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Grothendieck's existence theorem in formal geometry

Luc Illusie
Laboratoire de Mathématiques
Université Paris-Sud
Bâtiment 425
F-91405 Orsay
France

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The main theme of these notes is Grothendieck's exposé 182 at the Séminaire Bourbaki [G]. Most of the material in (loc. cit.) has been treated at length in [EGA III] and [SGA 1]. Our purpose here is to provide an introduction, explaining the proofs of the key theorems, discussing typical applications, and updating when necessary. The central results are the comparison theorem between formal and algebraic cohomology for proper morphisms and the existence theorem for formal sheaves. We give the highlights of the proofs in §§2, 4 after recalling some basic terminology on formal schemes in §1 (sticking to the locally noetherian context, which suffices here). In §3 we revisit some points of [EGA III 7] : base change formula and cohomological flatness. We believe that the use of derived categories and, especially, perfect complexes, simplifies the exposition. This section, however, is not essential for the sequel. In §5 we discuss several applications to the existence of formal or algebraic liftings, by combining Grothendieck's theorems of the preceding sections with basic results of deformation theory, mostly in the smooth case.

I am very grateful to Serre for a conversation about his examples in [S2] and for sending me a copy of Mumford's unpublished letter to him [M5]. Raynaud and Messing read a preliminary version of these notes and suggested several corrections and modifications. I thank them heartily, as well as the students of the school for numerous questions and comments.

Contents

1. Locally noetherian formal schemes

1.1. An *adic noetherian ring* is a noetherian ring A which is separated and complete for the I*-adic topology*, i. e. the topology given by the I*-adic filtration*, consisting of the powers of an ideal I, in other words, $A = \lim_{n \geq 0} A/I^{n+1}$. With such a ring is associated a topologically ringed space

$$\mathcal{X} = (\operatorname{Spf} A, \mathcal{O}_{\mathcal{X}}), \tag{1.1.1}$$

defined as follows. For $n \in \mathbb{N}$, let $X_n = \operatorname{Spec}(A/I^{n+1})$. These schemes form an increasing sequence of closed subschemes of $\operatorname{Spec} A$, (with closed, nilpotent immersions as transition maps)

$$X_0 = \operatorname{Spec} A/I \to X_1 \to \cdots \to X_n \to \cdots.$$

They all have the same underlying space $\mathcal{X}$, called the *formal spectrum* of A. Note that I is contained in the radical of A [EGA O_I 7.1.10], i. e. $1-x$ is invertible for all $x \in I$, which means that $\mathcal{X}$, as a closed subspace of $\operatorname{Spec} A$, contains all its closed points. Every open subset of $\operatorname{Spec} A$ containing $\mathcal{X}$ is equal to $\operatorname{Spec} A$. The sheaf of rings $\mathcal{O}_{\mathcal{X}}$ is defined to be the inverse limit of the sheaves $\mathcal{O}_{X_n}$ on $\mathcal{X}$, equipped with the natural topology such that on any open subset U of $\mathcal{X}$, $\Gamma(U, \mathcal{O}_{\mathcal{X}}) = \lim_n \Gamma(U, \mathcal{O}_{X_n})$, where $\Gamma(U, \mathcal{O}_{X_n})$ has the discrete topology. In particular, $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = A$, and for $f \in A$, and $\mathcal{D}(f) = \mathcal{X}_f$ the open subset of $\mathcal{X}$ where the image of f in A_0 is invertible, $\Gamma(\mathcal{X}_f, \mathcal{O}_{\mathcal{X}}) = A_{\{f\}}$, the completed fraction ring $\lim_n S_f^{-1} A / S_f^{-1} I^{n+1}$. The stalks $\mathcal{O}_{\mathcal{X},x} = \operatorname{colim}_{f \in A, f(x) \neq 0} A_{\{f\}}$ are local (noncomplete) rings.

The topologically ringed space (1.1.1) depends only on A as a *topological* ring. It doesn't change if one replaces I by any *ideal of definition* J of A, i. e. an ideal J such that $J \supset I^p \supset J^q$ for some positive integers p, q, or, equivalently, which is open and whose powers tend to zero for the I-adic topology. The space $\mathcal{X}$ is the subspace of $\operatorname{Spec} A$ consisting of *open* prime ideals, and $\mathcal{O}_{\mathcal{X}}$ is the inverse limit of the sheaves $(A/J)^{\sim}$ where J runs through the ideals of definition of A.

An *affine noetherian formal scheme* is a topologically ringed space isomorphic to one of the form (1.1.1). A *locally noetherian formal scheme* is a topologically ringed space such that any point has an open neighborhood which is an affine noetherian formal scheme. It is called *noetherian* if its underlying space is noetherian. A *morphism* $f : \mathcal{X} \to \mathcal{Y}$ between locally noetherian formal schemes is a morphism of ringed spaces which is *local* (i. e. such that for each point $x \in X$, the map $\mathcal{O}_{\mathcal{Y},f(x)} \to \mathcal{O}_{\mathcal{X},x}$ is local) and *continuous* (i. e. for every open affine $V \subset \mathcal{Y}$, the map $\Gamma(V, \mathcal{O}_V) \to \Gamma(f^{-1}(V), \mathcal{O}_{\mathcal{X}})$ is continuous). Locally noetherian formal schemes form a category in an obvious way.

As in the case of usual schemes, one checks that if $\mathcal{Y} = \operatorname{Spf}(A)$ is a noetherian affine formal scheme (in the sequel we will usuallly omit the sheaf of rings from the notation) and $\mathcal{X}$ is any locally noetherian formal scheme, then we have

$$\operatorname{Hom}(\mathcal{X}, \mathcal{Y}) = \operatorname{Hom}_{cont}(A, \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})), \tag{1.1.2}$$

where $\operatorname{Hom}_{cont}$ means the set of *continuous* ring homomorphisms. In particular, if $\mathcal{X}$ is affine, of ring B, then

$$\operatorname{Hom}(\mathcal{X}, \mathcal{Y}) = \operatorname{Hom}_{cont}(A, B).$$

1.2. Let $\mathcal{X} = \operatorname{Spf} A$ be an affine noetherian formal scheme, and let I be an ideal of definition of A. Let M be an A-module *of finite type*. With M is associated a coherent module $\tilde{M}$ on $X = \operatorname{Spec} A$. In an analogous way, one associates with M a module M^Δ on $\mathcal{X}$, defined as follows. For $n \in \mathbb{N}$, let $X_n = \operatorname{Spec} A/I^{n+1}$ as in 1.1. We put

$$M^\Delta = \lim_n \tilde{M}_n,$$

where $M_n = M/I^{n+1}M$. It is easily checked that M^Δ does not depend on the choice of I, that the functor $M \mapsto M^\Delta$ is exact, that

$$\Gamma(\mathcal{X}, M^\Delta) = M,$$

and that the formation of M^Δ commutes with tensor products and internal *Hom*. The main point is that, if

$$i : \mathcal{X} \to X$$

is the natural morphism, defined by the inclusion on the underlying topological spaces and the canonical map $\mathcal{O}_X \to \mathcal{O}_\mathcal{X}$ on the sheaves of rings, then, since M is of finite type, Krull's theorem implies that

$$M^\Delta = i^* \tilde{M}.$$

Since, for any $f \in A$, $A_{\{f\}}$ is adic noetherian, it follows that $\mathcal{O}_\mathcal{X}$ is a *coherent* sheaf of rings, M^Δ is *coherent*, and the coherent modules on $\mathcal{X}$ are exactly those of the form M^Δ for M of finite type over A.

1.3. Locally noetherian formal schemes are more conveniently described - and in practice usually appear - as colimits of increasing chains of nilpotent thickenings. By a *thickening* we mean a closed immersion of schemes $X \to X'$ whose ideal I is a nilideal ; the schemes X and X' then have the same underlying space. If X' is noetherian, so is X, and I is nilpotent ; conversely, if X is noetherian and I/I^2 coherent (as an $\mathcal{O}_X$-module), X' is noetherian [EGA O_I 7.2.6, I 10.6.4]. If X' is noetherian, X' is affine if and only if X is [EGA I 6.1.7]. We say that a thickening is *of order* n if $I^{n+1} = 0$.

Let $\mathcal{X}$ be a locally noetherian formal scheme. It follows from the discussion in the affine case that $\mathcal{O}_\mathcal{X}$ is a *coherent* sheaf of rings, and that the coherent modules on $\mathcal{X}$ are exactly the modules which are of finite presentation, or equivalently, which on any affine open $U = \operatorname{Spf} A$ are of the form M^Δ for an A-module M of finite type.

An *ideal of definition* of $\mathcal{X}$ is a coherent ideal $\mathcal{I}$ of $\mathcal{O}_\mathcal{X}$ such that, for any point $x \in \mathcal{X}$, there exists an affine neighborhood $U = \operatorname{Spf} A$ of x such that $\mathcal{I}|U$ is of the form I^Δ for an ideal of definition I of A. A coherent ideal $\mathcal{I}$ is an ideal of definition if and only if the ringed space $(\mathcal{X}, \mathcal{O}_\mathcal{X}/\mathcal{I})$ is a scheme having $\mathcal{X}$ as an underlying space. Ideals of definition of $\mathcal{X}$ exist. In fact, there is a *largest* one,

$$\mathcal{T} = \mathcal{T}_\mathcal{X}, \tag{1.3.1}$$

which is the unique ideal of definition $\mathcal{I}$ such that $(\mathcal{X}, \mathcal{O}_\mathcal{X}/\mathcal{I})$ is *reduced*. If $U = \operatorname{Spf} A$ is an affine open subset, then $\mathcal{T}|U = T^\Delta$, where T is the ideal of elements a of A which are *topologically nilpotent*, i. e. whose image in A/I is nilpotent. If $\mathcal{I}$ is an ideal of definition of $\mathcal{X}$, so is any power $\mathcal{I}^n$ for $n \geq 1$. If $\mathcal{X}$ is noetherian, then, as in the affine case, if $\mathcal{I}$ and $\mathcal{J}$ are ideals of definition of $\mathcal{X}$, there exists positive integers p, q such that $\mathcal{J} \supset \mathcal{I}^p \supset \mathcal{J}^q$.

Fix an ideal of definition $\mathcal{I}$ of $\mathcal{X}$. For $n \in \mathbb{N}$, the ringed space $(\mathcal{X}, \mathrm{O}_{\mathcal{X}}/\mathcal{I}^{n+1})$ is a *locally noetherian scheme* X_n, and we have an increasing chain of thickenings

$$X. = (X_0 \to X_1 \to \cdots \to X_n \to \cdots), \tag{1.3.2}$$

whose colimit (in the category of locally noetherian formal schemes) is $\mathcal{X}$: the thickenings induce the identity on the underlying spaces, which are all equal to the underlying space of $\mathcal{X}$, and we have

$$\mathcal{O}_{\mathcal{X}} = \lim_n \mathcal{O}_{X_n},$$

as topological rings ($\Gamma(U, \mathcal{O}_{X_n})$ having the discrete topology on any affine open U). Let J_n be the ideal of X_0 in X_n, i. e. $J_n = \operatorname{Ker} \mathcal{O}_{X_n} \to \mathcal{O}_{X_0}$. Then, for $m \leq n$, the ideal of X_m in X_n is J_n^{m+1} (in particular, $J_n^{n+1} = 0$), J_1 is a coherent module on X_0, and $J_n = \mathcal{I}/\mathcal{I}^{n+1}$.

Conversely, consider a sequence (1.3.2) of ringed spaces satisfying :

(i) X_0 is a locally noetherian scheme,

(ii) the underlying maps of topological spaces are homeomorphisms and, using them to identify the underlying spaces, the maps of rings $\mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_n}$ are surjective,

(iii) if $J_n = \operatorname{Ker} \mathcal{O}_{X_n} \to \mathcal{O}_{X_0}$, then for $m \leq n$, the $\operatorname{Ker} \mathcal{O}_{X_n} \to \mathcal{O}_{X_m} = J_n^{m+1}$

(iv) J_1 (as an $\mathcal{O}_{X_0}$-module) is coherent,

Then the topologically ringed space $\mathcal{X} = (X_0, \lim_n \mathcal{O}_{X_n})$ is a locally noetherian formal scheme, and if $\mathcal{I} = \operatorname{Ker} \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{X_0} = \lim J_n$, $\mathcal{I}$ is an ideal of definition of $\mathcal{X}$, and $\mathcal{I}^{n+1} = \operatorname{Ker} \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{X_n}$.

The verification is straightforward [EGA I 10.6.3 - 10.6.5], by reduction to the case where X_0 is *affine*, of ring A_0, in which case every X_n is automatically affine noetherian, of ring A_n, and $\mathcal{X} = \operatorname{Spf} A$, where $A = \lim_n A_n$.

1.4. Let $\mathcal{X}$ be a locally noetherian formal scheme, $\mathcal{I}$ an ideal of definition of $\mathcal{X}$, and consider the corresponding chain of thickenings (1.3.2). For $m \leq n$ denote by

$$u_{mn} : X_m \to X_n \quad , \quad u_n : X_n \to \mathcal{X}$$

the canonical morphisms. If E is a coherent module on $\mathcal{X}$, then $E_n := u_n^* E$ is a coherent module on X_n, and these modules form an inverse system, with $\mathcal{O}_{X_n}$-linear transition maps $E_n \to E_m$ inducing isomorphism $u_{mn}^* E_n \xrightarrow{\sim} E_m$, and $E = \lim_n E_n$. Conversely, let $F. = (F_n, f_{mn} : F_n \to F_m)$ be an inverse system of $\mathcal{O}_{X_n}$-modules, with transition maps f_{mn} which are $\mathcal{O}_{X_n}$-linear. We will say that $F.$ is *coherent* if each F_n is coherent and the transition maps f_{mn} induce isomorphisms $u_{mn}^* F_n \xrightarrow{\sim} F_m$. If $F.$ is coherent, and $F := \lim_n F_n$ is the corresponding $\mathcal{O}_{\mathcal{X}}$-module, then F is coherent and $F.$ is canonically isomorphic to the inverse system $(u_n^* F)$. The functor

$$\mathrm{Coh}(\mathcal{X}) \to \mathrm{Coh}(X.), \quad E \mapsto (u_n^* E) \tag{1.4.1}$$

from the category of coherent sheaves on $\mathcal{X}$ to the category $\mathrm{Coh}(X.)$ of coherent inverse systems (F_n) is an equivalence. For $E = \lim_n E_n \in \mathrm{Coh}(\mathcal{X})$ as above, the *support* of E is *closed* (as E is coherent) and coincides with that of E_0. By (a special case of) the flatness criterion [B, III, §5, th. 1], E is flat (equivalently, locally free of finite type) if and only if E_n is locally free of finite type for all n.

1.5. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of locally noetherian formal schemes, and let $\mathcal{J}$ be an ideal of definition of $\mathcal{Y}$. Since $\mathcal{J} \subset \mathcal{T}_{\mathcal{Y}}$, the continuity of f implies that the ideal $f^*(\mathcal{J})\mathcal{O}_{\mathcal{X}}$ is contained in $\mathcal{T}_{\mathcal{X}}$ (1.3.1). Fix an ideal of definition $\mathcal{I}$ such that $f^*(\mathcal{J})\mathcal{O}_{\mathcal{X}} \subset \mathcal{I}$ (e. g. $\mathcal{I} = \mathcal{T}_{\mathcal{X}}$), and consider the inductive systems $X_.$, $Y_.$ defined by $\mathcal{I}$ and $\mathcal{J}$ respectively, as in (1.3.2). Then, since $f^*(\mathcal{J}^{n+1})\mathcal{O}_{\mathcal{X}} \subset \mathcal{I}^{n+1}$, f induces a morphism of inductive systems

(1.5.1) $$f_. : X_. \to Y_.,$$

i. e. morphisms of schemes $f_n : X_n \to Y_n$ such that the squares

(1.5.2) $$\begin{array}{ccc} X_m & \longrightarrow & X_n \\ {\scriptstyle f_m}\downarrow & & \downarrow{\scriptstyle f_n} \\ Y_m & \longrightarrow & Y_n \end{array}$$

are commutative, and f is the colimit of the morphisms f_n, characterized by making the squares

(1.5.3) $$\begin{array}{ccc} X_n & \xrightarrow{u_n} & \mathcal{X} \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f} \\ Y_n & \xrightarrow{u_n} & \mathcal{Y} \end{array}$$

commutative. It is easily checked [EGA I 10.6.8] that $f \mapsto f_.$ defines a bijection from the set of morphisms from $\mathcal{X}$ to $\mathcal{Y}$ such that $f^*(\mathcal{J})\mathcal{O}_{\mathcal{X}} \subset \mathcal{I}$ to the set of morphisms of inductive systems of the type (1.5.1).

In general, $f^*(\mathcal{J})\mathcal{O}_{\mathcal{X}}$ is not an ideal of definition of $\mathcal{X}$. When this is the case, f is called an *adic morphism* (and $\mathcal{X}$ a $\mathcal{Y}$-adic formal scheme). One can then take $\mathcal{I} = f^*(\mathcal{J})\mathcal{O}_{\mathcal{X}}$, and the squares (1.5.2) are *cartesian.* Conversely any morphism of inductive systems (1.5.1) such that the squares (1.5.2) are cartesian define an adic morphism from $\mathcal{X}$ to $\mathcal{Y}$.

Let $f : \mathcal{X} \to \mathcal{Y}$ be an adic morphism, and let E be a coherent sheaf on $\mathcal{X}$. Then the following conditions are equivalent :

(i) E is *flat over* $\mathcal{Y}$ (or $\mathcal{Y}$-flat), i. e. for every point x of X, the stalk E_x is flat over $\mathcal{O}_{\mathcal{Y},f(x)}$;

(ii) with the notations of (1.5.3), $E_n = u_n^* E$ is Y_n-flat for all $n \geq 0$;

(iii) E_0 is Y_0-flat and the natural (surjective) map

$$\mathrm{gr}^n \mathcal{O}_{\mathcal{Y}} \otimes_{\mathrm{gr}^0 \mathcal{O}_{\mathcal{Y}}} \mathrm{gr}^0 E \to \mathrm{gr}^n E,$$

where the associated graded gr is taken with respect to the $\mathcal{J}$-adic filtration, is an isomorphism for all $n \geq 0$.

This is a consequence of the flatness criterion [B, III, §5, th. 2, prop. 2].

When $E = \mathcal{O}_{\mathcal{X}}$ satisfies the above equivalent conditions, we say that f is *flat.*

1.6. Let X be a locally noetherian *scheme*, and let X' be a closed subset of (the underlying space of) X. Choose a coherent ideal I of $\mathcal{O}_X$ such that the closed subscheme of X defined by I has X' as an underlying space. Such ideals exist, there is in fact a largest one, consisting of local sections of $\mathcal{O}_X$ vanishing on X' ; for this one, X' has the reduced scheme structure. Consider the inductive system of (locally noetherian) schemes, all having X' as underlying space,

$$X_0 \to X_1 \to \cdots \to X_n \to \cdots,$$

where X_n is the closed subscheme of X defined by I^{n+1}. It satisfies the conditions (i) - (iv) of 1.3 and therefore the colimit

$$X_{/X'} := \mathrm{colim}_n X_n \tag{1.6.1}$$

is a locally noetherian formal scheme, having X' as underlying space, called the *formal completion of X along X'*. It is sometimes denoted simply $\hat{X}$, when no confusion can arise. It is easily checked that $X_{/X'}$ does not depend on the choice of the ideal I. In fact, $\mathcal{O}_{\hat{X}}$ is the inverse limit of the rings $\mathcal{O}_X/J$, where J runs through all the coherent ideals of $\mathcal{O}_X$ such that the support of $\mathcal{O}_X/J$ is X' (on any noetherian open subset of X, the powers of I form a cofinal system). If X is affine, $X = \mathrm{Spec}\, A$, and $I = \tilde{J}$, then $\hat{X} = \mathrm{Spf}\, \hat{A}$, where $\hat{A}$ is the completion of A with respect to the J-adic topology.

The closed immersions $i_n : X_n \to X$ define a morphism of ringed spaces

$$i_X : \hat{X} \to X \tag{1.6.2}$$

(or i), which is *flat*, and for any *coherent* sheaf F on X, the natural map

$$i^*F \to F_{/X'} := \lim_n i_n^* F. \tag{1.6.3}$$

is an isomorphism. When $X = \mathrm{Spec}\, A$ and $F = \tilde{M}$, with M an A-module of finite type, then $F_{/X'} = M^{\Delta}$ (1.2). The above assertion follows from Krull's theorem : if A is noetherian, and J is an ideal of A, then the J-adic completion $\hat{A}$ is flat over A, and for any A-module M of finite type, $\hat{M} = M \otimes \hat{A}$. One writes sometimes $\hat{F}$ for $F_{/X'}$ when no confusion can arise. Note that if F is not coherent, (1.6.3) is not in general an isomorphism. One checks similarly that the kernel of the adjunction map

$$F \to i_* i^* F \tag{1.6.4}$$

consists of sections of F which are zero in a neighborhood of X'.

Let $f : X \to Y$ be a morphism of locally noetherian schemes, X' (resp. Y') a closed subset of X (resp. Y) such that $f(X') \subset Y'$. Choose coherent ideals $J \subset \mathcal{O}_X$, $K \subset \mathcal{O}_Y$ defining closed subschemes with underlying spaces X' and Y' respectively and such that $f^*(K)\mathcal{O}_X \subset J$ (one can take for example for K any ideal defining a closed subscheme with underlying space Y' and for J the ideal of sections of $\mathcal{O}_X$ vanishing on X'). Then f induces a morphism of inductive systems

$$f_. : X_. \to Y_.,$$

where X_n (resp. Y_n) is defined as above. By the correspondence explained in 1.5 we get from $f_.$ a morphism

$$\hat{f} : X_{/X'} \to Y_{/Y'}, \tag{1.6.5}$$

which does not depend on the choices of J, K, and is called the *extension* of f to the completions $X_{/X'}$ and $Y_{/Y'}$. This morphism sits in a commutative square

$$\begin{array}{ccc} X_{/X'} & \xrightarrow{i} & X \\ {\scriptstyle \hat{f}}\downarrow & & \downarrow{\scriptstyle f} \\ Y_{/Y'} & \xrightarrow{i} & Y \end{array} \quad , \tag{1.6.6}$$

where the horizontal maps are the canonical morphisms (1.6.2). When $X' = f^{-1}(Y')$, one can take $J = f^*(K)\mathcal{O}_X$, all the squares

$$\begin{array}{ccc} X_n & \longrightarrow & X \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f} \\ Y_n & \longrightarrow & Y \end{array}$$

are cartesian, hence the same holds for the square (1.5.2), and therefore $\hat{f}$ is an *adic* morphism.

2. The comparison theorem

2.1. Let $f : X \to Y$ be a morphism of locally noetherian schemes, let Y' be a closed subset of Y, $X' = f^{-1}(Y')$. Write $\hat{X} = X_{/X'}$, $\hat{Y} = Y_{/Y'}$. If F is an $\mathcal{O}_X$-module, the square (1.6.6) defines base change maps (see 2.19)

$$i^* R^q f_* F \to R^q \hat{f}_*(i^* F) \tag{2.1.1}$$

(for all $q \in \mathbb{Z}$), which are maps of $\mathcal{O}_{\hat{Y}}$-modules. If F is *coherent*, then $i^* F$ can be identified with $\hat{F} = F_{/X'}$ by (1.6.3), and similarly $i^* R^q f_* F$ can be identified with $(R^q f_* F)_{/Y'}$ if $R^q f_* F$ is coherent : this is the case when F is coherent and f is *proper* (or f is of finite type and *the support of F is proper over* Y, which means ([EGA II 5.4.10]) that there exists a closed subscheme Z of X which is proper over Y and whose underlying space is the support of F), by the finiteness theorem for proper morphisms [EGA III 3.2.1, 3.2.4]. In this case, (2.1.1) can be rewritten

$$(R^q f_* F)\hat{} \to R^q \hat{f}_* \hat{F}. \tag{2.1.2}$$

On the other hand, the squares (1.5.3), with $\mathcal{X} = \hat{X}$, $\mathcal{Y} = \hat{Y}$ define $\mathcal{O}_{Y_n}$-linear base change maps

$$u_n^* R^q \hat{f}_* \hat{F} \to R^q (f_n)_* F_n,$$

where $F_n = u_n^* \hat{F} = i_n^* F$ (in the notation of (1.6.2)). By adjunction, these maps can be viewed as $\mathcal{O}_{\hat{Y}}$-linear maps

$$R^q \hat{f}_* \hat{F} \to R^q (f_n)_* F_n,$$

hence define $\mathcal{O}_{\hat{Y}}$-linear maps

$$R^q \hat{f}_* \hat{F} \to \lim_n R^q (f_n)_* F_n. \tag{2.1.3}$$

Note that the base change map (2.1.1) is defined more generally for $F \in D^+(X, \mathcal{O}_X)$, as induced on the sheaves $\mathcal{H}^q$ from the base change map in $D^+(\hat{Y}, \mathcal{O}_{\hat{Y}})$

$$i^* Rf_* F \to R\hat{f}_* i^* F. \tag{2.1.4}$$

Theorem 2.2. *Let $f : X \to Y$ be a finite type morphism of noetherian schemes, Y' a closed subset of Y, $X' = f^{-1}(Y')$, $\hat{f} : \hat{X} \to \hat{Y}$ the extension of f to the formal completions of X and Y along X' and Y'. Let F be a coherent sheaf on X whose support is proper over Y. Then, for all q, the canonical maps (2.1.2), (2.1.3) are topological isomorphisms.*

Remarks 2.3. (a) Under the assumptions of 2.2 on f, it follows that for any $F \in D^+(X, \mathcal{O}_X)$ such that, for all i, $\mathcal{H}^i F$ is coherent and properly supported over Y, (2.1.4) is an isomorphism. Using that the natural functor from the bounded derived category $D^b(\mathrm{Coh}(X))$ of coherent sheaves on X to the full subcategory $D^b(X)_{coh}$ of $D^b(X) := D^b(X, \mathcal{O}_X)$ consisting of complexes with coherent cohomology is an equivalence [SGA 6 II 2.2.2.1], one can extend the isomorphism (2.1.3) of 2.2 to the case $F \in D^b(X)_{coh}$. We omit the details.

(b) By considering a closed subscheme Z of X whose underlying space is the support of F, 2.2 is reduced to the case where f is *proper*.

(c) Grothendieck's original proof has not been published. From [G, p. 05], one can guess that it consisted of two steps : (i) proof in the case where f is projective, using *descending* induction on q (see [H, III 11.1] for the case where Y' is a point) ; (ii) proof in the general case by reducing to the projective case via Chow's lemma and noetherian induction. The proof given in [EGA III 4.1.7, 4.1.8] follows an argument due to Serre.

(d) It is easily seen that 2.2 is actually equivalent to the following special case :

Corollary 2.4. *Under the assumptions of 2.2, suppose that $Y = \mathrm{Spec}\, A$, with A a noetherian ring, let I be an ideal of A such that $\mathrm{Supp}(\mathcal{O}_Y/\mathcal{I}) = Y'$, where $\mathcal{I} = \tilde{I}$. Let $Y_n = \mathrm{Spec}(A/I^{n+1})$, $X_n = Y_n \times_Y X$, $F_n = i_n^* F = F/\mathcal{I}^{n+1}F$. Then, for all q, the natural maps*

$$\varphi_q : H^q(X, F)\hat{} \to \lim_n H^q(X, F_n), \tag{2.4.1}$$

defined by the composition of (2.1.2) and (2.1.3), and

$$\psi_q : H^q(\hat{X}, \hat{F}) \to \lim_n H^q(X, F_n), \tag{2.4.2}$$

defined by (2.1.3), are topological isomorphisms.

The proof of 2.4 ([EGA III 4.1.7]) uses two ingredients, the first one is standard, elementary commutative and homological algebra, the second is much deeper : (a) the Artin-Rees lemma and the Mittag-Leffler conditions ; (b) the finiteness theorem for proper morphisms [EGA III 3.2], especially a *graded* variant [EGA III 3.3.2]. We will briefly review (a) and (b) and then give the highlights of the proof of 2.4.

2.5. *Artin-Rees and Mittag-Leffler.*

2.5.1. Let A be a noetherian ring, I an ideal of A, M a finitely generated A-module endowed with a decreasing filtration by submodules $(M_n)_{n\in\mathbb{Z}}$. The filtration (M_n) is called *I-good* if it is *exhaustive* (i. e. there exists n_1 such that $M_{n_1} = M$) and it satisfies the following two conditions :

(i) $IM_n \subset M_{n+1}$ for all $n \in \mathbb{Z}$ (which means that M, filtered by (M_n) is a filtered module over the ring A filtered by the I-adic filtration) ;

(ii) there exists an integer n_0 such that $M_{n+1} = IM_n$ for all $n \geq n_0$.

For example, the *I-adic filtration* of M, defined by $M_n = M$ for $n \leq 0$ and $M_n = I^nM$ for $n \geq 0$ is I-good. All I-good filtrations define on M the same topology, namely the I-adic topology.

Assume that condition (i) holds. Consider the graded ring

$$A' := \oplus_{n\in\mathbb{N}} I^n,$$

sometimes written $\oplus I^nt^n$, where t is an indeterminate, to make clear that $I^n = I^nt^n$ is the n-th component of A', and the graded module over A',

$$M' = \oplus_{n\in\mathbb{N}} M_n,$$

also sometimes written $\oplus_{n\in\mathbb{N}} M_nt^n$. A basic observation [B, III, §3, th. 1] - whose proof is straightforward - is that condition (ii) is equivalent to

(ii)' M' is a finitely generated A'-module.

Since A' is noetherian, this immediately implies the classical *Artin-Rees theorem* : if N is a submodule of M, then the filtration induced on N by the I-adic filtration of M is I-good, in other words, there exists $n_0 \geq 0$ such that, for all $n \geq n_0$,

$$I^nM \cap N = I^{n-n_0}(I^{n_0}M \cap N).$$

That (ii)' implies (ii) is a particular case of the following (equally straightforward) property [EGA II 2.1.6] :

(iii) Let A be a commutative ring, $S = \oplus_{n\in\mathbb{N}} S_n$ a graded A-algebra, of finite type over S_0 and generated by S_1, and $M = \oplus_{n\in\mathbb{Z}} M_n$ a graded S-module of finite type. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $M_{n+1} = S_1M_n$

2.5.2. Let A be a commutative ring. Let $M. = (M_n, u_{mn} : M_n \to M_m)$ be a projective system of A-modules, indexed by $\mathbb{N}$. We say that :

(i) $M.$ is *strict* if the transition maps u_{mn} are surjective,

(ii) $M.$ is *essentially zero* if for each m there exists $n \geq m$ such that $u_{mn} = 0$, in other words the *pro-object* defined by $M.$ is zero,

(iii) $M.$ satisfies the *Mittag-Leffler condition* (ML for short) if for each m there exists $n \geq m$ such that, for all $n' \geq n$, $\operatorname{Im} u_{mn'} = \operatorname{Im} u_{mn}$ in M_m.

It is sometimes useful to consider the following stronger conditions : we say that :

(ii)' $M.$ is *Artin-Rees zero* (AR zero for short) if there exist an integer $r \geq 0$ such that for all n, $u_{n,n+r} = 0$,

(iii)' $M.$ satisfies the *Artin-Rees-Mittag-Leffler condition* (ARML for short) if there exists an integer $r \geq 0$ such that, for all m and all $n' \geq m + r$, $\operatorname{Im} u_{mn'} = \operatorname{Im} u_{m,m+r}$.

We refer to [EGA 0_{III} 13] for a discussion of the Mittag-Leffler condition. Let us just recall two basic (easy) points :

(a) If $M.$ is essentially zero, then $\lim_n M_n = 0$.

(b) The functor $M. \mapsto \lim_n M_n$ is left exact. Moreover, let

$$0 \to L. \to M. \to N. \to 0$$

be an exact sequence of inverse systems of A-modules ; if $L.$ satisfies ML then the sequence

$$0 \to \lim_n L_n \to \lim_n M_n \to \lim_n N_n \to 0$$

is exact.

The stronger condition (iii)' (sometimes called *uniform* Mittag-Leffler condition) has a close relationship with the Artin-Rees theorem. See [SGA 5 V] for a discussion of this. The terminology *AR zero*, *ARML* is taken from there.

We will need a (very) particular case of a general result [EGA O_{III} 13.3.1] of commutation of $H^q(X, -)$ with inverse limits :

Proposition 2.5.3. *Let X be a scheme, and let $(F_n)_{n \in \mathbb{N}}$ be an inverse system of quasi-coherent sheaves on X with* surjective *transition maps. Assume that, for all $i \in \mathbb{Z}$, the inverse system (of $\mathbb{Z}$-modules) $H^i(X, F.)$ satisfies ML. Then, for all $i \in \mathbb{Z}$, the natural map*

$$H^i(X, \lim_n F_n) \to \lim_n H^i(X, F_n)$$

is an isomorphism.

The proof of the (more general) result of (loc. cit.) is elementary. One can give a shorter (but less elementary) proof of 2.5.3 using the derived functors of lim. The sheaves $R^q \lim F_n$ are associated with the presheaves $U \mapsto R^q \lim \Gamma(U, F_n)$. Since the F_n are quasi-coherent and the transition maps are surjective, if U is affine, the inverse system $\Gamma(U, F_n)$ is strict, hence $R^q \lim \Gamma(U, F_n) = 0$ for $q > 0$, so the natural map $F = \mathrm{limF_n} \to R \lim F_n$ is an isomorphism. Now, we have

$$(*) \qquad R\Gamma(X, R\lim F_n) = R\lim R\Gamma(X, F_n).$$

Since the inverse systems $H^i(X, F.)$ satisfy ML, we have $R^p \lim H^q(X, F_n) = 0$ for all $p > 0$, so the spectral sequence associated with (*) degenerates at E_2 and yields the desired isomorphisms.

2.6 *The finiteness theorem.*

The fundamental finiteness theorem for proper morphisms [EGA III 3.2.1] asserts that if $f : X \to Y$ is a proper morphism, with Y locally noetherian, and F is a coherent sheaf on X, then, for all $q \in \mathbb{Z}$, the sheaves $R^q f_* F$ on Y are coherent. We will need the following variant [EGA III 3.3.1] :

Theorem 2.6.1. *Let $f : X \to Y$ is a proper morphism, with Y noetherian. Let $S = \oplus_{n\in\mathbb{N}} S_n$ be a quasi-coherent, graded $\mathcal{O}_Y$-algebra of finite type over S_0 and generated by S_1. Let $M = \oplus_{n\in\mathbb{Z}} M_n$ be a quasi-coherent, graded $f^*(S)$-module of finite type. Then, for all $q \in \mathbb{Z}$,*

$$R^q f_* M := \oplus_{n\in\mathbb{Z}} R^q f_* M_n$$

is a graded S-module of finite type, and there exists an integer n_0 such that, for any $n \geq n_0$,

$$R^q f_* M_n = S_{n-n_0} R^q f_* M_{n_0}.$$

Here the structure of graded S-module on $R^q f_* M$ comes from the multiplication maps, which are the composites

$$S_k \otimes R^q f_* M_n \to f_* f^* S_k \otimes R^q f_* M_n \to R^q f_*((f^* S_k) \otimes M_n) \to R^q f_* M_{n+k}.$$

The last assertion in 2.6.1 is a consequence of the first one (thanks to 2.5.1 (iii)), and the first one follows from the finiteness theorem applied to the (proper) morphism $\tilde{f} : \tilde{X} \to \tilde{Y}$ defined by the cartesian square

$$\begin{array}{ccc} X & \longleftarrow & \tilde{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \tilde{f}} \\ Y & \longleftarrow & \tilde{Y} \end{array} ,$$

where $\tilde{Y} = \operatorname{Spec} S$, $\tilde{X} = \operatorname{Spec} f^*(S)$, and to the coherent module $\tilde{M}$ on $\tilde{X}$.

Corollary 2.6.2. *Under the assumptions of 2.4, let $B := \oplus_{n\in\mathbb{N}} I^n$. Then, for all q, $\oplus_{n\in\mathbb{N}} H^q(X, I^n F)$ is a finitely generated graded B-module, and there exists $n_0 \geq 0$ such that, for all $n \geq n_0$, $H^q(X, I^n F) = I^{n-n_0} H^q(X, I^{n_0} F)$.*

2.7. *Proof of 2.4.*

In contrast with Grothendieck's original proof, the proof given in [EGA III 4.1.7] does not go by descending induction on q. The integer q remains fixed in the whole proof, which consists of a careful analysis of the inverse system of maps

(2.7.1) $$H^q(F) \to H^q(F_n),$$

where $H^q = H^q(X, -)$ for brevity. The map (2.7.1) sits in a portion of the long exact sequence of cohomology associated with the short exact sequence

$$0 \to I^{n+1} F \to F \to F_n \to 0,$$

namely

(2.7.2) $$H^q(I^{n+1} F) \to H^q(F) \to H^q(F_n) \to H^{q+1}(I^{n+1} F) \to H^{q+1}(F).$$

We deduce from (2.7.2) an exact sequence

(2.7.3) $$0 \to R_n \to H^q(F) \to H^q(F_n) \to Q_n \to 0,$$

where

$$R_n = \mathrm{Im}\ H^q(I^{n+1}F) \to H^q(F),$$

and

$$Q_n = \mathrm{Im}\ H^q(F_n) \to H^{q+1}(I^{n+1}F) = \mathrm{Ker}\ H^{q+1}(I^{n+1}F) \to H^{q+1}(F).$$

The main points are the following :

(1) The filtration (R_n) on $H^q(F)$ is I-good (2.5.1) ; in particular, the topology defined by (R_n) on $H^q(F)$ is the I-adic topology.

(2) The inverse system $Q. = (Q_n)$ is AR zero (2.5.2 (ii)').

(3) The inverse system $H^q(F.) = (H^q(F_n))$ satisfies ARML (2.5.2 (iii)').

Let us first show that (1), (2), (3) imply 2.4. Consider the exact sequence of inverse systems defined by (2.7.3) :

$$(*) \qquad 0 \to H^q(F)/R_n \to H^q(F_n) \to Q_n \to 0.$$

By (2) we have $\lim_n Q_n = 0$ (2.5.2 (a)). By the left exactness of the functor $\lim_n$ we thus get an isomorphism

$$(**) \qquad \lim_n H^q(F)/R_n \xrightarrow{\sim} \lim_n H^q(F_n).$$

By (1) the map

$$(***) \qquad H^q(F)\hat{} = \lim_n H^q(F)/I^{n+1}H^q(F) \to \lim_n H^q(F)/R_n$$

deduced from the surjections $H^q(F)/I^{n+1}H^q(F) \to H^q(F)/R_n$ is an isomorphism. Putting (**) and (***) together, we get that (2.4.1) is an isomorphism. By definition, we have

$$H^q(\hat{X}, \hat{F}) = H^q(\hat{X}, \lim_n F_n) = H^q(X, i_* \lim F_n) = H^q(X, \lim(i_n)_* F_n).$$

Thanks to (3), the assumptions of 2.5.3 are satisfied, therefore 2.4.2 is an isomorphism.

It remains to show (1), (2), (3).

Proof of (1). We have $R_{-1} = H^q(F)$. The inclusions

$$I^m R_n \subset R_{m+n}$$

follow from the fact that the natural map

$$\oplus_{n\in\mathbb{N}} H^q(I^{n+1}F)t^n \to \oplus_{n\in\mathbb{N}} H^q(F)t^n$$

is a map of graded B-modules, where $B = \oplus_{n\in\mathbb{N}} I^n t^n$ (2.6.2). By 2.6.2 (applied to IF), $\oplus_{n\in\mathbb{N}} H^q(I^{n+1}F)t^n$ is of finite type over B, and therefore so is its quotient $R := \oplus_n R_n$, which proves (1), thanks to the equivalence between conditions (ii) and (ii)' in 2.5.1.

Proof of (2). This is the most delicate point. By 2.6.2 again, $N := \oplus_n H^{q+1}(I^{n+1}F)$ is finitely generated over B. Since B is noetherian, $Q := \oplus_n Q_n$, which is a (graded) sub-B-module of N is also finitely generated, and therefore there exists $r \geq 0$ such that $Q_{n+1} = IQ_n$ for all $n \geq r$. Since Q_k, as a quotient of $H^q(F_k)$ is killed by I^{k+1} (as an A-module), each Q_n is therefore killed by I^{r+1} (as an A-module). Now, for $a \in I^p$, the composition of the multiplication by a from $H^{q+1}(I^{n+1}F)$ to $H^{q+1}(I^{p+n+1}F)$ with the transition map from $H^{q+1}(I^{p+n+1}F)$ to $H^{q+1}(I^{n+1}F)$ is the multiplication by a in $H^{q+1}(I^{n+1}F)$. Since $Q_{n+r+1} = I^{r+1}Q_n$ for $n \geq r$, it follows that, for all $n \geq r$, the transition map $Q_{n+r+1} \to Q_n$ is zero, and hence, if $s = 2r+1$, for all n the transition map $Q_{n+s} \to Q_s$ is zero.

Proof of (3). This is a formal consequence of (2). In the exact sequence (*), the left term has surjective transition maps (thus trivially satisfies ARML) and the right one is AR zero so they both trivially satisfy ARML. Therefore the middle one satisfies ARML in view of the second assertion of the following lemma [SGA 5 V 2.1.2], whose proof is elementary :

Lemma 2.7.4. *Let*

$$0 \to L'_\cdot \to L_\cdot \to L''_\cdot \to 0$$

be an exact sequence of inverse systems of A-modules. If $L_\cdot$ satisfies ARML, so does $L''_\cdot$, and if $L'_\cdot$ and $L''_\cdot$ satisfy ARML, so does $L_\cdot$.

Remarks 2.8. (a) Property (2) in 2.7 is the key technical point in Deligne's construction of the $Rf_!$ functor from $\mathrm{pro}D^b(X)_{coh}$ to $\mathrm{pro}D^b(S)_{coh}$ for $f : X \to S$ a compactifiable morphism of noetherian schemes (i. e. of the form gi with g proper and i an open immersion) [D1, Prop. 5] (more precisely, if, in the situation of 2.4, f is assumed to induce an isomorphism from $X - X'$ to $Y - Y'$, then what is shown in (loc. cit.) is that the inverse systems $H^q(I^{n+1}F)$ are AR zero for $q > 0$, as follows from the fact that, for $q > 0$, $H^q(F)$ is killed by a power of I.

(b) The proof of 2.2 shows that if $f : X \to Y$, Y', X' are as in 2.2 and F is a coherent sheaf on X such that, for some integer q, the graded modules $\oplus_n R^q f_*(I^n F)$ and $\oplus_n R^{q+1} f_*(I^n F)$ over the graded $\mathcal{O}_Y$-algebra $\oplus_n I^n$ are finitely generated, then $R^q f_* F$ is coherent and (2.1.3) and (2.1.4) are isomorphisms. See [SGA 2 IX] for details and examples. This refined comparison theorem is a key tool in Grothendieck's Lefschetz type theorems for the fundamental group and the Picard group [SGA 2 X, XI].

The comparison theorem 2.2 has many corollaries and applications. We will mention only a few of them. The following one (for $r = 0, 1$) is the main ingredient in the proof of Grothendieck's existence theorem, which will be discussed in §3.

Corollary 2.9 [EGA III 4.5.1]. *Let A be a noetherian ring, I an ideal of A, $f : X \to Y$ a morphism of finite type, $\hat{f} : \hat{X} \to \hat{Y}$ its completion along $Y' = V(I)$ and $X' = f^{-1}(Y')$ as in 2.4. Let F, G be coherent sheaves on X whose supports have an intersection which is proper over Y. Then, for all $r \in \mathbb{Z}$,* $\mathrm{Ext}^r(F, G)$ *is an A-module of finite type, and the natural map* $\mathrm{Ext}^r(F, G) \to \mathrm{Ext}^r(\hat{F}, \hat{G})$ *induces an isomorphism*

$$\mathrm{Ext}^r(F, G)\hat{} \xrightarrow{\sim} \mathrm{Ext}^r(\hat{F}, \hat{G}). \tag{2.9.1}$$

We have

$$Ext^r(F, G) = H^r RHom(F, G) = H^r R\Gamma(X, R\mathcal{H}om(F, G)).$$

The hypotheses on F, G imply that the cohomology sheaves of $R\mathcal{H}om(F,G)$ are coherent and have proper support over Y. Therefore, by the finiteness theorem, the cohomology groups of $RHom(F,G) = R\Gamma(X, R\mathcal{H}om(F,G))$ are finitely generated over A, and by 2.3 (a), the base change map

$$R\Gamma(X, R\mathcal{H}om(F,G))\hat{} \to R\Gamma(\hat{X}, R\mathcal{H}om(F,G)\hat{}),$$

where $(-)\hat{} = i^*$, is an isomorphism. But, since i is flat,

$$R\mathcal{H}om(F,G)\hat{} = R\mathcal{H}om(\hat{F},\hat{G}),$$

and the conclusion follows.

The next corollary is very useful in geometric applications :

Corollary 2.10 [EGA III 4.2.1] (theorem on formal functions). *Let $f : X \to Y$ be a proper morphism of locally noetherian schemes, y a point of Y, $X_y = X \times_Y \operatorname{Spec} k(y)$ the fiber of f at y, F a coherent sheaf on X. Let $F_n = F \otimes \mathcal{O}_y/\mathbf{m}_y^{n+1}$ on $X_n = X \times_Y \operatorname{Spec} \mathcal{O}_y/\mathbf{m}_y^{n+1}$. Then, for all $q \in \mathbb{Z}$, the stalk $R^q f_*(F)_y$ is an $\mathcal{O}_y$-module of finite type, and the natural map*

$$(R^q f_*(F)_y)\hat{} = \lim_n (R^q f_*(F)_y/\mathbf{m}_y^{n+1} R^q f_*(F)_y) \to \lim_n H^q(X_y, F_n) \tag{2.10.1}$$

is an isomorphism.

The map (2.10.1) is defined by the base change maps $(R^q f_*(F)_y/\mathbf{m}^{n+1} R^q f_*(F)_y) \to H^q(X_y, F_n)$, where in the right hand side, X_y is viewed as the underlying space of the scheme X_n. When y is closed, 2.10 is a special case of 2.2. One reduces to this case by base changing by $\operatorname{Spec} \mathcal{O}_y \to Y$.

2.11. Let $f : X \to Y$ be a proper morphism of locally noetherian schemes. Then $f_*\mathcal{O}_X$ is a finite $\mathcal{O}_Y$-algebra. Its spectrum $Y' = \operatorname{Spec} f_*\mathcal{O}_X$ is a finite scheme over Y, and the identity map of $f_*\mathcal{O}_X$ defines a factorization of f into

$$X \xrightarrow{f'} Y' \xrightarrow{g} Y,$$

with f' proper and g finite, called the *Stein factorization* of f. Its main property is described in the following theorem :

Theorem 2.12 [EGA III 4.3.1] (*Zariski's connectedness theorem*). *With the assumptions and notations of 2.11, $f'_*\mathcal{O}_X = \mathcal{O}_{Y'}$, and the fibers of f' are connected and nonempty.*

The first assertion follows trivially from the definitions. For the second one, one first reduces to the case where $Y' = Y$ and y is a closed point of Y. Then, if $\hat{X}$ is the completion of X along X_y, by 2.10,

$$\mathcal{O}_{Y,y}\hat{} = (f_*\mathcal{O}_X)_y\hat{} = H^0(X_y, \mathcal{O}_{\hat{X}}),$$

which cannot be the product of two nonzero rings.

In particular, if $Y' = Y$, i. e. $f_*\mathcal{O}_X = \mathcal{O}_Y$, the fibers of f are connected and nonempty. It is not hard to see, using the base change formula 3.3 below, that they are in fact *geometrically connected* (i. e. are connected and remain so after any field extension) [EGA III 4.3.4].

The following corollaries are easy, see [EGA III 4.3, 4.4] for details.

Corollary 2.13. *Under the assumptions of 2.12, for every point y of Y, the connected components of the fiber X_y correspond bijectively to the points of Y'_y, i. e. to the maximal ideals of the finite $\mathcal{O}_y$-algebra $f_*(\mathcal{O}_X)_y$.*

This is because the underlying space $g^{-1}(y)$ of Y'_y is finite and discrete.

Corollary 2.14. *Let $f : X \to Y$ be a proper and surjective morphism of integral noetherian schemes, with Y normal. Assume that the generic fiber of f is geometrically connected. Then all fibers of f are geometrically connected.*

Let ζ (resp. η) be the generic point of X (resp. Y) (so that $f(\zeta) = \eta$). The hypothesis on the generic fiber means that the algebraic closure K' of $K = k(\eta)$ in $k(\zeta)$ is a (finite) radicial extension of K [EGA IV 4.5.15]. Let $y \in Y$. Since $\mathcal{O}_y$ is normal et K' is radicial over K, the normalization A of $\mathcal{O}_y$ in K' is a local ring. and the residue field extension is radicial [B chap. 5, §2, $n°3$, Lemme 4]. Since A contains $(f_*\mathcal{O}_X)_y$, the same holds for $(f_*\mathcal{O}_X)_y$. Therefore, by 2.12 (and the remark after it) the fiber X_y is geometrically connected.

Corollary 2.15. *Under the assumptions of 2.12, a point x of X is isolated in its fiber, i. e. is such that there exists an open neighborhood V of x such that $V \cap X_{f(x)} = \{x\}$, if and only if $f'^{-1}(f'(x)) = \{x\}$. The set U of such points is open in X, $U' = f'(U)$ is open in Y', and $f' : X \to Y'$ induces an isomorphism $f'_{U'} : U \xrightarrow{\sim} f'(U)$.*

Let $y = f(x)$, $y' = f'(x)$. Since $g^{-1}(y)$ is finite, discrete, x is isolated in $f^{-1}(y)$ if and only if it is in $f'^{-1}(y')$. So we may assume $Y' = Y$, i.e. $f_*\mathcal{O}_X = \mathcal{O}_Y$, and hence, by 2.13, $f^{-1}(y) = \{x\}$. Choose open affine neighborhoods $U = \operatorname{Spec} B$, $V = \operatorname{Spec} A$ of x and y respectively, such that $f(U) \subset V$. Since f is closed, $f(X - U)$ is a closed subset of Y which does not contain y. Therefore, there exists an open affine neighborhood of y of the form $V_s = \operatorname{Spec} A_s$ for some $s \in A$ such that $f^{-1}(V_s) \subset U$. Then $f^{-1}(V_s) = U_s = \operatorname{Spec} B_s$. Since $f_*\mathcal{O}_X = \mathcal{O}_Y$, f induces on V_s an isomorphism $U_s \xrightarrow{\sim} V_s$.

Corollary 2.16. *Let $f : X \to Y$ be a proper morphism of locally noetherian schemes. If f is quasi-finite (i. e. has finite fibers), then f is finite.*

Corollary 2.17 (Zariski's Main Theorem). *Let $f : X \to Y$ be a compactifiable morphism of locally noetherian schemes (2. 8 (a)) (e. g. a quasi-projective morphism, with Y noetherian [EGA II 5.3.2]). If f is quasi-finite, then f can be factored as $f = gj$, where $j : X \to Z$ is an open immersion and $g : Z \to Y$ is a finite morphism.*

If Y is noetherian, one can remove the hypothesis that f should be compactifiable, provided that f is assumed to be separated and of finite presentation, see [EGA IV 8.12.6], whose proof makes no use of the comparison theorem 2.2 but relies on deeper commutative algebra.

Finally, we mention a useful application of 2.13. If X is a locally noetherian scheme, we denote by $\pi_0(X)$ the set of its connected components.

Corollary 2.18. *Let A be a henselian noetherian local ring, $S = \operatorname{Spec} A$, s its closed point, X a proper scheme over S. Then the natural map*

$$\pi_0(X_s) \to \pi_0(X)$$

is bijective.

Consider the Stein factorization

$$X \xrightarrow{f'} S' \longrightarrow S$$

of the structural morphism $f : X \to S$. We have $S' = \operatorname{Spec} A'$, where A' is a finite A-algebra. Since A is henselian, A' decomposes as a product of local A-algebras A_i, parametrized by the points i of S'_s. Let $S'_i = \operatorname{Spec} A_i$ and $X_i = S'_i \times_S X$, so that X is the disjoint union of the X_i's. By 2.13 the fiber $(X_i)_i = f'^{-1}(i)$ of X_i at i is connected. Since X_i is proper over S'_i and S'_i is local, no component of X_i can be disjoint from its special fiber, hence X_i is connected. Hence the X_i's are the connected components of X and they correspond bijectively to the connected components of X_s by associating to a component its special fiber.

2.19. *Base change maps.* Let

(2.19.1)
$$\begin{array}{ccc} X' & \xrightarrow{h} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & X \end{array}$$

be a commutative square of ringed spaces and let F be an $\mathcal{O}_X$-module. Then there is a canonical map of $\mathcal{O}_{Y'}$-modules

(2.19.2)
$$\gamma : g^* f_* F \to f'_* h^* F,$$

called the *base change map*, which is defined in the following two equivalent ways. Let $a = gf' = fh$.

(a) By adjunction between g^* and g_*, defining γ is equivalent to defining

$$\gamma_1 : f_* F \to g_* f'_* h^* F = a_* h^* F.$$

One has $a_* h^* F = f_* h_* h^* F$, and one defines γ_1 by applying f_* to the adjunction map $F \to h_* h^* F$.

(b) By adjunction between f'^* and f'_*, defining γ is equivalent to defining

$$\gamma_2 : f'^* g^* f_* F = a^* f_* F \to h^* F.$$

One has $a^* f_* F = h^* f^* f_* F$, and one defines γ_2 by applying h^* to the adjunction map $f^* f_* F \to F$.

That these two definitions are equivalent is a nontrivial fact, proved by Deligne [SGA 4 XVII] in a much more general context.

Along the same lines, one defines, for all $q \in \mathbb{Z}$, a canonical map

(2.19.3)
$$\gamma : g^* R^q f_* F \to R^q f_* (h^* F),$$

also called *base change map*. Again, by adjunction between g^* and g_*, it is equivalent to define

$$\gamma_1 : R^q f_* F \to g_* R^q f_*(h^* F).$$

One defines γ_1 as the composition vu of the following two maps :

$$u : R^q f_* F \to R^q a_*(h^* F),$$

$$v : R^q a_*(h^* F) \to g_* R^q f_*(h^* F).$$

The map u is the classical *functoriality map* on cohomology. Namely, we have an adjunction map in $D^+(X)$:

$$\alpha : F \to R h_* h^* F,$$

defined as the composition $F \to h_* h^* F \to h_* \mathcal{C}(h^* F)$, where the first map is the classical adjunction map and the second one is given by the choice of a resolution $h^* F \to C(h^* F)$ of $h^* F$ by modules acyclic for h_*. Applying Rf_* to α, we get a map

$$Rf_*(\alpha) : Rf_* F \to Rf_* R h_* h^* F = R a_* h^* F,$$

giving u by passing to cohomology sheaves. In other words, if V is an open subset of Y and $U = f^{-1}(V)$, $U' = a^{-1}(V) = h^{-1}(U)$, $R^q f_* F$ is the sheaf associated to the presheaf $V \mapsto H^q(U, F)$, $R^q a_*(h^* F)$ is the sheaf associated to the presheaf $V \mapsto H^q(U', h^* F)$, and u is associated to the functoriality map $H^q(U, F) \to H^q(U', h^* F)$.

The map v is an edge homomorphism $H^q \to E_\infty^{0q} \to E_2^{0q}$ for the spectral sequence

$$E_2^{ij} = R^i g_* R^j f'_*(h^* F) \to R^{i+j} a_*(h^* F).$$

More explicity, with the above notations and $V' = g^{-1}(V)$, v is associated to the map

$$H^q(U', h^* F) \to H^0(V', R^q f'_*(h^* F))$$

obtained by restricting an element of $H^q(U', h^* F)$ to open subsets $f'^{-1}(W)$ for W open in V'.

Under suitable assumptions of cohomological finiteness, it is possible to define a base change map in $D(Y')$,

$$Lg^* Rf_* F \to Rf'_* L h^* F, \tag{2.19.4}$$

inducing (2.19.3) (cf. [SGA 4 XVII 4.1.5]). However, when (2.19.1) is a cartesian square of schemes and F is a quasi-coherent sheaf, this map has no good properties in general (see 3.5).

3. Cohomological flatness

3.1. The results of this section will not be used in §4. They complement those of §2. More precisely, following [EGA III 7] we address the following question : in the situation of 2.10, with F *flat* over Y, when can we assert that the individual base change maps

$$R^q f_*(F)_y/\mathbf{m}_y^{n+1} R^q f_*(F)_y \to H^q(X_y, F_n)$$

are isomorphisms ? More generally, when can we assert that the formation of $R^q f_*(F)$ commutes with any base change, when is $R^q f_*(F)$ locally free of finite type ? As was shown in [SGA 6 III], the use of derived categories simplifies the presentation given in [EGA III 7]. Other expositions are given in [H, III 12] and [M1, 5].

In what follows, if X is a scheme, we denote by $D(X)$ the derived category of the category of $\mathcal{O}_X$-modules. The main tool is the following *base change formula* :

Theorem 3.2. *Let*

(3.2.1)
$$\begin{array}{ccc} X' & \xrightarrow{h} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian square of schemes, with X and Y quasi-compact and separated. Let F (resp. G) be a quasi-coherent sheaf on X (resp. Y'). Assume that F and G are tor-independent *on Y, i. e. that for all points $x \in X$, $y' \in Y'$ such that $g(y') = f(x)$ we have*

$$Tor_q^{\mathcal{O}_{Y,y}}(G_{y'}, F_x) = 0$$

for all $q > 0$ (this is the case for example if F or G is flat *over Y). Then there is a natural isomorphism in $D(Y')$:*

(3.2.2)
$$G \otimes_Y^L Rf_*F \xrightarrow{\sim} Rf'_*(G \otimes_Y F),$$

where $G \otimes_Y^L Rf_*F := G \otimes^L Lg^* Rf_*F$ and $G \otimes_Y F = f'^*G \otimes h^*F$.

When $Y = Y'$ (resp. $G = \mathcal{O}_{Y'}$), the isomorphism (3.2.2) is called the *projection isomorphism* (resp. the *base change isomorphism*). When $G = \mathcal{O}_{Y'}$, one deduces from (3.2.2) a canonical map, for $q \in \mathbb{Z}$,

(3.2.3)
$$g^* R^q f_* F \to R^q f'_*(h^*F).$$

This map is the composition of the canonical map $g^* R^q f_* F \to H^q(Lg^* Rf_* F)$ and the isomorphism $H^q(Lg^* Rf_* F) \xrightarrow{\sim} R^q f'_*(h^*F)$ deduced from (3.2.2) by applying H^q. It will follow from the construction of (3.2.2) that this map is the base change map defined in (2.19.3). It is *not* an isomorphism in general. This question is addressed in 3.10-3.11.

The following corollaries are the most useful particular cases :

Corollary 3.3. *If, in the cartesian square (3.2.1), g is flat, then (3.2.2) gives a base change isomorphism*

$$g^* Rf_* F \xrightarrow{\sim} Rf'_* h^* F,$$

and the induced base change maps (3.2.3) are isomorphisms.

Corollary 3.4. *Let $f : X \to Y$ be a morphism between quasi-compact and separated schemes. Let y be a point of Y, denote by X_y the fiber of f at y, i. e.* $\operatorname{Spec} k(y) \times_Y X$, *and let F be a quasi-coherent sheaf on X, flat over Y. Then (3.2.2) gives a natural isomorphism (in the derived category of $k(y)$-vector spaces)*

$$k(y) \otimes^L_{\mathcal{O}_Y} Rf_* F \xrightarrow{\sim} R\Gamma(X_y, \mathcal{O}_{X_y} \otimes_{\mathcal{O}_X} F).$$

Let us prove 3.2. First, consider the case where X, Y, Y' are affine, with rings B, A, A' respectively, so that X' is affine of ring $B' = A' \otimes_A B$, and $F = \tilde{M}$, $G = \tilde{N}$ for some B-module M and A'-module N. Then $Rf_* F$ is represented by the underlying A-module $M_{[A]}$ of M, and $Rf'_*(G \otimes_Y F)$ by the underlying A'-module $(N \otimes_A M)_{[A']}$ of $(N \otimes B') \otimes_{B'} (B' \otimes M)$. On the other hand, $G \otimes^L_Y Rf_* F$ is represented by $N \otimes^L_A M_{[A]} := N \otimes^L_{A'} (A' \otimes^L_A M_{[A]})$, which can be calculated as $N \otimes_A P$ where P is a flat resolution of $M_{[A]}$. The tor-independence hypothesis says that $Tor^A_q(N, M_{[A]}) = 0$ for $q > 0$, i. e. the natural map

$$(*) \qquad N \otimes^L_A M_{[A]} \to N \otimes_A M_{[A]}$$

is an isomorphism (in $D(A')$). The isomorphism (3.2.2) is the composition of (*) and the (trivial) isomorphism

$$(**) \qquad N \otimes_A M_{[A]} \xrightarrow{\sim} (N \otimes_A M)_{[A']}.$$

Assume now that the morphism f (but not necessarily the scheme Y) is affine. Then $X = \operatorname{Spec} B$ for a quasi-coherent $\mathcal{O}_Y$-algebra B, and $F = \tilde{M}$ for a quasi-coherent B-module M. We have again $Rf_* F = f_* F$, which is represented by the underlying (quasi-coherent) $\mathcal{O}_Y$-module $M_{[A]}$ of M. The preceding discussion, applied to affine open subsets of Y' above affine open subsets of Y, shows that we have natural identifications

$$G \otimes^L_Y Rf_* F \xrightarrow{\sim} G \otimes g^* f_* F \xrightarrow{\sim} f'_*(G \otimes_Y F) \xrightarrow{\sim} Rf'_*(G \otimes_Y F).$$

Their composition defines the isomorphism (3.2.2).

In the general case, choose a finite open affine cover $\mathcal{U} = (U_i)_{i \in I}$ of X $(I = \{1, \cdots, r\}$. Since X and Y are separated, any finite intersection $U_{i_0 \cdots i_n} = U_{i_0} \cap \cdots \cap U_{i_n}$ $(i_0 < \cdots < i_n$ of the U_i's is affine over Y [EGA II 1.6.2]. Therefore (by [EGA III 1.4]) we have

$$Rf_* F = f_* \check{C}(\mathcal{U}, F),$$

where $\check{C}(\mathcal{U}, F)$ is the alternating Čech complex of $\mathcal{U}$ with values in F. By the discussion in the case f is affine, we get isomorphisms

$$(***) \qquad G \otimes^L_Y Rf_* F \xrightarrow{\sim} G \otimes g^* f_* \check{C}(\mathcal{U}, F) \xrightarrow{\sim} f'_* \check{C}(\mathcal{U}', G \otimes_Y F) \xrightarrow{\sim} Rf'_*(G \otimes_Y F),$$

where $\mathcal{U}'$ is the cover of X' formed by the inverse images of the U_i's. It is easy to check that the composition (***) does not depend on the choice of $\mathcal{U}$. We take this composition as the definition of the isomorphism (3.2.2).

The compatibility between (3.2.3) and (2.19.3) is left to the reader.

Remark 3.5. It is easy to generalize 3.2 to the case Y is quasi-compact and f is quasi-compact and quasi-separated (in the last part of the argument, the intersections $U_{i_0 \cdots i_n}$ are only quasi-compact, and one has to replace the Čech complex by a suitable "hyper Čech" variant).

It seems difficult, however, to get rid of the tor-independence assumption. For example, when (3.2.1) is a cartesian square of *affine* schemes, as in the beginning of the proof of 3.2, and $F = \mathcal{O}_X$, but no tor-independence assumption is made, we do have a base change map of the form (2.19.4), namely, the map corresponding to the map

$$A' \otimes^L_A B \to A' \otimes_A B$$

in $D(A')$, but this map is an isomorphism if and only if A' and B are tor-independent over A.

In order to obtain a satisfactory formalism one has to use some tools of homotopical algebra, such as derived tensor products of rings. No account has been written down as yet.

3.6. The main application of 3.2 is to the case f is a proper morphism of noetherian schemes and F is a coherent sheaf on X, which is flat over Y. In this case, the complex Rf_*F has nice properties, namely it's a *perfect* complex, and the base change formula 3.4 enables one to analyze the compatibility with base change of its cohomology sheaves R^qf_*F around a point y of Y.

We first recall some basic finiteness conditions on objects of $D(X)$, where X is a locally noetherian scheme. These are discussed in much greater generality in [SGA 6 I, II, III]. There are three main conditions : pseudo-coherence, finite tor-dimension, perfectness, the last one being a combination of the first two.

3.6.1. *Pseudo-coherence.* A complex $E \in D(X)$ is called *pseudo-coherent* if it is in $D^-(X)$ (i e. $H^q(E) = 0$ for $q >> 0$) and has coherent cohomology (i. e. $H^q(E)$ is coherent for all q). One usually denotes by $D^*(X)_{coh}$ the full subcategory of $D^*(X)$ ($* = -, b$) consisting of pseudo-coherent complexes. It is a triangulated subcategory. If X is affine and $H^q(E) = 0$ for $q > a$, then E is pseudo-coherent if and only if E is isomorphic, in $D(X)$, to a complex L such that $L^q = 0$ for $q > a$ and L^q is free of finite type for all q. In particular, on any locally noetherian scheme X, a pseudo-coherent complex is locally isomorphic, in the derived category, to a bounded above complex of $\mathcal{O}$-modules which are free of finite type, and for any point x of X, the stalk E_x, as a complex of $\mathcal{O}_{X,x}$-modules, is isomorphic (in the derived category $D(\mathcal{O}_{X,x})$) to a bounded above complex of $\mathcal{O}_{X,x}$-modules which are free of finite type.

The above assertion is proven by an easy step by step construction [EGA 0_{III}11.9.1].

3.6.2. *Finite tor-dimension.* Let $a, b \in \mathbb{Z}$ with $a \leq b$. A complex $E \in D(X)$ is said to be of *tor-amplitude* in $[a, b]$ if it satisfies the following equivalent conditions :

(i) E is isomorphic, in $D(X)$, to a complex L such that $L^q = 0$ for $q \notin [a, b]$ and L^q is *flat* for all q ;

(ii) for any $\mathcal{O}_X$-module M, one has $H^q(M \otimes^L E) = 0$ for $q \notin [a, b]$.

The proof of the equivalence of (i) and (ii) is straightforward. A complex E is said to be of *finite tor-dimension* (or of *finite tor-amplitude*) if it is of tor-amplitude in $[a, b]$ for some interval $[a, b]$. The full subcategory of $D(X)$ consisting of complexes of finite tor-dimension is a triangulated subcategory.

For a complex E to be of tor-amplitude in $[a, b]$ it is necessary and sufficient that, for all $x \in X$, the stalk E_x, as a complex of $\mathcal{O}_{X,x}$-modules, be of tor-amplitude in $[a, b]$, i. e. isomorphic, in $D(\mathcal{O}_{X,x})$, to a complex L concentrated in degree in $[a, b]$ and flat in each degree, or, equivalently, such that, for any $\mathcal{O}_{X,x}$-module M, $H^q(M \otimes^L E_x) = 0$ for $q \notin [a, b]$.

3.6.3. *Perfectness.* A complex $E \in D(X)$ is called *perfect* if it is pseudo-coherent and locally of finite tor-dimension. It is said to be of *perfect amplitude* in $[a, b]$ (for $a, b \in \mathbb{Z}$ with $a \leq b$) if it is pseudo-coherent and of tor-amplitude in $[a, b]$. A *strictly perfect* complex is a bounded complex of locally free of finite type modules. The full subcategory of $D(X)$ consisting of perfect complexes is a triangulated subcategory.

Since an $\mathcal{O}_X$-module is locally free of finite type if and only if it is coherent and flat, it follows from 3.6.1 and 3.6.2 that a complex E is perfect if and only if it is locally isomorphic, in the derived category, to a strictly perfect complex. In the same vein, we have the following useful criterion :

Proposition 3.6.4. *Let x be a point of X and E be a pseudo-coherent complex on X such that $H^q(E_x) = 0$ for $q \notin [a, b]$, for some interval $[a, b]$. Then the following conditions are equivalent :*

(i) locally around x, E is of perfect amplitude in $[a, b]$;

(ii) $H^{a-1}(k(x) \otimes^L E) = 0$;

(iii) locally around x, E is isomorphic, in the derived category, to a complex of free of finite type $\mathcal{O}$-modules concentrated in degree in $[a, b]$.

By 3.6.1 we may assume that E has coherent components and is concentrated in degree in $[a, b]$, with E^q free of finite type for $q > a$. We have to show that (ii) implies that E^a is locally free of finite type around x. From the exact sequence

$$0 \to E^{[a+1,b]} \to E \to E^a[-a] \to 0,$$

where $E^{[a+1,b]}$ is the naïve truncation of E in degree $\geq a + 1$, we deduce that

$$Tor_1^{\mathcal{O}_{X,x}}(k(x), E_x^a) = 0.$$

By the standard flatness criterion [B, III, §5, th. 3], this implies that E_x^a is free of finite type, hence that E^a is free of finite type in a neighborhood of x.

Corollary 3.6.5. *Let x be a point of X, $q \in \mathbb{Z}$, and E be a pseudo-coherent complex on X such that $H^i E = 0$ for $i > b$ for some integer b. For $i \in \mathbb{Z}$, let*

$$\alpha^i(x) : k(x) \otimes H^i(E) \to H^i(k(x) \otimes^L E)$$

denote the canonical map.

(a) The following conditions are equivalent :
(i) $\alpha^q(x)$ is surjective ;
(ii) $\tau_{>q}E$ is of perfect amplitude in $[q+1, b]$ in a neighborhood of x.
When these conditions are satisfied, there is an open neighborhood U of x such that $\alpha^q(y)$ is bijective for all $y \in U$, and such that for all quasi-coherent modules M on U, the natural map

$$\alpha^q(M) : M \otimes H^q(E) \to H^q(M \otimes^L E)$$

is bijective.
(b) Assume that (a) (i) holds. Then the following conditions are equivalent :
(i) $\alpha^{q-1}(x)$ is surjective ;
(ii) $H^q(E)$ is locally free of finite type in a neighborhood of x.

Here, if L is a complex (in an abelian category) and $i \in \mathbb{Z}$, $\tau_{\geq i}L$ denotes the *canonical truncation* of L in degree $\geq i$, defined as $0 \to L^i/dL^{i-1} \to L^{i+1} \to \cdots$, and $\tau_{>i} = \tau_{\geq i+1}$.

Let us prove (a). The projection $E \to \tau_{\geq q}E$ induces an isomorphism

$$(*) \qquad H^q(k(x) \otimes^L E) \xrightarrow{\sim} H^q(k(x) \otimes^L \tau_{\geq q}E).$$

Consider the canonical distinguished triangle

$$(**) \qquad H^q(E)[-q] \to \tau_{\geq q}E \to \tau_{>q}E \to .$$

Taking (*) into account, we get from (**)

$$(***) \qquad \mathrm{Coker}\ \alpha^q(x) = H^q(k(x) \otimes^L \tau_{>q}E).$$

The equivalence between (i) and (ii) thus follows from 3.6.4. Assume that these conditions hold. It suffices to show the last assertion of (a). Let U be an open neighborhood of x such that $\tau_{>q}E|U$ is of perfect amplitude in $[q+1, b]$. Let M be a quasi-coherent sheaf on U. Applying $M \otimes^L -$ to (**), and taking into account that $H^i(M \otimes^L \tau_{>q}E) = 0$ for $i \leq q$, we get that $\alpha^q(M)$ is bijective. Let us prove (b). Since $\tau_{>q}E$ is of perfect amplitude in $[q+1, b]$ in a neighborhood of x, the triangle (**) shows that $H^q(E)$ is locally free of finite type in a neigborhood of x if and only if $\tau_{\geq q}E$ is of perfect amplitude in $[q, b]$ in a neighborhood of x. But, by (i), this condition is equivalent to the surjectivity of $\alpha^{q-1}(x)$.

3.7. Let $f : X \to Y$ be a morphism of locally noetherian schemes and let F be a coherent sheaf of X. As said earlier, the main applications of 3.2 deal with the case where F is flat over Y. By definition, F is flat over Y if and only if for all $x \in X$, the $\mathcal{O}_{X,x}$-module F_x is flat over $O_{Y,y}$, where $y = f(x)$. It is often convenient to express this in the following way, given by the flatness criterion [B, III, §5, th. 3] : F is flat over Y if and only if, for all $y \in Y$, the natural (surjective) map

$$(3.7.1) \qquad \mathrm{gr}^n \mathcal{O}_{Y,y} \otimes_{k(y)} \mathrm{gr}^0 F_y \to \mathrm{gr}^n F_y$$

is an isomorphism for all $n \geq 0$, where gr means the associated graded for the $\mathbf{m}_y$-adic filtration on $\mathcal{O}_{Y,y}$ ($\mathbf{m}_y$ being the maximal ideal) and F_y, the inverse image of F on $\mathrm{Spec}\, \mathcal{O}_{Y,y} \times_Y X$. The bijectivity of (3.7.1) is also equivalent to the fact that $Tor_1^{\mathcal{O}_{Y,y}}(k(y), F_y) = 0$, i. e. the natural map $k(y) \otimes^L F \to k(y) \otimes F$ is an isomorphism, or to the fact that, for each $n \geq 0$, $F_y/\mathbf{m}_y^{n+1}F$ is flat over $\mathrm{Spec}\, \mathcal{O}_{Y,y}/\mathbf{m}_y^{n+1}$.

Theorem 3.8. *Let $f : X \to Y$ be a proper morphism of noetherian schemes, and let F be a coherent sheaf on X. Then Rf_*F is pseudo-coherent (3.6.1) on Y. If F is flat over Y, Rf_*F is perfect (3.6.3).*

The first assertion is just a rephrasing of Grothendieck's finiteness theorem [EGA III 3.2.1], which says that the sheaves R^qf_*F are coherent, together with [EGA III 1.4.12], which implies that Rf_*F belongs to $D^b(Y)$. To prove the second assertion, we may assume that Y is affine. Let N be an integer such that $R^qf_*E = 0$ for all quasi-coherent sheaves E on X and $q > N$ (one can take N such that there is a covering of X by $N + 1$ open affine subsets [EGA III 1.4.12]). By (3.2.2), for any quasi-coherent $\mathcal{O}_Y$-module G, we have

$$G \otimes^L Rf_*F \xrightarrow{\sim} Rf_*(G \otimes_Y F),$$

and in particular,

$$H^q(G \otimes^L Rf_*F) = 0$$

for $q \neq [0, N]$. *A fortiori*, for any point y of Y and any $\mathcal{O}_{Y,y}$-module M, we have

$$H^q(M \otimes^L (Rf_*F)_y) = 0$$

for $q \neq [0, N]$. By 3.6.2, this means that Rf_*F is of perfect amplitude in $[0, N]$.

3.9. Under the assumptions of 3.8, with F flat over Y, assume Y affine, $Y = \operatorname{Spec} A$, and let Y' be a closed subscheme of Ydefined by an ideal I. As explained in [EGA III 7.4.8], the pseudo-coherence of Rf_*F, together with the base change formula 3.2, gives another proof (in this particular case) of the fact that the maps φ_q (2.4.1) are isomorphisms.

By 3.2, whe have, for $n \geq 0$,

$$(*) \qquad A/I^{n+1} \otimes^L R\Gamma(X, F) \xrightarrow{\sim} R\Gamma(X, F_n).$$

The map φ_q is the inverse limit of the maps

$$\varphi_{q,n} : A/I^{n+1} \otimes H^q(X, F) \to H^q(X, F_n),$$

obtained by composing the natural map $A/I^{n+1} \otimes H^q(X, F) \to H^q(A/I^{n+1} \otimes^L R\Gamma(X, F))$ with the isomorphism $H^q(*)$. Since Rf_*F is pseudo-coherent, $R\Gamma(X, F)$ is isomorphic to a complex P of A-modules, which is bounded above and consists of free modules of finite type. The maps $\varphi_{q,n}$ can be rewritten

$$H^q(P)/I^{n+1}H^q(P) \to H^q(P/I^{n+1}P).$$

In general, none is an isomorphism, but it follows from Artin-Rees that the limit

$$\lim H^q(P)/I^{n+1}H^q(P) \to \lim H^q(P/I^{n+1}P).$$

is an isomorphism.

3.10. Let $f : X \to Y$ be a proper morphism of separated noetherian schemes, and let F be a coherent sheaf on X, flat over Y. Let $q \in \mathbb{Z}$. We say that F is *cohomologically flat over* Y *in degree* q if, for any morphism $g : Y' \to Y$, the base change map (3.2.3)

$$g^* R^q f_* F \to R^q f'_* F' \tag{3.10.1}$$

is an isomorphism, where, in the notations of (3.2.1), $F' = h^* F$. When $F = \mathcal{O}_X$ (i. e. f is flat), we just say that f is cohomologically flat in degree q.

If y is a point of Y and $X_y = \operatorname{Spec} k(y) \times_Y X$ is the fiber of f at y, the map (3.10.1) reads

$$k(y) \otimes R^q f_* F \to H^q(X_y, F/\mathbf{m}_y F). \tag{3.10.2}$$

We shall denote it by $\alpha^q(y)$ by analogy with the notation used in 3.6.5. The following criterion is a simple consequence of 3.6.5, applied to the pseudo-coherent complex $E = Rf_*F$ on Y (cf. [EGA III 7.8.4], [H, III, 12.11]) :

Corollary 3.11. *With $f : X \to Y$ and F as in 3.10, let $q \in \mathbb{Z}$ and let y be a point of Y. Let b be an integer such that $R^i f_* F = 0$ for $i > b$.*

(a) The following conditions are equivalent :

(i) the map $\alpha^q(y)$ (3.10.2) is surjective ;

(ii) $\tau_{>q} Rf_* F$ is of perfect amplitude in $[q+1, b]$ in a neighborhood of y.

When these conditions are satisfied, there is an open neighborhood U of y such that $\alpha^q(z)$ is bijective for all $z \in U$ and such that $F|f^{-1}(U)$ is cohomologically flat over U in degree q.

(b) Assume that (a) (i) holds. Then the following conditions are equivalent :

(i) $\alpha^{q-1}(y)$ is surjective ;

(ii) $R^q f_* F$ is locally free of finite type in a neighborhood of y.

Remark 3.11.1. Since condition (a) for $q = -1$ is trivially satisfied for all y, we get that Rf_*F is of perfect amplitude in $[0, b]$, as already observed at the end of the proof of 3.8.

On the other hand, condition (b) for $q = b$ is trivially satisfied for all y. Hence F is cohomologically flat in degree b.

Remark 3.11.2 (cf. [EGA III 4.6.1]). The following criterion is very useful : *if $H^{q+1}(X_y, F/\mathbf{m}_y F) = 0$, then $\alpha_q(y)$ is surjective ; in particular, as follows from (b), if $H^1(X_y, F/\mathbf{m}_y F) = 0$, then, in a neighborhood of y, f_*F is locally free of finite type and commutes with base change.*

Indeed, if $H^{q+1}(X_y, F/\mathbf{m}_y F) = 0$, then the theorem on formal functions 2.10 implies that $(R^q f_*(F)_y)\hat{} = 0$, hence $R^q f_*(F)_y = 0$, so that $\tau_{>q} Rf_*(F)_y = \tau_{>q+1} Rf_*(F)_y$. Since (trivially) $\alpha_{q+1}(y) = 0$, by (a) we have that $\tau_{>q} Rf_* F$ is of perfect amplitude in $[q+2, b]$, hence *a fortiori* in $[q+1, b]$, and therefore $\alpha_q(y)$ is surjective.

3.12. Assume that (a) (i) of 3.11 holds for all $y \in Y$, i. e. that $\tau_{>q} Rf_* F$ is of perfect amplitude in $[q+1, b]$. The dual

$$K = R\mathcal{H}om(\tau_{>q} Rf_* F, \mathcal{O}_Y)$$

is a perfect complex, of perfect amplitude in $[-b, -q-1]$. Let

$$Q := H^{-q-1}(K).$$

Then, for any quasi-coherent $\mathcal{O}_Y$-module M, there is a natural isomorphism

$$R^{q+1}f_*(M \otimes_Y F) \xrightarrow{\sim} \mathcal{H}om(Q, M). \tag{3.12.1}$$

Moreover, the formation of Q commutes with any base change. (This is the so-called *exchange property*, cf. [EGA III 7.7.5, 7.7.6, 7.8.9].)

The proof is again a simple application of 3.2. By 3.2, we have

$$R^{q+1}f_*(M \otimes_Y F) = H^{q+1}(M \otimes^L Rf_*F). \tag{$*$}$$

The projection $Rf_*F \to \tau_{>q}Rf_*F$ induces an isomorphism

$$H^{q+1}(M \otimes^L Rf_*F) \xrightarrow{\sim} H^{q+1}(M \otimes^L \tau_{>q}Rf_*F). \tag{$**$}$$

Let $L := \tau_{>q}Rf_*F$, so that $K = R\mathcal{H}om(L, \mathcal{O}_Y)$. Since L is perfect, we have a natural biduality isomorphism

$$L \xrightarrow{\sim} R\mathcal{H}om(K, \mathcal{O}_Y),$$

which induces an isomorphism

$$M \otimes^L L \xrightarrow{\sim} R\mathcal{H}om(K, M). \tag{$***$}$$

Composing (*), (**) and $H^{q+1}(***)$, we get

$$R^{q+1}f_*(M \otimes_Y F) \xrightarrow{\sim} H^{q+1}R\mathcal{H}om(K, M) = \mathcal{E}xt^{q+1}(K, M).$$

But, since K is of perfect amplitude in $[-b, -q-1]$, i. e. locally isomorphic to a complex of free modules concentrated in degree in $[-b, -q-1]$, we have

$$\mathcal{E}xt^{q+1}(K, M) = \mathcal{H}om(H^{-q-1}K, M),$$

which gives (3.12.1). The proof of the compatibility of Q with base change is left to the reader.

3.13. Under the hypotheses of 3.8, the perfectness of Rf_*F implies nice properties of the functions on Y :

$$y \mapsto \mathrm{rk}\ H^q(X_y, F/\mathbf{m}_yF)$$

(for a fixed q), and

$$y \mapsto \sum (-1)^q \mathrm{rk}\ H^q(X_y, F/\mathbf{m}_yF).$$

The first one is *upper semicontinuous*, while the second one is *locally constant*. This follows from 3.4. The verification is left to the reader. For a detailed discussion of these questions, see [EGA III 7.7, 7.9] and [SGA 6 III].

4. The existence theorem

4.1. Let A be an adic noetherian ring (1.1), I an ideal of definition of A, $Y = \operatorname{Spec} A$, $Y_n = \operatorname{Spec} A/I^{n+1}$, $\hat{Y} = \operatorname{colim}_n Y_n = \operatorname{Spf}(A)$. The problem which is addressed in this section is the following : given an adic noetherian $\hat{Y}$-formal scheme $\mathcal{Z} = \operatorname{colim}_n Z_n$ (1.5), when can we assert the existence (and uniqueness) of a (suitable) locally noetherian scheme X over Y whose I-adic completion $\hat{X} = \operatorname{colim}_n X_n$, where $X_n = X \times_Y Y_n$, is isomorphic to $\mathcal{Z}$? This is the so-called problem of *algebraization*. As for the analogous problem in complex analytic geometry (Serre's *GAGA*), Grothendieck's approach consists in first fixing X and comparing coherent sheaves on X and $\hat{X}$. The fundamental result is the following theorem :

Theorem 4.2 [EGA III 5.1.4]. *Let X be a noetherian scheme, separated and of finite type over Y, and let $\hat{X}$ be its I-adic completion as in 4.1. Then the functor $F \mapsto \hat{F}$ (1.6) from the category of coherent sheaves on X whose support is proper over Y to the category of coherent sheaves on $\hat{X}$ whose support is proper over $\hat{Y}$ is an equivalence.*

Recall that the support of a coherent sheaf $\mathcal{E}$ on $\hat{X}$ is the support of $E_0 = \mathcal{E} \otimes \mathcal{O}_{X_0}$ on X_0 (1.4). It is called proper over $\hat{Y}$ if it is proper over Y_0 as a closed subset of X_0.

4.3. *Proof of 4.2.* Let F, G be coherent sheaves on X with proper supports over Y. By 2.9, $\operatorname{Hom}(F, G)$ is an A-module of finite type, hence separated and complete for the I-adic topology, and therefore the natural map

$$\operatorname{Hom}(F, G) \to \operatorname{Hom}(\hat{F}, \hat{G})$$

is an isomorphism. This proves that the $(-)\hat{}$ functor is fully faithful. It remains to prove that it is essentially surjective. This is done in several steps. We will outline the main points.

(a) *Projective case.* Assume $f : X \to Y$ to be projective. Let L be an ample line bundle on X. If M is an $\mathcal{O}_X$-module (resp. $\mathcal{O}_{\hat{X}}$-module) and $n \in \mathbb{Z}$, write, as usual, $M(n)$ for $M \otimes L^{\otimes n}$ (resp. $M \otimes \hat{L}^{\otimes n}$). The main point is the following result, which is a particular case of [EGA III 5.2.4] :

Lemma 4.3.1. *Let E be a coherent sheaf on $\hat{X}$. Then there exist nonnegative integers m, r and a surjective homomorphism*

$$\mathcal{O}_{\hat{X}}(-m)^r \to E.$$

Assuming 4.3.1, let us show how to prove the essential surjectivity in this case. Let E be a coherent sheaf on $\hat{X}$. By 4.3.1 we can find an exact sequence

$$\mathcal{O}_{\hat{X}}(-m_1)^{r_1} \xrightarrow{u} \mathcal{O}_{\hat{X}}(-m_0)^{r_0} \longrightarrow E \longrightarrow 0 ,$$

for some nonnegative integers m_0, m_1, r_0, r_1. By the full faithfulness of $(-)\hat{}$, there exists a unique morphism $v : \mathcal{O}_X(-m_1)^{r_1} \to \mathcal{O}_X(-m_0)^{r_0}$ such that $u = \hat{v}$. Let $F := \operatorname{Coker} v$. Then, by the exactness of $(-)\hat{}$ (on the category of coherent sheaves on X), $E = \hat{F}$.

Let us now prove 4.3.1. Since L is ample, so is $L_0 = \mathcal{O}_{X_0}(1)$ on X_0. Consider the graded $\mathcal{O}_{Y_0}$-algebra $S = \mathrm{gr}_I \mathcal{O}_Y = \oplus_{n\in\mathbb{N}} \tilde{I}^n/\tilde{I}^{n+1} = \oplus_{n\in\mathbb{N}} \mathcal{I}^n/\mathcal{I}^{n+1}$, and the graded $f_0^*(S)$-module $M = \mathrm{gr}_I E = \oplus_{n\in\mathbb{N}} \mathcal{I}^n E/\mathcal{I}^{n+1}E$, where $\mathcal{I} = I^\Delta$. Since $\mathrm{gr}_0 E$ is coherent on X_0 and the canonical map $\mathrm{gr}_I \mathcal{O}_Y \otimes_{\mathrm{gr}_0 \mathcal{O}_Y} \mathrm{gr}_0 E \to \mathrm{gr}_I E$ is surjective, M is of finite type over $f_0^*(S)$, hence corresponds to a coherent module $\tilde{M}$ on $X' := \mathrm{Spec}\, f_0^*(S)$. Since the inverse image $\mathcal{O}_{X'}(1)$ of $\mathcal{O}_X(1)$ on X' is ample, applying Serre's vanishing theorem [EGA III 2.2.1] for $\tilde{M}$, $\mathcal{O}_{X'}(1)$ and the morphism $f' : X' \to Y' = \mathrm{Spec}\, S$ deduced from f_0 by base change by $Y' = \mathrm{Spec}\, S \to Y$, we find that there exists an integer n_0 such that, for all $n \geq n_0$, all $k \in \mathbb{N}$, and all $q > 0$,

$$H^q(X_0, \mathrm{gr}_k E(n)) = 0.$$

It follows that, for all $n \geq n_0$ and all $k \geq 0$, the transition map $H^0(X_0, E_{k+1}(n)) \to H^0(X_0, E_k(n))$ is surjective, and consequently the canonical map

$$H^0(\hat{X}, E(n)) = \lim_k H^0(X_0, E_k(n)) \to H^0(X_0, E_0(n))$$

is surjective. Since $\mathcal{O}_{X_0}(1)$ is ample, we may assume that n_0 has been chosen large enough for the existence of a finite number of global sections of $E_0(n_0)$ generating $E_0(n_0)$. Lifting these sections to $H^0(\hat{X}, E(n_0))$, we find a map

$$u : \mathcal{O}_{\hat{X}}(-n_0)^r \to E,$$

such that $u_0 = u \otimes \mathcal{O}_{X_0} : \mathcal{O}_{X_0}(-n_0)^r \to E_0$ is surjective. By Nakayama's lemma (since I is contained in the radical of A (1.1)), this implies that u is surjective.

Remark 4.3.2. The above proof shows that the conclusion of 4.3.1 still holds if $\hat{X}$ is replaced by an adic $\hat{Y}$-formal scheme $\mathcal{X}$ such that $X_0 = \mathcal{X} \times_Y Y_0$ is proper and $\mathcal{O}_{\hat{X}}(1)$ by an invertible $\mathcal{O}_{\mathcal{X}}$-module L such that $L_0 = L \otimes \mathcal{O}_{X_0}$ is ample. It also shows that, under these hypotheses, there exists an integer n_0 such that $\Gamma(\mathcal{X}, E(n)) \to \Gamma(X_0, E_0(n))$ is surjective for all $n \geq n_0$, with the usual notation $E(n) = E \otimes L^{\otimes n}$.

(b) *Quasi-projective case.* Assume that we have an open immersion $j : X \to Z$, with Z projective over Y. Let E be a coherent sheaf on $\hat{X}$ whose support T_0 is proper over $\hat{Y}$. Then, the extension by zero $\hat{j}_! E$ is coherent on $\hat{Z}$, hence, by (a), of the form $\hat{F}$ for a coherent sheaf F on Z. The support T of F is contained in X (because $X \cap T$ is open in T and contains T_0 hence is equal to T), so that $F = j_! j^* F$, and $E = (j^*F)\hat{}$.

(c) *General case.* We proceed by noetherian induction on X. We assume that for all closed subschemes T of X distinct of X, all coherent sheaves on $\hat{T}$ whose support is proper over $\hat{Y}$ are *algebraizable*, i. e. of the form $\hat{F}$ for some coherent sheaf F on T with proper support over Y, and we show that every coherent sheaf on $\hat{X}$ whose support is proper over $\hat{Y}$ is algebraizable. The main tool is Chow's lemma [EGA II 5.6.1] : assuming X nonempty, one can find morphisms

$$Z \xrightarrow{g} X \xrightarrow{f} Y$$

such that g is projective and surjective, fg quasi-projective, and there exists an open immersion $j : U \to X$, with U nonempty, such that g induces an isomorphism over U. Let

$T = X - U$ with the reduced scheme structure, and J be the ideal of T in X. Let E be a coherent sheaf on $\hat{X}$ whose support is proper over $\hat{Y}$. Consider the exact sequence

$$(*) \qquad 0 \to K \to E \to \hat{g}_* \hat{g}^* E \to C \to 0.$$

It suffices to show the following points :

(1) $\hat{g}_* \hat{g}^* E$ is algebraizable.

(2) K and C are killed by a positive power $\hat{J}^N$ of $\hat{J}$, hence can be viewed as coherent sheaves on $\hat{T}'$, where T' is the thickening of T defined by J^N.

(3) K and C, as coherent sheaves on $\hat{T}'$ are algebraizable.

(4) For a coherent sheaf on $\hat{X}$ whose support is proper over Y the property of being algebraizable is stable under kernel, cokernel and extension.

For (1), note that by case (b), $\hat{g}^* E$ (which has proper support over Y, g being proper) is algebraizable. The fact that $\hat{g}_* \hat{g}^* E$ is algebraizable then follows from the comparison theorem 2.2.

To prove (2), one may work locally on $\hat{X}$. One may replace X by $\operatorname{Spec} B$ with B adic noetherian such that IB is an ideal of definition of B. Then $E = \tilde{F}$ for a coherent sheaf F on $\operatorname{Spec} B$, and by 2.9, (2) follows from the fact that the kernel and the cokernel of $F \to g_* g^* F$ are killed by a positive power of J.

In view of (2), (3) follows from the noetherian induction assumption.

In (4), the stability under kernel and cokernel is immediate, and the stability under extension follows from 2.9 for $r = 1$.

This completes the proof of 4.2.

4.4. We will first give applications of 4.2 to the algebraization of closed formal subschemes, finite formal schemes, and morphisms between formal schemes. We need some definitions.

(a) *Closed formal subschemes.* Let $\mathcal{X}$ be a locally noetherian formal scheme. If $\mathcal{A}$ is a coherent ideal of $\mathcal{O}_{\mathcal{X}}$, the topologically ringed space $\mathcal{Y}$ consisting of the support $\mathcal{Y}$ of $\mathcal{O}_{\mathcal{X}}/\mathcal{A}$, which is a closed subset of $\mathcal{X}$, and the sheaf of rings $\mathcal{O}_{\mathcal{X}}/\mathcal{A}$, restricted to $\mathcal{Y}$, is a locally noetherian formal scheme, adic over $\mathcal{Y}$ (1.5), called the *closed formal subscheme* of $\mathcal{X}$ defined by $\mathcal{A}$. If $\mathcal{I}$ is an ideal of definition of $\mathcal{X}$ (1.3) and $X_n = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}/\mathcal{I}^{n+1})$, so that $\mathcal{X} = \operatorname{colim}_n X_n$, then $\mathcal{Y} = \operatorname{colim}_n Y_n$, where Y_n is the closed subscheme of X_n such that $\mathcal{O}_{Y_n} = \mathcal{O}_{X_n} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{Y}}$. Conversely, any morphism of inductive systems $Y_. \to X_.$ such that $Y_n \to X_n$ is a closed subscheme and $Y_n = X_n \times_{X_{n+1}} Y_{n+1}$ (cf. 1.5) defines a closed formal subscheme $\mathcal{Y} = \operatorname{colim}_n Y_n$ of $\mathcal{X}$ such that $X_n \times_{\mathcal{X}} \mathcal{Y} = Y_n$. If $\mathcal{X}$ is affine, $X = \operatorname{Spf} A$, then $\mathcal{A} = \mathbf{a}^{\Delta}$ for an ideal $\mathbf{a}$ of A, and $\mathcal{Y} = \operatorname{Spf}(A/\mathbf{a})$. Finally, if X is a locally noetherian scheme and $\hat{X}$ is its completion along a closed subscheme X_0, then if Y is a closed subscheme of X and $\hat{Y}$ its completion along $Y_0 = X_0 \times_X Y$, $\hat{Y}$ is a closed formal subscheme of $\hat{X}$.

(b) *Finite morphisms.* Let $\mathcal{X} = \operatorname{colim} X_n$ be a locally noetherian formal scheme as in (a). A morphism $f : \mathcal{Z} \to \mathcal{X}$ of locally noetherian formal schemes is called *finite* [EGA III 4.8.2] if f is an adic morphism (1.5) and $f_0 : Z_0 \to X_0$ is finite. By standard commutative algebra [B, III §2, 11] (or [EGA 0_I 7.2.9]), this is equivalent to saying that locally f is of the form $\operatorname{Spf}(\mathrm{B}) \to \operatorname{Spf}(A)$ with B finite over A and IB-adic, I being an ideal of definition of A, or that f is adic and each $f_n : Z_n = X_n \times_{\mathcal{X}} \mathcal{Z} \to X_n$ is finite. If f is finite, $f_* \mathcal{O}_{\mathcal{Z}}$ is a finite $\mathcal{O}_{\mathcal{Y}}$-algebra $\mathcal{B}$ such that $\mathcal{O}_{X_n} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{B} = f_* \mathcal{O}_{X_n}$ for all n. If $\mathcal{X} = \hat{X}$ with X as in (a), and Z is a finite scheme over X, then the completion $\hat{Z}$ of Z along $Z_0 = X_0 \times_X Z$ is finite over $\hat{X}$.

Corollary 4.5. *Let X/Y be as in 4.2. Then $Z \mapsto \hat{Z}$ is a bijection from the set of closed subschemes of X which are proper over Y to the set of closed formal subschemes of $\hat{X}$ which are proper over $\hat{Y}$ (4.4 (a)).*

The nontrivial point is the surjectivity. Let $\mathcal{Z} = \operatorname{colim} Z_n$ be a closed formal subscheme of $\hat{X}$ which is proper over $\hat{Y}$. It corresponds to a coherent quotient $\mathcal{O}_{\mathcal{Z}}$ of $\mathcal{O}_{\hat{X}}$ which has proper support over $\hat{Y}$. By 4.2 there exists a unique coherent $\mathcal{O}_X$-module F such that $\hat{F} = \mathcal{O}_{\mathcal{Z}}$. The problem is to algebraize the surjective map $u : \mathcal{O}_{\hat{X}} \to \mathcal{O}_{\mathcal{Z}}$. One cannot apply 4.2 because the support of $\mathcal{O}_X$ is not necessarily proper over Y. But the support of F, which is the intersection of the supports of $\mathcal{O}_X$ and F, is proper over Y. By 2.9, this is enough to ensure that the map $\operatorname{Hom}(\mathcal{O}_X, F) \to \operatorname{Hom}(\mathcal{O}_{\hat{X}}, \mathcal{O}_{\mathcal{Z}})$ is bijective. Therefore there exists a unique $v : \mathcal{O}_X \to F$ such that $\hat{v} = u$. Since $v_0 = u_0$ is surjective, so is v, hence $F = \mathcal{O}_Z$ for a closed subscheme Z of X which is proper over Y and such that $\hat{Z} = \mathcal{Z}$.

Corollary 4.6. *Let X/Y be as in 4.2. Then $Z \mapsto \hat{Z}$ is an equivalence from the category of finite X-schemes which are proper over Y to the category of finite $\hat{X}$-formal schemes which are proper over $\hat{Y}$ (4.4 (b)).*

By $Z \to g_*\mathcal{O}_Z$ (resp. $\mathcal{Z} \to g_*\mathcal{O}_{\mathcal{Z}}$), where g is the structural morphism, the first (resp. second) category is anti-equivalent to that of $\mathcal{O}_X$ (resp. $\mathcal{O}_{\hat{X}}$)-algebras which are finite and whose support is proper over Y (resp. $\hat{Y}$). If A and B are finite $\mathcal{O}_X$-algebras with proper supports over Y, and if $u : A \to B$ is a map of $\mathcal{O}_X$-modules such that $\hat{u}$ is a map of $\mathcal{O}_{\hat{X}}$-algebras, then, by 4.2, u is automatically a map of $\mathcal{O}_X$-algebras. The full faithfulness follows. If $\mathcal{A}$ is a finite $\mathcal{O}_{\hat{X}}$-algebra with proper support over $\hat{Y}$, then by 4.2, there exists a coherent $\mathcal{O}_X$-module A with proper support over Y such that $\hat{A} = \mathcal{A}$ as $\mathcal{O}_{\hat{X}}$-modules. But by g, the maps $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and $\mathcal{O}_{\hat{X}} \to \mathcal{A}$ giving the algebra structure on $\mathcal{A}$ uniquely algebraize to maps giving to A a structure of $\mathcal{O}_X$-algebra such that $\hat{A} = \mathcal{A}$ as $\mathcal{O}_{\hat{X}}$-algebras.

Corollary 4.7. *Let X be a* proper *Y-scheme and let Z be a noetherian scheme, separated and of finite type over Y. Then the application*

$$\operatorname{Hom}_Y(X, Z) \to \operatorname{Hom}_{\hat{Y}}(\hat{X}, \hat{Z}) \ , \ f \mapsto \hat{f}$$

is bijective. In particular, the functor $X \mapsto \hat{X}$ from the category of proper *Y-schemes to the category of $\hat{Y}$-formal schemes is fully faithful.*

If $\hat{f} = \hat{g}$, the remark about the kernel of (1.6.4) shows (cf. [EGA I 10.9.4]) that $f = g$ in a neighborhood of X_0, hence everywhere since $X \to Y$ is proper (and, in particular, closed) and I is contained in the radical of A. To show the surjectivity, one applies 4.5 to the *graph* of a given morphism $\hat{X} \to \hat{Z}$, viewed as a closed formal subscheme of $(X \times_Y Z)\hat{}$.

Remark 4.8. If, in 4.7, one drops the hypothesis of properness on X, the conclusion no longer holds in general. For example, if $X = Z = \operatorname{Spec} A[t]$, then $\hat{X} = \hat{Z} = \operatorname{Spf} A\{t\}$, and $\operatorname{Hom}_Y(X, Z) = A[t]$, while $Hom_{\hat{Y}}(\hat{X}, \hat{Z}) = \operatorname{Hom}_{A,cont}(A\{t\}, A\{t\}) = A\{t\}$, where

$A\{t\} = A[t]^{\frown}$ is the ring of restricted formal series $\sum a_n t^n$, i. e. such that a_n tends to 0 for the I-adic topology as n tends to infinity.

4.9. If X is a proper Y-scheme, $\hat{X}$ is a noetherian adic $\hat{Y}$-formal scheme, which is *proper* over $\hat{Y}$ (by which we mean that $X_0 = X \times_Y Y_0$ is proper over $Y_0 = \mathrm{Spec}(A/I)$). If $\mathcal{X}$ is a proper adic $\hat{Y}$-formal scheme, and if $\mathcal{X}$ is *algebraizable*, i. e. is of the form $\hat{X}$ for a proper Y-scheme, then by 4.7, X is unique (up to a unique isomorphism inducing the identity on $\hat{X}$). Deformation theory can produce proper adic $\hat{Y}$-formal schemes which are not algebraizable, cf. 5.24 (b). This, however, cannot happen in the *projective* formal case, as is shown by the next result, which is extremely useful.

Theorem 4.10 [EGA III 5.4.5]. *Let $\mathcal{X} = \mathrm{colim}\, X_n$ be a proper, adic $\hat{Y}$-formal scheme, where $X_n = \mathcal{X} \times_{\hat{Y}} Y_n$. Let L be an invertible $\mathcal{O}_{\mathcal{X}}$-module such that $L_0 = L \otimes \mathcal{O}_{X_0} = L/IL$ is ample (so that X_0 is projective over Y_0). Then $\mathcal{X}$ is algebraizable, and if X is a proper Y-scheme such that $\hat{X} = \mathcal{X}$, then there exists an unique line bundle M on X such that $L = \hat{M}$, and M is ample (in particular, X is projective over Y).*

Using 4.3.2, choose n such that :

(i) $L_0^{\otimes n}$ is very ample, i. e. of the form $i_0^* \mathcal{O}_{P_0}(1)$ for a standard projective space $P_0 = \mathbb{P}^r_{Y_0}$ and a closed immersion $i_0 : X_0 \to P_0$.

(ii) $\Gamma(\mathcal{X}, L^{\otimes n}) \to \Gamma(X_0, L_0^{\otimes n})$ is surjective.

Using (ii), lift the canonical epimorphism $u_0 : \mathcal{O}_{X_0}^{r+1} \to L_0^{\otimes n}$ given by i_0 to an $\mathcal{O}_{\mathcal{X}}$-linear map $u : \mathcal{O}_{\mathcal{X}}^{r+1} \to L^{\otimes n}$. By Nakayama, each $u_k = u \otimes \mathcal{O}_{X_k} : \mathcal{O}_{X_k}^{r+1} \to L_k^{\otimes n}$ $(k \in \mathbb{N})$ is surjective, hence corresponds to a morphism $i_k : X_k \to P_k = \mathbb{P}^r_{Y_k}$ of Y_k-schemes. such that $L_k^{\otimes n} = i_k^* \mathcal{O}_{P_k}(1)$. By 4.4 (b) i_k is finite, hence a closed immersion by Nakayama. These closed immersions i_k form an inductive system $i_. : X_. \to P_.$, with cartesian squares of the type (1.5.2), hence define a closed formal subscheme (4.4 (a)) $i : \mathcal{X} \to \hat{P}$, where $\hat{P}$ is the completion of the standard projective space $P = \mathbb{P}^r_Y$ over $Y = \mathrm{Spec}\, A$, and $L^{\otimes n} = i^* \mathcal{O}_{\hat{P}}(1)$. By 4.5, there exists a (unique) closed susbscheme $j : X \to P$ such that $\hat{X} = \mathcal{X}$. Moreover, by 4.2, there exists a (unique) line bundle M on X such that $L = \hat{M}$. Since $L^{\otimes n} = i^* \mathcal{O}_{\hat{P}}(1)$ and $(M^{\otimes n})^{\frown} = \hat{M}^{\otimes n}$, we get $(M^{\otimes n})^{\frown} = (j^* \mathcal{O}_P(1))^{\frown}$, hence (by 4.2) $M^{\otimes n} = j^* \mathcal{O}_P(1))$, and therefore M is ample.

Remarks 4.11. (a) The main theorems in §§2, 4 are analogous to the results of Serre on the comparison between algebraic and analytic geometry (GAGA). See [S1] and [SGA 1 XII].

(b) The results of §§2, 4 have been generalized by Knutson to algebraic spaces [K, chap. V]. It seems, however, that a generalization to *stacks* (Deligne-Mumford's stacks, or Artin's stacks [A2], [LM]) is still lacking. For a generalization of Zariski's main theorem (2.17), see [LM, 16.5]. For a generalization of the finiteness theorem for proper morphisms, see [F].

5. Applications to lifting problems

5.1. Let A be a local noetherian ring with maximal ideal $\mathbf{m}$ and residue field $k = A/\mathbf{m}$. Let $S = \mathrm{Spec}\, A$, with closed point $s = \mathrm{Spec}\, k$. Here is a prototype of lifting problems.

Given a scheme X_0 of finite type over s, can one find a scheme X, of finite type and *flat* over S, lifting X_0, i. e. such that $X_s \simeq X_0$? For example, A could be a discrete valuation ring of mixed characteristic, with k of characteristic $p > 0$ and the fraction field K of characteristic zero, and a scheme X as above would provide a "lifting of X_0 to characteristic zero" (namely, the generic fiber X_η, $\eta = \operatorname{Spec} K$). Usually X_0 satisfies additional assumptions (e. g. properness, smoothness, etc.) and is sometimes endowed with additional structures (e. g. group structure), which are to be preserved in the lifting. We ignore this here for simplicity. Grothendieck's strategy to attack the problem consists of several steps.

(1) Try to lift X_0 to an inductive system of (flat and of finite type) schemes X_n such that $X_{n+1} \times_{S_{n+1}} S_n = X_n$. The closed immersion $S_n \to S_{n+1}$ is a thickening of order 1 (1.3) : its ideal $\mathbf{m}^{n+1}\mathcal{O}_{S_{n+1}}$ is killed by $\mathbf{m}$, and *a fortiori* is of square zero. Suppose X_m has been constructed for $m \leq n$. To lift X_n to X_{n+1} over S_{n+1}, then one usually encounters an obstruction in a cohomology group of X_0, and when this obstruction vanishes, the set of isomorphism classes of such X_{n+1} is in bijection with another cohomology group of X_0. Automorphisms of a given X_{n+1} inducing the identity on X_n can also be described by a suitable cohomology group of X_0. Such a study is the object of *deformation theory.*

(2) Suppose that an inductive system $X.$ as in (1) has been found. It defines an adic (locally noetherian) formal scheme $\mathcal{X}$ over the completion $\hat{S} = \operatorname{Spf}\hat{\mathrm{A}}$ of S at s, which is (by definition) flat and of finite type over $\hat{S}$ (1.5). The next problem is to *algebraize $\mathcal{X}$ over* $\operatorname{Spec} \hat{A}$, i. e. find X of finite type over $\operatorname{Spec} \hat{A}$ such that $\hat{X} = \mathcal{X}$. Here one can try to apply the existence theorems of §4, assuming X_0 *proper*, in which case the algebraization is unique if it exists. The main tool - not to say the only one - is 4.10. For this we have first to assume that X_0 is *projective.* Let L_0 be an ample invertible sheaf on X_0. If such an L_0 can be chosen such that it lifts to $\mathcal{X}$, namely that there exists a projective system L_n of invertible sheaves on the X_n's such that $L_n = L_{n+1} \otimes \mathcal{O}_{X_n}$, then by 4. 10, we are done : there exists a projective scheme X over $\operatorname{Spec} \hat{A}$ such that $\hat{X} = \mathcal{X}$. Supposing that L_m has been constructed for $m \leq n$, there is a cohomological obstruction to lifting L_n to L_{n+1}, similar to that alluded to in (1) and closely related to it.

(3) Having found X over $\operatorname{Spec} \hat{A}$, one cannot in general go further, i. e. descend X to $S = \operatorname{Spec} A$. But sometimes, in moduli problems, one encounters a situation where $X/\operatorname{Spec} \hat{A}$ enjoys a *versal* property. If moreover k is separably closed, then Artin's *approximation theory* [A 1], [A 2] usually enables us to descend X at least to the *henselization* S^h of S, i. e. find Z over S^h such that $X = \operatorname{Spec} \hat{A} \times_{S^h} Z$. In a sense, Artin's theory answers Grothendieck's question in [G, p.15] : "Pour passer de résultats connus pour le complété d'un anneau local à des résultats correspondants pour cet anneau local lui-même, il faudrait un quatrième "théorème fondamental", dont l'énoncé définitif reste à trouver".

In this section we recall basic facts on deformation theory and give applications to problems related to (1) and (2).

A. Deformation of vector bundles

5.2. The simplest deformation problem is the problem of deformation of *vector bundles*, i. e. *locally free sheaves of finite rank.* Let $i : X_0 \to X$ be a thickening of order one (1.3), defined by an ideal I of square zero. Let E_0 be a vector bundle on X_0. We want to "deform" (or "extend") E_0 over X, i. e. find a vector bundle E on X such that $\mathcal{O}_{X_0} \otimes E = E/IE = E_0$. More precisely, by a deformation of E_0 over X we mean a pair of a vector bundle E on X and an $\mathcal{O}_X$-linear map $E \to i_* E_0$ inducing an isomorphism $i^*E \xrightarrow{\sim} E_0$. By a morphism $u : E' \to E$ of deformations we mean a morphism u such that $i^*u = Id_{E_0}$. Such a morphism is automatically an isomorphism.

Theorem 5.3. *Let $i : X_0 \to X$ be as in 5.2.*

(a) Let E, F be vector bundles on X, $E_0 = i^*E$, $F_0 = i^*F$, and $u_0 : E_0 \to F_0$ be an $\mathcal{O}_{X_0}$-linear map. There is an obstruction

$$o(u_0, i) \in H^1(X_0, I \otimes \mathcal{H}om(E_0, F_0))$$

to the existence of an $\mathcal{O}_X$-linear map $u : E \to F$ extending u_0. When $o(u_0, i) = 0$, the set of u extending u_0 is an affine space under $H^0(X_0, I \otimes \mathcal{H}om(E_0, F_0))$.

(b) Let E_0 be a vector bundle on X_0. There is an obstruction

$$o(E_0, i) \in H^2(X_0, I \otimes \mathcal{E}nd(E_0))$$

whose vanishing is necessary and sufficient for the existence of a deformation E of E_0 over X. When $o(E_0, i) = 0$, the set of deformations of E_0 over X is an affine space under $H^1(X_0, \mathcal{E}nd(E_0) \otimes I)$, and the group of automorphisms of a given deformation E is identified by $a \mapsto a - Id$ with $H^0(X_0, \mathcal{E}nd(E_0) \otimes I)$.

The proof is elementary and would work in a more general context (ringed spaces or topoi). One first proves (a). The second assertion is clear. Moreover, extensions u of u_0 exist locally. Therefore we get a *torsor* P under $I \otimes \mathcal{H}om(E_0, F_0)$ on X_0, whose sections over an open subset U of X_0 are the $\mathcal{O}_X$-linear extensions of $u_0|U$. The class of P in $H^1(X_0, I \otimes \mathcal{H}om(E_0, F_0))$ is the obstruction $o(u_0, i)$. To prove (b), assume first, for simplicity, that X_0 (or X, this is equivalent) is *separated*. Choose $(\mathcal{U} = (U_i)_{i \in K}, (E_i)_{i \in K}$, where $\mathcal{U}$ is an affine open cover of X_0 and E_i a deformation to $X \cap U_i$ of $E_0|U_i$. Since X_0 is separated, $U_{ij} = U_i \cap U_j$ is affine, so by (a) one can find an isomorphism $g_{ij} : E_i|U_{ij} \xrightarrow{\sim} E_j|U_{ij}$ (inducing the identity on X_0). Such an isomorphism is unique up to the addition of $h_{ij} \in H^0(U_{ij}, I \otimes \mathcal{E}nd(E_0))$. Then

$$(i, j, k) \mapsto c_{ijk} = g_{ij} g_{ik}^{-1} g_{jk} \in H^0(U_{ijk}, I \otimes \mathcal{E}nd(E_0))$$

is a 2-cocycle of $\mathcal{U}$ with values in $I \otimes \mathcal{E}nd(E_0)$, which is a coboundary $dh, h = (h_{ij})$ if and only if the g_{ij} can be modified into a gluing data for the (E_i), in other words if and only if E_0 can be deformed over X. Thus the class of c,

$$[c] = o(E_0, i) \in H^2(X_0, I \otimes \mathcal{E}nd(E_0))$$

is the desired obstruction, which can be checked to be independent of the choices. If E_1 and E_2 are two deformations of E_0 over X, then by (a) the local isomorphisms from E_1 to E_2 form a torsor under $I \otimes \mathcal{E}nd(E_0)$, whose class

$$[E_2] - [E_1] \in H^1(X_0, I \otimes \mathcal{E}nd(E_0))$$

depends only on the isomorphism classes $[E_i]$ of E_1 and E_2, and is zero if and only if $[E_1] = [E_2]$. One checks that this defines the desired affine structure on the set of isomorphism classes of deformations. That finishes the proof in the case X_0 is separated. In the general case, the data $(U_i), (E_i), (g_{ij})$ have to be replaced by data $(U_i), (E_i), (g_{ij}^\alpha)$ where g_{ij}^α is an isomorphism from $E_i|U_{ij}^\alpha$ to $E_i|U_{ij}^\alpha$, for an open cover $(U_{ij}^\alpha)_\alpha \in A_{ij}$ of U_{ij}. Then the g_{ij}^α provide a 2-cocycle of the *hypercovering* defined by $(U_i), (U_{ij}^\alpha)$ (cf. [SGA 4 V 7]) whose cohomology class in $H^2(X_0, I \otimes \mathcal{E}nd(E_0))$ is the desired obstruction, and the rest of the proof goes on with minor modifications.

A more intrinsic way of presenting the proof is to use Giraud's language of *gerbes* [Gi]. The deformations $U \mapsto \mathcal{E}(U)$ of E_0 over variable open subsets U of X_0 form a stack in groupoids, which is in fact a gerbe, i. e. has the following properties : two objects of $\mathcal{E}(U)$ are locally isomorphic, and for any U, there is an open cover (U_i) of U such that $\mathcal{E}(U_i)$ is nonempty. The sheaves of automorphisms of objects of $\mathcal{E}(U)$ form a global sheaf on X_0, namely $I \otimes \mathcal{E}nd(E_0)$, called the *band* ("lien") of the gerbe $\mathcal{E}$. The obstruction $o(E_0, i)$ is the *cohomology class* of $\mathcal{E}$. When this class is zero, the gerbe is *neutral*, which means that the choice of a global object E (a deformation of E_0 over X) identifies $\mathcal{E}$ to the gerbe of *torsors* under $I \otimes \mathcal{E}nd(E_0)$ (over variable open subsets of X_0). See [Gi, VII 1.3.1] for a generalization of the preceding discussion to the case $GL(n)$) is replaced by a smooth group scheme G (and locally free sheaves of rank n by torsors under G).

Remarks 5.4. (a) The construction of the cocycle c in 5.3 (b) shows that if L_0, M_0 are *line bundles* on X_0, then

$$o(L_0 \otimes M_0, i) = o(L_0, i) + o(M_0, i)$$

in $H^2(X_0, I)$. Thus, on line bundles, the obstruction behaves like a first Chern class. In fact, the class $o(E_0, i)$ in 5.3 (b) can be viewed as a kind of *Atiyah class*, similar to that defined by Atiyah in [At] to construct Chern classes in Hodge cohomology, see [I1, chap. IV, V] and [Ka-Sa, 1.4.1].

(b) In practice, the ideal I is killed by a bigger ideal J. More precisely, changing notations, let $X_0 \longrightarrow X_1 \xrightarrow{i_1} X_2$ be closed immersions, I (resp. J) the ideal of X_1 (resp. X_0) in X_2, and suppose that $I \cdot J = 0$. In particular, $I^2 = 0$ and I can be viewed not just as an $\mathcal{O}_{X_1}$-module, but as an $\mathcal{O}_{X_0}(= \mathcal{O}_{X_2}/J)$-module. For vector bundles E, F on X_2, the groups $H^q(X_1, I \otimes \mathcal{H}om(E_1, F_1))$ appearing in 5.3 (a), with i replaced by i_1 and $E_1 = i_1^*E$, $F_1 = i_1^*F$, can then be rewritten

$$H^q(X_1, I \otimes_{\mathcal{O}_{X_1}} \mathcal{H}om(E_1, F_1)) = H^q(X_0, I \otimes_{\mathcal{O}_{X_0}} \mathcal{H}om(E_0, F_0)),$$

with $E_0 = \mathcal{O}_{X_0} \otimes E$, $F_0 = \mathcal{O}_{X_0} \otimes F$. Similarly, for a vector bundle E_1 on X_1, the groups $H^q(X_1, I \otimes_{\mathcal{O}_{X_1}} \mathcal{E}nd(E_1))$ appearing in 5.3 (b) (with i replaced by i_1), can be rewritten

$$H^q(X_1, I \otimes_{\mathcal{O}_{X_1}} \mathcal{E}nd(E_1)) = H^q(X_0, I \otimes_{\mathcal{O}_{X_0}} \mathcal{E}nd(E_0)),$$

with $E_0 = \mathcal{O}_{X_0} \otimes E_1$.

Corollary 5.5. *Let A be a complete local noetherian ring, with maximal ideal* $\mathbf{m}$ *and residue field k. Let $S = \operatorname{Spec} A$, $\hat{S} = \operatorname{colim} S_n$, where $S_n = \operatorname{Spec} A/\mathbf{m}^{n+1}$. Let $\mathcal{X} = \operatorname{colim} X_n$, $X_n = S_n \times_{\hat{S}} \mathcal{X}$, be a* flat *adic locally noetherian formal scheme over $\hat{S}$ (1.5), and assume that $H^2(X_0, \mathcal{O}_{X_0}) = 0$. Then any line bundle L_0 on X_0 can be lifted to a line bundle L on $\mathcal{X}$. If moreover $H^1(X_0, \mathcal{O}_{X_0}) = 0$, then such a lifting L is unique up to a (non unique) isomorphism (inducing the identity on L_0).*

Suppose that L_0 has been lifted to L_n on X_n. Let I_n be the ideal of X_n in X_{n+1}. By the flatness of $\mathcal{X}$ over $\hat{S}$, we have

$$I_n = \mathcal{O}_{X_0} \otimes_k \mathbf{m}^{n+1}/\mathbf{m}^{n+2}.$$

Taking 5.4 (b) into account, we see that the obstruction to lifting L_n to L_{n+1} on X_{n+1} lies in

$$H^2(X_0, I_n) = H^2(X_0, \mathcal{O}_{X_0}) \otimes_k \mathbf{m}^{n+1}/\mathbf{m}^{n+2} = 0,$$

whence the first assertion. For the second one, suppose L and L' are two liftings of L_0 on $\mathcal{X}$. Assume that an isomorphism $u_m : L_m \xrightarrow{\sim} L'_m$ has been constructed for $m \leq n$, with $u_0 = Id$. Then, since $H^1(X_0, \mathcal{O}_{X_0}) = 0$, by 5.3 (a) there is an isomorphism $u_{n+1} : L_{n+1} \xrightarrow{\sim} L'_{n+1}$ extending u_n, and $u = \lim u_n$ is an isomorphism from L to L' inducing the identity on L_0.

Corollary 5.6. *Let $\mathcal{X}$ be a proper, flat adic locally noetherian formal scheme over $\hat{S}$. Then :*

(a) If X/S is a proper scheme such that $\mathcal{X} = \hat{X}$, X is flat over S. Moreover, if $H^2(X_0, \mathcal{O}_{X_0}) = 0$, any line bundle L_0 on X_0 can be lifted to a line bundle L on X, which is unique (up to an isomorphism) if $H^1(X_0, \mathcal{O}_{X_0}) = 0$

(b) If X_0 is projective *and an ample line bundle L_0 on X_0 can be lifted to a line bundle $\mathcal{L}$ on $\mathcal{X}$, there exists a projective and flat scheme X/S such that $\mathcal{X} = \hat{X}$ and an ample line bundle L on X such that $\hat{L} = \mathcal{L}$.*

Let us prove (a). Let x be a point of $X_0 = X_s$ ($s = S_0 = \operatorname{Spec} k$). For all $n \geq 0$, $\mathcal{O}_{X,x}/\mathbf{m}^{n+1}\mathcal{O}_{X,x} = \mathcal{O}_{X_n,x}$ is flat over $A_n = A/\mathbf{m}^{n+1}$, hence $\mathcal{O}_{X,x}$ is flat over A by the usual flatness criterion. As the set of points at which a morphism is flat is open, X is flat over S in a neighbordhood of the special fibre X_s, hence everywhere since X is proper over S. The second assertion follows from 4.2 and 5.5. Assertion (b) follows from (a) and 4.10.

B. Deformation of smooth schemes

5.7. We now turn to the problem of deforming schemes. Let $i : S_0 \to S$ be a thickening of order one (1.3), defined by an ideal I of square zero, and let X_0 be a *flat* scheme over S_0. By a deformation (or lifting) of X_0 over S we mean a cartesian square

(5.7.1)
$$\begin{array}{ccc} X_0 & \xrightarrow{j} & X \\ \downarrow & & \downarrow \\ S_0 & \longrightarrow & S \end{array},$$

with X flat over S. The flatness condition is expressed by the fact that the natural map

$$f_0^* I \to J, \tag{5.7.2}$$

where $f_0 : X_0 \to S_0$ is the structural morphism and J the ideal of X_0 in X, is an isomorphism. By a morphism of deformations we mean an S-morphism $u : X \to X'$ such that $uj = j'$ (where $j' : X_0 \to X'$). Such a morphism u is necessarily an isomorphism.

5.8. We will first discuss the smooth case, which is elementary. Let $f : X \to Y$ be a morphism of schemes. Recall that f is called *smooth* if f is locally of finite presentation (i. e., locally of finite type if Y is locally noetherian) and satisfies the equivalent conditions :

(i) f is flat and the geometric fibers $X_{\overline{y}}$ of X/Y are regular (here $\overline{y} \to y \in Y$ runs through the geometric points of Y, with $k(\overline{y})$ algebraically closed) ;

(ii) (*jacobian criterion*) for every point $x \in X$ there exist open affine neighborhoods U of x and V of $y = f(x)$ such that $f(U) \subset V$ and U is the closed subscheme of a standard affine space $\mathbb{A}^n_V = \operatorname{Spec} A[t_1, \cdots, t_n]$ (where $V = \operatorname{Spec} A$) defined by equations $g_1 = \cdots = g_r = 0$ ($g_i \in A[t_1, \cdots, t_n]$) such that $\operatorname{rk}(\partial g_i/\partial t_j)(x) = r$;

(iii) (*formal smoothness*) for every commutative square

$$\begin{array}{ccc} S_0 & \xrightarrow{g_0} & X \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle f} \\ S & \xrightarrow{h} & Y \end{array} ,$$

where i is a thickening of order 1, there exists, Zariski locally on S, a Y-morphism $g : S \to X$ extending g_0, i. e. such that $gi = g_0$.

(For the equivalence of these conditions and basic facts on smooth and étale morphisms, see [BLR], [I2], and [EGA IV §17], [SGA 1 I, II, III] for a more comprehensive treatment.)

Suppose f is smooth. Then the sheaf of relative differentials $\Omega^1_{X/Y}$ is locally free of finite type, as well as the tangent sheaf

$$T_{X/Y} = \mathcal{H}om(\Omega^1_{X/Y}, \mathcal{O}_X).$$

Their common rank $r(x)$ at a point x of X is the dimension at x of the fiber $X_{f(x)}$, the *relative dimension* of X at x. It is a locally constant function of x. A morphism $f : X \to Y$ is called *étale* if f is smooth and of relative dimension zero at all points, in other words, f is smooth and $\Omega^1_{X/Y} = 0$, or, equivalently, f is flat, locally of finite presentation, and $\Omega^1_{X/Y} = 0$. Smoothness (resp. étaleness) is stable under composition and base change.

We will also need the definition of smoothness in the context of *formal schemes*. Let $\mathcal{Y} = \operatorname{colim} Y_n$ be a locally noetherian formal scheme, with the notations of 1.5. An adic morphism $f : \mathcal{X} \to \mathcal{Y}$ is called *smooth* if f is flat (1.5) and each X_n is smooth over Y_n (or, equivalently, by 5.8 (i), if X_0 is smooth over Y_0). We will refer to $\mathcal{X}$ as a *smooth formal scheme over $\mathcal{Y}$ lifting X_0*.

The main results about deformations of smooth schemes are summed up in the following theorem.

Theorem 5.9. *(a) Let X and Y be schemes over a scheme S, with Y smooth over S, and let $j : X_0 \to X$ be a closed subscheme defined by an ideal J of square zero. Let $g : X_0 \to Y$ be an S-morphism. There is an obstruction*

$$o(g,j) \in H^1(X_0, J \otimes_{\mathcal{O}_{X_0}} g^*T_{Y/S})$$

whose vanishing is necessary and sufficient for the existence of an S-morphism $h : X \to Y$ extending g, i. e. such that $hj = g$. When $o(g,j) = 0$, the set of extensions h of g is an affine space under $H^0(X_0, J \otimes_{\mathcal{O}_{X_0}} g^*T_{Y/S})$.

(b) Let $i : S_0 \to S$ be a thickening of order one defined by an ideal I of square zero, and let X_0 be a smooth S_0-scheme. There is an obstruction

$$o(X_0, i) \in H^2(X_0, f_0^*I \otimes T_{X_0/S_0})$$

(where $f_0 : X_0 \to S_0$ is the structural morphism) whose vanishing is necessary and sufficient for the existence of a deformation X of X_0 over S (5.7). When $o(X_0, i) = 0$, the set of isomorphism classes of such deformations is an affine space under $H^1(X_0, f_0^*I \otimes T_{X_0/S_0})$, and the group of automorphism of a fixed deformation is isomorphic to $H^0(X_0, f_0^*I \otimes T_{X_0/S_0})$. In particular, if X_0 is étale *over S_0, there exists a deformation X of X_0 over S, which is unique up to a unique isomorphism.*

Note that if X_0 is smooth (resp. étale), any deformation of X_0 over S is smooth (resp. étale). This follows from 5.8 (i).

The proof of 5.9 is similar to that of 5.3. One first proves (a). Since Y is smooth over S, an extension h of g exists locally on X_0. Moreover, two such extensions differ by an S-derivation of $\mathcal{O}_Y$ into g_*J, i. e. a section of $J \otimes_{\mathcal{O}_{X_0}} g^*T_{Y/S}$. Therefore, the extensions h over variable open subsets of X form a torsor on X under $J \otimes_{\mathcal{O}_{X_0}} g^*T_{Y/S}$, and $o(g,j)$ is the class of this torsor. To prove (b), one first observes that deformations of X_0 exist locally on X_0. This follows from 5.8 (ii) (lift the polynomials g_i's). Moreover, (a) implies that two deformations are locally isomorphic, and that, for any open subset U_0 of X_0, the sheaf of automorphisms of a deformation U of U_0 is identified by $a \mapsto a - Id$ with $f_0^*I \otimes T_{X_0/S_0}$. Therefore (cf. [Gi, VII 1.2]) by associating to each open subset U_0 of X_0 the groupoid of deformations of U_0, we define a *gerbe* $\mathcal{G} = \mathcal{G}_{X_0}$ whose band is $f_0^*I \otimes T_{X_0/S_0}$. The class $o(X_0, i)$ of $\mathcal{G}$ in $H^2(X_0, f_0^*I \otimes T_{X_0/S_0})$ is the obstruction to the existence of an object of $\mathcal{G}(X_0)$, i. e. a deformation of X_0. When $o(X_0, i) = 0$, $\mathcal{G}$ is *neutral*, i. e. a (global) deformation X of X_0 exists. Once such an X has been chosen, one can identify $\mathcal{G}$ to the gerbe of torsors on X_0 under $f_0^*I \otimes T_{X_0/S_0}$ by associating to a deformation U of an open subset U_0 of X_0 the torsor of local isomorphisms between U and $X|U_0$. In particular, the set of isomorphism classes of deformations of X_0 is then identified to $H^1(X_0, f_0^*I \otimes T_{X_0/S_0})$.

As in the proof of 5.3 one can exhibit a 2-cocycle defining $o(X_0, i)$. Suppose, for simplicity, that X_0 is separated. Choose $(\mathcal{U} = ((U_0)_i)_{i \in K}, (U_i)_{i \in K})$ where $\mathcal{U}$ is an affine open cover of X_0 and U_i a deformation of $(U_0)_i$. Since X_0 is separated, $(U_0)_{ij} = (U_0)_i \cap (U_0)_j$ is affine, so by (a) there is an isomorphism of deformations $g_{ij} : U_i|(U_0)_{ij} \xrightarrow{\sim} U_j|(U_0)_{ij}$. Then

$$(i,j,k) \mapsto c_{ijk} = g_{ij} g_{ik}^{-1} g_{jk} - Id \in H^0((U_0)_{ijk}, f_0^*I \otimes T_{X_0/S_0})$$

is a 2-cocycle of $\mathcal{U}$ with values in $f_0^*I \otimes T_{X_0/S_0}$, whose class in $H^2(X_0, f_0^*I \otimes T_{X_0/S_0})$ represents $o(X_0, i)$.

Remarks 5.10. (a) The obstruction $o(X_0, i)$ satisfies the following functoriality property. Let $g_0 : Y_0 \to S_0$ be a smooth morphism and let $h_0 : X_0 \to Y_0$ be an S_0-morphism (so that $f_0 = g_0 h_0 : X_0 \to S_0$). Then $o(X_0, i)$ and $o(Y_0, i)$ have the same image in $H^2(X_0, f_0^*I \otimes h_0^*T_{Y_0/S_0})$ under the canonical maps

$$H^2(X_0, f_0^*I \otimes T_{X_0/S_0}) \to H^2(X_0, f_0^*I \otimes h_0^*T_{Y_0/S_0}) \leftarrow H^2(Y_0, g_0^*I \otimes T_{Y_0/S_0}).$$

Moreover, if X_0, Y_0 are smooth S_0-schemes, the obstruction $o(X_0 \times_{S_0} Y_0, i)$ to the lifting of $X_0 \times_{S_0} Y_0$ to S satisfies the formula

$$o(X_0 \times_{S_0} Y_0, i) = pr_1^* o(X_0, i) + pr_2^* o(Y_0, i),$$

where pr_1 (resp. pr_2) is the projection from $X_0 \times_{S_0} Y_0$ to X_0 (resp. Y_0) and pr_1^* is the composite

$$H^2(X_0, I \otimes T_{X_0/S_0}) \to H^2(X_0 \times_{S_0} Y_0, I \otimes pr_1^* T_{X_0/S_0}) \to H^2(X_0 \times_{S_0} Y_0, I \otimes T_{X_0 \times_{S_0} Y_0})$$

of the functoriality map and the inclusion of the first direct summand, and similarly for pr_2^*.

The obstructions $o(g, j)$ satisfy a compatibility with respect to the composition of morphisms : in the situation of 5.9 (b), if X, Y, Z are smooth schemes over S, and $f_0 : X_0 \to Y_0$, $g_0 : Y_0 \to Z_0$ S_0-morphisms between their pull-backs to S_0, then the obstruction to lifting $h_0 = g_0 f_0$ to $h : X \to Z$ is the pull-back by g_0^* of the obstruction to lifting g_0 to $g : Y \to Z$.

(b) As in 5.4 (b), suppose $S_0 \longrightarrow S_1 \xrightarrow{i_1} S_2$ are closed immersions, where the ideal I of i_1 is killed by the ideal J of S_0 in S_2. Then, if $f_1 : X_1 \to S_1$ is a smooth morphism, the groups appearing in 5.9 (b) relative to the deformation of X_1 over S_2 can be rewritten $H^q(X_0, f_0^*I \otimes T_{X_0/S_0})$ where $f_0 : X_0 \to S_0$ is deduced from f_1 by base change.

C. Specialization of the fundamental group

5.11. The combination of the existence theorem 4.2 with 5.3 and 5.9 has powerful applications. We will first discuss those pertaining to the *fundamental group.*

Let X be a locally noetherian scheme. By an *étale cover* ("revêtement étale" [SGA 1 I 4.9]) of X we mean a *finite* and *étale* morphism $Y \to X$. A morphism $Y' \to Y$ of étale covers is defined as an X-morphism from Y' to Y. It is automatically an étale cover of Y. We denote by

(5.11.1) $$Et(X)$$

the category of étale covers of X. Suppose X is *connected* and fix a *geometric point* $\overline{x}$ of X, localized at some point x, i. e. a morphism $\operatorname{Spec} k(\overline{x}) \to \operatorname{Spec} k(x)$, with $k(\overline{x})$ a separably closed field. Then there is defined a profinite group

(5.11.2) $$\pi_1(X, \overline{x}),$$

called the *fundamental group* of X at $\overline{x}$, and an equivalence of categories

$$Et(X) \xrightarrow{\sim} \{\pi_1(X,\overline{x}) - fsets\}, \tag{5.11.3}$$

where $\{\pi_1(X,\overline{x}) - fsets\}$ denotes the category of finite sets on which $\pi_1(X,\overline{x})$ acts continuously [SGA 1, V 7]. More precisely, the functor

$$Et(X) \to \{fsets\}, \quad Y \mapsto Y(\overline{x}) = Y_{\overline{x}},$$

associating to an étale cover Y of X the finite set of its points over $\overline{x}$, called *fiber functor at* $\overline{x}$, is pro-representable : there is a pro-object $P = (P_i)_{i \in I}$ of $Et(X)$, called a *universal (pro-) étale cover* of X, and an isomorphism

$$\mathrm{Hom}(P,Y) = \mathrm{colim}_i \mathrm{Hom}(P_i,Y) \xrightarrow{\sim} Y(\overline{x}) \tag{5.11.4}$$

functorial in $Y \in Et(X)$. The identity of P corresponds by (5.11.4) to a point $\xi \in P(\overline{x}) = \lim P_i(\overline{x})$, which in turn defines (5.11.4) by $(u : P \to Y) \mapsto u(\xi) \in Y(\overline{x})$. The P_i's which are *Galois*, i. e. are connected, nonempty and such that the natural map $\mathrm{Aut}(P_i) = \mathrm{Hom}(P_i,P_i) \to \mathrm{Hom}(P,P_i)(\simeq P_i(\overline{x}))$ is bijective form a cofinal system, and therefore we have

$$\mathrm{Hom}(P,P) = \mathrm{Aut}(P) = \lim_{i \in J} \mathrm{Aut}(P_i),$$

where J is the subset of I consisting of indices i for which P_i is Galois. The group opposite to the group $\mathrm{Aut}(P)$ of automorphisms of P is by definition $\pi_1(X,\overline{x})$. In other words, it is the group of automorphism of the fiber functor at $\overline{x}$. It acts continuously and functorially (on the left) on $Y(\overline{x})$, and this defines the equivalence (5.11.3). An étale cover Y is *connected* if and only if $\pi_1(X,\overline{x})$ acts transitively on $Y(\overline{x})$.

If $\overline{a} \to X$, $\overline{b} \to X$ are two geometric points, then, as X is connected, the fiber functors $F_{\overline{a}}$ at $\overline{a}$ and $F_{\overline{b}}$ at $\overline{b}$ are isomorphic [SGA 1 V 5.6]. The choice of an isomorphism from $F_{\overline{a}}$ to $F_{\overline{b}}$ is called a *path* from $\overline{a}$ to $\overline{b}$. Such a path induces an isomorphism

$$\pi_1(X,\overline{a}) \xrightarrow{\sim} \pi_1(X,\overline{b}). \tag{5.11.5}$$

If X is not assumed to be connected, one defines the fundamental group of X at $\overline{x}$ as the fundamental group of the connected component containing x. The fundamental group is in a natural way a functor on geometrically pointed locally noetherian schemes. If $f : X \to Z$ is a morphism between *connected* locally noetherian schemes, then the inverse image functor

$$f^* : Et(Z) \to Et(X), \ Z' \mapsto X \times_Z Z'$$

is an equivalence if and only if the homomorphism

$$f_* : \pi_1(X,\overline{x}) \to \pi_1(Z,\overline{z})$$

is an isomorphism, where $\overline{z}$ is the geometric point $\overline{x} \to x \to z$ image of $\overline{x}$ by f. The homomorphism f_* is surjective if and only if the functor f^* is fully faithful, or equivalently, if for any connected étale cover Z' of Z, f^*Z' is connected [SGA 1 V 6.9]. It is injective if and only if, for any étale cover X' of X, there exists an étale cover Z' of Z and a map from a connected component of f^*Z' to X' [SGA 1 V 6.8].

The following result complements 2.18.

Theorem 5.12 [SGA 1 IX 1.10]. *Let A be a complete local noetherian ring, with maximal ideal* $\mathbf{m}$ *and residue field k. Let $S = \operatorname{Spec} A$, $\hat{S} = \operatorname{Spf} A = \operatorname{colim} S_n$, where $S_n = \operatorname{Spec} A/\mathbf{m}^{n+1}$. Let X be a proper scheme over S. Then the inverse image functor*

$$Et(X) \to Et(X_s),$$

where $s = S_0 = \operatorname{Spec} k$, is an equivalence. In other words, for any geometric point $\overline{x}$ of X_s, the natural homomorphism

$$\pi_1(X_s, \overline{x}) \to \pi_1(X, \overline{x})$$

is an isomorphism.

Let $\hat{X}$ be the formal completion of X along X_s, so that $\hat{X} = \operatorname{colim}_n X_n$, where $X_n = S_n \times_S X$, $S_n = \operatorname{Spec} A/\mathbf{m}^{n+1}$. Consider the natural morphisms

$$X_s \xrightarrow{i} \hat{X} \xrightarrow{j} X .$$

We have inverse image functors

$$Et(X) \xrightarrow{j^*} Et(\hat{X}) \xrightarrow{i^*} Et(X_s) ,$$

where $Et(\hat{X})$ denotes the category of étale covers of $\hat{X}$, i. e. of finite formal schemes $\mathcal{Y} = \operatorname{colim} Y_n$ over $\hat{X}$ (4.4 (b)) which are *étale*, i. e. such that Y_n is étale over X_n for all $n \geq 0$. By 5.9 (b), i^* is an equivalence. On the other hand, by 4.6, $(ji)^*$ is fully faithful. It remains to show that j^* is essentially surjective. Let $\mathcal{Y}$ be an étale cover of $\hat{X}$. By 4.6 there exists a unique scheme Y finite over X such that $\hat{Y} = \mathcal{Y}$. If y is a point of Y_s and $n \geq 0$, $\mathcal{O}_{Y,y}/\mathbf{m}^{n+1}\mathcal{O}_{Y,y} = \mathcal{O}_{Y_n,y}$ is flat over $\mathcal{O}_{X,x}/\mathbf{m}^{n+1}\mathcal{O}_{X,x} = \mathcal{O}_{X_n,x}$, where x is the image of y in X_s. Therefore Y is flat over X in a neighborhood of Y_s, and consequently flat over X since Y is proper over S. Moreover,

$$j^*\Omega^1_{Y/X} = \lim_n \Omega^1_{Y_n/X_n} = 0$$

since Y_n is étale over X_n. Hence, by 4.2, $\Omega^1_{Y/X} = 0$, and therefore Y is étale over X (5.8).

Remark 5.13. It follows from Artin's approximation theorem that the conclusion of 5.12 still holds if A is only assumed to be henselian instead of complete, see [SGA 4 1/2, Cohomologie étale : les points de départ, IV 2.2]. Statements 2.18 and 5.12 are crucial in the proof of the *proper base change theorem* in étale cohomology ((loc. cit.) and [SGA 4 XII]).

5.14. Theorem 5.12 is the starting point of Grothendieck's theory of *specialization for the fundamental group* [SGA 1 X]. Let $f : X \to Y$ be a *proper* morphism of locally noetherian schemes, with connected geometric fibers. Let s and η be points of Y, such that $s \in \overline{\{\eta\}}$,

$\overline{s}$ (resp. $\overline{\eta}$) a geometric point over s (resp. η), a (resp. b) a geometric point of $X_{\overline{s}}$ (resp. $X_{\overline{\eta}}$). Then there is defined (loc. cit. 2.1, 2.4) a homomorphism

$$\pi_1(X_{\overline{\eta}}, b) \to \pi_1(X_{\overline{s}}, a), \tag{5.14.1}$$

called the *specialization homomorphism.* This homomorphism is well defined up to an inner automorphism of the target. If Y is the spectrum of a henselian local noetherian ring A, with closed point s such that $s = \overline{s}$, (5.14.1) is the composition

$$\pi_1(X_{\overline{\eta}}, b) \to \pi_1(X, b) \xrightarrow{\sim} \pi_1(X, a) \xrightarrow{\sim} \pi_1(X_s, a),$$

where the first map is the functoriality map, the second one an isomorphism associated to a *path* from a to b (5.11) (such a path exists because the hypotheses imply, by 2.18, that X is connected), and the last one is the inverse of the isomorphism of 5.12, 5.13. The definition in the general case is more delicate, see (loc. cit.). It uses the fact that for a proper and connected scheme X over an algebraically closed field k, the fundamental group of X is invariant under algebraically closed extension of k (this fact is a (nontrivial) consequence of 5.12). Grothendieck's main result about (5.14.1) is the following theorem :

Theorem 5.15 [SGA I X 2.4, 3.8] (Grothendieck's specialization theorem). *Let $f : X \to Y$ be as in 5.14.*

(a) If f is flat and has geometrically reduced fibers (i. e. for any morphism $\overline{y} \to y \in Y$ with $\overline{y}$ the spectrum of an algebraically *closed field, $X_{\overline{y}}$ is reduced), then (5.14.1) is surjective ;*

(b) If f is smooth and p is the characteristic exponent of s, then (5.14.1) induces an isomorphism on the largest prime to p quotients of the fundamental groups

$$\pi_1^{(p')}(X_{\overline{\eta}}, b) \xrightarrow{\sim} \pi_1^{(p')}(X_{\overline{s}}, a).$$

(We use the notation $\pi_1^{(p')}$ to denote the largest prime to p quotients ; this notation has become more common than the notation $\pi_1^{(p)}$ used in (loc. cit.).)

Let us prove (a) in the case Y is the spectrum of a henselian local noetherian ring A, with algebraically closed residue field k and $s = \operatorname{Spec} k$ (the general case can be reduced to this one). We have to show that if Z is a connected étale cover of X, then $Z_{\overline{\eta}}$ is connected. Note that Z is again proper and flat over Y with geometrically reduced fibers. As Z is connected, so is the special fibre Z_s by 2.18. Therefore $H^0(Z_s, \mathcal{O}_{Z_s})$ is an artinian local k-algebra with residue field k. Since Z_s is reduced, $H^0(Z_s, \mathcal{O}_{Z_s}) = k$. The composition of the canonical maps

$$\mathcal{O}_Y \otimes k \to g_*\mathcal{O}_Z \otimes k \to H^0(Z_s, \mathcal{O}_{Z_s}) = k,$$

where $g : Z \to Y$ is the structural morphism, is the identity, in particular

$$g_*\mathcal{O}_Z \otimes k \to H^0(Z_s, \mathcal{O}_{Z_s})$$

is surjective. Since Z is flat over Y, it follows from 3.11 that this map is in fact an isomorphism and that $g_*\mathcal{O}_Z$ is free of rank 1 and its formation commutes with arbitrary base change, in other words, $g_*\mathcal{O}_Z = \mathcal{O}_Y$ holds *universally* (i. e. after any base change). In particular, $Z_{\overline{\eta}}$ is connected. The proof of (b) is more delicate. It relies on Abhyankar's lemma and Zariski-Nagata's purity theorem. See [SGA 1 X 3] for details and [O-V] for a survey.

The argument sketched for the proof of (a) gives in fact the following result ([SGA 1 X 1.2], [EGA III 7.8.6]) :

Proposition 5.16. *Let $f : X \to Y$ be a proper and flat morphism of locally noetherian schemes, having geometrically reduced fibers, and let*

$$X \to Y' \to Y$$

be its Stein factorization (2.11). Then Y' is an étale cover of Y, and its formation commutes with any base change. In particular f is cohomologically flat in degree zero (3.-), and the following conditions are equivalent :

(i) $f_*\mathcal{O}_X = \mathcal{O}_Y$;

(ii) the geometric fibers of f are connected.

Remarks 5.17. (a) Under the assumptions of 5.15 (a), i. e. for $f : X \to Y$ proper and flat, with geometrically reduced and connected fibers, the specialization homomorphism (5.14.1) has been extensively studied in the past few years, especially in the case of relative *curves*. See [BLoR] for a discussion of some aspects of this.

(b) A variant of the theory of the fundamental group in "logarithmic geometry" has been constructed by Fujiwara-Kato [FK]. See [I]3 for an introduction and [Vi, I 2.2] for a generalization of Grothendieck's specialization theorem 5.15 in this context and an application [Ki] to the action by outer automorphisms of the wild inertia on the prime to p fundamental group of varieties over local fields.

D. Curves

5.18. We now turn to applications to liftings of curves. Let Y be a locally noetherian scheme. By a *curve* over Y we mean a morphism $f : X \to Y$ which is flat, separated and of finite type, with relative dimension 1. Assume f is *proper*. Then, for any coherent sheaf F on X, $R^q f_*F = 0$ for $q > 1$ by 2.10, and if moreover F is flat over Y, e. g. $F = \mathcal{O}_X$, the complex Rf_*F is perfect, of perfect amplitude in [0,1] (3.11). In general, f is cohomolologically flat neither in degree 0 nor 1, as simple examples show [H, III 12.9.2]. However, if f has geometrically reduced fibers, f is cohomologically flat in degree 0 by 5.16, hence also in degree 1 by 3.11, i. e. $R^q f_*\mathcal{O}_X$ is locally free of finite type for all q. When, moreover, f has connected geometric fibers, so that $f_*\mathcal{O}_X = \mathcal{O}_Y$, the rank of the locally free sheaf $R^1 f_*\mathcal{O}_X$ is called the *(arithmetic) genus* of the curve X over Y. If f is proper and smooth, then, by Grothendieck's duality theorem, the sheaves $R^q f_*\Omega^1_{X/Y}$ are also locally free of finite type, and there is defined a *trace map*

$$\mathrm{Tr} : R^1 f_*\Omega^1_{X/Y} \to \mathcal{O}_Y,$$

and the pairing

$$R^q f_*\mathcal{O}_X \otimes R^{1-q} f_*\Omega^1_{X/Y} \to \mathcal{O}_Y$$

obtained by composing the natural pairing to $R^1 f_* \Omega^1_{X/Y}$ with Tr is a perfect pairing between locally free sheaves of finite type. In particular, if f has connected geometric fibers and is of genus g, $f_* \Omega^1_{X/Y}$ is locally free or rank g. Finally, recall that any curve over a field is *quasi-projective*. See [H, III Ex. 5.8] for the case the curve is proper over an algebraically closed field (the general case can be reduced to this one).

The main result on liftings of curves is the following theorem :

Theorem 5.19 [SGA 1 III 7.3]. *Let A be a complete local noetherian ring, with residue field k. Let $S = \operatorname{Spec} A$, $s = \operatorname{Spec} k$, and let X_0 be a projective and smooth scheme over s satisfying*

$$(i) \qquad H^2(X_0, T_{X_0/s}) = 0.$$

Then there exists a proper and smooth formal scheme (5.8) $\mathcal{X}$ over $\hat{S}$ lifting X_0. If, in addition to (i), X_0 satisfies

$$(ii) \qquad H^2(X_0, \mathcal{O}_{X_0}) = 0,$$

then there exists a projective and smooth scheme X over S such that $X_s = X_0$.

Conditions (i) and (ii) are satisfied, for example, if X_0 is a proper and smooth curve over s. Note that, if X_0 is a proper, geometrically connected, smooth curve of genus g, then the same is true for the fibers of X over S.

Let $\hat{S} = \operatorname{Spf} A = \operatorname{colim} S_n$, where $S_n = \operatorname{Spec} A/\mathbf{m}^{n+1}$, $\mathbf{m}$ denoting the maximal ideal of A. Let us show that, under the assumption (i), there exists a (proper) and smooth formal scheme $\mathcal{X} = \operatorname{colim} X_n$ over $\hat{S}$ lifting X_0. Assume X_m, smooth over S_m, has been constructed for $m \leq n$ such that $X_m = S_m \times_{S_n} X_n$, and let $i_n : S_n \to S_{n+1}$ be the inclusion. Then, by 5.9, 5.10 (b), there is an obstruction

$$o(X_n, i_n) \in H^2(X_0, T_{X_0/s} \otimes \mathbf{m}^{n+1}/\mathbf{m}^{n+2})$$

to the existence of a smooth lifting X_{n+1} of X_n over S_{n+1}. But

$$H^2(X_0, T_{X_0/s} \otimes \mathbf{m}^{n+1}/\mathbf{m}^{n+2}) = H^2(X_0, T_{X_0/s}) \otimes \mathbf{m}^{n+1}/\mathbf{m}^{n+2}),$$

which is zero by (i). This shows the existence of $\mathcal{X}$. As for he second assertion of 5.19, we deduce from 5.6 the existence of a projective and flat scheme X over S such that $X_s = X_0$. Then X is smooth over S at each point of X_s, hence in an open neighborhood of X_s, which has to be equal to X since X is proper over S.

By 5.19, *proper smooth curves in positive characteristic can be lifted to characteristic zero "without ramification"* : if k is of characteristic $p > 0$, one can take for A a *Cohen ring* for k, i. e. a complete discrete valuation ring with residue field k, fraction field of characteristic zero, and maximal ideal generated by p (the ring $W(k)$ of Witt vectors on k when k is perfect). Using this and the known structure of the (topological) fundamental group of compact Riemann surfaces, Grothendieck was able to deduce from the specialization theorem 5.15 the following results about the (algebraic) fundamental group of proper smooth curves in positive characteristic :

Theorem 5.20 [SGA 1 3.9, 3.10]. *Let k be an algebraically closed field of characteristic exponent p and let C be a proper, smooth and connected curve over k, of genus g. Let x be a rational point of C. Denote by Π_g the group defined by generators a_i, b_i $(1 \leq i \leq g)$, subject to the relation $\prod_{1 \leq i \leq g}(a_i, b_i) = 1$, where $(a, b) := aba^{-1}b^{-1}$, and let $\widehat{\Pi_g}$ be its profinite completion. Then there exist a surjective homomorphism*

$$\widehat{\Pi_g} \to \pi_1(C, x),$$

inducing an isomorphism

$$\widehat{\Pi_g}^{(p')} \xrightarrow{\sim} \pi_1(C, x)^{(p')}$$

on the largest prime to p quotients.

Here is a sketch of the argument. One first treats the case where $k = \mathbb{C}$. Let C^{an} be the (compact, connected and of genus g) Riemann surface associated to C. By Riemann's existence theorem, the functor $C' \mapsto C'^{an}$ from the category of finite étale covers of C to that of finite étale covers of C^{an} is an equivalence. It follows that $\pi_1(C, x)$ is the profinite completion of $\pi_1(C^{an}, x)$. Topological arguments, using the representation of C^{an} as the quotient of a polygon with $4g$ edges $(e_i, e_i^{-1}, f_i, f_i^{-1})$ $(1 \leq i \leq g$ by the identification specified by the word $\prod_{1 \leq i \leq g}(e_i, f_i)$ shows that $\pi_1(C^{an}, x) = \Pi_g$. So the result is proven in this case. The case where $p = 1$ is reduced to this one by standard limit arguments using the invariance of the (algebraic) fundamental group (of proper schemes) under arbitrary extension of algebraically closed fields. Finally, suppose $p \geq 2$. In 5.19, take $X_0 = C$ and $A = W(k)$ the ring of Witt vectors on k. Let X be a projective and smooth scheme over S such that $X_s = C$. Then, by 5.18, X/S is a projective and smooth curve with connected geometric fibers of genus g. The conclusion thus follows from the case $p = 1$ and 5.15.

Remarks 5.21. (1) If $g = 0$, then C is isomorphic to $\mathbb{P}^1_k$, hence simply connected (by Riemann-Hurwitz). More generally, all projective spaces $\mathbb{P}^r_k$ are simply connected [SGA 1 XI 1.1].

(2) If $g = 1$, then C is an *elliptic curve* and $\pi_1(C)$ is the *Tate module* of C, $T(C) = \lim {}_nC(k)$, where n runs through all integers ≥ 1, ${}_nC(k)$ denotes the kernel of the multiplication by n on $C(k)$, and for $m = nd$, ${}_mC(k)$ is sent to ${}_nC(k)$ by multiplication by d. More generally, if A is an abelian variety over k, then

$$\pi_1(A) = T(A),$$

where $T(-)$ is the Tate module, defined similarly [M1, IV 18].

(3) By 5.20, $\pi_1(C, x)$ is topologically of finite type. As Grothendieck observed in [SGA 1 X 2.8] It seems unlikely that $\pi_1(C, x)$ could be topologically of finite presentation, but the question is still open. Using some Lefschetz type arguments for hyperplane sections, Grothendieck shows that more generally, for any proper connected scheme X over k, $\pi_1(X)$ is topologically of finite generation [SGA 1 2.9].

(4) There is a variant of the last assertion of 5.20 for *affine* curves. More precisely, let C be a proper, smooth and connected curve of genus $g \geq 0$ over k. Let n be an integer ≥ 1,

$x_1, \cdots, x_n$ be distinct rational points of C, let $X = C - \{x_1, \cdots, x_n\}$, and pick a rational point x of X. Then there is an isomorphism

$$\widehat{\Pi}_{g,n}^{(p')} \xrightarrow{\sim} \pi_1(X, x)^{(p')},$$

where $\widehat{\Pi}_{g,n}^{(p')}$ is the prime to p quotient of the profinite completion of the (free) group $\Pi_{g,n}$ defined by generators a_i, b_i $(1 \leq i \leq g)$, s_i $(1 \leq i \leq n$, subject to the relation $\prod_{1 \leq i \leq g}(a_i, b_i) \prod_{1 \leq i \leq n} s_i = 1$ [SGA 1 XIII 2.12]. However, $\pi_1(X, x)$ is not topologically of finite type, even for $X = \mathbb{A}^1_k$. A finite group G is the Galois group of a connected étale cover of $\mathbb{A}^1_k$ if and only if its largest prime to p quotient is trivial (*Abhyankar's conjecture*, proven by Raynaud [R2]).

(5) For C proper, connected and smooth of genus $g \geq 2$, the (full) fundamental group of C encodes an amazingly deep information about C. For example, let me mention the following striking result of Tamagawa :

Theorem [A. Tamagawa, *Finiteness of isomorphism classes of curves in positive characteristic with prescribed fundamental groups*, to appear in J. Alg. Geometry]. *Let k be an algebraically closed field of characteristic $p > 0$, $A = \operatorname{Spec} k[[t]]$, and X a proper and smooth curve over S with connected geometric fibers of genus $g \geq 2$. Let $s = \operatorname{Spec} k$. Assume that the special fiber X_s can be defined over* Spec k_0, *where k_0 is a finite subfield of k. Let $\overline{\eta}$ be a geometric point over the generic point η of S. Then, if the specialization homomorphism*

$$\pi_1(X_{\overline{\eta}}, b) \to \pi_1(X_s, a),$$

of (5.14.1) is an isomorphism, X is constant *over S, i. e. is isomorphic to $X_s \times_s S$.*

E. Abelian varieties

5.22. Let me now come to liftings of abelian varieties. Let S be a scheme. Recall that an *abelian scheme* over S (*abelian variety* when S is the spectrum of a field) is an S-group scheme, which is proper and flat, and whose geometric fibers are reduced and irreducible. Let X be an abelian scheme over S. Then X is automatically *smooth* and *commutative*, see [M1, II 4] for the case S is the spectrum of an algebraically closed field, and [M2, 6.5] for the general case. It is also known that if S is normal, or even geometrically unibranch, X is *projective* over S [Mu]. Counter-examples outside of these hypotheses have been given by Raynaud [R1].

Grothendieck has shown (unpublished) that abelian varieties admit formal liftings :

Theorem 5.23. *Let $\hat{S} = \operatorname{Spf} A$ be as in 5.12, and let X_0 be an abelian variety of $s = \operatorname{Spec} k$. Then :*

(a) There exists a proper and smooth formal scheme $\mathcal{X}$ over $\hat{S}$ such that $s \times_{\hat{S}} \mathcal{X} = X_0$ and a section e of $\mathcal{X}$ over $\hat{S}$ extending the unit section e_0 of X_0.

(b) Let $(\mathcal{X}, e)$ be a lifting of (X_0, e_0) over $\hat{S}$ as in (a), and let $X_n = S_n \times_{\hat{S}} \mathcal{X}$, with S_n as in 2.12. One can, in a unique way, inductively define a structure of abelian scheme on X_n over S_n having e_n as unit section and such that $X_n = S_n \times_{S_{n+1}} X_{n+1}$ as abelian schemes.

Assuming that, for a fixed integer n, an abelian scheme X_n lifting X_0 has been constructed (with unit section e_n), we have to show that :

(i) there exists a smooth scheme X_{n+1} over S_{n+1} lifting X_n and a lifting e_{n+1} of e_n ;

(ii) given a smooth lifting X_{n+1} of X_n as a scheme and a lifting e_{n+1} of e_n, there exists a unique group scheme structure on X_{n+1} over S_{n+1} lifting that of X_n over S_n and having e_{n+1} as unit section.

The proofs of (i) and (ii) are similar. In both cases one encounters an obstruction, which lives in a *nonzero* cohomology group. Using the *functoriality* (5.10 (a)) of the obstruction with respect to a suitable morphism, one shows that it is zero.

Let us sketch the proof of (i) (cf. [O, p. 238]). Consider the obstruction

$$o(X_n) \in H^2(X_0, T_{X_0}) \otimes I$$

to the lifting of X_n to S_{n+1} (5.9 (b)), where we write T_{X_0} for $T_{X_0/s}$ and I for $\mathbf{m}^n/\mathbf{m}^{\mathbf{n+1}}$ for brevity. Consider, too, the obstruction

$$o(X_n \times X_n) \in H^2(X_0 \times X_0, T_{X_0 \times X_0}) \otimes I$$

to the lifting of $X_n \times X_n$ to S_{n+1}. By the compatibility of obstructions with products (5.10 (a)) we have

$$o(X_n \times X_n) = pr_1^* o(X_n) + pr_2^* o(X_n). \tag{1}$$

Let

$$s : X_n \times X_n \to X_n \quad , \quad (x, y) \mapsto x + y$$

be the sum morphism. By functoriality of the obstructions (5.10 (a)), $o(X_n)$ and $o(X_n \times X_n)$ have the same image by the two maps

$$H^2(X_0, T_{X_0}) \otimes I \to H^2(X_0 \times X_0, s^* T_{X_0}) \otimes I \leftarrow H^2(X_0 \times X_0, T_{X_0 \times X_0}) \otimes I.$$

These two maps can be rewritten

$$H^2(X_0) \otimes t_{X_0} \otimes I \xrightarrow{s^* \otimes Id} H^2(X_0 \times X_0) \otimes t_{X_0} \otimes I \xleftarrow{Id \otimes s} H^2(X_0 \times X_0) \otimes t_{X_0 \times X_0} \otimes I,$$

where we have written $H^*(-)$ instead of $H^*(-, \mathcal{O})$, and t means the tangent space at the origin, pull-back of the tangent bundle T by the unit section. In other words, we have

$$(s^* \otimes Id)(o(X_n)) = (Id \otimes s)(o(X_n \times X_n). \tag{2}$$

On the other hand, we know [S3, VII 21] that

$$H^q(X_0) = \Lambda^q H^1(X_0) \quad , \quad H^q(X_0 \times X_0) = \Lambda^q H^1(X_0 \times X_0),$$

and that

$$s^* : H^1(X_0) \to H^1(X_0 \times X_0) = H^1(X_0) \oplus H^1(X_0)$$

is the diagonal map. Choose a basis (e_i) $(1 \leq i \leq g)$ $(g = \dim X_0)$ for $H^1(X_0)$ and a basis ε_k $(1 \leq k \leq g)$ for t_{X_0} (actually, $H^1(X_0)$ and t_{X_0} are naturally dual to each other, and we could take dual bases, but we don't need this). Write

$$o(X_n) = \sum_{1 \leq i < j \leq g, 1 \leq k \leq g} a_{ij}^k e_i \wedge e_j \otimes \varepsilon_k,$$

with $a_{ij}^k \in I$. Let $e_i' = pr_1^* e_i$, $e_i'' = pr_2^* e_i$, $\varepsilon_k' = (\varepsilon_k, 0)$, $\varepsilon_k'' = (0, \varepsilon_k)$. By (1) we have

$$o(X_n \times X_n) = \sum a_{ij}^k e_i' \wedge e_j' \otimes \varepsilon_k' + \sum a_{ij}^k e_i'' \wedge e_j'' \otimes \varepsilon_k''. \tag{3}$$

Using that s^* is the diagonal map, hence sends e_i to $e_i' + e_i''$, we get

$$(s^* \otimes Id)(o(X_n)) = \sum a_{ij}^k (e_i' \wedge e_j' + e_i'' \wedge e_j' + e_i' \wedge e_j'' + e_i'' \wedge e_j'') \otimes \varepsilon_k.$$

Finally, since $s : t_{X_0 \times X_0} \to t_{X_0}$ sends ε_k' and ε_k'' to ε_k, we deduce from (2) :

$$\sum a_{ij}^k (e_i' \wedge e_j' + e_i'' \wedge e_j' + e_i' \wedge e_j'' + e_i'' \wedge e_j'') \otimes \varepsilon_k = \sum a_{ij}^k (e_i' \wedge e_j' + e_i'' \wedge e_j'') \otimes \varepsilon_k.$$

Therefore

$$a_{ij}^k = 0$$

for all i, j, k, i. e. $o(X_n) = 0$. The existence of a lifting e_{n+1} of the unit section e_n is immediate.

For the proof of (ii), see [M2, 6.15] : one first shows that the obstruction to lifting the difference map $\mu_n : X_n \times_{S_n} X_n \to X_n$, $(x, y) \mapsto x - y$ is zero, using its compatibility (5.10 (a)) with composition with the diagonal map $x \mapsto (x, x)$ and the map $x \mapsto (x, 0)$; one normalizes the lifting of μ_n using e_{n+1} and one concludes by a rigidity lemma.

Remarks 5.24. (a) Using the arguments above, Grothendieck actually proved a more general and precise result than 5.23, namely : if $S_0 \to S$ is a closed immersion of *affine* schemes, defined by an ideal I of square zero, and if if X_0 is an abelian scheme over S_0, then there exists an abelian scheme X over S lifting X_0 ; moreover, the set of isomorphism classes of abelian schemes X over S lifting X_0 is an affine space under $\Gamma(S_0, t_{\hat{X}_0} \otimes t_{X_0} \otimes I)$, where $\hat{X}_0$ is the dual abelian scheme, and the group of automorphisms of any lifting X (inducing the identity on X_0) is zero. A different proof is given in [I4, A 1.1], using the theory of the cotangent complex, which provides an obstruction to the lifting of X_0 *as a flat commutative group scheme,* living in a cohomology group which is zero.

(b) Consider a formal abelian scheme $\mathcal{X} = \operatorname{colim} X_n$ as in 5.23 (b). It is not true in general that $\mathcal{X}$ is algebraizable : using the theory of formal moduli of abelian varieties, one can construct examples of nonalgebraizable $\mathcal{X}$ already for $k = \mathbb{C}$, $A = \mathbb{C}[[t]]$ and $g = \dim X_0 = 2$. In contrast with the case of curves, it is indeed not always possible to lift an ample invertible sheaf L_0 on X_0 to $\mathcal{X}$ (or even to X_1). The step by step obstructions to such liftings lie in a group of the form $H^2(X_0, \mathcal{O}) \otimes I$, which is not zero for $g \geq 2$, and they can be nonzero.

On the other hand, Mumford has proven that *any abelian variety in positive characteristic can be lifted to characteristic zero* [M3]. More precisely, if k is an algebraically closed field of characteristic $p > 0$ and X_0 is an abelian variety over k, there exists a complete discrete valuation ring A having k as residue field and with fraction field of characteristic zero and a (projective) abelian scheme X over $\operatorname{Spec} A$ such that $X \otimes k = X_0$ (the ring A is a finite extension of the ring $W(k)$ of Witt vectors on k, which is in general ramified).

F. Surfaces

5.25. Let Y be a locally noetherian scheme. By a *surface* over Y, we mean a scheme X over Y, which is flat, separated and of finite type and of relative dimension 2. We will be concerned only with *proper and smooth* surfaces. By a theorem of Zariski ([Z], [H1]), a proper, smooth surface over a field is *projective*. In contrast with the case of curves and abelian varieties, there are proper, smooth surfaces over a field which do not lift formally. More precisely, let k be an algebraically closed field of characteristic $p > 0$. There are two kinds of nonliftability phenomena.

(a) *Nonliftability to* W_2. Let $W = W(k)$ be the ring of Witt vectors on k, $W_n = W/p^n W$ the ring of Witt vectors of length n. Let X_0 be a proper and smooth surface over k having nonclosed global differential forms of degree 1. Examples of such surfaces have been constructed by Mumford [M4] and, later on, by Lang [L], Raynaud and Szpiro (see [Fo]). By a theorem of Deligne-Illusie [DI, 2.4] this pathology prevents X_0 from being liftable to W_2.

(b) *Nonliftability to characteristic zero.* Improving a result of Serre [S2], Mumford [M5] has constructed examples of proper and smooth surfaces X_0 over $s = \operatorname{Spec} k$ having the following property. Let A be *any* integral, complete local noetherian ring with residue field k and fraction field of characteristic zero. Then there exists no proper and smooth scheme X over $\operatorname{Spec} A$ such that $X_s = X_0$.

Using Hodge-Witt numbers, which are fine invariants of X_0 defined in terms of the de Rham-Witt complex, Ekedhal [E, p. 114] observed that similar examples are provided by suitable Raynaud's surfaces as mentioned in (a).

The relation between phonomena of types (a) and (b) is not well understood.

5.26. Here are some results in the positive direction. As in 5.12, let A be a complete local noetherian ring, with maximal ideal $\mathbf{m}$ and residue field k. Let $S = \operatorname{Spec} A$, $\hat{S} = \operatorname{Spf} A = \operatorname{colim} S_n$, where $S_n = \operatorname{Spec} A/\mathbf{m}^{n+1}$. Let X_0 be a proper and smooth surface over $s = \operatorname{Spec} k$. Using 5.19 and the general results of [H, IV 2, 5] it is easy to see that if X_0 is *rational* or *ruled*, then X_0 lifts to a projective surface over S. On the other hand, we have seen that if X_0 is an *abelian* surface, then X_0 admits a formal smooth lifting $\mathcal{X}$ over $\hat{S}$. The same is true if X_0 is a *K3 surface*, i . e. a proper, smooth, connected surface such that $\Omega^2_{X_0/s}$ is trivial and $H^1(X_0, \mathcal{O}_{X_0}) = 0$. More precisely, we have the following result, due to Rudakov-Shafarevitch and Deligne :

Theorem 5.27 [D2, 1.8]. *With the notations of 5.26, let X_0 be a K3 surface over an algebraically closed field k.*

(a) There exists a proper and smooth formal scheme $\mathcal{X}$ over $\hat{S}$ lifting X_0.

(b) Let L_0 be an ample line bundle on X_0. Then there exists a complete discrete valuation ring R finite over the ring of Witt vectors $W(k)$, a proper and smooth scheme X over

$T = \operatorname{Spec} R$ *lifting* X_0*, and a lifting of* L_0 *to an ample line bundle* L *on* X.

Let us prove (a). By a basic result of Rudakov-Shafarevitch [RS] (see also [N]), we have

$$H^0(X_0, T_{X_O/k}) = 0.$$

Since $\Omega^2_{X_0/k}$ is trivial, we have $T_{X_O/k} = \Omega^1_{X_O/k}$, hence by Serre duality, $H^2(X_0, T_{X_O/k}) = 0$. Therefore the conclusion follows from 5.19. The proof of (b) is much more difficult, since $H^2(X_0, \mathcal{O}) = k$ and one cannot apply 5.19. See [D2] for details.

Remarks 5.28. (a) As in the case of abelian varieties, in the situation of 5.27 (a) it may happen that a given polarization of X_0 can't lift to $\mathcal{X}$, see [D2, 1.6] for a more precise statement.

(b) For $p = 3$, M. Hirokado [Hi] has constructed a Calabi-Yau threefold X_0/k (i. e. a smooth projective scheme of dimension 3 such that $\Omega^3_{X_0/k} \simeq \mathcal{O}_{X_0}$ and $H^1(X_0, \mathcal{O}_{X_0}) = H^2(X_0, \mathcal{O}_{X_0}) = 0$) having $b_3 = 0$, where $b_3 = \dim H^3(X_0, \mathbb{Q}_\ell), \ell \neq p$. By Hodge theory, such a scheme admits no smooth *projective* lifting to characterisic zero. This Calabi-Yau threefold is constructed as a quotient of a blow-up of $\mathbb{P}^3_k$ by a certain vector field. Thus, as Calabi-Yau threefolds can be considered as analogues of K3 surfaces, Deligne's result 5.27 does not extend to dimension 3.

G. Cotangent complex

5.29. So far we have considered deformations of smooth morphisms only. To deal with more general morphisms, one must use the theory of the cotangent complex [I1]. For an extensive survey, see [I5]. We will just give very brief indications.

Let $f : X \to Y$ be a morphism of schemes. The *cotangent complex* of f (or X/Y), denoted

$$L_{X/Y},$$

is a complex of $\mathcal{O}_X$-modules, concentrated in ≤ 0 degrees, defined as follows. The pair of functors : free $f^{-1}(\mathcal{O}_Y)$-algebra generated by a sheaf of sets, sheaf of sets underlying an $f^{-1}(\mathcal{O}_Y)$-algebra, defines a Godement style, standard simplicial $f^{-1}(\mathcal{O}_Y)$-algebra P, augmented to $\mathcal{O}_X$, whose components are free $f^{-1}(\mathcal{O}_Y)$-algebras over sheaves of sets, and such that the chain complex of the underlying augmented simplicial $f^{-1}(\mathcal{O}_Y)$-module is acyclic. Applying the functor Ω^1 (Kähler differentials) componentwise, one obtains a simplicial $f^{-1}(\mathcal{O}_Y)$-module $\Omega^1_{P/f^{-1}(\mathcal{O}_Y)} \otimes_P \mathcal{O}_X$, whose corresponding chain complex is $L_{X/Y}$. This complex has a natural augmentation to $\Omega^1_{X/Y}$, which defines an isomorphism $\mathcal{H}^0(L_{X/Y}) \xrightarrow{\sim} \Omega^1_{X/Y}$. Its components are flat $\mathcal{O}_X$-modules. It depends functorially on X/Y. Moreover, a sequence of morphisms $X \xrightarrow{f} Y \longrightarrow Z$ gives rise to a distinguished triangle in $D(X)$, called the *transitivity triangle*

$$f^* L_{Y/Z} \to L_{X/Z} \to L_{X/Y} \to .$$

Suppose f is a morphism locally of finite type between locally noetherian schemes. Then $L_{X/Y}$ is *pseudo-coherent* (3.6.1). If f is *smooth*, the augmentation $L_{X/Y} \to \Omega^1_{X/Y}$

is a quasi-isomorphism. If f is a *closed immersion*, defined by an ideal I, then there is a natural augmentation $L_{X/Y} \to I/I^2[1]$, which is a quasi-isomorphism when f is a *regular immersion*, i. e. is locally defined by a regular sequence ; in this case I/I^2 is locally free. If f is *locally of complete intersection*, i. e. is locally (on X) the composition of a regular immersion and a smooth morphism, then $L_{X/Y}$ is perfect, of perfect amplitude in $[-1,0]$ (3.6.3).

5.30. The relation between cotangent complex and deformation theory comes from the following fact. Let $f : X \to Y$ be a morphism of schemes. If $i : X \to X'$ is a closed immersion into a Y-scheme defined by an ideal I of square zero, I is a quasi-coherent module on X. We call i (or X') a *Y-extension of X by I*. For fixed I, these Y-extensions form an abelian group, which is shown to be canonically isomorphic to $Ext^1(L_{X/Y}, I)$. This isomorphism is functorial in I. Using the transitivity triangle (5.29), one easily deduces the following generalization of 5.9 :

Theorem 5.31. *(a) Let X and Y be schemes over a scheme S, and let $j : X_0 \to X$ be a closed subscheme defined by an ideal J of square zero. Let $g : X_0 \to Y$ be an S-morphism. There is an obstruction*

$$o(g,j) \in Ext^1(g^* L_{Y/S}, J)$$

whose vanishing is necessary and sufficient for the existence of an S-morphism $h : X \to Y$ extending g, i. e. such that $hj = g$. When $o(g,j) = 0$, the set of extensions h of g is an affine space under $Ext^1(g^* L_{Y/S}, J) = Hom(g^* \Omega^1_{Y/S}, J)$.

(b) Let $i : S_0 \to S$ be a thickening of order one defined by an ideal I of square zero, and let X_0 be a flat *S_0-scheme. There is an obstruction*

$$o(X_0, i) \in Ext^2(L_{X_0/S_0}, f_0^* I)$$

(where $f_0 : X_0 \to S_0$ is the structural morphism) whose vanishing is necessary and sufficient for the existence of a deformation X of X_0 over S (5.7). When $o(X_0, i) = 0$, the set of isomorphism classes of such deformations is an affine space under $Ext^1(L_{X_0/S_0}, f_0^* I)$, and the group of automorphism of a fixed deformation is isomorphic to $Ext^0(L_{X_0/S_0}, f_0^* I) = Hom(\Omega^1_{X_0/S_0}, f_0^* I)$.

Here is an application to liftings of certain singular curves (generalizing the smooth case, dealt with in 5.19) :

Corollary 5.32. *Let $S = \mathrm{Spec}\, A$ be as in 5.19. Let X_0 be a proper curve over s (5.18). We assume that X_0 is locally of complete intersection over s and is smooth over s outside a finite set of closed points. Then there exists a projective and flat curve X over S such that $X_s = X_0$.*

Note that such a lifting X is automatically locally of complete intersection over S [EGA IV 11.3.8, 19.2.4], and is smooth over S outside a finite subscheme (the nonsmoothness locus of X/S is closed and its special fiber is finite, hence is finite by 2.15).

As in the proof of 5.19, we first show that there exists a (proper) and flat formal scheme $\mathcal{X} = \mathrm{colim}\, X_n$ over $\hat{S}$ lifting X_0. Assume X_m, flat over S_m, has been constructed for

$m \leq n$ such that $X_m = S_m \times_{S_n} X_n$, and let $i_n : S_n \to S_{n+1}$ be the inclusion. Then, by 5.31, there is an obstruction

$$o(X_n, i_n) \in Ext^2(L_{X_0/s}, \mathcal{O}_{X_0} \otimes \mathbf{m}^{n+1}/\mathbf{m}^{n+2}) = Ext^2(L_{X_0/s}, \mathcal{O}_{X_0}) \otimes_k \mathbf{m}^{n+1}/\mathbf{m}^{n+2}$$

to the existence of a flat lifting X_{n+1} of X_n over S_{n+1}. Therefore it suffices to show

$$(*) \qquad Ext^2(L_{X_0/s}, \mathcal{O}_{X_0}) = 0.$$

We have

$$Ext^2(L_{X_0/s}, \mathcal{O}_{X_0}) = H^2(X_0, R\mathcal{H}om(L_{X_0/s}, \mathcal{O}_{X_0})).$$

Since $L_{X_0/s}$ is of perfect amplitude in $[-1, 0]$, $R\mathcal{H}om(L_{X_0/s}, \mathcal{O}_{X_0})$ is of perfect amplitude in $[0, 1]$, in particular,

$$\mathcal{E}xt^i(L_{X_0/s}, \mathcal{O}_{X_0}) = 0$$

for $i \neq 0, 1$. Hence it suffices to show

$$(1) \qquad H^2(X_0, \mathcal{H}om(L_{X_0/s}, \mathcal{O}_{X_0})) = 0,$$

$$(2) \qquad H^1(X_0, \mathcal{E}xt^1(L_{X_0/s}, \mathcal{O}_{X_0})) = 0.$$

(1) trivially holds because X_0 is of dimension 1. Since X_0 is smooth over s outside a finite closed subset Σ, $\mathcal{E}xt^1(L_{X_0/s}, \mathcal{O}_{X_0})$ is concentrated on Σ, which implies (2), hence (*).

It remains to show that $\mathcal{X}$ is algebraizable to a projective scheme over S. If D is any effective divisor supported on the smooth locus of X_0 and meeting each irreducible component of X_0, then $\mathcal{O}_{X_0}(D)$ is ample, and since $H^2(X_0, \mathcal{O}_{X_0}) = 0$, the conclusion follows from 5.6.

6. Serre's examples [S2]

6.1. Let k be an algebraically closed field of characteristic $p > 0$, $n \geq 0$, $r \geq 1$ integers, G a finite group, and

$$\rho_0 : G \to PGL_{n+1}(k) \ (= GL_{n+1}(k)/k^*)$$

a representation. Let $P_0 = \mathbb{P}^n_k$. Since the group of k-automorphisms of P_0 is $PGL_{n+1}(k)$ [H, II 7.1.1], ρ_0 defines an action of G on P_0. For $g \in G$, denote by Fix(g) the (closed) subscheme of fixed points of g (intersection of the graph of g and the diagonal in $P_0 \times_k P_0$). Let $Q_0 \subset P_0$ be the union of the Fix(g)'s for $g \neq e$. Consider the condition

$$(6.1.1) \qquad r + \dim(Q_0) < n.$$

The starting point of Serre's construction is the following result [S4, Prop. 15], which Serre attributes to Godeaux :

Proposition 6.2. *Assume that (6.1.1) holds. Then there exists an integer $d_0 \geq 1$ such that, for any integer d divisible by d_0, one can find a smooth complete intersection $Y_0 = \mathrm{V}(h_1, \cdots, h_{n-r})$ of dimension r in P_0, with* $\deg(h_i) = d$ *for* $1 \leq i \leq r$, *which is stable under G, and on which G acts freely.*

By [SGA 1, V 1.8] the action of G on P_0 is admissible, in particular, the quotient $Z_0 = P_0/G$ exists. The projection $f : P_0 \to Z_0$ is finite, and $(f_*\mathcal{O}_{P_0})^G = \mathcal{O}_{Z_0}$. By [EGA II 6.6.4], Z_0 is projective (indeed, condition (II bis) of [EGA II 6.5.1] is satisfied : as P_0 is normal, Z_0 is normal, too, as follows from the above formula for $\mathcal{O}_{Z_0}$). Choose an embedding $i : Z_0 \to \mathbb{P}^s_k$. Then $(if)^*\mathcal{O}_{\mathbb{P}^s_k}(1) = \mathcal{O}_{P_0}(d_0)$ for some integer $d_0 > 0$. For any integer $m \geq 1$, denote by $i_m : Z_0 \to \mathbb{P}^{N(m)}_k$ ($N(m) = \binom{s+m}{s} - 1$) the m-th multiple of i. Then $(i_m f)^*\mathcal{O}_{\mathbb{P}^{N(m)}_k}(1) = \mathcal{O}_{P_0}(d)$ where $d = md_0$. Since f is finite, $f(Q_0)$ is closed in Z_0 and $\dim(f(Q_0)) = \dim(Q_0)$. Since (6.1.1) holds, by a theorem of Bertini [ ], there exists a linear subspace $L_0 = V(\ell_1, \cdots, \ell_{n-r})$ of $\mathbb{P}^{N(h)}_k$ of codimension $n - r$ (with $\deg(\ell_i) = 1$), such that $L_0 \cap Z_0$ is contained in $U_0 = Z_0 - f(Q_0)$ and L_0 is transversal to U_0. Since $f|U_0 : f^{-1}(U_0) \to U_0$ is étale, the forms $h_i = (i_m f)^*\ell_i \in \Gamma(P_0, \mathcal{O}(d))$ $(1 \leq i \leq n-r)$ define a smooth complete intersection Y_0 in P_0, which is stable under G, and does not meet Q_0, hence on which G acts freely.

6.3. Let d and Y_0 be as in (6.2), and let

$$X_0 = Y_0/G$$

be the quotient of Y_0 by G. As G acts freely on Y_0, X_0/k is a smooth, projective scheme of dimension r, and the projection

$$f : Y_0 \to X_0$$

is an étale cover of group G [SGA 1, V 2.3]. Moreover, since Y_0 is a complete intersection of dimension $r \geq 1$, Y_0 is *connected* [FAC, no 78, Prop. 5].

The main point in Serre's construction is the following result.

Proposition 6.3. *Assume $r \geq 3$, or $r = 2$, $(p, n+1) = 1$, and $p|d$. Let A be a complete local noetherian ring, with residue field k. Let $\mathcal{X}$ be a flat, formal scheme over A lifting X_0. Then $\mathcal{X}$ is algebraizable, i. e. (4.9) there exists a (unique) proper scheme X/A such that $\hat{X} = \mathcal{X}$. Moreover, X is projective and smooth over A and the representation ρ_0 (6.1) lifts to a representation*

$$\rho : G \to PGL_{n+1}(A) \ (= GL_{n+1}(A)/A^*).$$

The case $r \geq 3$ is dealt with in [S2]. The case $r = 2$ is due to Mumford [M5].

By 5.9 (b), Y_0 lifts uniquely (up to a unique isomorphism) to a formal étale cover $\mathcal{Y} = \mathrm{colim} \mathrm{Y_m}$ of $\mathcal{X} = \mathrm{colim}\, X_m$, i. e. such that Y_m is finite étale over X_m for all $m \geq 0$ (where $X_m = \mathcal{X} \otimes A/\mathbf{m}^{n+1}$). By 5.9 (b) again, the action of G on Y_0 extends (uniquely) to an action of G on $\mathcal{Y}$, making $\mathcal{Y}$ an étale Galois cover of $\mathcal{X}$ of group G (i. e. an inductive system of G-Galois étale covers $Y_n \to X_m$). Since $r \geq 2$, we have $H^1(Y_0, \mathcal{O}_{Y_0}) = 0$ and $H^0(Y_0, \mathcal{O}_{Y_0}) = k$ ([FAC no 78] or [SGA 7 XI]), so 3.11.2 implies that $H^0(Y_m, \mathcal{O}_{X_m}) = A_m$ for all m.

Let $i : Y_0 \to P_0$ be the inclusion and $L_0 = \mathcal{O}_{Y_0}(1) = i^*\mathcal{O}_{P_0}(1)$. We shall show :

(*) L_0 lifts to an invertible sheaf $\mathcal{L}$ on $\mathcal{Y}$, unique up to a (non unique) isomorphism (inducing the identity on L_0).

Assume first that $r \geq 3$. Then, by (loc. cit.), $H^2(Y_0, \mathcal{O}_{Y_0}) = 0$. Since $H^1(Y_0, \mathcal{O}_{Y_0}) = 0$, too, (*) is true by 5.5. Assume now that $r = 2$. Then it is no longer true that $H^2(Y_0, \mathcal{O}_{Y_0}) = 0$. To show that L_0 lifts (in which case it will lift uniquely as $H^1(Y_0, \mathcal{O}_{Y_0}) = 0$), Mumford argues as follows. We have

$$(**)_0 \qquad \Omega^2_{Y_0/k} = \mathcal{O}_{Y_0}(N),$$

with $N = (n-r)d - n - 1$. The hypotheses imply $(p, N) = 1$. Assume that, for $m \geq 0$, L_0 has been lifted to an invertible sheaf L_m on X_m, and the isomorphism $(**)_0$ lifted to an isomorphism

$$(**)_m \qquad L_m^{\otimes N} \simeq \Omega^2_{Y_m/A_m}.$$

Let $i_m : Y_m \to Y_{m+1}$ be the inclusion. Consider the obstruction $o(L_m, i_m)$ to lifting L_m to Y_{m+1} (5.3 (b)). By 5.4 (a), we have

$$o(L_m^{\otimes N}, i_m) = No(L_m, i_m).$$

Since $\Omega^2_{Y_{m+1}/A_{m+1}}$ lifts $\Omega^2_{Y_m/A_m}$, the isomorphism $(**)_m$ implies that $o(L_m^{\otimes N}, i_m) = 0$, hence $o(L_m, i_m) = 0$ as well, since p does not divide N. Hence L_m lifts to an invertible sheaf L_{m+1} on Y_{m+1}. Since $H^1(Y_0, \mathcal{O}_{Y_0}) = 0$, by 5.3 (a) the isomorphism $(**)_m$ lifts to an isomorphism $(**)_{m+1}$. Therefore L_0 lifts to an invertible sheaf $\mathcal{L}$ on $\mathcal{Y}$.

Since L_0 is ample, by 5.6 there exists a projective and flat scheme Y over A such that $\hat{Y} = \mathcal{Y}$ and an ample line bundle L on Y such that $\hat{L} = \mathcal{L}$. By [EGA II 6.6.1], the norm $E_0 = N_{Y/X} L_0$ of L_0 is an ample line bundle on X_0. For $m \in \mathbb{N}$, let $E_m = N_{Y_m/X_m} L_m$ and $\mathcal{E} = \lim E_m$. Then $\mathcal{E}$ lifts E_0, so by 4.10 there exists a projective scheme X/A such that $\hat{X} = \mathcal{X}$ and an ample line bundle E on X such that $\hat{E} = \mathcal{E}$. By 5.6 (and the argument at the end of the proof of 5.19), X is smooth over A. Moreover, by 4.7, the étale Galois cover $\hat{Y} \to \hat{X}$ is deduced by completion of a (unique) étale Galois cover $Y \to X$ of group G, and by 4.2, $E = N_{Y/X} L$.

It remains to show that ρ_0 lifts to A. By ([FAC no 78] or [SGA 7 XI]), we have $H^0(Y_0, L_0) = k^{n+1}$, $H^1(Y_0, L_0) = 0$. Therefore, by 3.11.2, $H^0(Y_m, L_m) = A_m^{n+1}$ for all m, hence $H^0(Y, L) = A^{n+1}$ by 2.4. Let $g \in G$. Since $H^1(Y_0, \mathcal{O}_{Y_0}) = 0$, by 5.3 (a) and 4.2 there is an isomorphism $a(g) : L \xrightarrow{\sim} L$ above $g : Y \xrightarrow{\sim} Y$, i. e. an isomorphism $g^*L \xrightarrow{\sim} L$, unique up to an automorphism of L, such that $a(g)_0$ is the isomorphism $L_0 \to L_0$ above $g : Y_0 \xrightarrow{\sim} Y_0$ given by the action of g on Y_0 (which is well defined up to an automorphism of L_0). For g and h in G, we have $a(h)a(g) = a(gh)$ and $a(e) = Id$ up to an automorphism of L. Therefore we get a representation

$$\rho : G \to PGL(H^0(Y, L)) = PGL_{n+1}(A),$$

associating to g the automorphism $\rho(g)$ of $H^0(Y, L) = A^{n+1}$ induced by the pair $(g, a(g))$, which automorphism is well defined up to multiplication by an element of A^*. This representation lifts ρ_0.

6.4. Let now r and n be integers with $1 \leq r < n$, and let G be a group of type $(p, \cdots, p)$ of order p^s, i. e. $G \simeq \mathbb{F}_p^s$, with $s \geq n+1$. Assume moreover that $p \geq n+1$. Choose an *injective* homomorphism $h : G \to k$ (where k is considered as an additive group). Let $N = (u_{ij})$ be the nilpotent matrix of order $n+1$ defined by $u_{ij} = 1$ if $j = i+1$ and $u_{ij} = 0$ otherwise. For $g \in G$, let

$$\tilde{\rho}_0(g) = \exp(h(g)N) \in GL_{n+1}(k)$$

(which makes sense since $p \geq n+1$), and let $\rho_0(g)$ be the image of $\tilde{\rho}_0(g)$ in $PGL_{n+1}(k)$. We thus get a representation

$$\rho_0 : G \to PGL_{n+1}(k), \tag{6.4.1}$$

which is *faithful*, as h is injective. For any $g \in G$, $g \neq e$, $\mathrm{Fix}(g)$ consists of the single rational point $(1, 0, \cdots, 0)$ of P_0. In particular, $\dim Q_0 = 0$, with the notations of 6.1, so the condition (6.1.1) is satisfied.

Proposition 6.5. *Assume that $p > n+1$. Let A be an integral, complete, local noetherian ring with residue field k and field of fractions K of characteristic zero. Then there exists no homomorphism $\rho : G \to PGL_{n+1}(A)$ lifting ρ_0 (6.4.1).*

The following argument is due to Serre (private communication). Suppose that such a homomorphism ρ exists. Since ρ_0 is injective, ρ is injective, too, and so is the composition, still denoted ρ, with the inclusion $PGL_{n+1}(A) \to PGL_{n+1}(K)$. Since K is of characteristic zero and p does not divide $n+1$, this representation lifts to a (faithful) representation $\rho' : G \to SL(V)$, where $V = K^{n+1}$. As G is commutative and K is of characteristic zero, up to extending the scalars to a finite extension of K, V decomposes into a sum

$$V = \bigoplus_{1 \leq i \leq n+1} V_i$$

of 1-dimensional subspaces stable under G, corresponding to characters $\chi_i : G \to Aut(V_i) = K^*$, whose product is 1. The kernel Z of ρ is the intersection of the kernels H_i of χ_i, for $1 \leq i \leq n$. Each χ_i is a homomorphism from G to $\mu_p(K)$, so can be viewed as a linear form on G considered as a vector space over $\mathbb{F}_p$. Since G is of dimension $s \geq n+1$ over $\mathbb{F}_p$, Z cannot be zero, which contradicts the faithfulness of ρ.

Corollary 6.6. *Let r, n be integers such that $2 \leq r < n$ and $p > n+1$. Let $G = \mathbb{F}_p^s$, with $s \geq n+1$. There exists a smooth, projective complete intersection Y_0 of dimension r in P_0, stable under the action of G on P_0 defined by the representation ρ_0 constructed in (6.4.1), and on which G acts freely, and such that the smooth, projective scheme $X_0 = Y_0/G$ has the following property. Let A be an integral, complete, local noetherian ring with residue field k and field of fractions K of characteristic zero. Then there exists no formal scheme $\mathcal{X}$, flat over A, lifting X_0.*

Let d_0 be an integer ≥ 1 having the properties stated in 6.2. If $r \geq 3$, take any nonzero multiple d of d_0, and if $r = 2$, take any nonzero multiple d of d_0 which is divisible by p. By 6.2, choose a smooth, complete intersection Y_0 in P_0, of degree $(d, \cdots, d)$, stable under the action of G on P_0 defined by the representation ρ_0 constructed in (6.4.1), and on which G acts freely. Let $X_0 = Y_0/G$. Assume that there exists a formal scheme $\mathcal{X}$, flat over A, lifting X_0. Since $p > n+1$ and, if $r = 2$, p divides d, the assumptions of 6.3 are satisfied, and its conclusion, together with 6.5, yields a contradiction.

The minimal examples are obtained for $r = 2$, $n = 3$, $s = 4$, $p = 5$. (In [S2], the minimal ones were fore $r = 3$, $n = 4$, $s = 5$, $p = 7$).

Remark 6.7. Let X_0 be the scheme considered in 6.6. Let A be a complete, local noetherian ring with residue field k, which is the base of a *formal versal deformation* $\mathcal{X}$ of X_0 [Sc]. Such a ring A is a W-algebra which is formally of finite type, where $W = W(k)$ the ring of Witt vectors on k. Let $K_0 = W[1/p]$ be the fraction field of W. It follows from 6.6 that $A \otimes_W K_0 = 0$, in other words *there exists an integer* $n_0 \geq 1$ *such that* $p^{n_0} A = 0$. Otherwise, one could find an integral closed subscheme $T = \operatorname{Spec} B$ of $\operatorname{Spec} A$ with generic point of characteristic zero. By pulling back $\mathcal{X}$ to $\operatorname{Spf} B$, we would obtain a contradiction.

We have the following question : can one have $n_0 = 1$?

References

[A1] M. Artin, Algebraization of formal moduli I, in Global Analysis (papers in honor of K. Kodaira), Univ. Tokyo P. (1969), 21-71.

[A2] M. Artin, Versal deformations and algebraic stacks, Inv. Math. 27 (1974), 65-189.

[At] M. F. Atiyah, Complex analytic connections in fibre bundles, in Symposium Internacional de Topologia Algebraica, Univ. México, 1958, 77-82.

[B] N. Bourbaki, *Algèbre commutative*, Actualités Sc. et Ind. 1290, 1293, Hermann, 1961.

[BLR] S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron Models*, Erg. der Math. 21, Springer-Verlag, 1990.

[BLoR] *Courbes semi-stables et groupe fondamental en géométrie algébrique*, Proc. conf. on fundamental group of curves, Luminy 1998, eds. J.-B. Bost, F. Loeser, M. Raynaud, Progress in Math. 187, Birkhäuser, 2000, 169-184.

[D1] P. Deligne, *Appendix : cohomologie à support propre et construction du foncteur* $f^!$, in R. Hartshorne, *Residues and Duality*, SLN 20, Springer-Verlag, 1966.

[D2] P. Deligne, Relèvement des surfaces K3 en caractéristique nulle, in *Surfaces algébriques*, eds. J. Giraud, L. Illusie, M. Raynaud, SLN 868, Springer-Verlag, 1981, 58-79.

[DI] P. Deligne, L. Illusie, Relèvements modulo p^2 et décomposition du complexe de de Rham, Inv. Math. 89 (1987), 247-270.

[E] T. Ekedahl, *Diagonal complexes and F-gauge structures*, Hermann, 1986.

[F] G. Faltings, Finiteness of coherent cohomology for proper fppf stacks, preprint, 2001.

[FK] K. Fujiwara, K. Kato, Logarithmic étale topology theory, (incomplete) preprint, 1995.

[Fo] R. Fossum, Formes différentielles non fermées, in Séminaire sur les pinceaux de courbes de genre au moins deux, by L. Szpiro, Astérisque 86, 90-96 (1981).

[G] A. Grothendieck, *Géométrie formelle et géométrie algébrique*, Séminaire Bourbaki, Mai

1959, n° 182, W. Benjamin, 1966.
[Gi] J. Giraud, *Cohomologie non abélienne*, Grundl. der math. 179, Springer-Verlag, 1971.
[H] R. Hartshorne, *Algebraic Geometry*, GTM 52, Springer-Verlag, 1977.
[H1] R. Hartshorne, *Ample Subvarieties of Algebraic Varieties*, SLN 156, Springer-Verlag, 1970.
[Hi] M. Hirokado, A non-liftable Calabi-Yau threefold in characteristic 3, Tohoku Math. J. 51 (1999), 479-487.
[I1] L. Illusie, *Complexe cotangent et déformations I*, SLN 239, Springer-Verlag, 1971.
[I2] L. Illusie, Frobenius and Hodge degeneration, in *Introduction to Hodge theory*, SMF/AMS Texts 8, 2002.
[I3] L. Illusie, An Overview of the Work of K. Fujiwara, K. Kato, and C. Nakayama on Logarithmic Étale Cohomology, in *Cohomologies p-adiques et applications arithmétiques (II)*, eds. P. Berthelot, J.-M. Fontaine, L. Illusie, K. Kato, M. Rapoport, Astérique 279, SMF 2002, 271-322.
[I4] L. Illusie, Déformations de groupes de Barsotti-Tate, d'après A. Grothendieck, in *Séminaire sur les pinceaux arithmétiques : la conjecture de Mordell*, dirigé par L. Szpiro, Astérisque 227, 1985, 151-198.
[I5] L. Illusie, Cotangent complex and deformations of torsors and group schemes, in Toposes, Algebraic Geometry and Logic, SLN 274, Springer-Verlag, 1971, 159-188
[K] D. Knutson, *Algebraic Spaces*, SLN 203, Springer-Verlag, 1971.
[Ka-Sa] K. Kato and T. Saito, Conductor formula of Bloch, preprint, 2003.
[Ki] M. Kisin, Prime to p fundamental groups and tame Galois actions, Ann Inst. Fourier 50 (1999), 1099-1126.
[L] W. Lang, Quasi-elliptic surfaces in characteristic three, Ann. Sci. ENS, IV, Ser. 12, 473-500 (1979)
[LM] G. Laumon, L. Moret-Bailly, *Champs algébriques*, Erg. der Math. 39, Springer-Verlag, 2000.
[M1] D. Mumford, *Abelian Varieties*, Oxford Univ. P., 1970.
[M2] D. Mumford, *Geometric invariant theory*, Erg. der Math. 34, Springer-Verlag, 1965.
[M3] D. Mumford, Bi-extensions of formal groups, Alg. Geom., Bombay Coll. 1968, Tata Inst., Oxford Univ. Press 1969, 307-322].
[M4] D. Mumford, Pathologies of modular surfaces, Amer. J. Math. 83, 339-342 (1961).
[M5] D. Mumford, Letter to Serre, May 1961.
[Mu] J. Murre, Representations of unramified functors. Applications, Séminaire Bourbaki 64/65, $n°$294.
[N] N. Nygaard, A p-adic proof of the Rudakov-Shafarevitch theorem, Ann. of Math. 110, 515-528 (1979).
[O] F. Oort, Finite group schemes, local moduli for abelian varieties and lifting problems, Proc. Nordic Summer School Oslo, 1970, 223-254.
[O-V] F. Orgogozo, I. Vidal, Le théorème de spécialisation du groupe fondamental, in *Courbes semi-stables et groupe fondamental en géométrie algébrique*, Proc. conf. on fundamental group of curves, Luminy 1998, eds. J.-B. Bost, F. Loeser, M. Raynaud, Progress in Math. 187, Birkhäuser, 2000, 169-184.
[R1] M. Raynaud, Faisceaux amples sur les schémas en groupes et les espaces homogènes,

SLN 119, Springer-Verlag, 1970.
[R2] M. Raynaud, Revêtements de la droite affine en caractéristique $p > 0$ et conjecture d'Abhyankar, Inv. Math. 116 (1994), 425-462.
[RS] A. Rudakov and I. Shafarevitch, Inseparable morphisms of algebraic surfaces, Akad. Sc. SSSR, t. 40, n° 6, 1264-1307 (1976).
[S1] J.-P. Serre, Géométrie algébrique et géométrie analytique, Ann. Inst. Fourier 6 (1956), 1-42.
[S2] J.-P. Serre, Exemples de variétés projectives en caractéristique p non relevables en caractéristique zéro, Proc. Nat. Ac. Sc. 47 (1961), 108-109.
[S3] J.-P. Serre, *Groupes algébriques et corps de classes*, Actualités Sc. et Ind. 1264, Hermann, 1959.
[Sc] M. Schlessinger, *Functors of Artin rings*, Trans. Amer. Soc. 130 (1968), 205-222.
[Vi] I. Vidal, Contributions à la cohomologie étale des schémas et des log schémas, Thèse (Paris XI Orsay), 2001.
[Z] O. Zariski, *Introduction to the Problem of Minimal Models in the Theory of Algebraic Surfaces*, Pub. Math. Soc. Japan 4, 1958.
[EGA] *Éléments de Géométrie Algébrique*, par A. Grothendieck, rédigés avec la collaboration de J. Dieudonné, Pub. Math. IHES 4, 8, 11, 17, 20, 24, 28, 32, and Grundlehren 166, Springer-Verlag, 1971.
[FAC] J.-P. Serre, *Faisceaux algébriques cohérents*, Ann. of Math. 61 (1955), 197-278.
[SGA 1] *Revêtements étales et groupe fondamental*, Séminaire de Géométrie Algébrique du Bois-Marie 1960/1961, dirigé par A. Grothendieck, SLN 224, Springer-Verlag, 1971.
[SGA 2] *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux*, Séminaire de Géométrie Algébrique du Bois-Marie 1962, dirigé par A. Grothendieck, North-Holland Pub. Comp., Masson et Cie, 1968.
[SGA 5] *Cohomologie ℓ-adique et fonctions L*, Séminaire de Géométrie Algébrique du Bois-Marie 1965/66, dirigé par A. Grothendieck, SLN 589, Springer-Verlag, 1977.
[SGA 6] *Théorie des intersections et théorème de Riemann-Roch*, Séminaire de Géométrie Algébrique du Bois-Marie 1966/67, dirigé par P. Berthelot, A. Grothendieck, L. Illusie, SLN 225, Springer-Verlag, 1971.
[SGA 7] *Groupes de monodromie en géométrie algébrique*, Séminaire de Géométrie Algébrique du Bois-Marie 1967-1969, I dirigé par A. Grothendieck, II par P. Deligne et N. Katz, SLN 288, 340, Springer-Verlag 1972, 1973.