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Notes on Grothendieck topologies, fibered categories and descent theory

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Introduction

Descent theory has a somewhat formidable and totally undeserved reputation among algebraic geometers. In fact, it simply says that under certain conditions local homomorphisms between quasicoherent can be constructed locally and then glued together if they satisfy a compatibility condition, while quasicoherent sheaves themselves can be constructed locally and then glued together via isomorphisms that satisfy a cocycle condition.

Of course, if "locally" were to mean "locally in the Zariski topology" this would be a formal statement, certainly useful, but hardly deserving the name of a theory. The point is that "locally" here means locally in the flat topology; and the flat topology is something that is not a topology, but what is called a *Grothendieck topology.* Here the coverings are, essentially, flat surjective maps satisfying a finiteness condition. So there are many more coverings in this topology than in the Zariski coverings, and the statement becomes highly nontrivial.

There is also an abstract notion of "category in which descent theory is possible"; the category of pairs consisting of a scheme and a quasicoherent sheaf on it is an example. These categories are known as *stacks.* The general formalism is quite useful, even outside of moduli theory, where the theory of algebraic stacks has become absolutely central.

The purpose of these notes is to provide an exposition of descent theory, to supplement my lecture on Grothendieck topologies and descent theory in the *Advanced School in Basic Algebraic Geometry*, 7–18 July 2003 at I.C.T.P., with a stress on the general formalism. They are not yet in finished form; section 4.2.3 on descent for morphisms of schemes is still only a rough sketch, and the proof of Lemma 4.12 is still incomplete. In the final version they should contain many more examples and explanatory material, and also a section on group actions and descent along torsors.

They are, undoubtedly, full errors of various levels of gravity.

All of the ideas and the results contained in these notes are due to Grothendieck. There is nothing in here that is not, in some form, either in [**SGA1**] or in [**SGA4**], so I do not claim any originality at all.

We will assume that the reader is acquainted with the language of schemes, at least at the level of [**Hart77**]. I use some concepts that are not contained in [**Hart77**], such that of a morphism locally of finite presentation; but I recall the main properties of these in Chapter 1.

We make heavy use of the categorical language: I assume that the reader is acquainted with the notions of category, functor and natural transformation, equivalence of categories. On the other hand, I do not use any advanced concepts, nor do I use any real results in category theory, with one single exception: the reader should know that a fully faithful essentially surjective functor is an equivalence.

The reader should also recall that a *groupoid* is a category in which every arrow is invertible.

CHAPTER 1

Preliminary notions

1.0.1. Algebraic geometry. In this chapter we recall, without proof, some basic notions of scheme theory that are used in the notes. All rings and algebras will be commutative.

We will follow the terminology of [EGA], with the customary exception of calling a "scheme" what is called there a "prescheme" (in [EGA], a scheme is assumed to be separated).

We start with some finiteness conditions. Recall if B is an algebra over the ring A, we say that B is *finitely presented* if it is the quotient of a polynomial ring $A[x_1, \dots, x_n]$ over A by a finitely generated ideal. If A is noetherian, every finitely generated algebra is finitely presented.

If B is finitely presented over A, whenever we write $B = A[x_1, \dots, x_n]/I$, I is always finitely generated in $A[x_1, \dots, x_n]$ ([**EGA IV**, Proposition 1.4.4]).

DEFINITION 1.1 (See [**EGA IV**, 1.4.2]). A morphism of schemes $f\colon X \to Y$ is *locally of finite presentation* if for any $x \in X$ there are affine neighborhoods U of x in X and V of $f(x)$ in V such that $f(U) \subseteq V$, and $\mathcal{O}(U)$ is finitely presented over $\mathcal{O}(V)$.

Clearly, if Y is locally noetherian, then f is locally of finite presentation if and only if it is locally of finite type.

DEFINITION 1.2 ([**EGA IV**, 1.4]).

(i) If $f\colon X \to Y$ is locally of finite presentation, U and V are an open affine subsets of X and Y respectively, and $f(U) \subseteq V$, then $\mathcal{O}(U)$ is finitely presented over $\mathcal{O}(V)$.

(ii) The composition of morphisms locally of finite presentation is locally of finite presentation.

(iii) Given a cartesian diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Y \end{tikzcd}$$

if $X \to Y$ is locally of finite presentation, so is $X' \to Y'$.

DEFINITION 1.3 (See [**EGA I**, 6.6.1]). A morphism of schemes $X \to Y$ is *quasi-compact* if the inverse image in X of a quasi-compact open subscheme of Y is quasi-compact.

A scheme is quasi-compact if and only if it is the finite union of open affine subschemes; using this, it is easy to prove the following.

PROPOSITION 1.4 ([**EGA I**, Proposition 6.6.4]).
Let $f\colon X \to Y$ be a morphism of schemes. The following are equivalent.

(i) *f is quasi-compact.*
(ii) *The inverse image of an open affine subscheme of Y is quasi-compact.*
(iii) *There exists a covering $Y = \cap_i V_i$ by open affine subschemes, such that the inverse image in X of each V_i is quasi-compact.*

PROPOSITION 1.5 ([**EGA I**, 6.6]).

(i) *The composition of quasi-compact morphisms is quasi-compact.*
(ii) *Given a cartesian diagram*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y \end{array}$$

if $X \to Y$ is quasi-compact, so is $X' \to Y'$.

Let us turn to flat morphism.

DEFINITION 1.6. A morphism of schemes $f\colon X \to Y$ is *flat* if for any $x \in X$, the local ring $\mathcal{O}_{X,x}$ is flat as a module over $\mathcal{O}_{Y,f(x)}$.

PROPOSITION 1.7 ([**EGA IV**, Proposition 2.1.2]). *Given a morphism of schemes $f\colon X \to Y$, the following are equivalent.*

(i) *f is flat.*
(ii) *For any $x \in X$, there are affine neighborhoods U of x in X and V of $f(x)$ in V such that $f(U) \subseteq V$, and $\mathcal{O}(U)$ is finitely presented over $\mathcal{O}(V)$.*
(iii) *For any open affine subsets U in X and V in V such that $f(U) \subseteq V$, $\mathcal{O}(U)$ is flat over $\mathcal{O}(V)$.*

PROPOSITION 1.8 ([**EGA IV**, 2.1]).

(i) *The composition of flat morphisms is flat.*
(ii) *Given a cartesian diagram*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y \end{array}$$

if $X \to Y$ is flat, so is $X' \to Y'$.

DEFINITION 1.9. A morphism of schemes $f\colon X \to Y$ is *faithfully flat* if it is flat and surjective.

Let B be an algebra over A. We say that B is faithfully flat if the associated morphism of schemes $\operatorname{Spec} B \to \operatorname{Spec} A$ is faithfully flat.

PROPOSITION 1.10 ([**Mat86**, Theorems 7.2 and 7.3]). *Let B be an algebra over A. The following are equivalent.*

(i) *B is faithfully flat over A.*
(ii) *A sequence of A-modules $M' \to M \to M''$ is exact if and only if the induced sequence of B-modules $M' \otimes_A B \to M \otimes_A B \to M'' \otimes_A B$ is exact.*
(iii) *B is flat over A, and if M is a module over A with $M \otimes_A B = 0$ we have $M = 0$.*
(iv) *B is flat over A, and $\mathfrak{m}B \neq B$ for all maximal ideals $\mathfrak{m}$ of A.*

The following fact is very important.

PROPOSITION 1.11 ([**EGA IV**, Proposition 2.4.6]). *A flat morphism that is locally of finite presentation is open.*

This is not true in general for flat morphisms that are not locally of finite presentation; however, we have a weaker version of this fact.

PROPOSITION 1.12 ([**EGA IV**, Corollaire 2.3.12]). *If $f \colon X \to Y$ is a faithfully flat morphism, a subset of Y is open if and only its inverse image in X is open in X.*

In other words, Y has the topology induced by that of X.

PROPOSITION 1.13 ([**EGA IV**, Proposition 2.7.1]). *Suppose that we have a cartesian diagram of schemes*

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

in which $Y' \to Y$ is faithfully flat, and either quasi-compact or locally of finite presentation. Suppose that $X' \to Y'$ has one of the following properties:

(a) *separated,*
(b) *quasi-compact,*
(c) *locally of finite presentation,*
(d) *proper,*
(e) *affine,*
(f) *finite,*
(g) *flat,*
(h) *smooth,*
(i) *unramified,*
(j) *étale.*

Then $X \to Y$ has the same property.

In [EGA] all these statements are prove when $Y' \to Y$ is quasi-compact. Using Proposition 1.11 it is not hard to prove the statement also when $Y' \to Y$ is locally of finite presentation.

1.0.2. Category theory. We will assume that the reader is familiar with the concepts of category, functor and natural transformation. The standard reference in category theory is [**MaL**].

We will not distinguish between small and large categories. More generally, we will ignore any set-theoretic difficulties.

If $F\colon \mathcal{A} \to \mathcal{B}$ is a functor, recall that F is called *fully faithful* when for any two objects A and A' of $\mathcal{A}$, the function

$$\mathrm{Hom}_{\mathcal{A}}(A, A') \longrightarrow \mathrm{Hom}_{\mathcal{B}}(FA, FA')$$

induced by F is a bijection. F is called *essentially surjective* if every object of $\mathcal{B}$ is isomorphic to the image of an object of $\mathcal{A}$.

Recall also that F is called an *equivalence* when there exists a functor $G\colon \mathcal{B} \to \mathcal{A}$, such that the composition $GF\colon \mathcal{A} \to \mathcal{A}$ is isomorphic to $\mathrm{id}_{\mathcal{A}}$, and $FG\colon \mathcal{B} \to \mathcal{B}$ is isomorphic to $\mathrm{id}_{\mathcal{B}}$.

The composition of two equivalences is again an equivalence. In particular, "being equivalent" is an equivalence relation among categories.

The following fact will be used very frequently.

PROPOSITION 1.14. *A functor is an equivalence if and only if it is both fully faithful and essentially surjective.*

We will also make considerable use of the notions of fibered product and cartesian diagram in an arbitrary category.

Also, we will manipulate some cartesian diagrams. In particular the reader will encounter diagrams of the type

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[d] \\ A \arrow[r] & B \arrow[r] & C \end{tikzcd};$$

we will say that this is cartesian when both squares are cartesian. This is equivalent to saying that the right hand square and the square

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & C' \arrow[d] \\ A \arrow[r] & C \end{tikzcd},$$

obtained by composing the rows, are cartesian. There will be other statements of the type "there is a cartesian diagram ...". These should all be straightforward to check.

For any category $\mathcal{C}$ and any object X of $\mathcal{C}$ we denote by $(\mathcal{C}/X)$ the *comma category*, whose objects are arrows $U \to X$ in $\mathcal{C}$, and whose arrows are commutative diagrams

$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & & V \arrow[dl] \\ & X & \end{tikzcd}.$$

In what follows there will usually be a base category $\mathcal{C}$. We will always assume that $\mathcal{C}$ has fiber products.

CHAPTER 2

Contravariant functors

2.1. Representable functors and the Yoneda lemma

2.1.1. Representable functors. Let us start by recalling a few basic notions of category theory.

Let $\mathcal{C}$ be a category; we will always assume that $\mathcal{C}$ has fiber products. Consider functors from $\mathcal{C}^{\mathrm{op}}$ to (Set). These are the objects of a category, denoted by

$$\mathrm{Func}\big(\mathcal{C}^{\mathrm{op}}, (\mathrm{Set})\big),$$

in which the arrows are the natural transformations. From now on we will refer to natural transformations of contravariant functors on $\mathcal{C}$ as *morphisms*.

Let X be an object of $\mathcal{C}$. There is a contravariant functor

$$\mathrm{h}_X \colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$$

to the category of sets, which sends an object U of $\mathcal{C}$ to the set

$$\mathrm{h}_X U = \mathrm{Hom}_{\mathcal{C}}(U, X).$$

If $\alpha \colon U' \to U$ is an arrow in $\mathcal{C}$, then $\mathrm{h}_X\alpha \colon \mathrm{h}_X U \to \mathrm{h}_X U'$ is defined to be composition with α.

Now, an arrow $f \colon X \to Y$ yields a function $\mathrm{h}_f U \colon \mathrm{h}_X U \to \mathrm{h}_X U$ for each object U of $\mathcal{C}$, obtained by composition with f. The important fact is that this is a morphism $\mathrm{h}_X \to \mathrm{h}_Y$, that is, for all arrows $\alpha \colon U' \to U$ the diagram

$$\begin{array}{ccc}
\mathrm{h}_X U & \xrightarrow{\mathrm{h}_f U} & \mathrm{h}_Y U \\
\downarrow{\scriptstyle \mathrm{h}_X \alpha} & & \downarrow{\scriptstyle \mathrm{h}_Y \alpha} \\
\mathrm{h}_X U' & \xrightarrow{\mathrm{h}_f U'} & \mathrm{h}_Y U'
\end{array}$$

commutes.

Sending each object X of $\mathcal{C}$ to h_X, and each arrow $f \colon X \to Y$ of $\mathcal{C}$ to $\mathrm{h}_f \colon \mathrm{h}_X \to \mathrm{h}_Y$ defines a functor $\mathcal{C} \to \mathrm{Func}\big(\mathcal{C}^{\mathrm{op}}, (\mathrm{Set})\big)$.

YONEDA LEMMA (WEAK VERSION). *Let X and Y be objects of $\mathcal{C}$. The function*

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{Hom}(\mathrm{h}_X, \mathrm{h}_Y)$$

that sends $f \colon X \to Y$ to $\mathrm{h}_f \colon \mathrm{h}_X \to \mathrm{h}_Y$ is bijective.

In other words, the functor $\mathcal{C} \to \mathrm{Func}\big(\mathcal{C}^{\mathrm{op}}, (\mathrm{Set})\big)$ is fully faithful. It fails to be an equivalence of categories, because in general it will not be essentially surjective. This means that not every functor $\mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$ is isomorphic to a functor of the form h_X. However, if we restrict to the full subcategory of $\mathrm{Func}\big(\mathcal{C}^{\mathrm{op}}, (\mathrm{Set})\big)$ consisting of functors $\mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$ which are isomorphic to a functor of the form h_X, we do get a category which is equivalent to $\mathcal{C}$.

DEFINITION 2.1. A *representable functor* on the category $\mathcal{C}$ is a functor

$$F\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$$

which is isomorphic to a functor of the form h_X for some object X of $\mathcal{C}$.

If this happens, we say that *F is represented by X*.

Given two isomorphisms $F \simeq \mathrm{h}_X$ and $F \simeq \mathrm{h}_Y$, we have that the resulting isomorphism $\mathrm{h}_X \simeq \mathrm{h}_Y$ comes from a unique isomorphism $X \simeq Y$ in $\mathcal{C}$, because of the weak form of Yoneda's lemma. Hence *two objects representing the same functor are canonically isomorphic.*

2.1.2. Yoneda's lemma. The condition that a functor be representable can be given a new expression with the more general version of Yoneda's Lemma. Let X be an object of $\mathcal{C}$ and $F\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$ a functor. Given a natural transformation $\tau\colon \mathrm{h}_X \to F$, one gets an element $\xi \in FX$, defined as the image of the identity map $\mathrm{id}_X \in \mathrm{h}_X X$ via the function $\tau_X\colon \mathrm{h}_X X \to FX$. This construction defines a function $\mathrm{Hom}(\mathrm{h}_X, F) \to FX$.

Conversely, given an element $\xi \in FX$, one can define a morphism $\tau\colon \mathrm{h}_X \to F$ as follows. Given an object U of $\mathcal{C}$, an element of $\mathrm{h}_X U$ is an arrow $f\colon U \to X$; this arrow induces a function $Ff\colon FX \to FU$. We define a function $\tau U\colon \mathrm{h}_X U \to FU$ by sending $f \in \mathrm{h}_X U$ to $Ff(\xi) \in FU$. It is straightforward to check that the τ that we have defined is in fact a morphism. In this way we have defined functions

$$\mathrm{Hom}(\mathrm{h}_X, F) \longrightarrow F(X)$$

and

$$F(X) \longrightarrow \mathrm{Hom}(\mathrm{h}_X, F).$$

YONEDA LEMMA. *These two functions are inverse to each other, and therefore establish a bijective correspondence*

$$\mathrm{Hom}(\mathrm{h}_X, F) \simeq FX.$$

The proof is easy and left to the reader. Yoneda's lemma is not a deep fact, but its importance cannot be overestimated.

Let us see how this form of Yoneda's lemma implies the weak form above. Suppose that $F = \mathrm{h}_Y$: the function $\mathrm{Hom}(X,Y) = \mathrm{h}_Y X \to \mathrm{Hom}(\mathrm{h}_X, \mathrm{h}_Y)$ constructed here sends each arrow $f\colon X \to Y$ to

$$\mathrm{h}_Y f(\mathrm{id}_Y) = \mathrm{id}_Y \circ f\colon X \to Y,$$

so it is exactly the function $\mathrm{Hom}(X,Y) \to \mathrm{Hom}(\mathrm{h}_X, \mathrm{h}_Y)$ appearing in the weak form of the result.

One way to think about Yoneda's lemma is as follows. The weak form says that the category $\mathcal{C}$ is embedded in the category $\mathrm{Func}\big(\mathcal{C}^{\mathrm{op}}, (\mathrm{Set})\big)$. The strong version says that, given a functor $F\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$, this can be extended to the representable functor $\mathrm{h}_F\colon \mathrm{Func}\big(\mathcal{C}^{\mathrm{op}}, (\mathrm{Set})\big) \to (\mathrm{Set})$.

We can use Yoneda's lemma to give a very important characterization of representable functors.

DEFINITION 2.2. Let $F\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$ be a functor. A *universal object* for F is a pair (X, ξ) consisting of an object X of $\mathcal{C}$, and an element $\xi \in FX$, with the property that for each object U of $\mathcal{C}$ and each $\sigma \in FU$, there is a unique arrow $f\colon U \to X$ such that $Ff(\xi) = \sigma \in FU$.

In other words: the pair (X, ξ) is a universal object if the morphism $\mathrm{h}_X \to F$ defined by ξ is an isomorphism. Since every natural transformation $\mathrm{h}_X \to F$ is defined by some object $\xi \in FX$, we get the following.

PROPOSITION 2.3. *A functor* $F\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$ *is representable if and only if it has a universal object.*

Also, if F has a universal object (X, ξ), then is represented by X.

Yoneda's lemma insures that the natural functor $\mathcal{C} \to \mathrm{Func}\big(\mathcal{C}^{\mathrm{op}}, (\mathrm{Set})\big)$ which sends an object X to the functor h_X is an equivalence of $\mathcal{C}$ with the category of representable functors. From now on we will not distinguish between an object X and the functor h_X it represents. So, if X and U are objects of $\mathcal{C}$, we will write $X(U)$ for the set $\mathrm{h}_X U = \mathrm{Hom}_{\mathcal{C}}(U, X)$ of arrows $U \to X$. Furthermore, if X is an object and $F\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$ is a functor, we will also identify the set $\mathrm{Hom}(X, F) = \mathrm{Hom}(\mathrm{h}_X, F)$ of morphisms from h_X to F with FX.

2.1.3. Examples. Here are some examples of representable functors.

(i) Consider the functor $\mathrm{P}\colon (\mathrm{Set})^{\mathrm{op}} \to (\mathrm{Set})$ that sends each set S into the set $\mathrm{P}(S)$ of subsets of S. If $f\colon S \to T$ is a function, then $\mathrm{P}(f)\colon \mathrm{P}(T) \to \mathrm{P}(S)$ is defined by $\mathrm{P}(f)\tau = f^{-1}\tau$ for all $\tau \subseteq T$.

Given a subset $\sigma \subseteq S$, there is a unique function $\chi_\sigma\colon S \to \{0,1\}$ such that $\chi_\sigma^{-1}(\{1\}) = \sigma$, namely the *characteristic function*, defined by

$$\chi_\sigma(s) = \begin{cases} 1 & \text{if } s \in \sigma \\ 0 & \text{if } s \notin \sigma. \end{cases}$$

Hence the pair $(\{0,1\}, \{1\})$ is a universal object, and the functor P is represented by $\{0,1\}$.

(ii) This example is similar to the previous one. Consider the category (Top) of all topological spaces, with the arrows being given by continuous functions. Define a functor $\mathrm{F}\colon (\mathrm{Top})^{\mathrm{op}} \to (\mathrm{Set})$ sending each topological space S to the collection $\mathrm{F}(S)$ of all its closed subspaces. Endow $\{0,1\}$ with the coarsest topology in which the subset $\{1\} \subseteq \{0,1\}$ is closed; the closed subsets in this topology are $\emptyset$, $\{1\}$ and $\{0,1\}$. A function $S \to \{0,1\}$ is continuous if and only if $f^{-1}(\{1\})$ is closed in S, and so one sees that the pair $(\{0,1\}, \{1\})$ is a universal object for this functor.

(iii) The next example may look similar, but the conclusion is very different. Let (HausTop) be the category of all Hausdorff topological spaces, and consider the restriction $\mathrm{F}\colon (\mathrm{HausTop}) \to (\mathrm{Set})$ of the functor above. I claim that this functor is not representable.

In fact, assume that (X, ξ) is a universal object. Let S be any set, considered with the discrete topology; by definition, there is a unique function $f\colon S \to X$ with $f^{-1}\xi = S$, that is, a unique function $S \to \xi$. This means that ξ can only have one element. Analogously, there is a unique function $S \to X \setminus \xi$, so $X \setminus \xi$ also has a unique element. But this means that X is a Hausdorff space with two elements, so it must have the discrete topology; hence ξ is also open in X. Hence, if S is any topological space with a closed subset σ that is not open, there is no continuous function $f\colon S \to X$ with $f^{-1}\xi = \sigma$.

(iv) Take (Grp) to be the category of groups, and consider the functor $\mathrm{Sgr}\colon (\mathrm{Grp})^{\mathrm{op}} \to (\mathrm{Set})$ that associates to each subgroup G the set of all its subgroups. If $f\colon G \to H$ is a group homomorphism, we take $\mathrm{Sgr}\, f\colon \mathrm{Sgr}\, H \to \mathrm{Sgr}\, G$ to be the function associating with each subgroup of H its inverse image in G.

This is not representable: there does not exist a group Γ, together with a subgroup $\Gamma_1 \subseteq \Gamma$, with the property that for all groups G with a subgroup $G_1 \subseteq G$, there is a unique homomorphism $f\colon G \to \Gamma$ such that $f^{-1}\Gamma_1 = G_1$. This can be checked in several ways; for example, if we take the subgroup $\{0\} \subseteq \mathbb{Z}$, there should be a unique homomorphism $f\colon \mathbb{Z} \to \Gamma$ such that $f^{-1}\Gamma_1 = \{0\}$. But given one such f, then the homomorphism $\mathbb{Z} \to \Gamma$ defined by $n \mapsto f(2n)$ also has this property, and is different, so this contradicts unicity.

(v) Here is a much more sophisticated example. Let (Hot) be the category of all CW complexes, with the arrows being given by continuous functions modulo homotopy. There is a functor $\mathrm{H}^n\colon (\mathrm{Hot}) \to (\mathrm{Set})$ that sends a CW complex S into its n^{th} cohomology group $\mathrm{H}^n(S, \mathbb{Z})$. Then it is a highly nontrivial fact that this functor is represented by a CW complex, known as a Eilenberg–Mac Lane space, usually denoted by $\mathrm{K}(\mathbb{Z}, n)$.

But we are really interested in algebraic geometry, so let's give some examples in this context. Let $S = \operatorname{Spec} R$ (this is only for simplicity of notation, if S is not affine, nothing substantial changes).

Example 2.4. Consider the affine line $\mathbb{A}^1_S$ over a base scheme S. We have a functor

$$\mathcal{O}\colon (\mathrm{Sch}/S)^{\mathrm{op}} \to (\mathrm{Set})$$

that sends each scheme S to the ring of global sections $\mathcal{O}(S)$. Then $x \in \mathcal{O}(\mathbb{A}^1_S)$, and given a scheme S over S, and an element $f \in \mathcal{O}(S)$, there is a unique morphism $S \to \mathbb{A}^1_S$ such that the pullback of x to S is precisely f. This means that the functor $\mathcal{O}$ is represented by $\mathbb{A}^1_S$.

More generally, the affine space $\mathbb{A}^n_S$ represents the functor $\mathcal{O}^n$ that sends each scheme S into the ring $\mathcal{O}(S)^n$.

EXAMPLE 2.5. Now we look at $\mathbb{G}_{\mathrm{m},S} = \mathbb{A}^1_S \setminus 0_S$. Here by 0_S we mean the image of the zero-section $S \to \mathbb{A}^1_S$. Now, a morphism of S-schemes $\mathbb{G}_{\mathrm{m},S} \to S$ is determined by the image of $x \in \mathcal{O}(\mathbb{G}_{\mathrm{m},S})$ in $\mathcal{O}(S)$; therefore $\mathbb{G}_{\mathrm{m},S}$ represents the functor $\mathcal{O}^*\colon (\mathrm{Sch}/^{\mathrm{op}}) \to (\mathrm{Set})$ that sends each scheme S to the group $\mathcal{O}^*(S)$ of invertible sections of the structure sheaf.

A much more subtle example is given by projective spaces.

EXAMPLE 2.6. On the projective space $\mathbb{P}^n_S = \operatorname{Proj} R[x_0, \dots, x_n]$ there is a line bundle $\mathcal{O}(1)$, with n sections $x_1, \dots, x_n$ which generate it.

Suppose that S is a scheme, and consider the set of sequences

$$(\mathcal{L}, s_0, \dots, s_n),$$

where $\mathcal{L}$ is an invertible sheaf on S, $s_0, \dots, s_n$ sections of $\mathcal{L}$ that generate it. We say that $(\mathcal{L}, s_0, \dots, s_n)$ is equivalent to $(\mathcal{L}', s'_0, \dots, s'_n)$ if there exists an isomorphism of invertible sheaves $\phi\colon \mathcal{L} \simeq \mathcal{L}'$ carrying each s_i into s'_i. Notice that, since the s_i generate $\mathcal{L}$, if ϕ exists than it is unique.

One can consider a function $Q_n\colon (\mathrm{Sch}/ \to)(\mathrm{Set})$ that associates to each scheme S the set of sequences $(\mathcal{L}, s_0, \dots, s_n)$ as above, modulo equivalence. If $f\colon T \to S$ is a morphism, and $(\mathcal{L}, s_0, \dots, s_n) \in F(S)$, then there are sections $f^*s_0, \dots, f^*s_n$ of $f^*\mathcal{L}$ that generate it; this gives the structure of a functor to Q_n.

Another description of the functor Q_n is as follows. Given a scheme Q_n and a sequence $(\mathcal{L}, s_0, \dots, s_n)$ as above, the s_i define a homomorphism $\mathcal{O}_S^{n+1} \to \mathcal{L}$, and the fact that the s_i generate is equivalent to the fact that this homomorphism is surjective. Then two sequences are equivalent if and only if the represent the same quotient of $\mathcal{O}^n_S$.

It is a very well known fact, and, indeed, one of the founding stones of algebraic geometry, that for any sequence $(\mathcal{L}, s_0, \dots, s_n)$ over a scheme S, there is exists a unique morphism $f\colon S \to \mathbb{P}^n_S$ such that $(\mathcal{L}, s_0, \dots, s_n)$ is equivalent to $(f^*\mathcal{O}(1), f^*x_0, \dots, f^*x_n)$. This means precisely that $\mathbb{P}^n_S$ represents the functor Q_n.

EXAMPLE 2.7. A generalization of the previous examples is given by grassmannians. Suppose that $\mathcal{E}$ is a locally free coherent sheaf on S, and fix a non-negative integer r. Here we are not going to assume that S is affine. Consider the functor $\mathbb{G}(r, \mathcal{E})\colon (\mathrm{Sch}/^{\mathrm{op}}) \to (\mathrm{Set})$ that sends each scheme $s\colon S \to S$ over S to the set of all locally free quotients of rank r of the pullback $s^*\mathcal{E}$. If $f\colon T \to S$ is a morphism from $t\colon T \to S$ to $s\colon S \to S$, and $\phi\colon s^*\mathcal{E} \twoheadrightarrow Q$ is an object of $\mathbb{G}(r, \mathcal{E})(S)$, then

$$f^*\phi\colon t^*\mathcal{E} = f^*s^*\mathcal{E} \twoheadrightarrow f^*Q$$

is an object of $\mathbb{G}(r, \mathcal{E})(T)$.

If $\mathcal{E}$ is the trivial locally free sheaf $\mathcal{O}^n_S$, we denote $\mathbb{G}(r, \mathcal{O}^n_S)$ by $\mathbb{G}(r, n)$.

Notice that in the previous example we have that $\mathbb{G}(1, \mathcal{O}^{n+1})$ is the functor Q_n represented by $\mathbb{P}^n_S$.

REMARK 2.8. There is a dual version of Yoneda's lemma, that will be used in Example 3.15. Each object X of $\mathcal{C}$ defines a functor

$$\mathrm{Hom}_{\mathcal{C}}(X, -)\colon \mathcal{C} \to (\mathrm{Set}).$$

This can be viewed as the functor $\mathrm{h}_X\colon (\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \to (\mathrm{Set})$; hence, from the usual form of Yoneda's lemma applied to $\mathcal{C}^{\mathrm{op}}$ for any two objects X and Y we get a canonical bijective correspondence between $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ and the set of natural transformations $\mathrm{Hom}_{\mathcal{C}}(Y, -) \to \mathrm{Hom}_{\mathcal{C}}(X, -)$.

2.1.4. Group objects and actions. In this section, as usual, the category $\mathcal{C}$ will have fiber products; we will also assume that it has a final object pt.

DEFINITION 2.9. A *group object* of $\mathcal{C}$ is an object G of $\mathcal{C}$, together with a functor $\mathcal{C}^{\mathrm{op}} \to (\mathrm{Grp})$ into the category of groups, whose composition with the forgetful functor $(\mathrm{Grp}) \to (\mathrm{Set})$ equals h_G.

Equivalently: a group object is an object G, together with a group structure on $G(U)$ for each object U of $\mathcal{C}$, so that the function $f^*\colon G(V) \to G(U)$ associated with an arrow $f\colon U \to V$ in $\mathcal{C}$ is always a homomorphism of groups.

This can be restated using Yoneda's lemma.

PROPOSITION 2.10. *To give a group scheme structure on an object G of $\mathcal{C}$ is equivalent to assigning three arrows $\mathrm{m}_G\colon G \times G \to G$ (the multiplication), $\mathrm{i}_G\colon G \to G$ (the inverse), and $\mathrm{e}_G\colon \mathrm{pt} \to G$ (the identity), such that the following diagrams commute.*

(i) *The identity is a left and right identity:*

$$\begin{tikzcd} \mathrm{pt} \times G \arrow[r, "\mathrm{e}_G \times \mathrm{id}_G"] \arrow[dr, equal] & G \times G \arrow[d, "\mathrm{m}_G"] \\ & G \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} G \times \mathrm{pt} \arrow[r, "\mathrm{id}_G \times \mathrm{e}_G"] \arrow[dr, equal] & G \times G \arrow[d, "\mathrm{m}_G"] \\ & G \end{tikzcd}$$

(ii) *Multiplication is associative:*

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mathrm{m}_G \times \mathrm{id}_G"] \arrow[d, "\mathrm{id}_G \times \mathrm{m}_G"] & G \times G \arrow[d, "\mathrm{m}_G"] \\ G \times G \arrow[r, "\mathrm{m}_G"] & G \end{tikzcd}$$

(iii) *The inverse is a left and right inverse:*

$$\begin{tikzcd} G \arrow[r, "\mathrm{i}_G \times \mathrm{id}_G"] \arrow[d] & G \times G \arrow[d, "\mathrm{m}_G"] \\ \mathrm{pt} \arrow[r, "\mathrm{e}_G"] & G \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} G \arrow[r, "\mathrm{id}_G \times \mathrm{i}_G"] \arrow[d] & G \times G \arrow[d, "\mathrm{m}_G"] \\ \mathrm{pt} \arrow[r, "\mathrm{e}_G"] & G \end{tikzcd}$$

PROOF. It is immediate to check that, if $\mathcal{C}$ is the category of sets, the commutativity of the diagram above gives the usual group axioms. Hence the result follows by evaluating the diagrams above (considered as diagrams of functors) at any object U of $\mathcal{C}$. ♠

Thus, for example, a group object in the category of topological spaces is simply a group, that has a structure of a topological space, such that the multiplication map and the inverse map are continuous (of course the identity map is automatically continuous).

There is an obvious notion of left action of a functor into groups on a functor into sets.

DEFINITION 2.11. A left action α of a functor $G\colon \mathcal{C} \to (\mathrm{Grp})$ on a functor $F\colon \mathcal{C} \to (\mathrm{Set})$ is a natural transformation $G \times F \to F$, such that for any object U of $\mathcal{C}$, the induced function $G(U) \times F(U) \to F(U)$ is an action of the group $G(U)$ on the set $F(U)$.

Equivalently, an action of G on F consists of an action of $G(U)$ on $F(U)$ for all objects U of $\mathcal{C}$, such that for any arrow $f\colon U \to V$ in $\mathcal{C}$, any $g \in G(V)$ and any $x \in F(V)$ we have

$$f^*g \cdot f^*x = f^*(g \cdot x) \in F(U).$$

Right actions are defined analogously.

We define an action of a group object G on an object X as an action of the functor $\mathrm{h}_G\colon \mathcal{C} \to (\mathrm{Grp})$ on $\mathrm{h}_X\colon \mathcal{C} \to (\mathrm{Set})$.

Again, we can reformulate this definition in terms of diagrams.

PROPOSITION 2.12. *To give a left action of a group object G on an object X is equivalent to assigning an arrow $\rho\colon G\times X \to X$, such that the following diagrams commute.*

(i) *The identity of G acts like the identity on X:*

$$\begin{tikzcd} \mathrm{pt} \times X \arrow[r, "\mathrm{e}_G \times \mathrm{id}_X"] \arrow[dr, equal] & G \arrow[d, "\rho"] \\ & X \end{tikzcd}$$

(ii) *The action is associative with respect to the multiplication on G:*

$$\begin{tikzcd} G \times G \times X \arrow[r, "\mathrm{m}_G \times \mathrm{id}_X"] \arrow[d, "\mathrm{id}_G \times \rho"] & G \times X \arrow[d, "\rho"] \\ G \times X \arrow[r, "\rho"] & X \end{tikzcd}$$

PROOF. It is immediate to check that, if $\mathcal{C}$ is the category of sets, the commutativity of the diagram above gives the usual axioms for a left action. Hence the result follows by evaluating the diagrams above (considered as diagrams of functors) at any object U of $\mathcal{C}$. ♠

2.2. Relative representability

2.2.1. Fiber products of functors. The category $\mathrm{Func}(\mathcal{C}^{\mathrm{op}}, (\mathrm{Set}))$ has fiber products. These are defined as follows. Suppose that we are given three functors F_1, F_2 and G from $\mathcal{C}^{\mathrm{op}}$ to (Set), together with two natural transformations $\alpha_1\colon F_1 \to G$ and $\alpha_2\colon F_2 \to G$. The fiber product $F_1 \times_G F_2$ sends each object U of $\mathcal{C}$ into the fiber product of sets $F_1U \times_{GU} F_2U$, where of course the functions $F_1U \to GU$ and $F_1U \to GU$ are induced respectively by α_1 and α_2. The action of $F_1 \times_G F_2$ on arrows is defined in the obvious fashion.

Since the category $\mathrm{Func}(\mathcal{C}^{\mathrm{op}}, (\mathrm{Set}))$ has terminal object, the functor that sends each object to a fixed set with one element, it also has products, defined by the usual formula $(F_1 \times F_2)U = F_1U \times F_2U$.

If $X_1 \to Y$ and $X_2 \to Y$ are arrows in $\mathcal{C}$, $\mathrm{h}_{X_1} \to \mathrm{h}_Y$ and $\mathrm{h}_{X_2} \to \mathrm{h}_Y$ are the induced morphisms, then the fiber product $X_1 \times_Y X_2$ represents the fiber product $\mathrm{h}_{X_1} \times_{\mathrm{h}_Y} \mathrm{h}_{X_2}$; so we can write $X_1 \times_Y X_2$ to mean either the fiber product as an object of $\mathcal{C}$ or the fiber product of contravariant functors on $\mathcal{C}$.

2.2.2. Representable natural transformations.

DEFINITION 2.13. Let F and G be functors in $\mathrm{Func}(\mathcal{C}^{\mathrm{op}}, (\mathrm{Set}))$. A morphism of functors $\phi\colon F \to G$ is *representable* if for any object Y of $\mathcal{C}$ and any morphism $Y \to G$, the fiber product $F \times_G Y$ is representable.

Equivalently, the morphism τ is representable if whenever $H \to Y$ is a morphism and H is representable, so is $F \times_G H$.

PROPOSITION 2.14. *If $\tau\colon F \to G$ is a morphism of contravariant functors $\mathcal{C} \to$ (Set) and G is representable, then τ is representable if and only if F is representable.*

PROOF. Since the category $\mathcal{C}$ has fibered products, the fiber products of two representable functors is representable; hence if F is representable so is the morphism τ.

Conversely, if τ is representable, so is the fiber product $F \times_G G \simeq F$. ♠

DEFINITION 2.15. Let $\mathbf{P}$ be a property of arrows in $\mathcal{C}$. We say that $\mathbf{P}$ is *stable* if whenever

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

is a cartesian diagram and f has $\mathbf{P}$, then f' also has $\mathbf{P}$.

Examples of stable properties of continuous maps are being an embedding, being injective, being surjective, being a local homeomorphism, being open, being a covering map. Being closed, on the other hand, is not a stable property.

Any stable property of arrows in $\mathcal{C}$ can be extended to a property of representable morphisms. If $\mathbf{P}$ is a stable property of arrows in $\mathcal{C}$, we say that a representable morphism $F \to G$ has $\mathbf{P}$ if whenever $Y \to G$ is a morphism, with Y an object of $\mathcal{C}$, then the projection $F \times_G Y \to Y$ has $\mathbf{P}$. This makes sense, because $F \times_G Y$ is representable.

Consider a functor $F \colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$; there is morphism $\delta_F \colon F \to F \times F$, the *diagonal of* F, defined by sending each object U of $\mathcal{C}$ into the diagonal function $FU \to FU \times FU = (F \times F)U$.

PROPOSITION 2.16. *Let* $F \colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$ *be a functor. Then the following three conditions are equivalent.*

(i) *The diagonal* $\delta_F \colon F \to F \times F$ *is representable.*
(ii) *If* $X \to F$ *and* $Y \to F$ *are morphisms, where* X *and* Y *are objects of* $\mathcal{C}$, *then the fiber product* $X \times_F Y$ *is representable.*
(iii) *All morphisms from representable functors into* F *are representable.*

PROOF. Parts (ii) and (iii) are equivalent by the definition of a representable morphism.

Assume that the diagonal $\delta_F \colon F \to F \times F$ is representable, and that $X \to F$ and $Y \to F$ are morphisms from objects of $\mathcal{C}$. It is a standard fact that there is a cartesian square

$$\begin{tikzcd} X \times_F Y \arrow[r] \arrow[d] & X \times Y , \arrow[d] \\ F \arrow[r, "\delta_F"] & F \times F \end{tikzcd}$$

which shows that the fiber product $X \times_F Y$ is representable. Hence (ii) holds.

Conversely, assume that (ii) holds, and that there is given a morphism $X \to F \times F$, where X is an object of $\mathcal{C}$. There is another cartesian diagram

$$\begin{tikzcd} F \times_{F \times F} X \arrow[r] \arrow[d] & X \arrow[d, "\delta_X"] \\ X \times_F X \arrow[r] \arrow[d] & X \times X \arrow[d] \\ F \arrow[r, "\delta_F"] & F \times F \end{tikzcd}$$

showing that $F \times_{F \times F} X$ is representable, as required. ♠

2.3. Sheaves in Grothendieck topologies

2.3.1. Grothendieck topologies. Now we need the notion of a sheaf on the category (Top). Consider a functor $F \colon (\mathrm{Top})^{\mathrm{op}} \to (\mathrm{Set})$; for each topological space X we can consider the restriction F_X to the subcategory of (Top) whose objects are open subspaces of X, and whose arrows are the inclusion maps; this is a presheaf on X. We say that F is a *sheaf* on (Top) if F_X is a sheaf on X for all X.

For later use we are going to need the more general notion of sheaf in a Grothendieck topology; in this section we review this theory.

In a Grothendieck topology the "open sets" of a space are *maps* into this space; instead of intersections we have to look at fiber products, while unions play no role. The axioms do not describe the "open sets", but the coverings of a space.

DEFINITION 2.17. Let $\mathcal{C}$ be a category with fiber products. A *Grothendieck topology* on $\mathcal{C}$ is the assignment to each object U of $\mathcal{C}$ of a collection of sets of arrows $\{V_i \to U\}$, called *coverings of* U, so that the following conditions are satisfied.

(i) If $V \to U$ is an isomorphism, then the set $\{V \to U\}$ is a covering.
(ii) If $\{V_i \to U\}$ is a covering and $U' \to U$ is any arrow, then the collection of projections $\{V_i \times_U U' \to U'\}$ is a covering.
(iii) If $\{V_i \to U\}$ is a covering, and for each index i we have a covering $\{W_{ij} \to V_i\}$ (here j varies on a set depending on i), the collection of compositions $\{W_{ij} \to V_i \to U\}$ is a covering of U.

A category with a Grothendieck topology is called a *site*.

Notice that from (ii) and (iii) it follows that if $\{V_i \to U\}$ and $\{W_j \to U\}$ are two coverings of the same object, then $\{V_i \times_U W_j \to U\}$ is also a covering.

REMARK 2.18. In fact what we have defined here is what is called a *pretopology* in [**SGA4**]; a pretopology defines a topology, and very different pretopologies can define the same topology. The point is that the sheaf theory only depends on the topology, and not on the pretopology. So, for example, if two pretopologies on the same category satisfy the conditions of Proposition 2.30 below, the two induced topologies are the same, so the conclusion follows immediately.

Despite its unquestionable technical advantages, I do not find the notion of topology, as defined in [**SGA4**], very intuitive, so I prefer to avoid its use (just a question of habit, undoubtedly).

Here are some examples.

EXAMPLE 2.19 (The site of a topological space). Let X be a fixed topological space; call X_{cl} the category in which the objects are the open subsets of X, and the arrows are given by inclusions. Then we get a Grothendieck topology on X_{cl} by associating with each open subset $U \subseteq X$ the set of open coverings of U.

In this case if $V_1 \to U$ and $V_2 \to U$ are arrows, the fiber product $V_1 \times_U V_2$ is the intersection $V_1 \cap V_2$.

EXAMPLE 2.20 (The global classical topology). Here $\mathcal{C}$ is the category (Top) of topological spaces. If U is a topological space, then a covering of U will be a collection of open embeddings $V_i \to U$ whose images cover U.

Notice here we must interpret "open embedding" as meaning an open continuous injective map $V \to U$; if by an open embedding we mean the inclusion of an open subspace, then condition (i) of Definition 2.17 is not satisfied.

EXAMPLE 2.21 (The global étale topology for topological spaces). Here $\mathcal{C}$ is the category (Top) of topological spaces. If U is a topological space, then a covering of U will be a collection of local homeomorphisms $V_i \to U$ whose images cover U.

Here are the basic examples in algebraic geometry. A scheme is endowed with the Zariski topology, so it yields a site, according to Example 2.19; but of course, if this where the only significant example, the formalism of Grothendieck topologies would be useless.

EXAMPLE 2.22 (The small étale site of a scheme). Consider a scheme X. We can form a category $X_{\text{ét}}$, the full subcategory of the category (Sch/X) whose objects are morphism $U \to X$ that are locally of finite presentation and étale.

A covering $U_i \to U$ is a collection of morphisms of X-schemes whose images cover U. Recall that if U and each of the U_i is locally of finite presentation and étale over X, then each of the morphisms $U_i \to U$ is locally of finite presentation and étale, hence it has an open image.

Here are four topologies that one can put on the category (Sch/S) of schemes over a fixed scheme S. Several more have been used in different contexts.

EXAMPLE 2.23 (The global Zariski topology). Here a covering $\{U_i \to U\}$ is a collection of open embeddings covering U. As in the example of the global classical topology, an open embedding must be defined as a morphism $V \to U$ that gives an isomorphism of V with an open subscheme of U, and not simply as the embedding of an open subscheme.

EXAMPLE 2.24 (The global étale topology). A covering $\{U_i \to U\}$ is a collection of étale maps of finite presentation whose images cover U.

EXAMPLE 2.25 (The fpqc topology). The coverings $\{U_i \to U\}$ are collections of flat morphisms, such that the induced morphism $\coprod U_i \to U$ is quasi-compact and surjective.

In other words, a fpqc covering $\{U_i \to U\}$ is a collection of flat quasi-compact morphisms, whose images cover U, such that for each affine subset $V \subseteq U$ the inverse image of V in U_i is not empty only for finitely many i.

EXAMPLE 2.26 (The fppf topology). A covering $\{U_i \to U\}$ is a collection of flat maps locally of finite presentation whose images cover U.

The acronyms fppf and fpqc stand for "fidèlement plat et de présentation finie" and "fidèlement plat et quasi-compact".

The fppf topology is finer than the étale topology, which is turn finer than the fppf topology. On the other hand the fpqc topology can not be compared with any of the others.

2.3.2. Sheaves. If X is a topological space, a presheaf of sets on X is a functor $X_{\mathrm{cl}}{}^{\mathrm{op}} \to (\mathrm{Set})$, where X_{cl} is the category of open subsets of X, as in Example 2.19. The condition that F be a sheaf can easily be generalized to any site, provided that we substitute intersections, that do not make sense, with fiber products.

DEFINITION 2.27. Let $\mathcal{C}$ be a site, $F\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$ a functor.

(i) F is *separated* if, given a covering $\{U_i \to U\}$ and two sections a and b in FU whose pullbacks to each FU_i coincide, it follows that $a = b$.

(ii) F is a *sheaf* if the following condition is satisfied. Suppose that we are given a covering $\{U_i \to U\}$ in $\mathcal{C}$, and a set of sections $a_i \in FU_i$. Call $\mathrm{pr}_1\colon U_i \times_U U_j \to U_i$ and $\mathrm{pr}_2\colon U_i \times_U U_j \to U_j$ the first and second projection respectively, and assume that $\mathrm{pr}_1^* a_i = \mathrm{pr}_2^* a_j \in F(U_i \times_U U_j)$ for all i and j. Then there is a unique section $a \in FU$ whose pullback to FU_i is a_i for all i.

If F and G are sheaves on a site $\mathcal{C}$, a *morphism of sheaves* $F \to G$ is simply a natural transformation of functors.

Of course one can also define sheaves of groups, rings, and so on, as usual: a functor from $\mathcal{C}^{\mathrm{op}}$ to the category of groups, or rings, is a sheaf if its composition with the forgetful functor to the category of sets is a sheaf.

The reader might find our definition of sheaf rather pedantic, and wonder why we did not simply say "assume that the pullbacks of a_i and a_j to $F(U_i \times_U U_j)$ coincide". The reason is the following: when $i = j$, in the classical case of a topological space we have $U_i \times_U U_i = U_i \cap U_i = U_i$, so the two possible pullbacks from $U_i \times_U U_i \to U_i$ coincide; but if the map $U_i \to U$ is not injective, then the two projections $U_i \times_U U_i \to U_i$ will be different. So, for example, in the classical case coverings with one subset are not interesting, and the sheaf condition is automatically verified for them, while in the general case this is very far from being true.

A sheaf on a site is clearly separated.

An alternative way to state the condition that F is a sheaf is the following.

Let A, B and C be sets, and suppose that we are given a diagram

$$A \xrightarrow{f} B \underset{h}{\overset{g}{\rightrightarrows}} C.$$

(that is, we are given a function $f\colon A \to B$ and two functions $f, g\colon B \to C$). We say that the diagram is an *equalizer* if f is injective, and maps A surjectively onto the subset $\{b \in B \mid g(b) = h(b)\} \subseteq B$.

Equivalently, the diagram is an equalizer if $g \circ f = h \circ f$, and every function $p\colon D \to B$ such that $g \circ p = h \circ p$ factors uniquely through A.

Now, take a functor $F\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$ and a covering $\{U_i \to U\}$ in $\mathcal{C}$. There is a diagram

$$FU \longrightarrow \prod_i FU_i \overset{\mathrm{pr}_1^*}{\underset{\mathrm{pr}_2^*}{\rightrightarrows}} \prod_{i,j} F(U_i \times_U U_j) \tag{2.3.1}$$

where the function $FU \to \prod_i FU_i$ is induced by the restrictions $FU \to FU_i$, while

$$\mathrm{pr}_1^*\colon \prod_i FU_i \longrightarrow \prod_{i,j} F(U_i \times_U U_j)$$

sends an element $(a_i) \in \prod_i FU_i$ into the element $\mathrm{pr}_1^*(a_i) \in \prod_{i,j} F(U_i \times_U U_j)$ whose component in $F(U_i \times_U U_j)$ is the pullback $\mathrm{pr}_1^* a_i$ of a_i along the first projection $U_i \times_U U_j \to U_i$. The function

$$\mathrm{pr}_2^*\colon \prod_i FU_i \longrightarrow \prod_{i,j} F(U_i \times_U U_j)$$

is defined similarly.

One immediately sees that F is a sheaf if and only if the diagram (2.3.1) is an equalizer for all coverings $\{U_i \to U\}$ in $\mathcal{C}$.

There is an interesting characterization of sheaves, that has been taken as the definition in [**ML-Mo92**]. Given a covering $\mathcal{U} = \{U_i \to U\}$ in $\mathcal{C}$, we define a subfunctor $\mathrm{h}_{\mathcal{U}} \subseteq \mathrm{h}_X$, by taking $\mathrm{h}_{\mathcal{U}}(T)$ to be the set of arrows $T \to U$ with the property that for some i there is a factorization $T \to U_i \to U$. In technical terms, $\mathrm{h}_{\mathcal{U}}$ is the *sieve* associated with the covering $\mathcal{U}$. Then we have the following fact.

PROPOSITION 2.28. *A functor $F\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$ is a sheaf if and only if for any covering $\mathcal{U} = \{U_i \to U\}$ in $\mathcal{C}$, the induced function*

$$FU \simeq \mathrm{Hom}(\mathrm{h}_U, F) \longrightarrow \mathrm{Hom}(\mathrm{h}_{\mathcal{U}}, F)$$

is bijective. Furthermore, F is separated if and only if this function is always injective.

In fact, it is not hard to see that $\mathrm{Hom}(\mathrm{h}_{\mathcal{U}}, F)$ is in a bijective correspondence with the set of elements (ξ_i) of the product $\prod_i F(U_i)$, such that for any pair of indices i and j the restrictions of ξ_i and ξ_j to U_{ij} coincide.

Sometimes two different topologies on the same category define the same sheaves.

DEFINITION 2.29. Let $\mathcal{C}$ be a category, $\{U_i \to U\}$ a set of arrows. A *refinement* $\{V_j \to U\}$ is a set of arrows such that for each index j there is some index i such that $V_j \to U$ factors through U_i.

PROPOSITION 2.30. *Let $\mathcal{T}$ and $\mathcal{T}'$ be two Grothendieck topologies on the same category $\mathcal{C}$. Suppose that every covering in $\mathcal{T}$ is also in $\mathcal{T}'$, and that every covering in $\mathcal{T}'$ has a refinement in $\mathcal{T}$. Then a functor $\mathcal{C} \to (\mathrm{Set})$ is a sheaf in the topology $\mathcal{T}$ if and only if it is a sheaf on the topology $\mathcal{T}'$.*

In particular, the sheaves on (Top) in the classical, the étale topology and the local fibration topology are the same.

PROOF. Since $\mathcal{T}'$ contains all the coverings of $\mathcal{T}$, clearly any functor $F\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$ that is a sheaf in the topology $\mathcal{T}'$ is also a sheaf in $\mathcal{T}$. On the other hand, assume that $F\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$ is a sheaf in the topology $\mathcal{T}$, and take a covering $\{U_i \to U\}$ of an object U in the topology $\mathcal{T}'$. There is a refinement $\{V_j \to U\}$ of $\{U_i \to U\}$ in $\mathcal{T}$; for each index j choose a factorization $V_j \to U_{\iota_j} \to U$. If two section of FU coincide when pulled back to each FU_i they also coincide when pulled back to each FV_j, and therefore they coincide; hence the functor F is separated in the topology $\mathcal{T}'$.

Now, assume that we are given a collection of sections $\{a_i\} \in \prod_i FU_i$, such that the pullbacks $\mathrm{pr}_1^* a_i$ and $\mathrm{pr}_2^* a_{i'}$ to $F(U_i \times_U U_{i'})$ coincide for all indices i and i'. For each j call b_j the pullback of a_{ι_j} to V_j through the arrow $V_j \to U_{\iota_j}$. I claim that for every pair of indices j and j' the pullbacks of b_j and $b_{j'}$ to $V_j \times_U V_{j'}$ coincide. In fact, the composition of $\mathrm{pr}_1\colon V_j \times_U V_{j'} \to V_j$ with the arrow $V_j \to U_{\iota_j}$ factors through $\mathrm{pr}_1\colon U_{\iota_j} \times U_{\iota_{j'}} \to U_{\iota_j}$; and analogously for the second projection. Since the pullbacks $\mathrm{pr}_1^* a_{\iota_j}$ and $\mathrm{pr}_2^* a_{\iota_{j'}}$ to $F(U_i \times_U U_{i'})$ coincide, the thesis follows.

Since F is a sheaf in $\mathcal{T}$, there will exist some a in FU whose pullback to FV_j is b_j for all j. Now we need to show that the pullback of a to FU_i is a_i for all i. For each j and i there is a commutative diagram

$$\begin{array}{ccccc} V_j \times_U U_i & \longrightarrow & U_{\iota_j} \times_U U_i & \longrightarrow & U_i \;; \\ \downarrow & & \downarrow & & \downarrow \\ V_j & \longrightarrow & U_{\iota_j} & \longrightarrow & U \end{array}$$

since the pullbacks of a_{ι_j} and a_i to $F(U_{\iota_j} \times_U U_i)$ coincide, this shows that the pullbacks of $a \in FU$ and $a_i \in FU_i$ to $F(V_j \times_U U_i)$ are the same. But $\{V_j \times_U U_i \to U_i\}$ is a covering of U_i in the topology $\mathcal{T}'$, and since F is separated in the topology $\mathcal{T}'$ we conclude that in fact the pullback of a to FU_i equals a_i. ♠

DEFINITION 2.31. A topology $\mathcal{T}$ on a category $\mathcal{C}$ is called *saturated* if, whenever $\{U_i \to U\}$ is a set of arrows, $\{V_{ij} \to U_i\}$ is a covering of U_i for each i, and the set $\{V_{ij} \to U\}$ of compositions is a covering, then $\{U_i \to U\}$ is a covering.

If $\mathcal{T}$ is a topology of $\mathcal{C}$, the *saturation* of $\mathcal{T}$ is the set $\widetilde{\mathcal{T}}$ of all sets of arrows $\{U_i \to U\}$ with the property that there exists a covering $\{V_{ij} \to U_i\}$ for each i such that the set $\{V_{ij} \to U\}$ of compositions is a covering.

PROPOSITION 2.32. *The saturation $\widetilde{\mathcal{T}}$ of a topology $\mathcal{T}$ is a saturated topology. Furthermore, a functor $\mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$ is a sheaf under $\widetilde{\mathcal{T}}$ if and only if it is a sheaf under $\mathcal{T}$.*

PROOF. We leave it as an exercise for the reader to prove that $\widetilde{\mathcal{T}}$ is a saturated topology. The last statement follows from Proposition 2.30. ♠

EXAMPLE 2.33. The global étale topology on (Top) is a saturated topology. It is the saturation of the classical topology; hence the global étale topology and the classical topology have the same sheaves.

In the category (Sch/S) the étale topology and the fppf topology are both saturated. On the other hand, in the category of schemes an étale morphism is not Zariski-locally an open embedding, hence the global étale topology is not the saturation of the global Zariski topology.

2.3.3. Sheaf conditions on representable functors.

PROPOSITION 2.34. *A representable functor* $(\mathrm{Top})^{\mathrm{op}} \to (\mathrm{Set})$ *is a sheaf in the classical topology.*

The proof is straightforward. It is similarly easy to show that a representable functor in the category (Sch/S) over a base scheme S is a sheaf in the Zariski topology. On the other hand the following is not straightforward.

PROPOSITION 2.35 (Grothendieck). *A representable functor in* (Sch/S) *is a sheaf in the fpqc and in the fppf topologies.*

So, in particular, it is also a sheaf in the étale topology.

DEFINITION 2.36. A topology $\mathcal{T}$ on a category $\mathcal{C}$ is called *subcanonical* if every representable functor in $\mathcal{C}$ is a sheaf with respect to $\mathcal{T}$.

A *subcanonical site* is a category endowed with a subcanonical topology.

There are examples of sites that are not subcanonical, but I have never had dealings with any of them.

The name "subcanonical" comes from the fact that on a category $\mathcal{C}$ there is a topology, known as the *canonical topology*, which is the finest topology in which every representable functor is a sheaf. We will not be needing this fact.

PROOF OF PROPOSITION 2.35. We will use the following useful criterion.

LEMMA 2.37. *Let S be a scheme, $F\colon (\mathrm{Sch}/S)^{\mathrm{op}} \to (\mathrm{Set})$ a functor. Suppose that F satisfies the following two conditions.*

(i) *F is a sheaf in the global Zariski topology.*

(ii) *Whenever $V \to U$ is a flat surjective morphism of affine S-schemes, the diagram*

$$FU \longrightarrow FV \underset{\mathrm{pr}_2^*}{\overset{\mathrm{pr}_1^*}{\rightrightarrows}} F(V \times_U V)$$

is an equalizer.

Then F is a sheaf in both the fppf and the fpqc topology.

Conversely, one sees easily that if F is a sheaf in both the fppf and the fpqc topology, then it satisfies the conditions of the Proposition.

PROOF. Take a covering $\{U_i \to U\}$ of schemes over S, in either the fppf or the fpqc topology, and set $V = \coprod_i U_i$. The induced morphism $V \to U$ is flat, surjective and either of finite presentation (in the case of the fppf topology) or quasi-compact (in the case of the fpqc topology). Since F is a Zariski sheaf, the function $FV \to \prod_i FU_i$ induced by restrictions is an isomorphism. We have a commutative diagram of functions

$$\begin{tikzcd}
FU \arrow[r] \arrow[d] & FV \arrow[r, shift left, "\mathrm{pr}_1^*"] \arrow[r, shift right, "\mathrm{pr}_2^*"'] \arrow[d] & F(V \times_U V) \arrow[d] \\
FU \arrow[r] & \prod_i FU_i \arrow[r, shift left, "\mathrm{pr}_1^*"] \arrow[r, shift right, "\mathrm{pr}_2^*"'] & \prod_{i,j} F(U_i \times_U U_j)
\end{tikzcd}$$

where the columns are bijections; hence to show that the bottom row is an equalizer it is enough to show that the top row is an equalizer. In other words, we have shown that it is enough to consider coverings $\{V \to U\}$ consisting of a single morphism. Similarly, to check that F is separated we may limit ourselves to considering coverings consisting of a single morphism.

This argument also shows that if $\{U_i \to U\}$ is a finite covering, such that U and the U_i are affine, then the diagram

$$FU \longrightarrow \prod_i FU_i \overset{\mathrm{pr}_1^*}{\underset{\mathrm{pr}_2^*}{\rightrightarrows}} \prod_{i,j} F(U_i \times_U U_j)$$

is an equalizer. In fact, in this case the finite disjoint union $\coprod_i U_i$ is also affine.

Now we are given a morphism $f \colon V \to U$ that is flat and surjective, and either quasi-compact or locally finitely presented.

LEMMA 2.38. *We can write U as an union of affine subschemes $U = \cup_{i \in I} U_i$, in such a way that for each i there is a* finite *number of open affine subscheme V_{ia} of in $f^{-1}U_i$, so that that the V_{ia}, taken all together, form a covering of V, and $\{V_{ia} \to U_i\}$ is a covering for all i.*

PROOF. This is clear in the quasi-compact case: just write U as a union of affine subschemes $U = \cup_i U_i$; then each inverse image $f^{-1}U_i$ is quasi-compact, hence it is the union of finitely many open affine subscheme V_{ia}.

In the finitely presented case one may need infinitely many affines to cover each $f^{-1}U_i$; but the projection $f^{-1}U_i \to U_i$ is open, while U_i is quasi-compact, so we can choose finitely many open affine subscheme V_{ia} that cover U_i. These, taken all together, need not cover all of V; but we can change the set of indices, defining $\overline{I}$ to be the set of pairs (i, L), where $i \in I$ and L is a finite set of open affine subschemes of V whose images cover

U_i. The projection $\overline{I} \to I$ is surjective, and we obtain a covering with the desired properties by setting $U = \cap_{(i,L)\in\overline{I}} U_i$ and $\{V_{(i,L),a}\} = L$. ♠

Now consider the diagram

$$\begin{tikzcd}
FU \arrow[r] \arrow[d] & FV \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & F(V \times_U V) \arrow[d] \\
\prod_i FU_i \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \prod_i \prod_a FV_{ia} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \prod_i \prod_{a,b} F(V_{ia} \times_U V_{ib}) \\
\prod_{i,j} F(U_i \cap U_j) \arrow[r] & \prod_{i,j} \prod_{a,b} F(V_{ia} \cap V_{jb}) &
\end{tikzcd}$$

Its columns are equalizers, because F is a sheaf in the Zariski topology. On the other hand, the second row is an equalizer, because each diagram

$$FU_i \longrightarrow \prod_k FV_{ia} \rightrightarrows \prod_{a,b} F(V_{ia} \times_U V_{ib})$$

is an equalizer, and a product of equalizers is easily seen to be an equalizer. Hence the restriction function $FU \to FV$ is injective, so F is separated. But this implies that the bottom row is injective, and with an easy diagram chasing one shows that the top row is exact. ♠

To prove Proposition 2.35 we need to check that if $F = h_X$, where X is an S-scheme, then the second condition of the Proposition is satisfied. First of all, let us notice that it is enough to prove the result in case $S = \operatorname{Spec} \mathbb{Z}$, that is, when (Sch/S) is simply the category of all schemes.

In fact, suppose that the result holds for $S = \operatorname{Spec} \mathbb{Z}$; we need to show that for any fppf of fpqc covering $\{U_i \to U\}$ of S-schemes the sequence

$$\operatorname{Hom}_S(U, X) \longrightarrow \prod_i \operatorname{Hom}_S(U_i, X) \rightrightarrows \prod_{i,j} \operatorname{Hom}_S(U_i \times_U U_i, X)$$

is an equalizer. The injectivity of the function

$$\operatorname{Hom}_S(U, X) \longrightarrow \prod_i \operatorname{Hom}_S(U_i, X)$$

is clear, since $\operatorname{Hom}_S(U, X)$ injects into $\operatorname{Hom}_(U, X)$, $\prod_i \operatorname{Hom}_S(U_i, X)$ injects into $\prod_i \operatorname{Hom}(U_i, X)$, and $\operatorname{Hom}_(U, X)$ injects into $\prod_i \operatorname{Hom}(U_i, X)$, because $\operatorname{Hom}(-, X)$ is a sheaf. On the other hand, let us suppose that we are given an element (a_i) of $\prod_i \operatorname{Hom}_S(U_i, X)$, with the property that for all pairs i, j of indices $\operatorname{pr}_1^* a_i = \operatorname{pr}_2^* a_j$ in $\operatorname{Hom}_S(U_i \times_U U_j, X)$. Then there exists a morphism $a \in \operatorname{Hom}(U, X)$ such that the composition $U_i \to U \xrightarrow{a} X$ coincides with a_i for all, and we only have to check that a is a morphism of S-schemes. But the composition $U_i \to U \xrightarrow{a} X \to S$ coincides with the structure morphism $U_i \to S$ for all i; since $\operatorname{Hom}(-, S)$ is a sheaf on the category of schemes. so that $\operatorname{Hom}(U, S)$ injects into $\prod_i \operatorname{Hom}(U_i, S)$, this implies that $U \xrightarrow{a} X \to S$ is the structure morphism of U, and this completes the proof.

So for the rest of the proof we only need to work with morphism of schemes, without worrying about base schemes. We will assume at first that X is affine. Set $U = \operatorname{Spec} A$, $V = \operatorname{Spec} B$, $X = \operatorname{Spec} R$. In this case the result is an easy consequence of the following lemma. Consider the ring homomorphism $f\colon A \to B$ corresponding to the morphism $V \to U$, and the two homomorphism of A-algebras $e_0, e_1\colon B \to B \otimes_A B$ defined by $e_1(b) = b \otimes 1$ and $e_2(b) = 1 \otimes b$; these correspond to the two projections $V \times_U V \to V$.

LEMMA 2.39. *The sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{e_1 - e_2} B \otimes_A B$$

is exact.

PROOF. The injectivity of f is clear, because B is faithfully flat over A. Also, it is clear that the image of f is contained in the kernel of $e_1 - e_2$, so we have only to show that the kernel of $e_1 - e_2$ is contained in the image of f.

Assume that there exists a homomorphism of A-algebras $g\colon B \to A$ (in other words, assume that the morphism $V \to U$ has a section). Then the composition $g \circ f\colon A \to A$ is the identity. Take an element $b \in \ker(e_0 - e_1)$; by definition, this means that $b \otimes 1 = 1 \otimes b$ in $B \otimes_A B$. By applying the homomorphism $g \otimes \mathrm{id}_B\colon B \otimes_A B \to A \otimes_A B = B$ to both members of the equality we obtain that $f(gb) = b$, hence $b \in \operatorname{im} f$.

In general, there will be no section $U \to V$; however, suppose that there exists a faithfully flat A algebra $A \to A'$, such that the homomorphism $f \otimes \mathrm{id}_{A'}\colon A' \to B \otimes A'$ obtained by base change has a section $B \otimes A' \to A'$ as before. Set $B' = B \otimes A'$. Then there is a natural isomorphism of A'-algebras $B' \otimes_{A'} B' \simeq (B \otimes_A B) \otimes_A A'$, making the diagram

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r, "f\otimes \mathrm{id}_{A'}"] \arrow[d, equal] & B' \arrow[r, "e_1' - e_2'"] \arrow[d, equal] & B' \otimes_{A'} B' \arrow[d] \\
0 \arrow[r] & A' \arrow[r, "f\otimes \mathrm{id}_{A'}"] & B' \arrow[r, "(e_1 - e_2)\otimes \mathrm{id}_{A'}"] & (B \otimes_A B) \otimes_A A'
\end{tikzcd}$$

commutative. The bottom row is exact, because of the existence of a section, and so the top row is exact.

But to find such homomorphism $A \to A'$ it is enough to set $A' = B$; the product $B \otimes_A A' \to A'$ defined by $b \otimes b' \mapsto bb'$ gives the desired section. In geometric terms, the diagonal $V \to V \times_U V$ gives a section of the first projection $V \times_U V \to V$. ♠

To finish the proof of Proposition 2.35 in the case that X is affine, recall that morphisms of schemes $U \to X$, $V \to X$ and $V \times_U V \to X$ correspond to ring homomorphisms $R \to A$, $R \to B$ and $R \to B \otimes_A B$; then the result is immediate from the lemma above. This proves that h_X is a sheaf when X is affine.

If X is not necessarily affine, write $X = \cup_i X_i$ as a union of affine open subschemes.

First of all let us show that h_X is separated. Given a covering $V \to U$, take two morphisms $f, g\colon U \to X$ such that the two compositions $V \to U \to X$ are equal. Since $V \to U$ is surjective, f and g coincide set-theoretically, so we can set $U_i = f^{-1}X_i = g^{-1}X_i$, and call V_i the inverse image of U_i in V. The two compositions

$$V_i \longrightarrow U_i \overset{f|_{U_i}}{\underset{g|_{U_i}}{\rightrightarrows}} X_i$$

coincide, and X_i is affine; hence $f\mid_{U_i} = g\mid_{U_i}$ for all i, so $f = g$, as desired.

To complete the proof, suppose that $g\colon V \to X$ is a morphism with the property that the two compositions

$$V \times_U V \overset{\mathrm{pr}_1}{\underset{\mathrm{pr}_2}{\rightrightarrows}} V \xrightarrow{g} X$$

are equal; we need to show that g factors through U. First of all, I claim that g factors through U set-theoretically.

For this, take two points v_1 and v_2 with the same image u in U. I claim that there exists $w \in V \times_U V$ such that $\mathrm{pr}_1(w) = v_1$ and $\mathrm{pr}_2(w) = v_2$. For this, consider the extensions $k(u) \subseteq k(v_1)$ and $k(u) \subseteq k(v_2)$; the tensor product $k(v_1) \otimes_{k(u)} k(v_2)$ is not 0, hence it has a maximal ideal. If we call K the quotient field, K is an extension of $k(u)$ containing both $k(v_1)$ and $k(v_2)$. The two compositions $\operatorname{Spec} K \to \operatorname{Spec} k(v_1) \to V \xrightarrow{g} U$ and $\operatorname{Spec} K \to \operatorname{Spec} k(v_2) \to V \xrightarrow{g} U$ coincide, so get a morphism $\operatorname{Spec} K \to V \times_U V$. We take w to be the image of $\operatorname{Spec} K$ in $V \times_U V$.

But $V \to U$ is surjective, so g factors through U set-theoretically, as claimed. Since U has the quotient topology induced by the morphism $V \to U$ (Proposition 1.12), we get that the resulting function $f\colon U \to X$ is continuous.

Set $U_i = f^{-1}X_i$ and $V_i = g^{-1}V_i$ for all i. The compositions

$$V_i \times_U V_i \overset{\mathrm{pr}_1}{\underset{\mathrm{pr}_2}{\rightrightarrows}} V_i \xrightarrow{g|_{V_i}} V_i \longrightarrow X_i$$

coincide, and X_i is affine, so $g\mid_{V_i}\colon V_i \to X$ factors uniquely through a morphism $f_i\colon U_i \to X_i$. We have

$$f_i\mid_{U_i\cap U_j} = f_j\mid_{U_i\cap U_j}\colon U_i \cap U_j \longrightarrow X,$$

because h_X is separated; hence the f_i glue together to give the desired factorization $V \to U \to X$. ♠

2.3.4. The sheafification of a functor. The usual construction of the sheafification of a presheaf of sets on a topological space carries over to this more general context.

DEFINITION 2.40. Let $\mathcal{C}$ be a site, $F\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$ a functor. A *sheafification* of F is a sheaf $F^{\mathrm{a}}\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$, together with a natural transformation $F \to F^{\mathrm{a}}$, such that:

(i) given an object U of $\mathcal{C}$ and two objects ξ and η of $F(U)$ whose images ξ^{a} and η^{a} in $F^{\mathrm{a}}(U)$ are isomorphic, there exists a covering $\{\sigma_i\colon U_i \to U\}$ such that $\sigma_i^*\xi = \sigma_i^*\eta$, and

(ii) for each object U of $\mathcal{C}$ and each $\overline{\xi} \in F^{\mathrm{a}}(U)$, there exists a covering $\{\sigma_i\colon U_i \to U\}$ and elements $\xi_i \in F(U_i)$ such that $\xi_i^{\mathrm{a}} = \sigma_i^*\xi$.

THEOREM 2.41. *Let $\mathcal{C}$ be a site, $F\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$ a functor.*

(i) *If $F^{\mathrm{a}}\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$ is a sheafification of F, any morphism from F to a sheaf factors uniquely through F^{a}.*

(ii) *There exists a sheafification $F \to F^{\mathrm{a}}$, which is unique up to a canonical isomorphism.*

(iii) *The natural transformation $F \to F^{\mathrm{a}}$ is injective if and only if F is separated.*

SKETCH OF PROOF. For part (i), let $\phi\colon F \to G$ be a natural transformation from F to a sheaf $G\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$.

Let us prove the first part. For each object U of $\mathcal{C}$, we define an equivalence relation $\sim$ on FU as follows. Given two sections a and b in FU, we write $a \sim b$ if there is a covering $U_i \to U$ such that the pullbacks of a and b to each U_i coincide. We check easily that this is an equivalence relation, and we define $F^{\mathrm{s}}U = FU/\sim$. We also verify that if $V \to U$ is an arrow in $\mathcal{C}$, the pullback $FU \to FV$ is compatible with the equivalence relations, yielding a pullback $F^{\mathrm{s}}U \to F^{\mathrm{s}}V$. This defines the functor F^{s} with the surjective morphism $F \to F^{\mathrm{s}}$. It is straightforward to verify that F^{s} is separated, and that every natural transformation from F to a separated functor factors uniquely through F^{s}.

To construct F^{a}, we take for each object U of $\mathcal{C}$ the set of pairs $(\{U_i \to U\}, \{a_i\})$, where $\{U_i \to U\}$ is a covering, and $\{a_i\}$ is a set of sections with $a_i \in F^{\mathrm{s}}U_i$ such that the pullback of a_i and a_j to $F^{\mathrm{s}}(U_i \times_U U_j)$, along the first and second projection respectively, coincide. On this set we impose an equivalence relation, by declaring $(\{U_i \to U\}, \{a_i\})$ to be equivalent to $(\{V_j \to U\}, \{b_j\})$ when the restrictions of a_i and b_j to $F^{\mathrm{s}}(U_i \times_U V_j)$, along the first and second projection respectively, coincide. To verify the transitivity of this relation we need to use the fact that the functor F^{s} is separated.

For each U we call $F^{\mathrm{a}}U$ the set of equivalence classes. If $V \to U$ is an arrow, we define a function $F^{\mathrm{a}}U \to F^{\mathrm{a}}V$ by associating with the class of a pair $(\{U_i \to U\}, \{a_i\})$ in $F^{\mathrm{a}}U$ the class of the pair $(\{U_i \times_U V\}, p_i^*a_i)$, where $p_i\colon U_i \times_U V \to U_i$ is the projection. Once we have checked that this is well defined, we obtain a functor $F^{\mathrm{a}}\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$. There is also a natural transformation $F^{\mathrm{s}} \to F^{\mathrm{a}}$, obtained by sending an element $a \in F^{\mathrm{s}}U$ into $(\{U = U\}, a)$. Then one verifies that F^{a} is a sheaf, and that the composition of the natural transformations $F \to F^{\mathrm{s}}$ and $F^{\mathrm{s}} \to F^{\mathrm{a}}$ has the desired universal property.

The unicity up to a canonical isomorphism follows immediately from part (i). Part (iii) follows easily from the definition. ♠

CHAPTER 3

Fibered categories

3.1. Fibered categories

3.1.1. Definition and first properties. In this section we will fix a category $\mathcal{C}$ with products and fiber products; the topology will play no role. We will study categories over $\mathcal{C}$, that is, categories $\mathcal{F}$ equipped with a functor $\mathrm{p}_{\mathcal{F}} \colon \mathcal{F} \to \mathcal{C}$.

We will draw several commutative diagrams involving objects of $\mathcal{C}$ and $\mathcal{F}$; an arrow going from an object ξ of $\mathcal{F}$ to an object U of $\mathcal{C}$ will be of type "$\xi \mapsto U$", and will mean that $\mathrm{p}_{\mathcal{F}}\xi = U$. Furthermore the commutativity of the diagram

$$\begin{tikzcd}
\xi \arrow[r, "\phi"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\
U \arrow[r, "f"] & V
\end{tikzcd}$$

will mean that $\mathrm{p}_{\mathcal{F}}\phi = f$.

Definition 3.1. Let $\mathcal{F}$ be a category over $\mathcal{C}$. An arrow $\phi \colon \xi \to \eta$ of $\mathcal{F}$ is *cartesian* if for any arrow $\psi \colon \zeta \to \eta$ in $\mathcal{F}$ and any arrow $h \colon \mathrm{p}_{\mathcal{F}}\zeta \to \mathrm{p}_{\mathcal{F}}\xi$ in $\mathcal{C}$ with $\mathrm{p}_{\mathcal{F}}\phi \circ h = \mathrm{p}_{\mathcal{F}}\psi$, there exists a unique arrow $\theta \colon \zeta \to \xi$ with $\mathrm{p}_{\mathcal{F}}\theta = h$ and $\theta \circ \phi = \zeta$, as in the commutative diagram

$$\begin{tikzcd}
\zeta \arrow[rrd, bend left, "\psi"] \arrow[rd, dashed, "\theta"'] \arrow[dd, mapsto] & & \\
& \xi \arrow[r, "\phi"'] \arrow[dd, mapsto] & \eta \arrow[dd, mapsto] \\
\mathrm{p}_{\mathcal{F}}\zeta \arrow[rd, "h"'] \arrow[rrd, bend left] & & \\
& \mathrm{p}_{\mathcal{F}}\xi \arrow[r] & \mathrm{p}_{\mathcal{F}}\eta
\end{tikzcd}$$

If $\xi \to \eta$ is a cartesian arrow of $\mathcal{F}$ mapping to an arrow $U \to V$ of $\mathcal{C}$, we also say that ξ is *a pullback of η to U*.

Remark 3.2. The definition of cartesian arrow we give is more restrictive than the definition in **[SGA1]**; however, the resulting notions of fibered category coincide.

REMARK 3.3. Given two pullbacks $\phi\colon \xi \to \eta$ and $\widetilde{\phi}\colon \widetilde{\xi} \to \eta$ of η to U, the unique arrow $\theta\colon \widetilde{\xi} \to \xi$ that fits into the diagram

$$
\begin{tikzcd}
\widetilde{\xi} \arrow[dr, dashed, "\theta"] \arrow[drr, bend left, "\widetilde{\phi}"] \arrow[dd, mapsto] & & \\
& \xi \arrow[r, "\phi"'] \arrow[dd, mapsto] & \eta \arrow[dd, mapsto] \\
U \arrow[dr, equal] \arrow[drr, bend left] & & \\
& U \arrow[r] & V
\end{tikzcd}
$$

is an isomorphism. In other words, a pullback is unique, up to a unique isomorphism.

The following facts are easy to prove, and are left to the reader.

PROPOSITION 3.4.

(i) *If $\mathcal{F}$ is a category over $\mathcal{C}$, the composition of cartesian arrows in $\mathcal{F}$ is cartesian.*
(ii) *A cartesian arrow of $\mathcal{F}$ whose image in $\mathcal{C}$ is an isomorphism is also an isomorphism.*
(iii) *If $\xi \to \eta$ and $\eta \to \zeta$ are arrows in $\mathcal{F}$ and $\eta \to \zeta$ is cartesian, then $\xi \to \eta$ is cartesian if and only if the composition $\xi \to \zeta$ is cartesian.*
(iv) *Let $\mathrm{p}_{\mathcal{F}}\colon \mathcal{F} \to \mathcal{C}$ and $\mathrm{p}_{\mathcal{G}}\colon \mathcal{G} \to \mathcal{C}$ be categories over $\mathcal{C}$. If $F\colon \mathcal{F} \to \mathcal{G}$ is a functor with $\mathrm{p}_{\mathcal{G}} \circ F = \mathrm{p}_{\mathcal{F}}$, $\xi \to \eta$ is an arrow in $\mathcal{F}$ that is cartesian over its image $F\xi \to F\eta$ in $\mathcal{F}$, and $F\xi \to F\eta$ is cartesian over its image $\mathrm{p}_{\mathcal{G}}\xi \to \mathrm{p}_{\mathcal{G}}\eta$ in $\mathcal{C}$, then $\xi \to \eta$ is cartesian over $\mathrm{p}_{\mathcal{G}}\xi \to \mathrm{p}_{\mathcal{G}}\eta$.*

DEFINITION 3.5. A *fibered category over $\mathcal{C}$* is a category $\mathcal{F}$ over $\mathcal{C}$, such that given an arrow $f\colon U \to V$ in $\mathcal{C}$ and an object η of $\mathcal{F}$ mapping to V, there is a cartesian arrow $\phi\colon \xi \to \eta$ with $\mathrm{p}_{\mathcal{F}}\phi = f$.

If $\mathcal{F}$ and $\mathcal{G}$ are fibered categories over $\mathcal{C}$, then a *morphism of fibered categories* $F\colon \mathcal{F} \to \mathcal{G}$ is a functor such that:

(i) F is *base-preserving*, that is, $\mathrm{p}_{\mathcal{G}} \circ F = \mathrm{p}_{\mathcal{F}}$;
(ii) F sends cartesian arrows to cartesian arrows.

In other words, in a fibered category $\mathcal{F} \to \mathcal{C}$ we can pull back objects of $\mathcal{F}$ along any arrow of $\mathcal{C}$.

Notice that in the definition above the equality $\mathrm{p}_{\mathcal{G}} \circ F = \mathrm{p}_{\mathcal{F}}$ must be interpreted as an actual equality. In other words, the existence of an isomorphism of functors between $\mathrm{p}_{\mathcal{G}} \circ F$ and $\mathrm{p}_{\mathcal{F}}$ is not enough.

PROPOSITION 3.6. *Let $\mathrm{p}_{\mathcal{F}}\colon \mathcal{F} \to \mathcal{C}$ and $\mathrm{p}_{\mathcal{G}}\colon \mathcal{G} \to \mathcal{C}$ be categories over $\mathcal{C}$, $F\colon \mathcal{F} \to \mathcal{G}$ a functor with $\mathrm{p}_{\mathcal{G}} \circ F = \mathrm{p}_{\mathcal{F}}$. Assume that $\mathcal{G}$ is fibered over $\mathcal{C}$.*

(i) *If $\mathcal{F}$ is fibered over $\mathcal{G}$, then it also fibered over $\mathcal{C}$.*
(ii) *If F is an equivalence of categories, then $\mathcal{F}$ is fibered over $\mathcal{C}$.*

PROOF. Part (i) follows from Proposition 3.4 (iv). Part (ii) follows from the easy fact that if F is an equivalence then $\mathcal{F}$ is fibered over $\mathcal{G}$. ♠

3.1.2. Fibered categories as pseudo-functor.

DEFINITION 3.7. Let $\mathcal{F}$ be a fibered category over $\mathcal{C}$. Given an object U of $\mathcal{C}$, the *fiber* $\mathcal{F}(U)$ of $\mathcal{F}$ over U is the subcategory of $\mathcal{F}$ whose objects are the objects ξ of $\mathcal{F}$ with $p_{\mathcal{F}}\xi = U$, and whose arrows are arrows ϕ in $\mathcal{F}$ with $p_{\mathcal{F}}\phi = \mathrm{id}_U$.

By definition, if $F\colon \mathcal{F} \to \mathcal{G}$ is a morphism of fibered categories over $\mathcal{C}$ and U is an object of $\mathcal{C}$, the functor F sends $\mathcal{F}(U)$ to $\mathcal{G}(U)$, so we have a restriction functor $F_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$.

Notice that formally we could give the same definition of a fiber for any functor $p_{\mathcal{F}}\colon \mathcal{F} \to \mathcal{C}$, without assuming that $\mathcal{F}$ is fibered over $\mathcal{C}$. However, we would end up with a useless notion. For example, it may very well happen that we have two objects U and V of $\mathcal{C}$ which are isomorphic, but such that $\mathcal{F}(U)$ is empty while $\mathcal{F}(V)$ is not. This kind of pathology does not arise for fibered categories, and here is why.

Let $\mathcal{F}$ be a category fibered over $\mathcal{C}$, and $f\colon U \to V$ an arrow in $\mathcal{C}$. For each object η over V, we choose a pullback $\phi_\eta\colon f^*\eta \to \eta$ of η to U. We define a functor $f^*\colon \mathcal{F}(V) \to \mathcal{F}(U)$ by sending each object η of $\mathcal{F}(V)$ to $f^*\eta$, and each arrow $\beta\colon \eta \to \eta'$ of $\mathcal{F}(U)$ to the unique arrow $f^*\beta\colon f^*\eta \to f^*\eta'$ in $\mathcal{F}(V)$ making the diagram

$$\begin{tikzcd} f^*\eta \arrow[r] \arrow[d, dashed, "f^*\beta"] & \eta \arrow[d, "\beta"] \\ f^*\eta' \arrow[r] & \eta' \end{tikzcd}$$

commute.

DEFINITION 3.8. A *cleavage* of a fibered category $\mathcal{F} \to \mathcal{C}$ consists of a class K of cartesian arrows in $\mathcal{F}$ such that for each arrow $f\colon U \to V$ in $\mathcal{C}$ and each object η in $\mathcal{F}(V)$ there exists a unique arrow in K with target η mapping to f in $\mathcal{C}$.

By the axiom of choice, every fibered category has a cleavage. Given a fibered category $\mathcal{F} \to \mathcal{C}$ with a cleavage, we associate with each object U of $\mathcal{C}$ a category $\mathcal{F}(U)$, and to each arrow $f\colon U \to V$ a functor $f^*\colon \mathcal{F}(V) \to \mathcal{F}(U)$, constructed as above. It is very tempting to believe that in this way we have defined a functor from $\mathcal{C}$ to the category of categories; however, this is not quite correct. First of all, pullbacks $\mathrm{id}_U^*\colon \mathcal{F}(U) \to \mathcal{F}(U)$ are not necessarily identities. Of course we could just choose all pullbacks along identities to be identities on the fiber categories: this would certainly work, but it is not very natural, as there are often natural defined pullbacks where this does not happen (in Example 3.11 and many others). What happens in general is that, when U is an object of $\mathcal{C}$ and ξ an object of $\mathcal{F}(U)$, we have the pullback $\epsilon_U(\xi)\colon \mathrm{id}_U^*\xi \to \xi$ is an isomorphism, because of Proposition 3.4 (ii), and this defines an isomorphism of functors $\epsilon_U\colon \mathrm{id}_U^* \simeq \mathrm{id}_{\mathcal{F}(U)}$.

A more serious problem is the following. Suppose that we have two arrows $f\colon U \to V$ and $g\colon V \to W$ in $\mathcal{C}$, and an object ζ of $\mathcal{F}$ over W.

Then $f^*g^*\zeta$ is a pullback of ζ to U; however, pullbacks are not unique, so there is no reason why $f^*g^*\zeta$ should coincide with $(gf)^*\zeta$. However, there is a canonical isomorphism $\alpha_{f,g}(\zeta)\colon f^*g^*\zeta \simeq (gf)^*\zeta$ in $\mathcal{F}(U)$, because both are pullbacks, and this gives an isomorphism $\alpha_{f,g}\colon f^*g^* \simeq (gf)^*$ of functors $\mathcal{F}(W) \to \mathcal{F}(U)$.

So, after choosing a cleavage a fibered category almost gives a functor from $\mathcal{C}$ to the category of categories, but not quite. The point is that the category of categories is not just a category, but what is known as a 2-category; that is, its arrows are functors, but two functors between the same two categories in turn form a category, the arrows being natural transformations of functors. Thus there are 1-arrows (functors) between objects (categories), but there are also 2-arrows (natural transformations) between 1-arrows.

What we get instead of a functor is what is called a *pseudo-functor*, or, in a more modern terminology, a *lax 2-functor*.

DEFINITION 3.9. A *pseudo-functor* Φ on $\mathcal{C}$ consists of the following data.

(i) For each object U of $\mathcal{C}$ a category ΦU.
(ii) For each arrow $f\colon U \to V$ a functor $f^*\colon \Phi V \to \Phi U$.
(iii) For each object U of $\mathcal{C}$ an isomorphism $\epsilon_U\colon \mathrm{id}_U^* \simeq \mathrm{id}_{\Phi U}$ of functors $\Phi U \to \Phi U$.
(iv) For each pair of arrows $U \xrightarrow{f} V \xrightarrow{g} W$ an isomorphism $\alpha_{f,g}\colon f^*g^* \simeq (gf)^*\colon \Phi W \to \Phi U$ of functors $\Phi W \to \Phi U$.

These data are required to satisfy the following conditions.

(a) If $f\colon U \to V$ is an arrow in $\mathcal{C}$ and η is an object of ΦV, we have

$$\alpha_{\mathrm{id}_U,f}(\eta) = \epsilon_U(f^*\eta)\colon \mathrm{id}_U^* f^*\eta \to f^*\eta$$

and

$$\alpha_{f,\mathrm{id}_V}(\eta) = f^*\epsilon_V(\eta)\colon f^*\mathrm{id}_V^*\eta \to f^*\eta.$$

(b) Whenever we have arrows $U \xrightarrow{f} V \xrightarrow{g} W \xrightarrow{h} T$ and an object θ of $\mathcal{F}(T)$, the diagram

$$\begin{tikzcd}
f^*g^*h^*\theta \arrow[r, "\alpha_{f,g}(h^*\theta)"] \arrow[d, "f^*\alpha_{g,h}(\theta)"] & (gf)^*h^*\theta \arrow[d, "\alpha_{gh,f}(\theta)"] \\
f^*(hg)^*\theta \arrow[r, "\alpha_{f,hg}(\theta)"] & (hgf)^*\theta
\end{tikzcd}$$

commutes.

We have seen how to associate to a fibered category over $\mathcal{C}$, equipped with a cleavage, the data for a pseudo-functor; we still have to check that the two conditions of the definition are satisfied.

PROPOSITION 3.10. *A fibered category over $\mathcal{C}$ with a cleavage defines a pseudo-functor on $\mathcal{C}$.*

PROOF. We have to check that the two conditions are satisfied. Let us do this for condition (b) (the argument for condition (a) is very similar). The point is that $f^*g^*h^*\zeta$ and $(hgf)^*\zeta$ are both pullbacks of ζ, and so, by the definition of cartesian arrow, there is a unique arrow $f^*g^*h^*\zeta \to (hgf)^*\zeta$ lying over the identity on U, and making the diagram

$$\begin{tikzcd}
f^*g^*h^*\zeta \arrow[rr] \arrow[dr] & & (hgf)^*\zeta \arrow[dl] \\
& \zeta &
\end{tikzcd}$$

commutative. But one sees immediately that both $\alpha_{gh,f}(\zeta) \circ \alpha_{f,g}(h^*\zeta)$ and $\alpha_{f,hg}(\zeta) \circ f^*\alpha_{g,h}(\zeta)$ satisfy this condition. ♠

A functor $\Phi\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Cat})$ from $\mathcal{C}$ into the category of categories gives rise to a pseudo-functor on $\mathcal{C}$, simply by defining all ϵ_U and all $\alpha_{f,g}$ to be identities.

3.1.3. The fibered category associated with a pseudo-functor. Conversely, from a pseudo-functor on $\mathcal{C}$ one gets a fibered category over $\mathcal{C}$ with a cleavage. First of all, let us analyze the case that the pseudo-functor is simply a functor $\Phi\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Cat})$ into the category of categories, considered as a 1-category. This means that with each object U of $\mathcal{C}$ we associate a category ΦU, and for each arrow $f\colon U \to V$ gives a functor $\Phi f\colon \Phi V \to \Phi U$, in such a way that $\Phi \mathrm{id}_U\colon \Phi U \to \Phi U$ is the identity, and $\Phi(g \circ f) = \Phi f \circ \Phi g$ every time we have two composable arrows f and g in $\mathcal{C}$.

To this Φ we can associate a fibered category $\mathcal{F} \to \mathcal{C}$, such that for any object U in $\mathcal{C}$ the fiber $\mathcal{F}(U)$ is canonically equivalent to the category ΦU. An object of $\mathcal{F}$ is a pair (ξ, U) where U is an object of $\mathcal{C}$ and ξ is an object of $\mathcal{F}(U)$. An arrow $(a, f)\colon (\xi, U) \to (\eta, V)$ in $\mathcal{F}$ consists of an arrow $f\colon U \to V$ in $\mathcal{C}$, together with an arrow $a\colon \xi \to \Phi f(\eta)$ in ΦU.

The composition is defined as follows: if $(a, f)\colon (\xi, U) \to (\eta, V)$ and $(b, g)\colon (\eta, V) \to (\zeta, W)$ are two arrows, then

$$(b, g) \circ (a, f) = (\Phi b \circ a, g \circ f)\colon (\xi, U) \to (\zeta, W).$$

There is an obvious functor $\mathcal{F} \to \mathcal{C}$ that sends an object (ξ, U) into U and an arrow (a, f) into f; I claim that this functor makes $\mathcal{F}$ into a fibered category over $\mathcal{C}$. In fact, given an arrow $f\colon U \to V$ in $\mathcal{C}$ and an object (η, V) in $\mathcal{F}(V)$, then $\big(\Phi f(\eta), U\big)$ is an object of $\mathcal{F}(U)$, and it is easy to check that the pair $\big(f, \mathrm{id}_{\Phi f(\eta)}\big)$ gives a cartesian arrow $\big(\Phi f(\eta), U\big) \to (\eta, V)$.

The fiber of $\mathcal{F}$ is canonically equivalent to the category ΦU: the equivalence $\mathcal{F}(U) \to \Phi U$ is obtained at the level of objects by sending (ξ, U) to ξ, and at the level of arrows by sending (a, id_U) to a. The collection of all the arrows of type $\big(f, \mathrm{id}_{\Phi f(\eta)}\big)$ gives a cleavage.

The general case is similar, only much more confusing. Consider a pseudo-functor Φ on $\mathcal{C}$. As before, we define the objects of $\mathcal{F}$ to be pairs (ξ, U) where U is an object of $\mathcal{C}$ and ξ is an object of $\mathcal{F}(U)$. Again, an arrow

$(a, f)\colon (\xi, U) \to (\eta, V)$ in $\mathcal{F}$ consists of an arrow $f\colon U \to V$ in $\mathcal{C}$, together with an arrow $a\colon \xi \to f^*(\eta)$ in ΦU.

Given two arrows $(a, f)\colon (\xi, U) \to (\eta, V)$ and $(b, g)\colon (\eta, V) \to (\zeta, W)$, we define the composition $(b, g) \circ (a, f)$ as the pair $(b \cdot a, gf)$, where $b \cdot a = \alpha_{f,g}(\zeta) \circ f^*b \circ a$ is the composition

$$\xi \xrightarrow{a} f^*\zeta \xrightarrow{f^*b} f^*g^*\zeta \xrightarrow{\alpha_{f,g}(\zeta)} (gf)^*\zeta$$

in ΦU.

Let us check that composition is associative. Given three arrows

$$(\xi, U) \xrightarrow{(a,f)} (\eta, V) \xrightarrow{(b,g)} (\zeta, W) \xrightarrow{(c,h)} (\theta, T)$$

we have to show that

$$(c, h) \circ \big((b, g) \circ (a, f)\big) \stackrel{\text{def}}{=} \big(c \cdot (b \cdot a), hgf\big)$$

equals

$$\big((c, h) \circ (b, g)\big) \circ (a, f) \stackrel{\text{def}}{=} \big((c \cdot b) \cdot a, hgf\big).$$

By the definition of the composition, we have

$$\begin{aligned} c \cdot (b \cdot a) &= \alpha_{gf,h}(\theta) \circ (gf)^*c \circ (b \cdot a) \\ &= \alpha_{gf,h}(\theta) \circ (gf)^*c \circ \alpha_{f,g}(\zeta) \circ f^*b \circ a \end{aligned}$$

while

$$\begin{aligned} (c \cdot b) \cdot a &= \alpha_{f,hg}(\theta) \circ f^*(c \cdot b) \circ a \\ &= \alpha_{f,hg}(\zeta) \circ f^*\alpha_{g,h}(\theta) \circ f^*g^*c \circ f^*b \circ a; \end{aligned}$$

hence it is enough to show that the diagram

$$\begin{tikzcd} f^*g^*\zeta \arrow[r, "f^*g^*c"] \arrow[d, "\alpha_{f,g}(\zeta)"] & f^*g^*h^*\theta \arrow[r, "f^*\alpha_{g,h}(\theta)"] \arrow[d, "\alpha_{gf,h}(\theta)"] & f^*(hg)^*\theta \arrow[d, "\alpha_{f,hg}(\theta)"] \\ (gf)^*\zeta \arrow[r, "(gf)^*c"] & (gf)^*h^*\theta \arrow[r, "\alpha_{gf,h}(\theta)"] & (hgf)^*\theta \end{tikzcd}$$

commutes. But the commutativity of the first square follows from the fact that $\alpha_{f,g}$ is a natural transformation of functor, while that of the second is condition (b) in Definition 3.9.

Given an object (ξ, U) of $\mathcal{F}$, we have the isomorphism $\epsilon_U(\xi)\colon \mathrm{id}_U^*\xi \to \xi$; we define the identity $\mathrm{id}_{(\xi,U)}\colon (\xi, U) \to (\xi, U)$ as $\mathrm{id}_{(\xi,U)} = \big(\epsilon_U(\xi)^{-1}, \mathrm{id}_U\big)$. To check that this is neutral with respect to composition, take an arrow $(a, f)\colon (\xi, U) \to (\eta, V)$; we have

$$(a, f) \circ \big(\epsilon_U(\xi)^{-1}, \mathrm{id}_U\big) = \big(a \cdot \epsilon_U(\xi)^{-1}, f\big)$$

and

$$a \cdot \epsilon_U(\xi)^{-1} = \alpha_{\mathrm{id}_U, f}(f^*\eta) \circ \mathrm{id}_U^* a \circ \epsilon_U(\xi)^{-1}.$$

But condition (a) of Definition 3.9 says that $\alpha_{\mathrm{id}_U, f}(f^*\eta)$ equals $\epsilon_U(f^*\eta)$, while the diagram

$$
\begin{tikzcd}
\mathrm{id}_U^*\xi \arrow[r, "\epsilon_U(\xi)"] \arrow[d, "\mathrm{id}_U^* a"] & \xi \arrow[d, "a"] \\
\mathrm{id}_U^* f^*\xi \arrow[r, "\epsilon_U(f^*\eta)"] & f^*\eta
\end{tikzcd}
$$

commutes, because ϵ_U is a natural transformation. This implies that $a \cdot \epsilon_U(\xi)^{-1} = a$, and therefore $(a, f) \circ \bigl(\epsilon_U(\xi)^{-1}, \mathrm{id}_U\bigr) = (a, f)$.

A similar argument shows that $\bigl(\epsilon_U(\xi)^{-1}, \mathrm{id}_U\bigr)$ is also a left identity.

Hence $\mathcal{F}$ is a category. There is an obvious functor $\mathrm{p}_{\mathcal{F}} \colon \mathcal{F} \to \mathcal{C}$ sending an object (ξ, U) into U and an arrow (a, f) into f. I claim that this makes $\mathcal{F}$ into a category fibered over $\mathcal{C}$.

Take an arrow $f \colon U \to V$ of $\mathcal{C}$, and an object (η, V) of $\mathcal{F}$ over V. I claim that the arrow

$$(\mathrm{id}_{f^*\eta}, f) \colon (f^*\eta, U) \longrightarrow (\eta, V)$$

is cartesian. To prove this, suppose that we are given a diagram

$$
\begin{tikzcd}
(\zeta, W) \arrow[dr, dashed, "(c,h)"'] \arrow[drr, bend left, "(b,g)"] \arrow[dd, mapsto] & & \\
& (f^*\eta, U) \arrow[r, "(\mathrm{id}_{f^*\eta}, f)"'] \arrow[dd, mapsto] & (\eta, V) \arrow[dd, mapsto] \\
W \arrow[dr, "h"'] \arrow[drr, bend left, "g"] & & \\
& U \arrow[r, "f"'] & V
\end{tikzcd}
$$

(without the dotted arrow); we need to show that there is a unique arrow (c, h) that can be inserted in the diagram. But it is easy to show that

$$(\mathrm{id}_{f^*\eta}, f) \circ (c, h) = (\alpha_{h,f}(\eta) \circ c, hg),$$

and this tells us that the one and only arrow that fits into the diagram is $(\alpha_{h,f}(\eta)^{-1} \circ b, h)$.

This shows that $\mathcal{F}$ is fibered over $\mathcal{C}$, and also gives us a cleavage.

Finally, let us notice that for all objects U of $\mathcal{C}$ there is functor $\mathcal{F}(U) \to \Phi U$, sending an object (ξ, U) to ξ and an arrow (a, f) into a. This is an isomorphism of categories.

The cleavage constructed above gives, for each arrow $f \colon U \to V$, functors $f^* \colon \mathcal{F}(V) \to \mathcal{F}(U)$. If we identify each $\mathcal{F}(U)$ with ΦU via the isomorphism above, then these functors correspond to the $f^* \colon \Phi V \to \Phi U$. Hence if we start with a pseudo-functor, we construct the associated fibered category with a cleavage, and then we take the associated pseudo-functor, this is isomorphic to the original pseudo-functor (in the obvious sense).

Conversely, it is easy to see that if we start from a fibered category with a cleavage, construct the associated pseudo-functor, and then take the associated fibered category with a cleavage, we get something isomorphic to the original fibered category with a cleavage (again in the obvious sense). So really giving a pseudo-functor is the same as giving a fibered category with a cleavage.

On the other hand, since cartesian pullbacks are unique up to a unique isomorphism (Remark 3.2), also cleavages are unique up to a unique isomorphism. This means that, in a sense that one could make precise, the theory of fibered categories is equivalent to the theory of pseudo-functors. On the other hand, as was already remarked in [**SGA1**, Remarque, pp. 193–194], often the choice of a cleavage hinders more that it helps.

3.2. Examples of fibered categories

EXAMPLE 3.11. Let $\operatorname{Arr}\mathcal{C}$ be the category of arrows in $\mathcal{C}$; its objects are the arrows in $\mathcal{C}$, while an arrow from $X\colon S \to U$ to $Y\colon T \to V$ is a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & Y \arrow[d, "g"] \\ U \arrow[r] & V \end{tikzcd} .$$

The functor $\mathrm{p}_{\operatorname{Arr}\mathcal{C}}\colon \operatorname{Arr}\mathcal{C} \to \mathcal{C}$ sends each arrow $S \to U$ to its codomain U, and each commutative diagram to its bottom row.

I claim that $\operatorname{Arr}\mathcal{C}$ is a fibered category over $\mathcal{C}$. In fact, it easy to check that the cartesian diagrams are precisely the cartesian squares, so the statement follows from the fact that $\mathcal{C}$ has fibered products.

EXAMPLE 3.12. As a variant of the example above,let $\mathcal{P}$ a class of arrows that is stable under pullback. This means that if we have a cartesian square in $\mathcal{C}$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ U \arrow[r] & V \end{tikzcd}$$

and $Y \to V$ is in $\mathcal{P}$, then $X \to U$ is also in $\mathcal{P}$. The arrows in $\mathcal{P}$ are the objects in a category, again denoted by $\mathcal{P}$, in which an arrow from $X \to U$ to $Y \to V$ is a cartesian square as above.

It is easy to see that this is a category fibered in groupoids over $\mathcal{C}$

EXAMPLE 3.13. Let G a topological group. The *classifying stack* of G is the fibered category $\mathcal{B}G \to (\mathrm{Top})$ over the category of topological spaces, whose objects are principal G bundles $P \to S$, and whose arrows (ϕ, f) from $P \to U$ to $Q \to V$ are commutative diagrams

$$\begin{tikzcd} P \arrow[r, "\phi"] \arrow[d] & Q \arrow[d] \\ U \arrow[r, "f"] & V \end{tikzcd}$$

where the function ϕ is G-equivariant. The functor $\mathcal{B}G \to$ (Top) sends a principal bundle $P \to U$ into the topological space U, and an arrow (ϕ, f) into f.

Contrary to the usual convention, in a principal G-bundle $P \to S$ we will write the action of G on the left.

It is important to notice that any such diagram is cartesian; so $\mathcal{B}G \to$ (Top) has the property that each of its arrows is cartesian.

We are mostly interested in categories of sheaves. The simplest example is the fibered category of sheaves on objects of a site, defined as follows.

EXAMPLE 3.14. Let $\mathcal{C}$ be a site, and call $\mathcal{T}$ its topology. For each object X of $\mathcal{C}$ there is an induced topology $\mathcal{T}_X$ on the category $(\mathcal{C}/X)$, in which a set of arrows

$$\left\{\begin{tikzcd}[column sep=small] U_i \arrow[rr] \arrow[dr] & & U \arrow[dl] \\ & X & \end{tikzcd}\right\}$$

is a covering if and only if the set $\{U_i \to U\}$ is a covering in $\mathcal{C}$. We will refer to a sheaf in the site $(\mathcal{C}/X)$ as a *sheaf on* X, and denote the category of sheaves on X by $\operatorname{Sh} X$.

If $f\colon X \to Y$ is an arrow in $\mathcal{C}$, there is a corresponding restriction functor $f^*\colon \operatorname{Sh} Y \to \operatorname{Sh} X$, defined as follows.

If G is a sheaf on Y and $U \to X$ is an object of $(\mathcal{C}/X)$, we define $f^*G(U \to Y) = G(U \to Y)$, where $U \to Y$ is the composition of $U \to X$ with f.

If $U \to X$ and $V \to X$ are objects of $(\mathcal{C}/X)$ and $\phi\colon U \to V$ is an arrow in $(\mathcal{C}/X)$, then ϕ is also an arrow from $U \to Y$ to $V \to Y$, hence it induces a function $\phi^*\colon F^*(V \to X) = F(U \to Y) \to F(V \to Y) = f^*F(V \to X)$. This gives f^*F the structure of a functor $(\mathcal{C}/X)^{\mathrm{op}} \to$ (Set). One checks immediately that f^*F is a sheaf on $(\mathcal{C}/X)$.

If $\phi\colon F \to G$ is a natural transformation of sheaves on $(\mathcal{C}/Y)$, there is an induced natural transformation $f^*\phi\colon f^*F \to f^*G$ of sheaves on $(\mathcal{C}/X)$, defined in the obvious way. This defines a functor $f^*\colon \operatorname{Sh} Y \to \operatorname{Sh} X$.

It is immediate to check that, if $f\colon X \to Y$ and $g\colon Y \to Z$ are arrows in $\mathcal{C}$, we have an *equality* of functors $(gf)^* = f^*g^*\colon (\mathcal{C}/Z) \to (\mathcal{C}/X)$. Furthermore $\mathrm{id}_X^*\colon \operatorname{Sh} X \to \operatorname{Sh} X$ is the identity. This means that we have defined a functor from $\mathcal{C}$ to the category of categories, sending an object X into the category of categories. According to the result of 3.1.3, this yields a category $(\mathrm{Sh}/\mathcal{C}) \to \mathcal{C}$, whose fiber over X is $\operatorname{Sh} X$.

There are many variants on this example, by considering sheaves in abelian groups, rings, and so on.

This example is particularly simple, because it is defined by a functor. In most of the cases that we are interested in, the sheaves on a given object will be defined in a site that is not the one inherited from the base category $\mathcal{C}$; this creates some difficulties, and forces one to use the unpleasant machinery of pseudo-functors. On the other hand, this discrepancy between the base topology and the topology on which the sheaves is what makes descent theory for quasicoherent sheaves so much more than an exercise in formalism.

Let us consider directly the example we are interested in, that is, fibered categories of quasicoherent sheaves.

EXAMPLE 3.15. Here $\mathcal{C}$ will be the category (Sch/S) of schemes over a fixed base scheme S. For each scheme U we define $\mathrm{QCoh}\,U$ to be the category of quasicoherent sheaves on U. Given a morphism $f\colon U \to V$, we have a functor $f^*\colon \mathrm{QCoh}\,V \to \mathrm{QCoh}\,U$. Unfortunately, given two morphisms $U \xrightarrow{f} V \xrightarrow{g} W$, the pullback $(gf)^*\colon \mathrm{QCoh}\,W \to \mathrm{QCoh}\,U$ does not coincide with the composition $f^*g^*\colon \mathrm{QCoh}\,W \to \mathrm{QCoh}\,U$, but it is only canonically isomorphic to it. This may induce one to suspect that we are in the presence of a pseudo-functor; and this is indeed the case.

The neatest way to prove this is probably by exploiting the fact that the pushforward $f_*\colon \mathrm{QCoh}\,U \to \mathrm{QCoh}\,V$ is functorial, that is, $(gf)_*$ equals g_*f_* on the nose, and f^* is a left adjoint to f_*. This means that, given quasicoherent sheaves $\mathcal{N}$ on U and $\mathcal{M}$ on V, there is a canonical isomorphism of groups

$$\Theta_f(\mathcal{N},\mathcal{M})\colon \mathrm{Hom}_{\mathcal{O}_V}(\mathcal{N}, f_*F) \simeq \mathrm{Hom}_{\mathcal{O}_U}(f^*\mathcal{N}, \mathcal{M})$$

that is natural in $\mathcal{M}$ and $\mathcal{N}$. More explicitly, there are two functors

$$\mathrm{QCoh}\,U^{\mathrm{op}} \times \mathrm{QCoh}\,V \longrightarrow (\mathrm{Grp})$$

defined by

$$(\mathcal{M},\mathcal{N}) \mapsto \mathrm{Hom}_{\mathcal{O}_V}(G, f_*\mathcal{M})$$

and

$$(\mathcal{M},\mathcal{N}) \mapsto \mathrm{Hom}_{\mathcal{O}_U}(f^*\mathcal{N}, \mathcal{M});$$

then Θ_f defines a natural transformation from the first to the second.

Equivalently, if $\alpha\colon \mathcal{M} \to \mathcal{M}'$ and $\beta\colon \mathcal{N} \to \mathcal{N}'$ are homomorphism of quasicoherent sheaves on U and V respectively, the diagrams

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{O}_V}(\mathcal{N}, f_*\mathcal{M}) \arrow[r, "\Theta_f(\mathcal{N},\mathcal{M})"] \arrow[d, "f_*\alpha\circ-"] & \mathrm{Hom}_{\mathcal{O}_U}(f^*\mathcal{N}, \mathcal{M}) \arrow[d, "\alpha\circ-"] \\
\mathrm{Hom}_{\mathcal{O}_V}(\mathcal{N}, f_*\mathcal{M}') \arrow[r, "\Theta_f(\mathcal{N}',\mathcal{M})"] & \mathrm{Hom}_{\mathcal{O}_U}(f^*\mathcal{N}, \mathcal{M}')
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{O}_V}(\mathcal{N}', f_*\mathcal{M}) \arrow[r, "\Theta_f(\mathcal{N},\mathcal{M}')"] \arrow[d, "-\circ\beta"] & \mathrm{Hom}_{\mathcal{O}_U}(f^*\mathcal{N}', \mathcal{M}) \arrow[d, "-\circ f^*\beta"] \\
\mathrm{Hom}_{\mathcal{O}_V}(\mathcal{N}, f_*\mathcal{M}) \arrow[r, "\Theta_f(\mathcal{N},\mathcal{M})"] & \mathrm{Hom}_{\mathcal{O}_V}(\mathcal{N}, f_*\mathcal{M})
\end{tikzcd}$$

commute.

If U is a scheme over S and $\mathcal{N}$ a quasicoherent sheaf on U, then the pushforward functor $(\mathrm{id}_U)_*\colon \mathrm{QCoh}\,U \to \mathrm{QCoh}\,U$ is the identity (this has to be interpreted literally, I am not simply asserting the existence of a canonical isomorphism between $(\mathrm{id}_U)_*$ and the identity on $\mathrm{QCoh}\, U$). Now, if $\mathcal{M}$ is a quasicoherent sheaf on U, there is a canonical adjunction isomorphism

$$\Theta_{\mathrm{id}_U}(\mathcal{M}, -)\colon \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{M}, (\mathrm{id}_U)_*-) = \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{M}, -) \simeq \operatorname{Hom}_{\mathcal{O}_U}(\mathrm{id}_U^*\mathcal{M}, -)$$

of functors from $\mathrm{QCoh}\, U$ to (Set). By the dual version of Yoneda's lemma (Remark 2.8) this corresponds to an isomorphism $\epsilon_U(\mathcal{M})\colon \mathrm{id}_U^*\mathcal{M} \simeq \mathcal{M}$. This is easily seen to be functorial, and therefore defines an isomorphism

$$\epsilon_U\colon \mathrm{id}_U^* \simeq \mathrm{id}_{\mathrm{QCoh}\, U}$$

of functors from $\mathrm{QCoh}\, U$ to itself. This is the first piece of data that we need.

For the second, consider two morphisms $U \xrightarrow{f} V \xrightarrow{g} W$ and a quasicoherent sheaf $\mathcal{P}$ on W. We have the chain of isomorphism of functors $\mathrm{QCoh}\, U \to (\mathrm{Grp})$

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{O}_U}\big((gf)^*\mathcal{P}, -\big) &\simeq \operatorname{Hom}_{\mathcal{O}_W}\big(\mathcal{P}, (gf)_*-\big) && \big(\text{this is } \Theta_{gf}(\mathcal{P}, -)^{-1}\big)\\
&= \operatorname{Hom}_{\mathcal{O}_W}\big(\mathcal{P}, g_*f_*-\big)\\
&\simeq \operatorname{Hom}_{\mathcal{O}_V}(g^*\mathcal{P}, f_*-) && \big(\text{this is } \Theta_g(\mathcal{P}, f_*-)\big)\\
&\simeq \operatorname{Hom}_{\mathcal{O}_U}(g^*f^*\mathcal{P}, -) && \big(\text{this is } \Theta_f(g^*\mathcal{P}, -)\big);
\end{aligned}$$

the composition

$$\Theta_f(g^*\mathcal{P}, -) \circ \Theta_g(\mathcal{P}, f_*-) \circ \Theta_{gf}(\mathcal{P}, -)^{-1}\colon$$
$$\operatorname{Hom}_{\mathcal{O}_U}\big((gf)^*\mathcal{P}, -\big) \simeq \operatorname{Hom}_{\mathcal{O}_U}(g^*f^*\mathcal{P}, -)$$

corresponds, again because of the dual Yoneda lemma, to an isomorphism $\alpha_{f,g}(\mathcal{P})\colon f^*g^*\mathcal{P} \simeq (gf)^*\mathcal{P}$. These give an isomorphism $\alpha_{f,g}\colon f^*g^* \simeq (gf)^*$ of functors $\mathrm{QCoh}\, W \to \mathrm{QCoh}\, U$. We have to check that the ϵ_U and $\alpha_{f,g}$ satisfy the conditions of Definition 3.9.

Take a morphism of schemes $f\colon U \to V$. We need to prove that for any quasicoherent sheaf $\mathcal{N}$ on V, we have the equality

$$\alpha_{id_U,f}(\mathcal{N}) = \epsilon_U(f^*\mathcal{N})\colon \mathrm{id}_U^*f^*\mathcal{N} \to f^*\mathcal{N}.$$

This is straightforward: by the dual Yoneda lemma, it is enough to show that $\alpha_{id_U,f}(\mathcal{N})$ and $\epsilon_U(f^*\mathcal{N})$ induce the same natural transformation

$$\operatorname{Hom}_{\mathcal{O}_U}(f^*\mathcal{N}, -) \longrightarrow \operatorname{Hom}_{\mathcal{O}_U}(\mathrm{id}_U^*f^*\mathcal{N}, -).$$

But by definition the natural transformation induced by $\epsilon_U(f^*\mathcal{N})$ is $\Theta_{\mathrm{id}_U}(f^*\mathcal{N}, -)$, while that induced by $\alpha_{id_U,f}(\mathcal{N})$ is

$$\Theta_{\mathrm{id}_u}(f^*\mathcal{N}, -) \circ \Theta_f(\mathcal{N}, (\mathrm{id}_U)_*-) \circ \Theta_f(\mathcal{N}, -)^{-1} = \Theta_{\mathrm{id}_u}(f^*\mathcal{N}, -).$$

A similar arguments works for the second part of the first condition.

For the second condition the argument is similar, and left to the reader.

There are many variants of this example. For example, one can define the fibered category of sheaves of $\mathcal{O}$-modules over the category of ringed spaces in exactly the same way.

3.3. Categories fibered in groupoids

DEFINITION 3.16. A *category fibered in groupoids over* $\mathcal{C}$ is a category $\mathcal{F}$ fibered over $\mathcal{C}$, such that the category $\mathcal{F}(U)$ is a groupoid for any object U of $\mathcal{C}$.

In the literature one often finds a different definition of a category fibered in groupoids.

PROPOSITION 3.17. *Let $\mathcal{F}$ be a category over $\mathcal{C}$. Then $\mathcal{F}$ is fibered in groupoids over $\mathcal{C}$ if and only if the following two conditions hold.*

(i) *Every arrow in $\mathcal{F}$ is cartesian.*
(ii) *Given an object η of $\mathcal{F}$ and an arrow $f\colon U \to p_{\mathcal{F}}\eta$ of $\mathcal{C}$, there exists an arrow $\phi\colon \xi \to \eta$ of $\mathcal{F}$ with $\mathrm{p}_{\mathcal{F}}\phi = f$.*

PROOF. Suppose that these two conditions hold. Then it is immediate to see that $\mathcal{F}$ is fibered over $\mathcal{C}$. Also, if $\phi\colon \xi \to \eta$ is an arrow of $\mathcal{F}(U)$ for some object U of $\mathcal{C}$, then we see from condition 3.17 (i) that there exists an arrow $\psi\colon \eta \to \xi$ with $\mathrm{p}_{\mathcal{F}}\psi = \mathrm{id}_U$ and $\phi\psi = \mathrm{id}_\eta$; that is, every arrow in $\mathcal{F}(U)$ has a right inverse. But this right inverse ψ also must also have a right inverse, and then the right inverse of ψ must be ϕ. This proves that every arrow in $\mathcal{F}(U)$ is invertible.

Conversely, assume that $\mathcal{F}$ is fibered over $\mathcal{C}$, and each $\mathcal{F}(U)$ is a groupoid. Condition (ii) is trivially verified. To check condition (i), let $\phi\colon \xi \to \eta$ be an arrow in $\mathcal{C}$ mapping to $f\colon U \to V$ in $\mathcal{C}$. Choose a pullback $\phi'\colon \xi' \to \eta$ of η to U; by definition there will be an arrow $\alpha\colon \xi \to \xi'$ in $\mathcal{F}(U)$ such that $\phi'\alpha = \phi$. Since $\mathcal{F}(U)$ is a a groupoid, α will be an isomorphism, and this implies that ϕ is cartesian. ♠

COROLLARY 3.18. *Any base preserving functor from a fibered category to a category fibered in groupoids is a morphism.*

PROOF. This is clear, since every arrow in a category fibered in groupoids is cartesian. ♠

Of the examples of Section 3.1, 3.11 and 3.12 are not in general fibered in groupoids, while the classifying stack of a topological group introduced in 3.13 is always fibered in groupoids.

Give a fibered category $\mathcal{F} \to \mathcal{C}$, the subcategory $\mathcal{F}_{\mathrm{cart}}$ whose objects are the same as the objects of $\mathcal{F}$, but the arrows are the cartesian arrows, is fibered in groupoids. Any morphism $\mathcal{G} \to \mathcal{F}$ of fibered categories, where $\mathcal{G}$ is fibered in groupoids, factors uniquely through $\mathcal{F}_{\mathrm{cart}}$.

3.4. Functors and categories fibered in sets

The notion of category generalizes the notion of set: a set can be thought of as a category in which every arrow is an identity. Furthermore functors between sets are simply functions.

Similarly, fibered categories are generalizations of functors.

DEFINITION 3.19. A *category fibered in sets over* $\mathcal{C}$ is a category $\mathcal{F}$ fibered over $\mathcal{C}$, such that for any object U of $\mathcal{C}$ the category $\mathcal{F}(U)$ is a set.

Here is an useful characterization of categories fibered in sets.

PROPOSITION 3.20. *Let $\mathcal{F}$ be a category over $\mathcal{C}$. Then $\mathcal{F}$ is fibered in sets if and only if for any object η of $\mathcal{F}$ and any arrow $f\colon U \to \mathrm{p}_{\mathcal{F}}\eta$ of $\mathcal{C}$, there is a unique arrow $\phi\colon \xi \to \eta$ of $\mathcal{F}$ with $\mathrm{p}_{\mathcal{F}}\phi = f$.*

PROOF. Suppose that $\mathcal{F}$ is fibered in sets. Given η and $f\colon U \to \mathrm{p}_{\mathcal{F}}\eta$ as above, pick a cartesian arrow $\xi \to \eta$ over f. If $\xi' \to \eta$ is any other arrow over f, by definition there exists an arrow $\xi' \to \xi$ in $\mathcal{F}(U)$ making the diagram

$$\begin{tikzcd}
\xi' \arrow[rr, dashed] \arrow[dr] & & \xi \arrow[dl] \\
& \eta &
\end{tikzcd}$$

commutative. Since $\mathcal{F}(U)$ is a set, it follows that this arrow $\xi' \to \xi$ is the identity, so the two arrows $\xi \to \eta$ and $\xi' \to \eta$ coincide.

Conversely, assume that the condition holds. Given a diagram

$$\begin{tikzcd}
\zeta \arrow[dr, dashed, "\theta"'] \arrow[drr, bend left, "\psi"] \arrow[dd, mapsto] & & \\
& \xi \arrow[r, "\phi"'] \arrow[dd, mapsto] & \eta \arrow[dd, mapsto] \\
\mathrm{p}_{\mathcal{F}}\zeta \arrow[dr, "h"'] \arrow[drr, bend left] & & \\
& \mathrm{p}_{\mathcal{F}}\xi \arrow[r] & \mathrm{p}_{\mathcal{F}}\eta
\end{tikzcd}$$

the condition implies that the only arrow $\theta\colon \zeta \to \xi$ over h makes the diagram commutative; so the category $\mathcal{F}$ is fibered.

It is obvious that the condition implies that $\mathcal{F}(U)$ is a set for all U. ♠

So, for categories fibered in sets the pullback of an object of $\mathcal{F}$ along an arrow of $\mathcal{C}$ is strictly unique. It follows from this that when $\mathcal{F}$ is fibered in sets over $\mathcal{C}$ and $f\colon U \to V$ is an arrow in $\mathcal{C}$, the pullback map $f^*\colon \mathcal{F}(V) \to \mathcal{F}(U)$ is uniquely defined, and the composition rule $f^*g^* = (gf)^*$ holds. Also for any object U of $\mathcal{C}$ we have that $\mathrm{id}_U^*\colon \mathcal{F}(U) \to \mathcal{F}(U)$ is the identity. This means that we have defined a functor $\Phi_{\mathcal{F}}\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$ by sending each object U of $\mathcal{C}$ to $\mathcal{F}(U)$, and each arrow $f\colon U \to V$ of $\mathcal{C}$ to the function $f^*\colon \mathcal{F}(V) \to \mathcal{F}(U)$.

Furthermore, if $F\colon \mathcal{F} \to \mathcal{G}$ is a morphism of categories fibered in sets, because of the condition that $\mathrm{p}_{\mathcal{G}} \circ F = \mathrm{p}_{\mathcal{F}}$, then every arrow in $\mathcal{F}(U)$, for some object U of $\mathcal{C}$, will be send to $\mathcal{F}(U)$ itself. So we get a function $F_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$. It is immediate to check that this gives a natural transformation $\phi_F\colon \Phi_{\mathcal{F}} \to \Phi_{\mathcal{G}}$.

There is a category of categories fibered in sets over $\mathcal{C}$, where the arrows are morphisms of fibered categories; the construction above gives a functor from this category to the category of functors $\mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$.

PROPOSITION 3.21. *This is an equivalence of the category of categories fibered in sets over $\mathcal{C}$ and the category of functors $\mathcal{C}^{\mathrm{op}} \to$ (Set).*

PROOF. The inverse functor is obtained by the construction of 3.1.3. If $\Phi\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$ is a functor, we construct a category fibered in sets $\mathcal{F}_\Phi$ as follows. The objects of $\mathcal{F}_\Phi$ will be pairs (U, ξ), where U is an object of $\mathcal{C}$, and $\xi \in \Phi U$. An arrow from (U, ξ) to (V, η) is an an arrow $f\colon U \to V$ of $\mathcal{C}$ with the property that $\Phi f \eta = \xi$. It follows from Proposition 3.20 that $\mathcal{F}_\Phi$ is fibered in sets over $\mathcal{C}$.

To each natural transformation of functors $\phi\colon \Phi \to \Phi'$ we associate a morphism $F_\phi\colon \mathcal{F}_\Phi \to \mathcal{F}_{\Phi'}$. An object (U, ξ) of $\mathcal{F}_\Phi$ will be sent to $(U, \phi_U \xi)$. If $f\colon (U, \xi) \to (V, \eta)$ is an arrow in $\mathcal{F}_\Phi$, then f is simply an arrow $f\colon U \to V$ in $\mathcal{C}$, with the property that $\Phi f(\eta) = \xi$. This implies that $\Phi'(f)\phi_V(\eta) = \phi_U \Phi(f)(\eta) = \phi_V \xi$, so the same f will yield an arrow $f\colon (U, \phi_U \xi) \to (V, \phi_V \eta)$.

We leave it the reader to check that this defines a functor from the category of functors to the category of categories fibered in sets. ♠

So, any functor $\mathcal{C}^{\mathrm{op}} \to$ (Set) will give an example of a fibered category over $\mathcal{C}$.

In particular, given an object X of $\mathcal{C}$, we have the representable functor $\mathrm{h}_X\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$, defined on objects by the rule $\mathrm{h}_X U = \mathrm{Hom}_{\mathcal{C}}(U, X)$. The category in sets over $\mathcal{C}$ associated with this functor is the category $(\mathcal{C}/X)$, whose objects are arrows $U \to X$, and whose arrows are commutative diagrams

$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & & V \arrow[dl] \\ & X & \end{tikzcd}.$$

So the situation is the following. From Yoneda's lemma we see that the category $\mathcal{C}$ is embedded into the category of functors $\mathcal{C}^{\mathrm{op}} \to$ (Set), while the category of functors is embedded into the category of fibered categories.

From now we will identify a functor $F\colon \mathcal{C}^{\mathrm{op}} \to$ (Set) with the corresponding category fibered in sets over $\mathcal{C}$, and we will (inconsistently) call a category fibered in sets simply "a functor".

3.4.1. Categories fibered over an object.

PROPOSITION 3.22. *Let $\mathcal{G}$ be a category fibered in sets over $\mathcal{C}$, $\mathcal{F}$ another category, $F\colon \mathcal{F} \to \mathcal{G}$ a functor. Then $\mathcal{F}$ is fibered over $\mathcal{G}$ if and only if it is fibered over $\mathcal{C}$ via the composition $\mathrm{p}_{\mathcal{G}} \circ F\colon \mathcal{F} \to \mathcal{C}$.*

Furthermore, $\mathcal{F}$ is fibered in groupoids over $\mathcal{G}$ if and only if it fibered in groupoids over $\mathcal{C}$, and is fibered in sets over $\mathcal{G}$ if and only if it fibered in sets over $\mathcal{C}$.

PROOF. One sees immediately that an arrow of $\mathcal{G}$ is cartesian over its image in $\mathcal{F}$ if and only if it is cartesian over its image in $\mathcal{C}$, and the first statement follows from this.

Furthermore, one sees that the fiber of $\mathcal{F}$ over an object U of $\mathcal{C}$ is the disjoint union, as a category, of the fibers of $\mathcal{F}$ over all the objects of $\mathcal{G}$ over U; if these fiber are groupoids, or sets, so is their disjoint union. ♠

This is going to be used as follows. Suppose that S is an object of $\mathcal{C}$, and consider the category fibered in sets $(\mathcal{C}/S) \to \mathcal{C}$, corresponding to the representable functor $\mathrm{h}_S \colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$. By Proposition 3.22, a fibered category $\mathcal{F} \to (\mathcal{C}/S)$ is the same as a fibered category $\mathcal{F} \to \mathcal{C}$, together with a morphism $\mathcal{F} \to \mathcal{C}$.

It is interesting to describe this process for functors. Given a functor $F \colon (\mathcal{C}/S)^{\mathrm{op}} \to (\mathrm{Set})$, this corresponds to a category fibered in sets $F \to (\mathcal{C}/S)$; this can be composed with the forgetful functor $(\mathcal{C}/S) \to \mathcal{C}$ to get a category fibered in sets $F \to \mathcal{C}$, which in turn corresponds to a functor $F' \colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$. What is this functor? One minute's thought will convince you that it can be described as follows: $F'(U)$ is the disjoint union of the $F(U \xrightarrow{u} S)$ for all the arrows $u \colon U \to S$ in $\mathcal{C}$. The action of F' on arrows is the obvious one.

3.5. Equivalences of fibered categories

3.5.1. Natural transformations of functors. The fact that fibered categories are categories, and not functors, has strong implications, and does cause difficulties. As usual, the main problem is that functors between categories can be isomorphic without being equal; in other words, functors between two fixed categories form a category, the arrows being given by natural transformations.

DEFINITION 3.23. Let $\mathcal{F}$ and $\mathcal{G}$ be two categories fibered over $\mathcal{C}$, F, $G \colon \mathcal{F} \to \mathcal{G}$ two morphisms. A *base-preserving natural transformation* $\alpha \colon F \to G$ is a natural transformation such that for any object ξ of $\mathcal{F}$, the arrow $\alpha_\xi \colon F\xi \to G\xi$ is in $\mathcal{G}(U)$, where $U \overset{\mathrm{def}}{=} \mathrm{p}_{\mathcal{F}}\xi = \mathrm{p}_{\mathcal{G}}(F\xi) = \mathrm{p}_{\mathcal{G}}(G\xi)$.

An *isomorphism of* F *with* G is a base-preserving natural transformation $F \to G$ which is an isomorphism of functors.

It is immediate to check that the inverse of a base-preserving isomorphism is also base-preserving.

There is a category whose objects are the morphism from a $\mathcal{F}$ to $\mathcal{G}$, and the arrows are base-preserving natural transformations; we denote it by $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{G})$.

3.5.2. Equivalences.

DEFINITION 3.24. Let $\mathcal{F}$ and $\mathcal{G}$ be two fibered categories over $\mathcal{C}$. An *equivalence*, or *isomorphism*, of $\mathcal{F}$ with $\mathcal{G}$ is a morphism $F \colon \mathcal{F} \to \mathcal{G}$, such that there exists another morphism $G \colon \mathcal{G} \to \mathcal{F}$, together with isomorphisms of $G \circ F$ with $\mathrm{id}_{\mathcal{F}}$ and of $F \circ G$ with $\mathrm{id}_{\mathcal{G}}$.

We call G simply an *inverse* to F.

PROPOSITION 3.25. *Suppose that $\mathcal{F}$, $\mathcal{F}'$, $\mathcal{G}$ and $\mathcal{G}'$ are categories fibered over $\mathcal{C}$. Suppose that $F\colon \mathcal{F}' \to \mathcal{F}$ and $G\colon \mathcal{G} \to \mathcal{G}'$ are equivalences. Then there an equivalence of categories*

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{F},\mathcal{G}) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}',\mathcal{G}')$$

that sends each $\Phi\colon \mathcal{F} \to \mathcal{G}$ into the composition

$$G \circ \Phi \circ F\colon \mathcal{F}' \to \mathcal{G}'.$$

The proof is left as an exercise to the reader.

The following is the basic criterion for checking whether a morphism of fibered categories is an equivalence.

PROPOSITION 3.26. *Let $F\colon \mathcal{F} \to \mathcal{G}$ be a morphism of fibered categories. Then F is an equivalence if and only if the restriction $F_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is an equivalence of categories for any object U of $\mathcal{C}$.*

PROOF. Suppose that $G\colon \mathcal{G} \to \mathcal{F}$ is an inverse to F; the two isomorphisms $F \circ G \simeq \mathrm{id}_{\mathcal{G}}$ and $G \circ F \simeq \mathrm{id}_{\mathcal{F}}$ restrict to isomorphisms $F_U \circ G_U \simeq \mathrm{id}_{\mathcal{G}(U)}$ and $G_U \circ F_U \simeq \mathrm{id}_{\mathcal{F}(U)}$, so G_U is an inverse to F_U.

Conversely, we assume that $F_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is an equivalence of categories for any object U of $\mathcal{C}$, and construct an inverse $G\colon \mathcal{G} \to \mathcal{F}$. Here is the main fact that we are going to need.

LEMMA 3.27. *Let $F\colon \mathcal{F} \to \mathcal{G}$ a morphism of fibered categories such that every restriction $F_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is fully faithful. Then the functor F is fully faithful.*

PROOF. We need to show that, given two objects ξ' and η' of $\mathcal{F}$ and an arrow $\phi\colon F\xi' \to F\eta'$ in $\mathcal{G}$, there is a unique arrow $\phi'\colon \xi' \to \eta'$ in $\mathcal{F}$ with $F\phi' = \phi$. Set $\xi = F\xi'$ and $\eta = F\eta'$. Let $\eta'_1 \to \eta'$ be a pullback of η' to U, $\eta_1 = F\eta'_1$. Then the image $\eta_1 \to \eta$ of $\eta'_1 \to \eta'$ is cartesian, so every morphism $\xi \to \eta$ factors uniquely as $\xi \to \eta_1 \to \eta$, where the arrow $\xi \to \xi_1$ is in $\mathcal{G}(U)$. Analogously all arrows $\xi' \to \eta'$ factor uniquely through through η_1; since every arrow $\xi \to \eta_1$ in $\mathcal{G}(U)$ lifts uniquely to an arrow $\xi' \to \eta'_1$ in $\mathcal{F}(U)$, we have proved the Lemma. ♠

For any object ξ of $\mathcal{G}$ pick an object $G\xi$ of $\mathcal{F}(U)$, where $U = \mathrm{p}_{\mathcal{G}}\xi$, together with an isomorphism $\alpha_\xi\colon \xi \simeq F(G\xi)$ in $\mathcal{G}(U)$; these $G\xi$ and α_ξ exist because $F_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is an equivalence of categories.

Now, if $\phi\colon \xi \to \eta$ is an arrow in $\mathcal{G}$, by the Lemma there is a unique arrow $G\phi\colon G\xi \to G\eta$ such that $F(G\phi) = \alpha_\eta \circ \phi \circ \alpha_\xi^{-1}$, that is, such that the diagram

$$\begin{tikzcd} \xi \arrow[r, "\phi"] \arrow[d, "\alpha_\xi"] & \eta \arrow[d, "\alpha_\eta"] \\ F(G\xi) \arrow[r, "F(G\phi)"] & F(G\eta) \end{tikzcd}$$

commutes.

These operations define a functor $G\colon \mathcal{G} \to \mathcal{F}$. It is immediate to check that by sending each object ξ to the isomorphism $\alpha_\xi\colon \xi \simeq F(G\xi)$ we define an isomorphism of functors $\mathrm{id}_{\mathcal{F}} \simeq F \circ G\colon \mathcal{G} \to \mathcal{G}$.

We only have left to check that $G \circ F\colon \mathcal{F} \to \mathcal{F}$ is isomorphic to the identity $\mathrm{id}_{\mathcal{F}}$.

Fix an object ξ' of $\mathcal{F}$ over an object U of $\mathcal{C}$; we have a canonical isomorphism $\alpha_{F\xi'}\colon F\xi' \simeq F(G(F\xi'))$ in $\mathcal{G}(U)$. Since F_U is fully faithful there is a unique isomorphism $\beta_{\xi'}\colon \xi' \simeq G(F\xi')$ in $\mathcal{F}(U)$ such that $F\beta_{\xi'} = \alpha_{F\xi'}$; one checks easily that this defines an isomorphism of functors $\beta\colon G \circ F \simeq \mathrm{id}_{\mathcal{G}}$. ♠

3.5.3. Quasi-functors. As we remarked in 3.4, the notion of category generalizes the notion of set.

It is also possible to characterize the categories that are equivalent to a set: these are the equivalence relations.

Suppose that $R \subseteq X \times X$ is an equivalence relation on a set X. We can produce a category (X, R) in which X is the set of objects, R is the set of arrows, and the source and target maps $R \to X$ are given by the first and second projection. Then given x and y in X, there is precisely one arrow (x, y) if x and y are in the same equivalence class, while there is none if they are not. Then transitivity assures us that we can compose arrows, while reflexivity tell us that over each object $x \in X$ there is a unique arrow (x, x), which is the identity. Finally symmetry tells us that any arrow (x, y) has an inverse (y, x). So, (X, R) is groupoid such that from a given object to another there is at most one arrow.

Conversely, given a groupoid such that from a given object to another there is at most one arrow, if we call X the set of objects and R the set of arrows, the source and target maps induce an injective map $R \to X \times X$, that gives an equivalence relation on X.

So an equivalence relation can be thought of as a groupoid such that from a given object to another there is at most one arrow. Equivalently, an equivalence relation is a groupoid in which the only arrow from an object to itself is the identity.

PROPOSITION 3.28. *A category is equivalent to a set if and only if it is an equivalence relation.*

PROOF. If a category is equivalent to a set, it is immediate to see that it is an equivalence relation. If (X, R) is an equivalence relation and X/R is the set of isomorphism classes of objects, that is, the set of equivalence classes, one checks immediately that the function $X \to X/R$ gives a functor that is fully faithful and essentially surjective, so it is an equivalence. ♠

There is an analogous result for fibered categories.

DEFINITION 3.29. A category $\mathcal{F}$ over $\mathcal{C}$ is a *quasi-functor*, or that it is *fibered in equivalence relations*, if it is fibered, and each fiber $\mathcal{F}(U)$ is an equivalence relation.

We have the following characterization of quasi-functors.

PROPOSITION 3.30. *A category $\mathcal{F}$ over $\mathcal{C}$ is a quasi-functor if and only if the following two conditions hold.*

(i) *Given an object η of $\mathcal{F}$ and an arrow $f\colon U \to p_{\mathcal{F}}\eta$ of $\mathcal{C}$, there exists an arrow $\phi\colon \xi \to \eta$ of $\mathcal{F}$ with $\mathrm{p}_{\mathcal{F}}\phi = f$.*

(ii) *Given two objects ξ and η of $\mathcal{F}$ and an arrow $f\colon \mathrm{p}_{\mathcal{F}}\xi \to \mathrm{p}_{\mathcal{F}}\eta$ of $\mathcal{C}$, there exists at most one arrow $\xi \to \eta$ over f.*

The easy proof is left to the reader.

PROPOSITION 3.31. *A fibered category over $\mathcal{C}$ is a quasi-functor if and only if it is equivalent to a functor.*

PROOF. This is an application of Proposition 3.26.

Suppose that a fibered category $\mathcal{F}$ is equivalent to a functor Φ; then every category $\mathcal{F}(U)$ is equivalent to the set ΦU, so $\mathcal{F}$ is fibered in equivalence relations over $\mathcal{C}$ by Proposition 3.28.

Conversely, assume that $\mathcal{F}$ is fibered in equivalence relations. In particular it is fibered in groupoid, so every arrow in $\mathcal{F}$ is cartesian, by Proposition 3.17. For each object U of $\mathcal{C}$, denote by ΦU the set of isomorphism classes of elements in $\mathcal{F}(U)$. Given an arrow $f\colon U \to V$ in $\mathcal{C}$, two isomorphic object η and η' of $\mathcal{F}(V)$, and two pullbacks ξ and ξ' of η and η' to $\mathcal{F}(U)$, we have that ξ and ξ' are isomorphic in $\mathcal{F}(U)$; this gives a well defined function $f^*\colon \Phi V \to \Phi U$ that sends an isomorphism class $[\eta]$ in $\mathcal{F}(V)$ into the isomorphism class of pullbacks of η. It is easy to see that this gives Φ the structure of a functor $\mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$. If we think of Φ as a category fibered in sets, we get by construction a morphism $\mathcal{F} \to \Phi$. Its restriction $\mathcal{F}(U) \to \Phi U$ is an equivalence for each object U of $\mathcal{C}$, so by Proposition 3.26 the morphism $\mathcal{F} \to \Phi$ is an equivalence. ♠

Here are a few useful facts.

PROPOSITION 3.32.

(i) *If $\mathcal{G}$ is fibered in groupoid, then is $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F},\mathcal{G})$ is a groupoid.*

(ii) *If $\mathcal{G}$ is a quasi-functor, then $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F},\mathcal{G})$ is an equivalence relation.*

(iii) *If $\mathcal{G}$ is a functor, then $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F},\mathcal{G})$ is a set.*

We leave the easy proofs to the reader.

In 2-categorical terms, part (iii) says that the 2-category of categories fibered in sets is in fact just a 1-category, while part (ii) says that the 2-category of quasi-functors is equivalent to a 1-category.

3.6. Objects as fibered categories and the 2-Yoneda lemma

3.6.1. Representable fibered categories. In 2.1 we have seen how we can embed a category $\mathcal{C}$ into the functor category $\mathrm{Func}(\mathcal{C}^{\mathrm{op}}, (\mathrm{Set}))$, while in 3.4 we have seen how to embed the category $\mathrm{Func}(\mathcal{C}^{\mathrm{op}}, (\mathrm{Set}))$ into the 2-category of fibered categories over $\mathcal{C}$. By composing these embeddings we have embedded $\mathcal{C}$ into the 2-category of fibered categories: an object X of $\mathcal{C}$ is sent to the fibered category $(\mathcal{C}/X) \to \mathcal{C}$. Furthermore, an arrow $f\colon X \to Y$ goes to the morphism of fibered categories $(\mathcal{C}/f)\colon (\mathcal{C}/X) \to (\mathcal{C}/Y)$ that sends an object $U \to X$ of $(\mathcal{C}/X)$ to the composition $U \to X \xrightarrow{f} Y$. The functor $(\mathcal{C}/f)$ sends an arrow

$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & & V \arrow[dl] \\ & X & \end{tikzcd}$$

of $(\mathcal{C}/X)$ to the commutative diagram obtained by composing both sides with $f\colon X \to Y$.

This is the 2-categorical version of the weak Yoneda lemma.

THE WEAK 2-YONEDA LEMMA. *The function that sends each arrow $f\colon X \to Y$ to the functor $(\mathcal{C}/f)\colon (\mathcal{C}/X) \to (\mathcal{C}/Y)$ is a bijection.*

DEFINITION 3.33. A fibered category over $\mathcal{C}$ is *representable* if it is equivalent to a category of the form $(\mathcal{C}/X)$.

So a representable category is necessarily a quasi-functor, by Proposition 3.31. However, we should be careful: if $\mathcal{F}$ and $\mathcal{G}$ are fibered categories, equivalent to $(\mathcal{C}/X)$ and $(\mathcal{C}/Y)$ for two objects X and Y of $\mathcal{C}$, then

$$\operatorname{Hom}(X,Y) = \operatorname{Hom}\bigl((\mathcal{C}/X),(\mathcal{C}/Y)\bigr),$$

and according to Proposition 3.25 we have an equivalence of categories

$$\operatorname{Hom}\bigl((\mathcal{C}/X),(\mathcal{C}/Y)\bigr) \simeq \operatorname{Hom}_{\mathcal{C}}(\mathcal{F},\mathcal{G});$$

but $\operatorname{Hom}_{\mathcal{C}}(\mathcal{F},\mathcal{G})$ need not be a set, it could very well be an equivalence relation.

3.6.2. The 2-categorical Yoneda lemma. As in the case of functors, we have a stronger version of the 2-categorical Yoneda lemma. Suppose that $\mathcal{F}$ is a category fibered over $\mathcal{C}$, and that X is an object of $\mathcal{C}$. Suppose that we are given a morphism $F\colon (\mathcal{C}/X) \to \mathcal{F}$; to this we can associate an object $F(\mathrm{id}_X) \in \mathcal{F}(X)$. Also, to each base-preserving natural transformation $\alpha\colon F \to G$ of functors $F, G\colon (\mathcal{C}/X) \to \mathcal{F}$ we associate the arrow $\alpha_{\mathrm{id}_X}\colon F(\mathrm{id}_X) \to G(\mathrm{id}_X)$. This defines a functor

$$\operatorname{Hom}_{\mathcal{C}}\bigl((\mathcal{C}/X),\mathcal{F}\bigr) \longrightarrow \mathcal{F}(X).$$

Conversely, given an object $\xi \in \mathcal{F}(X)$ we get a functor $F_\xi\colon (\mathcal{C}/X) \to \mathcal{F}$ as follows. Given an object $\phi\colon U \to X$ of $(\mathcal{C}/X)$, we define $F_\xi(\phi) = \phi^*\xi \in \mathcal{F}(U)$; to an arrow

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr, "\phi"'] & & V \arrow[dl, "\psi"] \\ & X & \end{tikzcd}$$

in $(\mathcal{C}/X)$ we associate the only arrow $\theta\colon \phi^*\xi \to \psi^*\xi$ in $\mathcal{F}(U)$ making the diagram

$$\begin{tikzcd}
\phi^*\xi \arrow[dd] \arrow[dr, dashed, "\theta"'] \arrow[drr, bend left] & & \\
& \psi^*\xi s \arrow[r] \arrow[dd] & \xi \arrow[dd] \\
U \arrow[dr, "h"'] \arrow[drr, bend left, "\phi"] & & \\
& V \arrow[r, "\psi"'] & X
\end{tikzcd}$$

commutative. We leave it to the reader to check that F_ξ is indeed a functor.

2-YONEDA LEMMA. *The two functors above define an equivalence of categories*

$$\mathrm{Hom}_{\mathcal{C}}\big((\mathcal{C}/X), \mathcal{F}\big) \simeq \mathcal{F}(X).$$

PROOF. To check that the composition

$$\mathcal{F}(X) \longrightarrow \mathrm{Hom}_{\mathcal{C}}\big((\mathcal{C}/X), \mathcal{F}\big) \longrightarrow \mathcal{F}(X)$$

is isomorphic to the identity, notice that for any object $\xi \in \mathcal{F}(X)$, the composition applied to ξ yields $F_\xi(\xi) = \mathrm{id}_X^*\xi$, which is canonically isomorphic to ξ. It is easy to check that this defines an isomorphism of functors.

For the composition

$$\mathrm{Hom}_{\mathcal{C}}\big((\mathcal{C}/X), \mathcal{F}\big) \longrightarrow \mathcal{F}(X) \longrightarrow \mathrm{Hom}_{\mathcal{C}}\big((\mathcal{C}/X), \mathcal{F}\big)$$

take a morphism $F\colon (\mathcal{C}/X) \to \mathcal{F}$ and set $\xi = F(\mathrm{id}_X)$. We need to produce a base-preserving isomorphism of functors of F with F_ξ. The identity id_X is a terminal object in the category $(\mathcal{C}/X)$, hence for any object $\phi\colon U \to X$ there is a unique arrow $\phi\colon \mathrm{id}_X$, which is clearly cartesian. Hence it will remain cartesian after applying F, because F is a functor: this means that $F(\phi)$ is a pullback of $\xi = F(\mathrm{id}_X)$ along $\phi\colon U \to X$, so there is a canonical isomorphism $F_\xi(\phi) = \phi^*\xi \simeq F(\phi)$ in $\mathcal{F}(U)$. It is easy to check that this defines a base-preserving isomorphism of functors, and this ends the proof. ♠

We have identified an object X with the functor $\mathrm{h}_X\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$ it represents, and we have identified the functor h_X with the corresponding category $(\mathcal{C}/X)$: so, to be consistent, we have to identify X and $(\mathcal{C}/X)$. So, we will write X for $(\mathcal{C}/X)$.

As for functors, the strong form of the 2-Yoneda lemma can be used to reformulate the condition of representability. A morphism $(\mathcal{C}/X) \to \mathcal{F}$ corresponds to an object $\xi \in \mathcal{F}(X)$, which in turn defines the functor $F'\colon (\mathcal{C}/X) \to \mathcal{F}$ described above; this is isomorphic to the original functor F. Then F' is an equivalence if and only if for each object U of $\mathcal{C}$ the restriction

$$F'_U\colon \mathrm{Hom}_{\mathcal{C}}(U, X) = (\mathcal{C}/X)(U) \longrightarrow \mathcal{F}(U)$$

that sends each $f\colon U \to X$ to the pullback $f^*\xi \in \mathcal{F}(U)$, is an equivalence of categories. Since $\mathrm{Hom}_{\mathcal{C}}(U, X)$ is a set, this is equivalent to saying that $\mathcal{F}(U)$ is a groupoid, and each object of $\mathcal{F}(U)$ is isomorphic to the image of a unique element of $\mathrm{Hom}_{\mathcal{C}}(U, X)$ via a unique isomorphism. Since the isomorphisms $\rho \simeq f^*\xi$ in U correspond to cartesian arrows $\rho \to \xi$, and in a groupoid all arrows are cartesian, this means that $\mathcal{F}$ is fibered in groupoids, and for each $\rho \in \mathcal{F}(U)$ there exists a unique arrow $\rho \to \xi$. We have proved the following.

PROPOSITION 3.34. *A category fibered in groupoids $\mathcal{F}$ in $\mathcal{C}$ is representable if and only if $\mathcal{F}$ is fibered in groupoids, and there is an object X of $\mathcal{C}$ and an object ξ of $\mathcal{F}(U)$, such that for any object ρ of $\mathcal{F}$ there exists a unique arrow $\rho \to \xi$ in $\mathcal{F}$.*

3.7. The functors of arrows of a fibered category

Suppose that $\mathcal{F} \to \mathcal{C}$ is a fibered category; if U is an object in $\mathcal{C}$ and ξ, η are objects of $\mathcal{F}(U)$, we denote by $\mathrm{Hom}_U(\xi, \eta)$ the set of arrow from ξ to η in $\mathcal{F}(U)$.

Let ξ and η be two objects of $\mathcal{F}$ over the same object S of $\mathcal{C}$. Let $u_1 \colon U_1 \to S$ and $u_2 \colon U_2 \to S$ be arrows in $\mathcal{C}$; these are objects of the category $(\mathcal{C}/S)$. Suppose that $\xi_i \to \xi$ and $\eta_i \to \eta$ are pullbacks along $u_i \colon U_i \to S$ for $i = 1$, 2. For each arrow $f \colon U_1 \to U_2$ in $(\mathcal{C}/S)$, by definition of pullback there are two arrows, each unique, $\alpha_f \colon \xi_1 \to \xi_2$ and $\beta_f \colon \eta_1 \to \eta_2$, such that and the two diagrams

$$\begin{tikzcd} \xi_1 \arrow[rr, "\alpha_f"] \arrow[dr] & & \xi_2 \arrow[dl] \\ & \xi & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \eta_1 \arrow[rr, "\beta_f"] \arrow[dr] & & \eta_2 \arrow[dl] \\ & \xi & \end{tikzcd}$$

commute. By Proposition 3.4 (iii) the arrows α_f and β_f are cartesian; we define a pullback function

$$f^* \colon \mathrm{Hom}_{U_2}(\xi_2, \eta_2) \longrightarrow \mathrm{Hom}_{U_1}(\xi_1, \eta_1)$$

in which $f^*\phi$ is defined as the only arrow $f^*\phi \colon \xi_1 \to \eta_1$ in $\mathcal{F}(U_1)$ making the diagram

$$\begin{tikzcd} \xi_1 \arrow[r, "f^*\phi"] \arrow[d, "\alpha_f"] & \eta_1 \arrow[d, "\beta_f"] \\ \xi_2 \arrow[r, "\phi"] & \eta_2 \end{tikzcd}$$

commute. If we are given a third arrow $g \colon U_2 \to U_3$ in $(\mathcal{C}/S)$ with pullbacks $\xi_3 \to \xi$ and $\eta_3 \to \eta$, we have arrows $\alpha_g \colon \xi_2 \to \xi_3$ and $\beta_g \colon \eta_2 \to \eta_3$; it is immediate to check that

$$\alpha_{gf} = \alpha_g \circ \alpha_f \colon \xi_1 \to \xi_3 \quad \text{and} \quad \beta_{gf} = \beta_g \circ \beta_f \colon \eta_1 \to \eta_3$$

and this implies that

$$(gf)^* = f^*g^* \colon \mathrm{Hom}_{U_3}(\xi_3, \eta_3) \longrightarrow \mathrm{Hom}_{U_1}(\xi_1, \eta_1).$$

After choosing a cleavage for $\mathcal{F}$, we can define a functor

$$\underline{\mathrm{Hom}}_S(\xi,\eta)\colon (\mathcal{C}/S)^{\mathrm{op}} \to (\mathrm{Set})$$

by sending each object $u\colon U \to S$ into the set $\mathrm{Hom}_U(u^*\xi, u^*\eta)$ of arrows in the category $\mathcal{F}(U)$. An arrow $f\colon U_1 \to U_2$ from $u_1\colon U_1 \to S$ to $u_2\colon U_2 \to S$ yields a function

$$f^*\colon \mathrm{Hom}_{U_2}(u_2^*\xi, u_2^*\eta) \longrightarrow \mathrm{Hom}_{U_1}(u_1^*\xi, u_1^*\eta);$$

and this defines the effect of $\underline{\mathrm{Hom}}_S(\xi,\eta)$ on arrows.

It is easy to check that the functor $\underline{\mathrm{Hom}}_S(\xi,\eta)$ is independent of the choice of a cleavage, in the sense that cleavages give canonically isomorphic functors. Suppose that we have chosen for each $f\colon U \to V$ and each object ζ in $\mathcal{F}(V)$ another pullback $f^\vee\zeta \to \zeta$: then there is a canonical isomorphism $u^*\eta \simeq u^\vee\eta$ in $\mathcal{F}(U)$ for each arrow $u\colon U \to S$, and this gives a bijective correspondence

$$\mathrm{Hom}_U(u^*\xi, u^*\eta) \simeq \mathrm{Hom}_S(u^\vee\xi, u^\vee\eta),$$

yielding an isomorphism of the functors of arrows defined by the two pullbacks.

In fact, $\underline{\mathrm{Hom}}_S(\xi,\eta)$ can be more naturally defined as a quasi-functor $\mathcal{H}om_S(\xi,\eta) \to (\mathcal{C}/S)$; this does not require any choice of cleavages.

From this point of view, the objects of $\mathcal{H}om_S(\xi,\eta)$ over some object $u\colon U \to S$ of $(\mathcal{C}/S)$ are triples

$$(\xi_1 \to \xi, \eta_1 \to \eta, \phi),$$

where $\xi_i \to \xi$ and $\eta_i \to \eta$ are cartesian arrows of $\mathcal{F}$ over $u\colon U \to S$, and $\phi\colon \xi_1 \to \eta_1$ is an arrow in $\mathcal{F}(U)$. An arrow from $(\xi_1 \to \xi, \eta_1 \to \eta, \phi_1)$ over $u_1\colon U_1 \to S$ and $(\xi_2 \to \xi, \eta_2 \to \eta, \phi_2)$ over $u_2\colon U_2 \to S$ is an arrow $f\colon U_1 \to U_2$ in $(\mathcal{C}/S)$ such that $f^*\phi_2 = \phi_1$.

From Proposition 3.30 we see that $\mathcal{H}om_S(\xi,\eta)$ is a quasi-functor over $\mathcal{C}$, and therefore, by Proposition 3.31, it is equivalent to a functor: of course this is the functor $\underline{\mathrm{Hom}}_S(\xi,\eta)$ obtained by the previous construction.

This can be proved as follows: the objects of $\underline{\mathrm{Hom}}_S(\xi,\eta)$, thought of as a category fibered in sets over $(\mathcal{C}/S)$ are pairs $(\phi, u\colon U \to S)$, where $u\colon U \to S$ is an object of $(\mathcal{C}/S)$ and $\phi\colon u^*\xi \to u^*\eta$ is an arrow in $\mathcal{F}(U)$; this also gives an object $(u^*\xi \to \xi, u^*\eta \to \eta, \phi)$ of $\mathcal{H}om_S(\xi,\eta)$ over U. The arrows between objects of $\underline{\mathrm{Hom}}_S(\xi,\eta)$ to another are precisely the arrows between the corresponding objects of $\mathcal{H}om_S(\xi,\eta)$, so we have an embedding of $\underline{\mathrm{Hom}}_S(\xi,\eta)$ into $\mathcal{H}om_S(\xi,\eta)$. But every object of $\mathcal{H}om_S(\xi,\eta)$ is isomorphic to an object of $\underline{\mathrm{Hom}}_S(\xi,\eta)$, hence the two fibered categories are equivalent.

CHAPTER 4

Stacks

4.1. Descent of objects of fibered categories

4.1.1. Glueing continuous maps and topological spaces. The following is the archetypical example of descent. Take (Cont) to be the category of continuous maps (that is, the category of arrows in (Top), as in Example 3.11); this category is fibered on (Top) via the functor $\mathrm{p}_{(\mathrm{Cont})}\colon (\mathrm{Cont}) \to (\mathrm{Top})$ sending each continuous map to its codomain. Now, suppose that $f\colon X \to U$ and $g\colon Y \to U$ are two objects of (Cont) mapping to the same object U in (Top); we want to construct a continuous map $\phi\colon X \to Y$ over U, that is, an arrow in $(\mathrm{Cont})(U) = (\mathrm{Top}/U)$. Suppose that we are given an open covering $\{U_i\}$ of U, and continuous maps $\phi_i\colon f^{-1}U_i \to g^{-1}U_i$ over U_i; assume furthermore that the restriction of ϕ_i and ϕ_j to $f^{-1}(U_i \cap U_j) \to g^{-1}(U_i \cap U_j)$ coincide. Then there is a unique continuous map $\phi\colon X \to Y$ over U whose restriction to each $f^{-1}U_i$ coincides with f_i.

This can be written as follows. The category (Cont) is fibered over (Top), and if $f\colon V \to U$ is a continuos map, $X \to U$ an object of $(\mathrm{Cont})(U) = (\mathrm{Top}/U)$, then a pullback of $X \to U$ to V is given by the projection $V \times_U X \to V$. The functor $f^*\colon (\mathrm{Cont})(U) \to (\mathrm{Cont})(V)$ sends each object $X \to U$ to $V \times_U X \to V$, and each arrow in (Top/U), given by continuous function $\phi\colon X \to Y$ over U, to the continuous function $f^*\phi = \mathrm{id}_V \times_U f\colon V \times_U X \to V \times_U Y$.

Suppose that we are given two topological spaces X and Y with continuous maps $X \to S$ and $Y \to S$. Consider the functor

$$\underline{\mathrm{Hom}}_S(X,Y)\colon (\mathrm{Top}/S) \to (\mathrm{Set})$$

from the category of topological spaces over S, defined in Section 3.7. This sends each arrow $U \to S$ to the set of continuous maps $\mathrm{Hom}_U(U \times_S X, U \times_S Y)$ over U. The actions on arrows is obtained as follows: Given a continuous function $f\colon V \to U$, we send each continuous function $\phi\colon U \times_S X \to U \times_S Y$ to the function

$$f^*\phi = \mathrm{id}_V \times \phi\colon V \times_S X = V \times_U (U \times_S X) \to V \times_U (U \times_S Y) = V \times_S Y$$

.

Then the fact that continuous functions can be constructed locally and then glued together can be expressed by saying that the functor

$$\underline{\mathrm{Hom}}_S(X,Y)\colon (\mathrm{Top}/S)^{\mathrm{op}} \to (\mathrm{Set})$$

is a sheaf in the classical topology of (Top).

But there is more: not only we can construct continuous functions locally: we can also do this for spaces, although this is more complicated.

PROPOSITION 4.1. *Suppose that we are given a topological space U with an open covering $\{U_i\}$; for each triple of indices i, j and k set $U_{ij} = U_i \cap U_j$ and $U_{ijk} = U_i \cap U_j \cap U_k$. Assume that for each i we have a continuous map $u_i\colon X_i \to U_i$, and that for each pair of indices i and j we have a homeomorphism $\phi_{ij}\colon u_j^{-1}U_{ij} \simeq u_i^{-1}U_{ij}$ over U_{ij}, satisfying the cocycle condition*

$$\phi_{ik} = \phi_{ij} \circ \phi_{jk}\colon u_k^{-1}U_{ijk} \to u_j^{-1}U_{ijk} \to u_i^{-1}U_{ijk}.$$

Then there exists a continuous map $u\colon X \to U$, together with isomorphisms $\phi_i\colon u^{-1}U_i \simeq X_i$, such that $\phi_{ij} = \phi_i \circ \phi_j^{-1}\colon u_j^{-1}U_{ij} \to u^{-1}U_{ij} \to u_i U_{ij}$ for all i and j.

PROOF. Consider the disjoint union U' of the U_i; the fiber product $U' \times_U U'$ is the disjoint union of the U_{ij}. The disjoint union X' of the X_i, maps to U'; consider the subset $R \subseteq X' \times X'$ consisting of pairs $(x_i, x_j) \in X_i \times X_j \subseteq X' \times X'$ such that $x_i = \phi_{ij} x_j$. I claim that R is an equivalence relation in X'. Notice that the cocycle condition $\phi_{ii} = \phi_{ii} \circ \phi_{ii}$ implies that ϕ_{ii} is the identity on X_i, and this show that the equivalence relation is reflexive. The fact that $\phi_{ii} = \phi_{ij} \circ \phi_{ji}$, and therefore $\phi_{ji} = \phi_{ij}^{-1}$, prove that it is symmetric; and transitivity follows directly from the general cocycle condition. We define X to be the quotient X'/R.

If two points of X' are equivalent, then their images in U coincide; so there is an induced continuous map $u\colon X \to U$. The restriction to $X_i \subseteq X'$ of the projection $X' \to X$ gives a continuous map $\phi_i\colon X_i \to u^{-1}U_i$, that's easily checked to be a homeomorphism. One also sees that $\phi_{ij} = \phi_i \circ \phi_j^{-1}$, and this completes the proof. ♠

The facts that we can glue continuous maps and topological spaces say that (Cont) is a *stack* over (Top).

4.1.2. The category of descent data. Let $\mathcal{C}$ be a site. We have seen that a fibered category over $\mathcal{C}$ should be thought of as a contravariant functor from $\mathcal{C}$ to the category of categories, that is, presheaves of categories over $\mathcal{C}$. A stack is, morally, a sheaf of categories over $\mathcal{C}$.

Let $\mathcal{F}$ be a category fibered over $\mathcal{C}$. We will assume that it has given a cleavage; but we also indicate how the definitions can be given without resorting to the choice of a cleavage.

Given a covering $\{\sigma\colon U_i \to U\}$, set $U_{ij} = U_i \times_U U_j$ and $U_{ijk} = U_i \times_U U_j \times_U U_k$ for each triple of indices i, j and k.

DEFINITION 4.2. Let $\mathcal{U} = \{\sigma_i\colon U_i \to U\}$ be a covering in $\mathcal{C}$. An *object with descent data* $(\{\xi_i\}, \{\phi_{ij}\})$ on $\mathcal{U}$, is a collection of objects $\xi_i \in \mathcal{F}(U_i)$, together with isomorphisms $\phi_{ij}\colon \mathrm{pr}_2^* \xi_j \simeq \mathrm{pr}_1^* \xi_i$ in $\mathcal{F}(U_i \times_U U_j)$, such that the following cocycle condition is satisfied.

For any triple of indices i, j and k, we have the equality

$$\mathrm{pr}_{13}^*\phi_{ik} = \mathrm{pr}_{12}^*\phi_{ij} \circ \mathrm{pr}_{23}^*\phi_{jk} \colon \mathrm{pr}_3^*\,\xi_k \longrightarrow \mathrm{pr}_1^*\,\xi_i$$

where the pr_{ab} and pr_a are projections on the a^{th} and b^{th} factor, or the a^{th} factor respectively.

The isomorphisms ϕ_{ij} are called *transition isomorphisms* of the object with descent data.

An arrow between objects with descent data

$$\{\alpha_i\} \colon (\{\xi_i\}, \{\phi_{ij}\}) \longrightarrow (\{\eta_i\}, \{\psi_{ij}\})$$

is a collection of arrows $\alpha_i \colon \xi_i \to \eta_i$ in $\mathcal{F}(U_i)$, with the property that for each pair of indices i, j, the diagram

$$\begin{tikzcd}
\mathrm{pr}_2^*\,\xi_j \arrow[r, "\mathrm{pr}_2^*\,\alpha_j"] \arrow[d, "\phi_{ij}"] & \mathrm{pr}_2^*\,\eta_j \arrow[d, "\psi_{ij}"] \\
\mathrm{pr}_1^*\,\xi_i \arrow[r, "\mathrm{pr}_1^*\,\alpha_i"] & \mathrm{pr}_1^*\,\eta_i
\end{tikzcd}$$

commutes.

In understanding the definition above it may be useful to contemplate the cube

(4.1.1)
$$\begin{tikzcd}
& U_{ijk} \arrow[rr, "\mathrm{pr}_{23}"] \arrow[dl, "\mathrm{pr}_{12}"'] \arrow[dd, "\mathrm{pr}_{13}"] & & U_{jk} \arrow[dl] \arrow[dd] \\
U_{ij} \arrow[rr] \arrow[dd] & & U_j \arrow[dd] & \\
& U_{ik} \arrow[rr] \arrow[dl] & & U_k \arrow[dl] \\
U_i \arrow[rr] & & U &
\end{tikzcd}$$

in which all arrows are given by projections, and every face is cartesian.

There is an obvious way of composing morphisms, that makes objects with descent data the objects of a category, denoted by $\mathcal{F}(\mathcal{U}) = \mathcal{F}(\{U_i \to U\})$.

For each object ξ of $\mathcal{F}(U)$ we can construct an object with descent data on a covering $\{\sigma_i \colon U_i \to U\}$ as follows. The objects are the pullbacks $\sigma_i^*\xi$; the isomorphisms $\phi_{ij} \colon \mathrm{pr}_2^*\,\sigma_j^*\xi \simeq \mathrm{pr}_1^*\,\sigma_i^*\xi$ are the isomorphisms that come from the fact that both $\mathrm{pr}_2^*\,\sigma_j^*\xi$ and $\mathrm{pr}_1^*\,\sigma_i^*\xi$ are pullbacks of ξ to U_{ij}. If we identify $\mathrm{pr}_2^*\,\sigma_j^*\xi$ with $\mathrm{pr}_1^*\,\sigma_i^*\xi$, as is commonly done, then the ϕ_{ij} are identities.

Given an arrow $\alpha \colon \xi \to \eta$ in $\mathcal{F}(U)$, we get arrows $\sigma_i^* \colon \sigma_i^*\xi \to \sigma_i^*\eta$, yielding an arrow from the object with descent associated with ξ to the associated with η. This defines a functor $\mathcal{F}(U) \to \mathcal{F}(\{U_i \to U\})$.

It is important to notice that these construction do not depend on the choice of a cleavage, in the following sense. Given a different cleavage, for each covering $\{U_i \to U\}$ there is a canonical equivalence of the resulting categories $\mathcal{F}(\{U_i \to U\})$; and the functors $\mathcal{F}(U) \to \mathcal{F}(\{U_i \to U\})$ commute with these equivalences.

There is a very elegant treatment of descent data that does not depend on choosing of a cleaving; we will not use this approach, so we will be a little sketchy. Let $\{U_i \to U\}_{i \in I}$ be a covering. We define an object with descent data to be a triple of sets

$$(\{\xi_i\}_{i \in I}, \{\xi_{ij}\}_{i,j \in I}, \{\xi_{ijk}\}_{i,j,k \in I})$$

in each ξ_α is an object of $\mathcal{F}(U_\alpha)$, plus a commutative diagram

$$\begin{tikzcd}
& \xi_{ijk} \arrow[rr] \arrow[dl] \arrow[dd] & & \xi_{jk} \arrow[dl] \arrow[dd] \\
\xi_{ij} \arrow[rr] \arrow[dd] & & \xi_j & \\
& \xi_{ik} \arrow[rr] \arrow[dl] & & \xi_k \\
\xi_i & & &
\end{tikzcd}$$

in $\mathcal{F}$ for all the triples of indices in which every arrow is cartesian, and such that when applying $\mathrm{p}_{\mathcal{F}}$ every arrow maps to the appropriate projection in the diagram (4.1.1). These form the objects of a category $\mathcal{F}_{\mathrm{desc}}(\{U_i \to U\})$.

An arrow

$$\{\phi_i\}_{i \in I} \colon (\{\xi_i\}, \{\xi_{ij}\}, \{\xi_{ijk}\}) \longrightarrow (\{\eta_i\}, \{\eta_{ij}\}, \{\eta_{ijk}\})$$

consists of set of arrows with $\phi_i : \xi_i \to \eta_i$ in $\mathcal{F}(U_i)$, such that for every pair of indices i and j we have

$$\mathrm{pr}_1^* \phi_i = \mathrm{pr}_2^* \phi_j \colon \xi_{ij} \to \eta_{ij}.$$

Alternatively, and perhaps more naturally, we could define an arrow as a triple $((\{\phi_i\}_{i \in I}, \{\phi_{ij}\}_{i,j \in I}, \{\phi_{ijk}\}_{i,j,k \in I}))$, where $\phi_\alpha \colon \xi_\alpha \to \eta_\alpha$ is an arrow in $\mathcal{F}(U_\alpha)$ for each α, with the obvious compatibility conditions with the various arrows involved in the definition of an object. We leave it to the reader to check that these two definition of an arrow are equivalent.

Once we have chosen a cleaving, there is a functor from $\mathcal{F}_{\mathrm{desc}}(\{U_i \to U\})$ to $\mathcal{F}(\{U_i \to U\})$. Given an object $(\{\xi_i\}, \{\xi_{ij}\}, \{\xi_{ijk}\})$ of $\mathcal{F}_{\mathrm{desc}}(\{U_i \to U\})$, the arrows $\xi_{ij} \to \xi_i$ and $\xi_{ij} \to \xi_j$ induce isomorphisms $\xi_{ij} \simeq \mathrm{pr}_1^* \xi_i$ and $\xi_{ij} \simeq \mathrm{pr}_2^* \xi_j$; the resulting isomorphism $\mathrm{pr}_2^* \xi_j \simeq \mathrm{pr}_1^* \xi_i$ is easily seen to satisfy the cocycle condition, thus defining an object of $\mathcal{F}(\{U_i \to U\})$. An arrow $\{\phi_i\}$ in $\mathcal{F}_{\mathrm{desc}}(\{U_i \to U\})$ is already an arrow in $\mathcal{F}(\{U_i \to U\})$.

It is not hard to check that this functor is an equivalence of categories.

We can not define a functor $\mathcal{F}(U) \to \mathcal{F}_{\mathrm{desc}}(\{U_i \to U\})$ directly, without the choice of a cleaving. However, let us define another category

$$\mathcal{F}_{\mathrm{comp}}(\{U_i \to U\}),$$

in which the objects are quadruples $(\xi, \{\xi_i\}, \{\xi_{ij}\}, \{\xi_{ijk}\})$, where ξ is an object of $\mathcal{F}(U)$ and each ξ_α is an object of $\mathcal{F}(U_\alpha)$, plus a commutative cube

$$\begin{tikzcd}
& \xi_{ijk} \arrow[rr] \arrow[dl] \arrow[dd] & & \xi_{jk} \arrow[dl] \arrow[dd] \\
\xi_{ij} \arrow[rr] \arrow[dd] & & \xi_j \arrow[dd] & \\
& \xi_{ik} \arrow[rr] \arrow[dl] & & \xi_k \arrow[dl] \\
\xi_i \arrow[rr] & & \xi &
\end{tikzcd}$$

in $\mathcal{F}$ for all the triples of indices, in which all the arrows are cartesian, and whose image in $\mathcal{C}$ is the cube (4.1.1). An arrow from $(\xi, \{\xi_i\}, \{\xi_{ij}\}, \{\xi_{ijk}\})$ to $(\eta, \{\eta_i\}, \{\eta_{ij}\}, \{\eta_{ijk}\})$ can be indifferently defined as an arrow $\phi\colon \xi \to \eta$ in $\mathcal{F}(U)$, or as collections of arrows $\xi \to \eta$, $\xi_i \to \eta_i$, $\xi_{ij} \to \eta_{ij}$ and $\xi_{ijk} \to \eta_{ijk}$ satisfying the obvious commutativity conditions.

There is a functor from $\mathcal{F}_{\mathrm{comp}}(\{U_i \to U\})$ to $\mathcal{F}(U)$ that sends a whole object $(\xi, \{\xi_i\}, \{\xi_{ij}\}, \{\xi_{ijk}\})$ into ξ, and is easily seen to be an equivalence. There is also a functor from $\mathcal{F}_{\mathrm{comp}}(\{U_i \to U\})$ to $\mathcal{F}_{\mathrm{desc}}(\{U_i \to U\})$ that forgets the object of $\mathcal{F}(U)$. This takes the place of the functor from $\mathcal{F}(U)$ to $\mathcal{F}(\{U_i \to U\})$ defined using cleavages.

Yet another definition of objects with descent data can be given as follows (this was pointed out to me by Behrang Noohi). Let $\mathcal{U} = \{U_i \to U\}$ a covering in $\mathcal{C}$. Define $(\mathcal{C}/\mathcal{U})$ to be the category fibered in sets on $\mathcal{C}$ associated with the functor $\mathrm{h}_{\mathcal{U}}\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$, defined immediately before Proposition 2.28. This is the full subcategory of $(\mathcal{C}/U)$, whose objects are arrows $T \to U$ that factor through some $U_i \to U$.

There is a functor $\mathrm{Hom}_{\mathcal{C}}\bigl((\mathcal{C}/\mathcal{U}), \mathcal{F}\bigr) \to \mathcal{F}_{\mathrm{desc}}(\mathcal{U})$, defined as follows. Suppose that we are given a morphism $F\colon (\mathcal{C}/\mathcal{U}) \to (\mathrm{Set})$. For any triple of indices i, j and k we have objects $U_i \to U$, $U_{ij} \to U$ and $U_k \to U$ of $(\mathcal{C}/\mathcal{U})$; and each of the projections of (4.1.1) not landing in U is an arrow in $(\mathcal{C}/\mathcal{U})$. Hence we can apply F and get a diagram

$$\begin{tikzcd}
& F(U_{ijk}) \arrow[rr] \arrow[dl] \arrow[dd] & & F(U_{jk}) \arrow[dl] \arrow[dd] \\
F(U_{ij}) \arrow[rr] \arrow[dd] & & F(U_j) & \\
& F(U_{ik}) \arrow[rr] \arrow[dl] & & F(U_k) \\
F(U_i) & & &
\end{tikzcd}$$

giving an object of $\mathcal{F}_{\mathrm{desc}}(\mathcal{U})$. This extends to a functor in the obvious way, and, using the 2-Yoneda lemma, it is not hard to see that this functor is an equivalence of categories.

The embedding $(\mathcal{C}/\mathcal{U}) \subseteq (\mathcal{C}/U)$ induces a functor

$$\mathrm{Hom}_{\mathcal{C}}\bigl((\mathcal{C}/U), \mathcal{F}\bigr) \longrightarrow \mathrm{Hom}_{\mathcal{C}}\bigl((\mathcal{C}/\mathcal{U}), \mathcal{F}\bigr);$$

if we choose a cleavage, the composition of functors

$$\mathrm{Hom}_{\mathcal{C}}\big((\mathcal{C}/U), \mathcal{F}\big) \longrightarrow \mathrm{Hom}_{\mathcal{C}}\big((\mathcal{C}/\mathcal{U}), \mathcal{F}\big) \simeq \mathcal{F}_{\mathrm{desc}}(\mathcal{U}) \simeq \mathcal{F}(\mathcal{U})$$

is isomorphic to the composition

$$\mathrm{Hom}_{\mathcal{C}}\big((\mathcal{C}/U), \mathcal{F}\big) \simeq \mathcal{F}(U) \longrightarrow \mathcal{F}(\mathcal{U})$$

where the first is the equivalence of the 2-Yoneda lemma.

4.1.3. Fibered categories with descent.

DEFINITION 4.3. Let $\mathcal{F} \to \mathcal{C}$ be a fibered category on a site $\mathcal{C}$.

(i) $\mathcal{F}$ is a *prestack* over $\mathcal{C}$ if for each covering $\{U_i \to U\}$ in $\mathcal{C}$, the functor $\mathcal{F}(U) \to \mathcal{F}(\{U_i \to U\})$ is fully faithful.

(ii) $\mathcal{F}$ is a *stack* over $\mathcal{C}$ if for each covering $\{U_i \to U\}$ in $\mathcal{C}$, the functor $\mathcal{F}(U) \to \mathcal{F}(\{U_i \to U\})$ is an equivalence of categories.

This condition can be restated using the functor of arrows of Section 3.7.

The category $(\mathcal{C}/S)$ inherits a Grothendieck topology from the given Grothendieck topology on $\mathcal{C}$; simply, a covering of an object $U \to S$ of $(\mathcal{C}/S)$ is a collection of arrows

$$\begin{tikzcd} U_i \arrow[rr, "f_i"] \arrow[dr] & & U \arrow[dl] \\ & S & \end{tikzcd}$$

such that the collection $\{f_i \colon U_i \to U\}$ is a covering in $\mathcal{C}$. In other words, the coverings of $U \to S$ are simply the coverings of U.

Finally, we have the following definition.

DEFINITION 4.4. An object with descent data $(\{\xi_i\}, \{\phi_{ij}\})$ in $\mathcal{F}(\{U_i \to U\})$ is *effective* if it is isomorphic to the image of an object of $\mathcal{F}(U)$.

Another way of saying this is as follows: an object with descent data $(\{\xi_i\}, \{\phi_{ij}\})$ in $\mathcal{F}(\{U_i \to U\})$ is effective if there exists an object ξ of $\mathcal{F}(U)$, together with cartesian arrows $\xi_i \to \xi$ over $\sigma\colon U_i \to U$, such that the diagram

$$\begin{tikzcd} \mathrm{pr}_2^* \xi_j \arrow[rr, "\phi_{ij}"] \arrow[d] & & \mathrm{pr}_1^* \xi_i \arrow[d] \\ \xi_j \arrow[dr] & & \xi_i \arrow[dl] \\ & \xi & \end{tikzcd}$$

commutes for all i and j. In fact, the cartesian arrows $\xi_i \to \xi$ correspond to isomorphisms $\xi_i \simeq \sigma_i^* \xi$ in $\mathcal{F}(U_i)$; and the commutativity of the diagram above

PROPOSITION 4.5. *Let $\mathcal{F}$ be a fibered category over a site $\mathcal{C}$.*

(i) *$\mathcal{F}$ is a prestack if and only if for any object S of $\mathcal{C}$ and any two objects ξ and η in $\mathcal{F}(S)$, the functor $\underline{\mathrm{Hom}}_S(\xi, \eta)\colon (\mathcal{C}/S) \to (\mathrm{Set})$ is a sheaf.*

(ii) *$\mathcal{F}$ is a stack if and only if it is a prestack, and all objects with descent data in $\mathcal{F}$ are effective.*

PROOF. Let us prove the first part. Assume that for any object S of $\mathcal{C}$ and any two objects ξ and η in $\mathcal{F}(S)$, the functor $\underline{\mathrm{Hom}}_S(\xi,\eta)\colon (\mathcal{C}/S) \to (\mathrm{Set})$ is a sheaf. Take an object U of $\mathcal{C}$, a covering $\{U_i \to U\}$, and two objects ξ and η of $\mathcal{F}(U)$. If we call $(\{\xi_i\},(\alpha_{ij}))$ and $(\{\eta_i\},(\beta_{ij}))$ the descent data associated with ξ and η respectively, we see easily that the arrows in $\mathcal{F}(\{U_i \to U\})$ are the collections of arrows $\{\phi_i\colon \xi_i \to \eta_i\}$ such that the restrictions of ϕ_i and ϕ_j to the pullbacks of ξ and η to U_{ij} coincide. The fact that $\underline{\mathrm{Hom}}_U(\xi,\eta)$ insures that this comes from a unique arrow $\xi \to \eta$ in $\mathcal{F}(U)$; but this means precisely that the functor $\mathcal{F}(U) \to \mathcal{F}(\{U_i \to U\})$ is fully faithful.

The proof of the opposite implication is similar, and left to the reader.

The second part follows immediately from the first. ♠

Using the description of the category of objects with descent data given at the end of 4.1.2 we can given the following very elegant characterization of stacks (this was explained to me by Behrang Noohi).

PROPOSITION 4.6. *A fibered category $\mathcal{F} \to \mathcal{C}$ is a stack if and only if for any covering $\mathcal{U}$ in $\mathcal{C}$ the functor*

$$\mathrm{Hom}_{\mathcal{C}}\big((\mathcal{C}/U),\mathcal{F}\big) \longrightarrow \mathrm{Hom}_{\mathcal{C}}\big((\mathcal{C}/\mathcal{U}),\mathcal{F}\big)$$

induced by the embedding $(\mathcal{C}/\mathcal{U}) \subseteq (\mathcal{C}/U)$ is an equivalence.

An equivalence of fibered categories $\mathcal{F} \simeq \mathcal{G}$ induces an equivalence of categories $\mathcal{F}(\mathcal{U}) \simeq \mathcal{F}(\mathcal{U})$ for all coverings $\mathcal{U}$ in $\mathcal{C}$. From this it is not hard to show the following fact.

PROPOSITION 4.7. *If two fibered categories over a site are equivalent, and one of them is a stack, the other is also a stack.*

Stacks are the correct generalization of sheaves, and give the right notion of "sheaf of categories". We should of course prove the following statement.

PROPOSITION 4.8. *Let $\mathcal{C}$ be a site, $F\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$ a functor; we can also consider it as a category fibered in sets $F \to \mathcal{C}$.*

(i) *F is a prestack if and only if it is a separated functor.*
(ii) *F is stack if and only if it is a sheaf.*

PROOF. Consider a covering $\{U_i \to U\}$. The fiber of the category $F \to \mathcal{C}$ over U is precisely the set $F(U)$, while the category $F(\{U_i \to U\})$ is the set of elements $(\xi_i) \in \prod_i F(U_i)$ such that the pullbacks of ξ_i and ξ_j to $F(U_i \times_U U_j)$, via the first and second projections $U_i \times_U U_j \to U_i$ and $U_i \times_U U_j \to U_i$, coincide. The functor $F(U) \to F(\{U_i \to U\})$ is the function that sends each element $\xi \in F(U)$ to the collection of restrictions $(\xi\mid_{U_i})$.

Now, to say that a function, thought of as a functor between discrete categories, is fully faithful is equivalent to saying that it is injective; while to say that it is an equivalence it is like saying that it is a bijection. From this both statements follow. ♠

4.2. Descent theory for quasicoherent sheaves

4.2.1. Descent for modules over commutative rings. Here we develop an affine version of the descent theory for quasicoherent sheaves. It is only needed to prove Theorem 4.11 below, so it may be a good idea to postpone reading it until it is necessity arises.

If A is a commutative ring, we will denote by Mod_A the category of modules over A. If N is an A-module, we denote by $\iota_N\colon N \otimes_A B \simeq B \otimes_A N$ the usual isomorphism defined by $\iota_N(n \otimes b) = b \otimes n$. Furthermore, we denote by $\alpha_M\colon M \to B \otimes M$ the homomorphism defined by $\alpha_M(m) = 1 \otimes m$.

Consider a ring homomorphism $f\colon A \to B$. For each $r \geq 0$ set

$$B^{\otimes r} = \overbrace{B \otimes_A B \otimes_A \cdots \otimes_A B}^{r \text{ times}}.$$

A B-module N becomes a module over $B^{\otimes 2}$ in two different ways, as $N \otimes_A B$ and $B \otimes_A N$; in both cases the multiplication is defined by the formula $(b_1 \otimes b_2)(x_1 \otimes x_2) = b_1 x_1 \otimes b_2 x_2$. Analogously, N becomes a module over $B^{\otimes 3}$ as $N \otimes_A B \otimes_A B$, $B \otimes_A N \otimes_A B$ and $B \otimes_A B \otimes_A N$ (more generally, N becomes a module over $B^{\otimes r}$ in r different ways; but we will not need this).

Let us assume that we have a homomorphism of $B^{\otimes 2}$-modules $\psi\colon N \otimes_A B \longrightarrow B \otimes_A N$. Then there are three associated homomorphism of $B^{\otimes 3}$-modules

$$\begin{aligned}
\psi_1 &\colon B \otimes_A N \otimes_\alpha B \longrightarrow B \otimes_A B \otimes_A N,\\
\psi_2 &\colon N \otimes_A B \otimes_A B \longrightarrow B \otimes_A B \otimes_A N,\\
\psi_3 &\colon N \otimes_A B \otimes_A B \longrightarrow B \otimes_A N \otimes B
\end{aligned}$$

by inserting the identity in the first, second and third position, respectively. More explicitly, we have $\psi_1 = \mathrm{id}_B \otimes \psi$, $\psi_2 = \psi \otimes \mathrm{id}_B$, while we have $\psi_2(x_1 \otimes x_2 \otimes x_3) = \sum_i y_i \otimes x_2 \otimes z_i$ if $\psi(x_1 \otimes x_3) = \sum_i y_i \otimes z_i \in B \otimes_A N$. Alternatively, $\psi_2 = (\mathrm{id}_B \otimes \iota_N) \circ \psi \circ (\mathrm{id}_N \otimes \iota_B)$.

Let us define a category $\mathrm{Mod}_{A \to B}$ as follows. Its objects are pairs (N, ψ), where N is a B-module and $\psi\colon N \otimes_A B \simeq B \otimes_A N$ is an isomorphism of $B^{\otimes 2}$-modules such that

$$\psi_2 = \psi_1 \circ \psi_3\colon N \otimes_A B \otimes_A B \longrightarrow B \otimes_A B \otimes_A N.$$

An arrow $\beta\colon (N, \psi) \to (N', \psi')$ is a homomorphism of B-modules $\beta\colon N \to N'$, making the diagram

$$\begin{tikzcd}
N \otimes_A B \arrow[r, "\psi"] \arrow[d, "\beta \otimes \mathrm{id}_B"] & B \otimes_A N \arrow[d, "\mathrm{id}_B \otimes \beta"] \\
N' \otimes_A B \arrow[r, "\psi'"] & B \otimes_A N'
\end{tikzcd}$$

commutative.

We have a functor $F\colon \mathrm{Mod}_A \to \mathrm{Mod}_{A\to B}$, sending an A-module M into the pair $(B \otimes_A M, \psi_M)$, where

$$\psi_M\colon (B \otimes_A M) \otimes_A B \longrightarrow B \otimes_A (B \otimes_A M)$$

is defined by the rule

$$\psi_M(b \otimes m \otimes b') = b \otimes b' \otimes m.$$

It is easily checked that ψ_M is an isomorphism of $B^{\otimes 2}$-modules, and that $(M \otimes_A B, \psi_M)$ is in fact an object of $\mathrm{Mod}_{A\to B}$.

If $\alpha\colon M \to M'$ is a homomorphism of A-modules, one sees immediately that $\mathrm{id}_B \otimes \alpha\colon B \otimes_A M \to B \otimes_A M'$ is an arrow in $\mathrm{Mod}_{A\to B}$. This defines the desired functor F.

THEOREM 4.9. *If B is faithfully flat over A, the functor*

$$F\colon \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_{A\to B}$$

defined above is an equivalence of categories.

PROOF. Let us define a functor $G\colon \mathrm{Mod}_{A\to B} \to \mathrm{Mod}_A$. We send an object (N, ψ) into the A-submodule $GN \subseteq N$ consisting of elements $n \in N$ such that $1 \otimes n = \psi(n \otimes 1)$.

Given an arrow $\beta\colon (N, \psi) \to (N', \psi')$ in $\mathrm{Mod}_{A\to B}$, it follows from the definition of an arrow that β takes GN to GN'; this defines the functor G.

We need to check that the compositions GF and FG are isomorphic to the identity. For this we need the following generalization of Lemma 2.39. Recall that we have defined the two homomorphism of A-algebras

$$e_1, e_2\colon B \to B \otimes_A B$$

by $e_1(b) = b \otimes 1$ and $e_2(b) = 1 \otimes b$.

LEMMA 4.10. *Let M be a A-module. Then the sequence*

$$0 \longrightarrow M \xrightarrow{\alpha_M} B \otimes_A M \xrightarrow{(e_1-e_2)\otimes \mathrm{id}_M} B^{\otimes 2} \otimes M$$

is exact.

The proof is a simple variant of the proof of Lemma 2.39.

Now notice that

$$(e_1 - e_2) \otimes \mathrm{id}_M(b \otimes m) = b \otimes 1 \otimes m - 1 \otimes b \otimes m = \psi_M(b \otimes m \otimes 1) - 1 \otimes b \otimes m$$

for all m and b; and this implies that

$$(e_1 - e_2) \otimes \mathrm{id}_M(x) = \psi_M(x \otimes 1) - 1 \otimes x$$

for all $x \in B \otimes_A M$. Hence $G(B \otimes_A M, \psi_M)$ is the kernel of d_1, and the homomorphism $M \to B \otimes M$ establishes a natural isomorphism between M and $G(M \otimes_A B) = GF(M)$, showing that GF is isomorphic to the identity.

Now take an object (N, ψ) of $\mathrm{Mod}_{A\to B}$, and set $M = G(N, \psi)$. The fact that M is an A-submodule of the B-module N induces a homomorphism of

B-modules $\theta\colon B\otimes_A M \to N$ with the usual rule $\theta(m\otimes b) = bm$. It is straightforward to check that θ is an arrow in $\mathrm{Mod}_{A\to B}$, hence it defines a natural transformation $\mathrm{id} \to FG$. We have to check that θ is an isomorphism.

Call α and β the homomorphisms $N \to B\otimes N$ defined by $\alpha(n) = 1\otimes n$ and $\beta(n) = \psi(n\otimes 1)$; by definition, M is the kernel of $\alpha - \beta$. There is a diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & M\otimes B \arrow[r, "i\otimes\mathrm{id}_B"] \arrow[d, "\theta\circ\iota_M"] & N\otimes_A B \arrow[r, "(\alpha-\beta)\otimes\mathrm{id}_B"] \arrow[d, "\psi"] & B\otimes N\otimes B \arrow[d, "\psi_1"] \\
0 \arrow[r] & N \arrow[r, "\alpha_M"] & B\otimes N \arrow[r, "(e_2-e_1)\otimes\mathrm{id}_N"] & B\otimes B\otimes N
\end{tikzcd}$$

where $i\colon M \hookrightarrow N$ denotes the inclusion. Let us show that it is commutative. For the first square, we have

$$\alpha_M\theta\iota_M(m\otimes b) = 1\otimes bm$$

while

$$\begin{aligned}
\psi(i\otimes\mathrm{id}_B)(m\otimes b) &= \psi(m\otimes b)\\
&= \psi\big((1\otimes b)(m\otimes 1)\big)\\
&= (1\otimes b)\psi(m\otimes 1)\\
&= (1\otimes b)(1\otimes m)\\
&= 1\otimes bm.
\end{aligned}$$

For the second square, it is immediate to check that $\psi_1\circ\alpha\otimes\mathrm{id}_B = e_2\otimes\mathrm{id}_N\circ\psi$. On the other hand

$$\begin{aligned}
\psi_1(\beta\otimes\mathrm{id}_B)(n\otimes b) &= \psi_1\big(\psi(n\otimes 1)\otimes b\big)\\
&= \psi_1\psi_3(n\otimes 1\otimes b)\\
&= \psi_2(n\otimes 1\otimes b)\\
&= (e_1\otimes\mathrm{id}_N)\psi(n\otimes b).
\end{aligned}$$

Both ψ and ψ_1 are isomorphisms; hence $\theta\circ\iota_N$ is an isomorphism, so θ is an isomorphism, as desired.

This finishes the proof of Theorem 4.9. ♠

4.2.2. Descent for quasicoherent sheaves. Here is the main result of descent theory for quasicoherent sheaves. It states that quasicoherent sheaves satisfy descent with respect to the fppf and the fpqc topology; in other words, they form a stack with respect to either topology. This is quite remarkable, because quasicoherent sheaves are sheaves in that Zariski topology, that is much coarser, so a priori one would not expect this to happen.

Given a scheme S, recall that in Example 3.15 we have constructed the fibered category (QCoh/S) of quasicoherent sheaves, whose fiber of a scheme U over S is the category $\mathrm{QCoh}\,U$ of quasicoherent sheaves on U.

THEOREM 4.11. *Let S be a scheme. The fibered category* (QCoh/S) *over* (Sch/S) *is stack with respect to the fppf and the fpqc topology.*

For the proof we will use the following criterion, a generalization of that of Lemma 2.37.

LEMMA 4.12. *Let S be a scheme, $\mathcal{F}$ be a fibered category over the category* (Sch/S)*. Suppose that the following conditions are satisfied.*

(i) *$\mathcal{F}$ is a stack with respect to the Zariski topology.*

(ii) *Whenever $V \to U$ is a flat surjective morphism of affine S-schemes, the functor*
$$\mathcal{F}(U) \longrightarrow \mathcal{F}(V \to U)$$
is an equivalence of categories.

Then $\mathcal{F}$ is a stack with respect to the fppf and the fpqc topologies.

PROOF. First of all, let us show that $\mathcal{F}$ is a prestack. Given an S scheme $T \to S$ and two objects ξ and η of $\mathcal{F}(T)$, consider the functor
$$\underline{\mathrm{Hom}}_T(\xi, \eta)\colon (\mathrm{Sch}/T) \longrightarrow (\mathrm{Set}).$$
We see immediately that the two conditions of Lemma 2.37 is satisfied, so $\underline{\mathrm{Hom}}_T(\xi, \eta)$ is a sheaf, and $\mathcal{F}$ is a prestack both in the fppf and the fpqc topologies.

Now we have to check that every object with descent data is effective. We start by analyzing the sections of $\mathcal{F}$ over the empty scheme $\emptyset$.

LEMMA 4.13. *The category $\mathcal{F}(\emptyset)$ is equivalent to the category with one object and one morphism.*

Equivalently, between any two object of $\mathcal{F}(\emptyset)$ there is a unique arrow.

PROOF. The scheme $\emptyset$ has the empty Zariski covering $\mathcal{U} = \emptyset$. By this I really mean the empty set, consisting of no morphisms at all, and not the set consisting of the embedding of $\emptyset \subseteq \emptyset$. There is only one object with descent data $(\emptyset, \emptyset)$ in $\mathcal{F}(\mathcal{U})$, and one one morphism $\emptyset$ from $(\emptyset, \emptyset)$ to itself. Hence $\mathcal{F}(\mathcal{U})$ is equivalent to the category with one object and one morphism; but $\mathcal{F}(\emptyset)$ is equivalent to $\mathcal{F}(\mathcal{U})$, because $\mathcal{F}$ is a stack in the Zariski topology. ♠

LEMMA 4.14. *If a scheme U is a disjoint union of open subschemes $\{U_i\}_{i\in I}$, then the functor $\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i)$ obtained from the various restriction functors $\mathcal{F}(U) \to \mathcal{F}(U_i)$ is an equivalence of categories.*

PROOF. Let ξ and η be objects of $\mathcal{F}(U)$; call ξ_i and η_i their restrictions to U_i. The fact that $\underline{\mathrm{Hom}}_U(\xi, \eta)\colon (\mathrm{Sch}/T) \to (\mathrm{Set})$ is a sheaf insures that the function
$$\mathrm{Hom}_{\mathcal{F}(U)}(\xi, \eta) \longrightarrow \prod_i \mathrm{Hom}_{\mathcal{F}(U_i)}(\xi_i, \eta_i)$$
is a bijection; but this means precisely that the functor is fully faithful.

To check that it is essentially surjective, take an object (ξ_i) in $\mathcal{F}(U_i)$. We have $U_{ij} = \emptyset$ when $i \neq j$, and $U_{ij} = U_i$ when $i = j$; we can define transition isomorphisms $\phi_{ij}\colon \mathrm{pr}_2^*\xi_j \simeq \mathrm{pr}_1^*\xi_i$ as the identity when $i = j$, and as the only arrow from $\mathrm{pr}_2^*\xi_j$ to $\mathrm{pr}_1^*\xi_i$ in $\mathcal{F}(U_{ij}) = \mathcal{F}(\emptyset)$ when $i \neq j$. These satisfy the cocycle condition; hence there is an object ξ of $\mathcal{F}(U)$ whose restriction to each U_i is isomorphic to ξ_i. Then the image of ξ into $\prod_{i\in I}\mathcal{F}(U_i)$ is isomorphic to (ξ_i), and the functor is essentially surjective. ♠

Given an arbitrary covering $\{U_i \to U\}$, set $V = \coprod_i U_i$, and call $f\colon V \to U$ the induced morphism. I claim that the functor $\mathcal{F}(U) \to \mathcal{F}(V \to U)$ is an equivalence if and only if $\mathcal{F}(U) \to \mathcal{F}(\{U_i \to U\})$ is. In fact, we will show that there is an equivalence of categories

$$\mathcal{F}(V \to U) \longrightarrow \mathcal{F}(\{U_i \to U\})$$

such that the composition

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(V \to U) \longrightarrow \mathcal{F}(\{U_i \to U\})$$

is isomorphic to the functor

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(\{U_i \to U\}).$$

This is obtained as follows. We have a natural isomorphism of U-schemes

$$V \times_U V \simeq \coprod_{i,j} U_i \times_U U_j,$$

so Lemma 4.14 gives us equivalences of categories

$$\mathcal{F}(V) \longrightarrow \prod_i \mathcal{F}(U_i) \tag{4.2.1}$$

and

$$\mathcal{F}(V \times_U V) \longrightarrow \prod_{i,j} \mathcal{F}(U_i \times_U U_j). \tag{4.2.2}$$

An object of $\mathcal{F}(V \to U)$ is a pair (η, ϕ), where η is an object of $\mathcal{F}(V)$ and $\phi\colon \mathrm{pr}_2^*\eta \simeq \mathrm{pr}_1^*\eta$ in $\mathcal{F}(V \times_U V)$ satisfying the cocycle condition. If η_i denotes the restriction of η to U_i for all i and $\phi_{ij}\colon \mathrm{pr}_2^*\eta \simeq \mathrm{pr}_1^*\eta$ the arrow pulled back from ϕ, the image of ϕ in $\prod_{i,j}\mathcal{F}(U_i \times_U U_j)$ is precisely the collection (ϕ_{ij}); it is immediate to see that (ϕ_{ij}) satisfies the cocycle condition.

In this way we associate to each object (η, ϕ) of $\mathcal{F}(V \to U)$ an object $(\{\eta_i\}, \{\phi_{ij}\})$ of $\mathcal{F}(\{U_i \to U\})$. An arrow $\alpha\colon (\eta, \phi) \to (\eta', \phi')$ is an arrow $\alpha\colon \eta \to \eta'$ in $\mathcal{F}(V)$ such that

$$\mathrm{pr}_1^*\alpha \circ \phi = \phi \circ \mathrm{pr}_2^*\alpha\colon \mathrm{pr}_2^*\eta \longrightarrow \mathrm{pr}_1^*\eta';$$

then one checks immediately that the collection of restrictions $\{\alpha_i\colon \eta_i \to \eta_i'\}$ gives an arrow $\{\alpha_i\}\colon (\{\eta_i\}, \{\phi_{ij}\}) \to (\{\eta_i'\}, \{\phi_{ij}'\})$.

Conversely, one can use the inverses of the functors (4.2.1) and (4.2.2) to define the inverse of the functor constructed above, thus showing that it is an equivalence (we leave the details to the reader). This equivalence has the desired properties.

This means that to check that $\mathcal{F}$ is a stack we can restrict consideration to coverings consisting of one object. Also, if U is an affine scheme over S, and $\{U_i \to U\}$ is a finite coverings by affine schemes, the disjoint union $V = \coprod_i U_i$ is also affine, hence the functor $\mathcal{F}(U) \to \mathcal{F}(V \to U)$ is an equivalence. We can conclude that the functor $\mathcal{F}(U) \to \mathcal{F}(\{U_i \to U\})$ is also an equivalence.

Given a covering $V \to U$, we choose a Zariski open covering $\{U_i\}_{i \in I}$ of U by open affine subschemes, and for each i a finite number of affine open subschemes V_{ia}, $a \in A_i$, of $f^{-1}U_i$, so that that the V_{ia}, taken all together, form a covering of V, and $\{V_{ia} \to U_i\}$ is a covering for all i (Lemma 2.38). If (η, ϕ) is an object with descent data for this covering, denote by $\eta_{i\alpha}$ the restriction of η to $\mathcal{F}(V_{ia})$. We also set $V_{iajb} = V_{ia} \times_U V_{jb}$, and denote by ϕ_{iajb} the arrow in $\mathcal{F}(V_{iajb})$ obtained by restricting ϕ.

Fix an index i. The covering $\{V_{ia} \to U_i\}_{a \in K_i}$ is a finite covering by affine schemes of an affine scheme, hence, because of the discussion above, every object with descent data is effective. Clearly $(\{\eta_{ia}\}_{a \in K_i}, \{\phi_{iaib}\})_{a,b \in K_i}$ forms an object with descent data; hence there exists an object ξ_i of $\mathcal{F}(U_i)$, together with cartesian arrows $\eta_{ia} \to \xi_i$ over $V_{ia} \to U_i$, making the diagram

$$\begin{tikzcd}
\operatorname{pr}_2^* \eta_{jb} \arrow[rr, "\phi_{iajb}"] \arrow[d] & & \operatorname{pr}_1^* \eta_{ia} \arrow[d] \\
\eta_{jb} \arrow[dr] & & \eta_{ia} \arrow[dl] \\
& \xi_i &
\end{tikzcd}$$

commutative for all pairs of indices a and b.

Now we want to glue the ξ_i together to give an object ξ of $\mathcal{F}(U)$; for this we need descent data for the Zariski covering $\{U_i \subseteq U\}$.

The rest of the proof will be in the final version of the notes. ♠

It is a standard fact that (QCoh/S) is a stack in the Zariski topology; so we only need to check that the second condition of Lemma 4.12 is satisfied; for this, we use the theory of 4.2.1. Take a flat surjective morphism $V \to U$, corresponding to a faithfully flat ring homomorphism $f\colon A \to B$. We have the standard equivalence of categories $\mathrm{QCoh}\, U \simeq \mathrm{Mod}_A^{\mathrm{op}}$; I claim that there is also an equivalence of categories $\mathrm{QCoh}(V \to U) \simeq \mathrm{Mod}_{A \to B}^{\mathrm{op}}$. A quasicoherent sheaf $\mathcal{M}$ on U corresponds to an A-module M. The inverse images $\operatorname{pr}_1^* \mathcal{M}$ and $\operatorname{pr}_2^* \mathcal{M}$ in $V \times_U B = \operatorname{Spec} B \otimes_A B$ correspond to the modules $M \otimes_A B$ and $B \otimes_A M$, respectively; hence an isomorphism $\phi\colon \operatorname{pr}_2^* \mathcal{M} \simeq \operatorname{pr}_1^* \mathcal{M}$ corresponds to an isomorphism $\psi\colon M \otimes_A B B \otimes_A M$. It is easy to see that ϕ satisfies the cocycle condition, so that $(\mathcal{M}, \phi)$ is an object of $\mathrm{QCoh}(V \to U)$, if and only if ψ satisfies the condition $\psi_1 \psi_3 = \psi_2$; this gives us the equivalence $\mathrm{QCoh}(V \to U) \simeq \mathrm{Mod}_{A \to B}^{\mathrm{op}}$. The functor $\mathrm{QCoh}\, U \to \mathrm{QCoh}(V \to U)$ corresponds to the functor $\mathrm{Mod}_A \to \mathrm{Mod}_{A \to B}$ defined in 4.2.1, in the sense that the compositions

$$\operatorname{QCoh} U \to \operatorname{QCoh}(V \to U) \simeq \operatorname{Mod}^{\mathrm{op}}_{A\to B}$$

and

$$\operatorname{QCoh} U \simeq \operatorname{Mod}^{\mathrm{op}}_{A} \simeq \operatorname{Mod}^{\mathrm{op}}_{A\to B}$$

are isomorphic. Since $\operatorname{Mod}_A \to \operatorname{Mod}_{A\to B}$ is an equivalence, this finishes the proof of Theorem 4.11.

4.2.3. Descent for morphisms of schemes.

Consider a site $\mathcal{C}$, a class $\mathcal{P}$ of arrows that is stable under pullback, and the associate category fibered in groupoids $\mathcal{P} \to \mathcal{C}$, as in Example 3.12.

DEFINITION 4.15. A class of arrows $\mathcal{P}$ in $\mathcal{C}$ is *local* if it is stable under pullback, and the following condition holds. Suppose that you are given a covering $\{U_i \to U\}$ in $\mathcal{C}$ and an arrow $X \to U$. Then, if the projections $U_i \times_U X \to U_i$ are in $\mathcal{P}$ for all i, $X \to U$ is also in $\mathcal{P}$.

The following fact is often useful.

PROPOSITION 4.16. *Let $\mathcal{C}$ be a subcanonical site, $\mathcal{P}$ a local class of arrow. Then $\mathcal{P} \to \mathcal{C}$ is a prestack.*

Recall (Definition 2.36) that a site is subcanonical when every representable functor is a sheaf.

PROOF. Let $\{U_i \to U\}$ be a covering, $X \to U$ and $Y \to V$ two arrows in $\mathcal{P}$. The arrows in $\mathcal{P}(U)$ are the isomorphisms $X \simeq Y$ in $\mathcal{C}$ that commute with the projections to U. Set $X_i = U_i \times_U X$ and $X_{ij} = U_{ij} \times_U X = X_i \times_X X_j$, and analogously for Y_i and Y_{ij}. Suppose that we have isomorphisms $f_i\colon X_i \simeq Y_i$ in $\mathcal{P}(U_i)$, such that the isomorphism $X_{ij} \simeq Y_{ij}$ induce by f_i and f_j coincide; we need to show that there is a unique isomorphism $f\colon X \to Y$ in $\mathcal{P}(U)$ whose restriction $X_i \to Y_i$ coincides with f_i for each i.

The compositions $X_i \xrightarrow{f_i} Y_i \to Y$ give sections $g_i \in \mathrm{h}_Y(X_i)$, such that the pullbacks of g_i and g_j to X_{ij} coincide. Since h_Y is a sheaf, $\{X_i \to X\}$ is a covering, and $X_{ij} = X_i \times_X X_j$ for any i and j, there is a unique arrow $f\colon X \to Y$ in $\mathcal{C}$, such the composition $X_i \to X \xrightarrow{f} Y$ is g_i, so that the diagram

$$\begin{tikzcd} X_i \arrow[r, "f_i"] \arrow[d] & Y_i \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes for all i. We only have to check that f is in $\mathcal{P}(U)$.

The fact that f is an isomorphism is proved easily; one constructs the inverse $f^{-1}\colon Y \to X$ by patching together the f_i^{-1}. It is also clear that the arrows $X \to U$ and $X \xrightarrow{f} Y \to U$ coincide, since they coincide when composed with $U_i \to U$ for all i, and since h_U is a sheaf, and in particular a separated functor. ♠

Most of the interesting properties of morphism of schemes are local in the fppf and in the fpqc, such as for example being flat, being of finite presentation, being quasi-compact, being proper, being smooth, being affine, and so on. For each of this properties we get a prestack of morphisms of schemes over (Sch/S).

The issue of effectiveness of descent data is much more delicate, however. One can give examples to show that it fails even proper and smooth morphisms. However, using descent theory for quasicoherent sheaves one can show that it holds for affine morphisms, and also with morphisms equipped with an invertible sheaf.

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