4,\ z5,\ z1\}$$

$$z2^3,\ \{z5,\ z2\}$$

$$z3\ z2,\ \{z4,\ z5,\ z2,\ z3\}$$

$$z2\ z4,\ \{z4,\ z5,\ z2\}$$

$$z2\ z5^6,\ \{z5,\ z2\}$$

$$z3^3,\ \{z4,\ z5,\ z3\}$$

$$z4\ z3,\ \{z4,\ z5,\ z3\}$$

$$z3\ z5^6,\ \{z4,\ z5,\ z3\}$$

$$z4^6,\ \{z4,\ z5\}$$

$$z4^3\ z5^6,\ \{z4,\ z5\}$$

$$z4^2\ z5^{12},\ \{z4,\ z5\}$$

$$z5^{48},\ \{z5\}$$

5.9 Résolution

Since the system has a reasonable number of complex roots, small coefficients, and far from the shape position, we choose the triangular sets based resolution method

```
> gb_drl:=gbasis(sys,DRL(vars)):
> gb_lex:=fglm(gb_drl):
> tri:=lextriangular(gb_lex):l_tri:=convert(tri,gbasis):
> sols:=[]:
> for i from 1 to nops(l_tri) do
> sols:=[op(sols),op(isolecoords(rur(pretend(l_tri[i],Lex(vars)))))]:
> od:
> sols;
```

$[\{z5 = -3.732050808,\ z4 = -0.2679491924,\ z3 = 1.000000000,\ z2 = 1.000000000,$
$z1 = 1.000000000,\ z0 = 1.000000000\}, \{z2 = -1.000000000,\ z0 = 3.732050808,$
$z4 = -1.000000000,\ z3 = -1.000000000,\ z1 = -1.000000000,\ z5 = 0.2679491924\},$
$\{z3 = 1.000000000,\ z2 = 1.000000000,\ z1 = 1.000000000,\ z0 = 1.000000000,$
$z4 = -3.732050808,\ z5 = -0.2679491924\}, \{z4 = 1.000000000,$
$z0 = -0.2679491924,\ z5 = -3.732050808,\ z3 = 1.000000000,\ z2 = 1.000000000,$
$z1 = 1.000000000\}, \{z4 = 1.000000000,\ z3 = 1.000000000,\ z2 = 1.000000000,$
$z1 = 1.000000000,\ z5 = -0.2679491924,\ z0 = -3.732050808\}, \{z2 = -1.000000000,$
$z3 = -1.000000000,\ z0 = -1.000000000,\ z1 = -1.000000000,\ z5 = 0.2679491924,$
$z4 = 3.732050808\}, \{z2 = -1.000000000,\ z0 = 0.2679491924,\ z4 = -1.000000000,$
$z3 = -1.000000000,\ z1 = -1.000000000,\ z5 = 3.732050808\}, \{z2 = -1.000000000,$
$z3 = -1.000000000,\ z0 = -1.000000000,\ z1 = -1.000000000,\ z4 = 0.2679491924,$
$z5 = 3.732050808\}, \{z4 = -3.732050807,\ z2 = 1.000000000,\ z1 = 1.000000000,$
$z0 = 1.000000000,\ z3 = -0.2679491924,\ z5 = 1.000000000\}, \{$
$z4 = -0.2679491925,\ z2 = 1.000000000,\ z1 = 1.000000000,\ z0 = 1.000000000,$
$z5 = 1.000000000,\ z3 = -3.732050808\}, \{z4 = 3.732050807,\ z2 = -1.000000000,$
$z5 = -1.000000000,\ z0 = -1.000000000,\ z1 = -1.000000000,\ z3 = 0.2679491924\},$
$\{z4 = 0.2679491925,\ z2 = -1.000000000,\ z5 = -1.000000000,\ z0 = -1.000000000,$
$z1 = -1.000000000,\ z3 = 3.732050808\}, \{z5 = -1.000000000,\ z4 = -1.000000000,$
$z0 = -1.000000000,\ z2 = 3.732050808,\ z3 = 0.2679491925,\ z1 = -1.000000000\}, \{$
$z5 = -1.000000000,\ z4 = -1.000000000,\ z2 = 0.2679491924,\ z0 = -1.000000000,$
$z3 = 3.732050807,\ z1 = -1.000000000\}, \{z4 = 1.000000000,\ z1 = 1.000000000,$
$z0 = 1.000000000,\ z3 = -3.732050807,\ z5 = 1.000000000,\ z2 = -0.2679491924\}, \{$
$z4 = 1.000000000,\ z1 = 1.000000000,\ z0 = 1.000000000,\ z3 = -0.2679491925,$
$z5 = 1.000000000,\ z2 = -3.732050808\}, \{z1 = 0.2679491924,\ z5 = -1.000000000,$
$z4 = -1.000000000,\ z3 = -1.000000000,\ z0 = -1.000000000,\ z2 = 3.732050807\}, \{$
$z4 = 1.000000000,\ z3 = 1.000000000,\ z0 = 1.000000000,\ z5 = 1.000000000,$
$z1 = -3.732050808,\ z2 = -0.2679491925\}, \{z4 = 1.000000000,\ z3 = 1.000000000,$
$z0 = 1.000000000,\ z5 = 1.000000000,\ z1 = -0.2679491924,\ z2 = -3.732050807\}, \{$
$z1 = 3.732050808,\ z5 = -1.000000000,\ z4 = -1.000000000,\ z3 = -1.000000000,$
$z0 = -1.000000000,\ z2 = 0.2679491925\}, \{z4 = 1.000000000,\ z3 = 1.000000000,$
$z2 = 1.000000000,\ z0 = -3.732050808,\ z5 = 1.000000000,\ z1 = -0.2679491925\}, \{$
$z2 = -1.000000000,\ z0 = 3.732050808,\ z5 = -1.000000000,\ z4 = -1.000000000,$
$z3 = -1.000000000,\ z1 = 0.2679491925\}, \{z2 = -1.000000000,\ z0 = 0.2679491924,$
$z5 = -1.000000000,\ z4 = -1.000000000,\ z3 = -1.000000000,\ z1 = 3.732050807\}, \{$
$z4 = 1.000000000,\ z0 = -0.2679491924,\ z3 = 1.000000000,\ z2 = 1.000000000,$
$z5 = 1.000000000,\ z1 = -3.732050807\}]$

```
> nops(sols);
```

$$24$$