The Picard scheme

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Abstract. We develop in detail most of the theory of the Picard scheme that Grothendieck sketched in two Bourbaki talks and in commentaries on them. Also, we review in brief much of the rest of the theory developed by Grothendieck and by others. But we begin with a historical introduction.

Contents

1. Introduction

On any ringed space X, the isomorphism classes of invertible sheaves form a group; it is denoted by $\mathrm{Pic}(X)$, and called the (absolute) *Picard group.* Suppose X is a "projective variety"; in other words, X is an integral scheme that is projective over an algebraically closed field k. Then, as is proved in these notes, the group $\mathrm{Pic}(X)$ underlies a natural k-scheme, which is a disjoint union of quasi-projective schemes, and the operations of multiplying and of inverting are given by k-maps. This scheme is denoted by $\mathbf{Pic}_{X/k}$, and called the *Picard scheme.* It is reduced in characteristic zero, but not always in positive characteristic. When X varies in an algebraic family, correspondingly, $\mathbf{Pic}_{X/k}$ does too.

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The Picard scheme was introduced in 1962 by Grothendieck. He sketched his theory in two Bourbaki talks, nos. 232 and 236, which were reprinted along with his commentaries in [**FGA**]. But Grothendieck advanced an old subject, which was actively being developed by many others at the time. Nevertheless, Grothendieck's theory was revolutionary, both in concept and in technique.

In order to appreciate Grothendieck's contribution fully, we have to review the history of the Picard scheme. Reviewing this history serves as well to introduce and to motivate Grothendieck's theory. Furthermore, the history is rich and fascinating, and it is a significant part of the history of algebraic geometry.

So let us now review the history of the Picard scheme up to 1962. We need only summarize and elaborate on scattered parts of Brigaglia, Ciliberto, and Pedrini's article [**BCP**] and of the author's article [**Kl04**]. Both articles give many precise references to the original sources and to the secondary literature; so few references are given here.

The Picard scheme has roots in the 1600s. Over the course of that century, the Calculus was developed, through the efforts of many individuals, in order to design lenses, to aim cannons, to make clocks, to hang cables, and so on. Thus interest arose in the properties of functions appearing as indefinite integrals.

Notably, in 1694, James Bernoulli analyzed the way rods bend, and was led to introduce the "lemniscate," a figure eight with equation $(x^2 + y^2)^2 = a^2(x^2 - y^2)$ where a is nonzero. In polar coordinates, he found the arc length s to be given by

$$s = \int_0^r \frac{a^2\,dr}{\sqrt{a^4 - r^4}}.$$

He surmised that s can not be expressed in terms of the elementary functions. Similar integrals had already arisen in attempts to rectify elliptical orbits; so these integrals became known as "elliptic integrals."

In 1698, James's brother, John, recalled there are algebraic relations among the arguments of sums and differences of logarithms and of the inverse trigonometric functions. Then he showed that, similarly, given two arcs from the origin on the cubical parabola $y = x^3$, their lengths differ by the length of a certain third such arc. And he posed the problem of finding more cases of this phenomenon.

Sure enough, between 1714 and 1720, Fagnano found, in an ad hoc manner, similar relations for the cords and arcs of ellipses, hyperbolas, and lemniscates. In turn, Fagnano's work led Euler in 1757 to discover the "addition formula"

$$\int_0^{x_1} \frac{dx}{\sqrt{1-x^4}} \pm \int_0^{x_2} \frac{dx}{\sqrt{1-x^4}} = \int_0^{x_3} \frac{dx}{\sqrt{1-x^4}}$$

where the variables x_1, x_2, x_3 must satisfy the symmetric relation

$$\begin{aligned} x_1^4x_2^4x_3^4 + 2x_1^4x_2^2x_3^2 + 2x_1^2x_2^4x_3^2 + 2x_1^2x_2^2x_3^4 \\ + x_1^4 + x_2^4 + x_3^4 - 2x_1^2x_2^2 - 2x_1^2x_3^2 - 2x_2^2x_3^2 = 0. \end{aligned}$$

In 1759, Euler generalized this formula to some other elliptic integrals. Specifically, Euler found the sum or difference of two to be equal to a certain third plus an elementary function. Moreover, he expressed regret that he could handle only square roots and fourth powers, but not higher roots or powers.

In 1826, Abel made a great advance: he discovered an addition theorem of sweeping generality. It concerns certain algebraic integrals, which soon came to be called "Abelian integrals." They are of the following form:

$$\psi x := \int_{x_0}^{x} R(x,y)\,dx$$

where x is an independent complex variable, R is a rational function, and $y = y(x)$ is an integral algebraic function; that is, y is the implicit multivalued function defined by an irreducible equation of the form

$$f(x,y) := y^n + f_1(x)y^{n-1} + \cdots + f_n(x) = 0$$

where the $f_i(x)$ are polynomials in x.

Let p be the genus of the curve $f = 0$, and let $h_1, \ldots, h_\alpha$ be rational numbers. Then Abel's addition theorem asserts that

$$h_1\psi x_1 + \cdots + h_\alpha \psi x_\alpha = v + \psi x'_1 + \cdots + \psi x'_p$$

where v is an elementary function of the independent variables $x_1, \ldots, x_\alpha$ and where $x'_1, \ldots, x'_p$ are algebraic functions of them. More precisely, v is a complex-linear combination of one algebraic function of $x_1, \ldots, x_\alpha$ and of logarithms of others; moreover, $x'_1, \ldots, x'_p$ work for every choice of ψx. Lastly, p is minimal: given algebraic functions $x'_1, \ldots, x'_{p-1}$ of $x_1, \ldots, x_p$, there exists an integral ψx such that, for any elementary function v,

$$\psi x_1 + \cdots + \psi x_p \neq v + \psi x'_1 + \cdots + \psi x'_{p-1}.$$

Abel finished his 61-page manuscript in Paris, and submitted it in person on 30 October 1826 to the Royal Academy of Sciences, which appointed Cauchy and Legendre as referees. However, the Academy did not publish it until 1841, long after Abel's death from tuberculosis on 6 April 1829.

Meanwhile, Abel feared his manuscript was lost forever. So in Crelle's Journal, **3** (1828), he summarized his general addition theorem informally. Then he treated in detail a major special case, that in which $f(x,y) := y^2 - \varphi(x)$ where $\varphi(x)$ is a squarefree polynomial of degree $d \geq 1$. In particular, Abel found

$$p = \begin{cases} (d-1)/2, & \text{if } d \text{ is odd;} \\ (d-2)/2, & \text{if } d \text{ is even.} \end{cases}$$

Thus, if $d \geq 5$, then $p \geq 2$, and so Euler's formula does not extend.

With Jacobi's help, Legendre came to appreciate the importance of this case. To it, Legendre devoted the third supplement to his long treatise on elliptic integrals, which are recovered when $d = 3, 4$. For $d \geq 5$, the integrals share many of the same formal properties. So Legendre termed them "ultra-elliptiques."

Legendre sent a copy of the supplement to Crelle for review on 24 March 1832, and Crelle asked Jacobi to review it. Jacobi translated "ultra-elliptiques" by "hyperelliptischen," and the prefix "hyper" has stuck. In his cover letter, Legendre praised Abel's addition theorem, calling it, in the immortal words of Horace's Ode 3, XXX.1, "a monument more lasting than bronze" (monumentum aere perennius). In his review, Jacobi said that the theorem would be a most noble monument were it to acquire the name **Abel's Theorem.** And it did!

Jacobi was inspired to give, a few months later, the first of several proofs of Abel's Theorem in the hyperelliptic case. Furthermore, he posed the famous problem, which became known as the "Jacobi Inversion Problem." He asked, "what, in the general case, are those functions whose inverses are Abelian integrals, and what does Abel's theorem show about them?"

Jacobi solved the inversion problem when $d = 5, 6$. Namely, he formed

$$\psi x := \int_{x_0}^{x} \frac{dx}{\sqrt{\varphi(x)}} \text{ and } \psi_1 x := \int_{x_0}^{x} \frac{x\,dx}{\sqrt{\varphi(x)}},$$

and he set

$$\psi x + \psi y = u \text{ and } \psi_1 x + \psi_1 y = v.$$

He showed $x + y$ and xy are single-valued functions of u and v with four periods.

Some historians have felt Abel had this inversion in mind, but ran out of time. At any rate, in 1827, Abel had originated the idea of inverting elliptic integrals, obtaining what became known as "elliptic functions." He and Jacobi studied them extensively. Moreover, Jacobi introduced "theta functions" as an aid in the study; they were generalized by Riemann in 1857, and used to solve the inversion problem in arbitrary genus.

Abel's paper on hyperelliptic integrals fills twelve pages. Eight are devoted to a computational proof of a key intermediate result. A half year later, in Crelle's Journal, **4** (1829), Abel published a 2-page paper with a conceptual proof of this result for any Abelian integral ψx. The result says that $\psi x_1 + \cdots + \psi x_\mu$ is equal to an elementary function v if $x_1, \ldots, x_\mu$ are not independent, but are the abscissas of the variable points of intersection of the curve $f = 0$ with a second plane curve that varies in a linear system — although this geometric formulation is Clebsch's.

In each of the first two papers, Abel addressed two more, intermediate questions: First, when is the sum $\psi x_1 + \cdots + \psi x_\mu$ constant? Second, what is the number α of x_i that can vary independently? Remarkably, the answers involve the genus p.

For hyperelliptic integrals, Abel found $\psi x_1 + \cdots + \psi x_\mu$ is constant if ψx is a linear combination of the following p integrals:

$$\int_{x_0}^{x} \frac{dx}{\sqrt{\varphi(x)}},\ \int_{x_0}^{x} \frac{x\,dx}{\sqrt{\varphi(x)}}, \ldots, \int_{x_0}^{x} \frac{x^{p-1}\,dx}{\sqrt{\varphi(x)}}.$$

Here $x_1, \ldots, x_\mu$ are the abscissas of the variable points of intersection of the curve $y^2 = \varphi(x)$ and the curve $\theta_1(x)y = \theta_2(x)$, whose coefficients vary, but $\theta_2(x)$ and $\varphi(x)$ retain a fixed common factor $\varphi_1(x)$. Furthermore, Abel found

$$\mu - \alpha \geq p; \tag{1.1}$$

equality does not always hold, but can be achieved, given d and α, by choosing the degrees of $\theta_1(x)$ and $\varphi_1(x)$ appropriately.

Suppose $\mu - \alpha = p$. Then

$$\psi x_1 + \cdots + \psi x_\alpha = v - (\psi x_{\alpha+1} + \cdots + \psi x_{\alpha+p}$$

where $x_{\alpha+1}, \ldots, x_{\alpha+p}$ are algebraic functions of $x_1, \ldots, x_\alpha$. Similarly, given any $x'_1, \ldots, x'_{\alpha'}$, we get

$$\psi x'_1 + \cdots + \psi x'_{\alpha'} + \psi x_{\alpha+1} + \cdots + \psi x_{\alpha+p} = v' - (\psi x''_1 + \cdots + \psi x''_p).$$

Subtract this formula from the one above, and set $V := v - v'$. The result is

$$\psi x_1 + \cdots + \psi x_\alpha - \psi x'_1 - \cdots - \psi x'_{\alpha'} = V + \psi x''_1 + \cdots + \psi x''_p,$$

namely, the addition theorem with $h_i = \pm 1$. This result is essentially Abel's main theorem on hyperelliptic integrals.

In his Paris manuscript, Abel addressed the two intermediate questions for an arbitrary f. However, his computations are more involved, and his results, less definitive. He found constancy holds when ψx is of the form

$$\psi x := \int_{x_0}^{x} \frac{h(x,y)}{\partial f/\partial y}\, dx \tag{1.2}$$

where $\deg h \leq \deg f - 3$. Also, h must satisfy certain linear conditions; namely, h must vanish suitably everywhere $\partial f/\partial y$ does on the curve $f = 0$, at finite distance and at infinity. Abel took the msximum number of independent h as the genus p.

Furthermore, Abel found that there exists an $i \geq 0$ such that

$$\mu - \alpha = p - i. \tag{1.3}$$

This equation does not contradict (1.1) as the two α's differ; in (1.1), the linear system of intersections is incomplete, whereas in (1.3), the system is complete.

Abel's ideas have been clarified and completed over the course of time through the efforts of many. Doubtless, Riemann made the greatest contribution in his revolutionary 1857 paper on Abelian functions. In his thesis of 1851, he had developed a way of extending complex analysis to a multivalued function y of a single variable x by viewing y as a single-valued function on an abstract multisheeted covering of the x-plane, the "Riemann surface" of y. In 1857, he treated the case where the surface is compact, and showed this case is precisely the case where y is algebraic.

Riemann defined the genus p topologically, essentially as half the first Betti number of the surface. However, the term "genus" is not Riemann's, but Clebsch's. Clebsch introduced it in 1865 to signal his aim of using p in order to classify algebraic curves. And he showed that every curve of genus 0 is birationally equivalent to a line, and every curve of genus 1, to a nonsingular plane cubic.

Also in 1865, Clebsch gave an algebro-geometric formula for the genus p of a plane curve: if the curve has degree d and, at worst, δ nodes and κ cusps, then

$$p = (d-1)(d-2)/2 - \delta - \kappa.$$

The next year, Clebsch and Gordan employed this formula to prove the birational invariance of p; they determined how d, δ, and κ change.

Plainly, birationally equivalent curves have homeomorphic Riemann surfaces, and so the same genus p. But Clebsch was no longer satisfied in just showing the consequences of Riemann's work. He now wanted to establish the theory of Abelian integrals on the basis of the algebraic theory of curves as developed by Cayley, Salmon, and Sylvester. At the time, Riemann's theory was strange and suspect; there was, as yet, no theory of manifolds, and no proof of the Dirichlet principle. Clebsch's efforts led to a sea change in algebraic geometry, which turned toward the study of birational invariants.

Riemann defined an integral to be of the "first kind" if it is finite everywhere. He proved these integrals form a vector space of dimension p. Furthermore, each can be expressed in the form (1.2) provided the curve $f = 0$ has, at worst, double points; if so, the linear conditions on h just require h to vanish at these double points. In 1874, Brill and M. Noether generalized this result to ordinary m-fold points: h must vanish to order $m - 1$. They termed such h "adjoints." Meanwhile, starting with Kronecker in 1858 and Noether in 1871 and continuing through Muhly and Zariski in 1938, many algebraic geometers developed corresponding ways of reducing the singularities of a given plane curve by means of birational transformations.

Euler noted the integral $\int_0^x dx/\sqrt{1-x^4}$ has a "modulus of multivaluedness" like that of the inverse trigonometric functions. Abel noted an arbitrary Abelian integral has a similar ambiguity, but viewed it as a sort of constant of integration, and avoided it by keeping the domain small. Riemann clarified the issue completely. He proved every integral of the first kind has $2p$ "periods," which are numbers that generate all possible changes in the value of the integral arising from changes in the path of integration.

Riemann, in effect, did as follows. He fixed a basis $\psi_1 x, \dots, \psi_p x$ of the integrals of the first kind, and he fixed a homology basis of $2p$ paths. Then, inside the vector space $\mathbb{C}^p$, he formed the lattice $\mathbb{L}$ generated by the $2p$ corresponding p-vectors of periods. And he proved the quotient is a p-dimensional complex torus

$$J := \mathbb{C}^p/\mathbb{L}.$$

Later J was termed the "Jacobian" to honor Jacobi's work on inversion.

Let C be the curve $f = 0$, or better, the associated Riemann surface. Let $C^{(\mu)}$ be its μ-fold symmetric product. Riemann, in effect, formed the following map:

$$\Psi_\mu \colon C^{(\mu)} \to J \text{ given by } \Psi_\mu(x_1, \dots, x_\mu) = \Bigl(\sum_{i=1}^{\mu} \psi_1 x_i, \dots, \sum_{i=1}^{\mu} \psi_p x_i\Bigr).$$

This map Ψ_μ is rather important. It has been called the "Abel–Jacobi map" and the "Abel map." The latter name is historically more correct and shorter, so better.

Riemann, in effect, studied the fibers of the Abel map Ψ_μ. He proved that, if two divisors $x_1 + \dots + x_\mu$ and $x'_1 + \dots + x'_\mu$ are linearly equivalent, then

$$\Psi_\mu(x_1, \dots, x_\mu) = \Psi_\mu(x'_1, \dots, x'_\mu).$$

Riemann called this result "Abel's Addition Theorem," and cited Jacobi's 1832 proof of it in the hyperelliptic case.

The converse of this result holds too. But Abel did not recognize it, and it lies, at best, between the lines of Riemann's paper. The converse was first explicitly stated by Clebsch in 1864, and first proved in full generality some time later by Weierstrass. In 1913, in Weyl's celebrated book on Riemann surfaces, Weyl combined the result and its converse under the heading of Abel's Theorem. Ever since then, most mathematicians have done the same, even though Weyl explained it is not historically correct to do so.

Together, the above result and its converse imply the fiber $\Psi_\mu^{-1}\Psi_\mu(x_1, \dots, x_\mu)$ is the complete linear system determined by $x_1 + \dots + x_\mu$. Its dimension is just Abel's α, the number of x_i that can vary independently in the system. Furthermore, in effect, Riemann rediscovered Abel's formula (1.3), and in 1864, Roch identified i as the number of independent adjoints vanishing on $x_1, \dots, x_\mu$. In 1874, Brill and Noether, inspired by Clebsch, gave the first algebro-geometric treatment of the formula, whose statement they named the "Riemann–Roch Theorem."

Finally, Riemann treated the inversion problem. In effect, he proved that the Abel map Ψ_p is biholomorphic on a certain saturated Zariski open subset $U \subset C^{(p)}$; namely, U is the complement of the image of $C^{(p-1)}$ in $C^{(p)}$ under the map

$$(x_2, \dots, x_p) \mapsto (x_0, x_2, \dots, x_p).$$

The inverse map $\Psi_p U \to U$ can be expressed using the coordinate functions on $C^{(p)}$, so in terms of functions on $\Psi_p U$. Since $J := \mathbb{C}^p/\mathbb{L}$, these functions can be lifted to an open subset of $\mathbb{C}^p$, and then continued to meromorphic functions on $\mathbb{C}^p$ with $2p$ periods. Riemann termed these special functions "Abelian functions."

Two years later, in 1859, Riemann proved that every meromorphic function F

in p variables has at most $2p$ independent period vectors. Those F with exactly $2p$ soon became known as "Abelian functions." They were studied by many, including Weierstrass, Frobenius, and Poincaré. In particular, in 1869, Weierstrass observed that not every F comes from a curve.

Form the set K of all "Abelian functions" whose group of periods contains a given lattice $\mathbb{L}$ in $\mathbb{C}^p$ of rank $2p$. It turns out K is a field of transcendence degree p over $\mathbb{C}$. Hence K is the field of rational functions on a p-dimensional projective algebraic variety A, which is parameterized on a Zariski open set by p of them. There are many such A, and all were called "Abelian varieties" at first. In 1919, Lefschetz proved there is a distinguished A, whose underlying set can be identified with $\mathbb{C}^p/\mathbb{L}$ in a natural way, and he restricted the term "Abelian variety" to it.

Not only do the points of an Abelian variety A form a group, but the operations of adding and of inverting are given by polynomials. Thus A is a complete algebraic group, or an "Abelian variety" in Weil's sense of 1948. Weil proved each such abstract Abelian variety is commutative. Earlier, in 1889, Picard had, in effect, proved this commutativity in the case of a surface.

Every connected projective algebraic group is parameterized globally by Abelian functions with a common lattice of periods. This fact was proved by Picard for surfaces in 1889 and, assuming the group is commutative, in any dimension in 1895. His proof was completed at certain points of analysis in 1903 by Painlevé. Thus the two definitions of Abelian variety agree, Lefschetz's and Weil's; however, Weil worked in arbitrary characteristic.

In the case of C above, its Jacobian J is thus an Abelian variety. Moreover, J is the quotient of $C^{(\mu)}$ for any $\mu \geq p$ by linear equivalence. So J and Ψ_μ are defined by integrals, but given by polynomials! And addition on J corresponds to addition of divisors. Therefore, J and Ψ_μ can be constructed algebro-geometrically just by forming the quotient. Severi attributed this construction to Castelnuovo.

In 1905, Castelnuovo generalized the construction to surfaces. To set the stage, he reviewed the case of curves, calling it very well known (notissimo). His work is a milestone in the history of irregular surfaces, which began in 1868.

In 1868, Clebsch generalized Abel's formula (1.2) to a surface $f(x, y, z) = 0$ with ordinary singularities and no point at infinity on the z-axis; in other words, $f = 0$ is a general projection of a smooth surface. Clebsch showed that every double integral of the first kind is of the form

$$\iint \frac{h(x, y, z)}{\partial f / \partial z}\, dx\, dy$$

where $\deg h = \deg f - 4$ and h vanishes when $\partial f/\partial z$ does. The number of independent integrals became known as the "geometric genus" and denoted by p_g.

In 1870, M. Noether found an algebro-geometric proof that p_g is a birational invariant, as conjectured by Clebsch. In 1871, Cayley found a formula for the expected number of independent h, later called the "arithmetic genus" and denoted by p_a. He observed that, if $f = 0$ is a ruled surface over a base curve of genus p, then $p_a = -p$, but $p_g = 0$. Later in 1871, Zeuthen used Cayley's formula to give an algebro-geometric proof that p_a too is a birational invariant.

In 1875, Noether explained the unexpected discrepancy between p_g and p_a: the vanishing conditions on h need not be independent; in any case, $p_g \geq p_a$. He noted that, if the surface $f = 0$ in 3-space is smooth or rational, then $p_g = p_a$. It was expected that equality usually holds; so when it did, $f = 0$ was termed "regular."

The difference $p_g - p_a$ gives a quantitative measure of the failure of $f = 0$ to be regular; so $p_g - p_a$ was termed its "irregularity."

In 1884, Picard was led to study simple integrals on the surface $f = 0$,

$$\int P(x, y, z)\, dx + \int Q(x, y, z)\, dy;$$

that are closed ($\partial P/\partial y = \partial Q/\partial x$); they became known as "Picard integrals." And q was used to denote the number of independent Picard integrals of the first kind. Picard noted that, if $f = 0$ is smooth, then $q = 0$. In 1894, Humbert proved that, if $q = 0$, then every algebraic system of curves is contained in a linear system.

Inspired by Humbert's result, in 1896, Castelnuovo proved that, if $p_g - p_a = 0$, then every algebraic system of curves on $f = 0$ is contained in a linear system under a certain restriction. In 1899, Enriques removed the restriction. For a modern version of this result and of its converse, which together characterize regular surfaces, see Exercises 5.16 and 5.17.

In 1897, Castelnuovo fixed a linear system of curves on the surface $f = 0$, and studied the "characteristic" linear system cut out on a general member by the other members. Let δ be the amount, termed the "deficiency," by which the dimension of the characteristic system falls short of the dimension of its complete linear system. Castelnuovo proved that

$$\delta \leq p_g - p_a,$$

and equality holds for the system cut out by the surfaces of suitably high degree.

In February 1904, Severi extended Castelnuovo's work. Severi took a complete algebraic system of curves, without repetition, on the surface $f = 0$, say with parameter space Σ of dimension R. Let $\sigma \in \Sigma$ be a general point, and C_σ the corresponding curve. Say C_σ moves in a complete linear system of dimension r. Now, to each tangent direction at $\sigma \in \Sigma$, Severi associated, in an injective fashion, a member of the complete characteristic system on C_σ. Thus he got an R-dimensional "characteristic" linear subsystem. The complete system has dimension $r+\delta$. Hence,

$$R \leq r + p_g - p_a.$$

A few months later, Enriques and, shortly afterward, Severi gave proofs that, if C_σ is sufficiently positive, then equality holds above; in other words, then the characteristic system of Σ is complete. Both proofs turned out to have serious gaps, as Severi himself observed in 1921. Meanwhile, in 1910, Poincaré gave an analytic construction of a family with $R = p_g - p_a$ and $r = 0$. It follows formally, by means of the Riemann–Roch Theorem for surfaces, that whenever C_σ is sufficiently positive, the characteristic system is complete. After Severi's criticism, it became a major open problem to find a purely algebro-geometric treatment of this issue. But see Corollary 5.5 and Remark 5.18: the solution finally came forty years later with Grothendieck's systematic use of nilpotents!

In mid January 1905, Severi proved that $p_g - p_a \geq q$ and that $p_g - p_a = b - q$ where b is the number of independent Picard integrals of the second kind, which is equal to the first Betti number. Simultaneously and independently, Picard too proved that $p_g - p_a = b - q$.

A week later and more fully that May and June, Castelnuovo took the last step in this direction. He fixed C_σ sufficiently positive, and formed the quotient, P say, of Σ modulo linear equivalence. So P is projective, and $\dim P = p_g - p_a$. Furthermore, since two sufficiently positive curves sum to a third, it follows that P is independent of the choice of C_σ, and is a commutative group variety. Hence P is an Abelian variety by the general theorem of Picard, completed by Painlevé, mentioned above. Hence P is parameterized by $\dim P$ Abelian functions. Castelnuovo proved they induce independent Picard integrals on $f = 0$. Therefore, $p_g - p_a \le q$. Thus Castelnuovo obtained the Fundamental Theorem of Irregular Surfaces:

$$\dim P = p_g - p_a = q = b/2. \tag{1.4}$$

For a modern discussion of the result, see Remark 5.15 and Exercise 5.16. In 1905, the term "Abelian variety" was not yet in use, and so, naturally enough, Castelnuovo termed P the "Picard variety" of the surface $f = 0$.

Picard also studied Picard integrals of the third kind on the surface $f = 0$. In 1901, he proved that there is a smallest integer ϱ such that any $\varrho + 1$ curves are the logarithmic curves of some such integral. On the basis this result, in 1905, Severi proved, in effect, that ϱ is the rank of the group of all curves modulo algebraic equivalence. In 1952, Néron proved the result in arbitrary characteristic. So the group is now called the "Néron–Severi group," and ϱ is called the "Picard number."

In 1908 and 1910, Severi studied, in effect, the torsion subgroup of the Néron–Severi group, notably proving it is finite. In 1957, Matsusaka proved this finiteness in arbitrary characteristic. However, there is no special name for this subgroup or for its order. For more about them and ϱ, see Corollary (6.17) and Remark (6.19).

The impetus to work in arbitrary characteristic came from developments in number theory. In 1921, E. Artin developed, in his thesis, an analogue of the Riemann Hypothesis, in effect, for a hyperelliptic curve over a prime field of odd characteristic. In 1929, F. K. Schmidt generalized Artin's work to all curves over all finite fields, and recast it in the geometric style of Dedekind and Weber. In 1882, they had viewed a curve as the set of discrete valuation rings in a finitely generated field of transcendence degree 1 over $\mathbb{C}$, and they had given an abstract algebraic treatment of the Riemann–Roch Theorem. Schmidt observed that their treatment works with little change in arbitrary characteristic, and he used the Riemann–Roch Theorem to prove that Artin's zeta function satisfies a natural functional equation.

In 1936, Hasse proved Artin's Riemann hypothesis in genus 1 using an analogue of the theory of elliptic functions. Then he and Deuring noted that to extend the proof to higher genus would require developing a theory of correspondences between curves analogous to that developed by Hurwitz and others. This work inspired Weil to study the fixed points of the Frobenius correspondence, and led to his announcement in 1940 and to his two great proofs in 1948 of Artin's Riemann hypothesis for the zeta function of an arbitrary curve and also to his proof of the integrality of his analogue of Artin's L-functions of 1923 and 1930.

First, in 1946, Weil carefully rebuilt the foundation of algebraic geometry from scratch. Following in the footsteps of E. Noether, van der Waerden, and Schmidt, Weil took a variable coefficient field of arbitrary characteristic inside a fixed algebraically closed coordinate field of infinite transcendence degree. Then he formed "abstract" varieties by patching pieces of projective varieties, and said when these varieties are "complete." Finally, he developed a calculus of cycles.

In 1948, Weil published two exciting monographs. In the first, he reproved the Riemann–Roch theorem for (smooth complete) curves, a theorem he regarded as fundamental (see [**We79**, I, p. 562, top; II, p. 541, top]). Then he developed an elementaty theory of correspondences between curves, which included Castelnuovo's theorem of 1906 on the positive definiteness of the equivalence defect of a correspondence. Of course, Castelnuovo's proof was set over $\mathbb{C}$, but "its translation into abstract terms was essentially a routine matter once the necessary techniques had been created," as Weil put it in his 1954 ICM talk. Finally, Weil derived the Riemann hypothesis.

In the second monograph, Weil established the abstract theory of Abelian varieties. He constructed the Jacobian J of a curve C of genus p by patching together copies of an open subset of the symmetric product $C^{(p)}$. Then taking a prime l different from the characteristic, he constructed, out of the points on J of order l^n for all $n \geq 1$, an l-adic representation of the ring of correspondences, equivalent to the representation on the first cohomology group of C. Finally, he proved the positive definiteness of the trace of this representation, reproved the Riemann hypothesis for the zeta function, and completed the proof of his analogue of Artin's conjectured integrality for L-functions of number fields.

Weil left open two questions: Do a curve and its Jacobian have the same coefficient field? Is every Abelian variety projective? Both questions were soon answered in the affirmative by Chow and Matsusaka. However, there has remained some general interest in constructing nonprojective varieties and in finding criteria for projectivity. Furthermore, Weil was led in 1956 to study the general question of descent of the coefficient field, and this work in turn inspired Grothendieck's general descent theory, which is outlined [**FGA**, no. 236].

In 1949, Weil published his celebrated conjectures about the zeta function of a variety of arbitrary dimension. Weil did not explicitly explain these conjectures in terms of a hypothetical cohomology theory, but such an explanation lies between the lines of his paper. Furthermore, it was credited to him explicitly in Serre's 1956 "Mexico paper" [**Sr56**, p. 24] and in Grothendieck's 1958 ICM talk.

In his talk, Grothendieck announced that he had found a new approach to developing the desired "Weil cohomology." He wrote: "it was suggested to me by the connections between sheaf-theoretic cohomology and cohomology of Galois groups on the one hand, and the classification of unramified coverings of a variety on the other ..., and by Serre's idea that a 'reasonable' algebraic principal fiber space ..., if it is not locally trivial, should become locally trivial on some covering unramified over a given point." This is the announcement of Grothendieck topology.

In 1960, Grothendieck and Dieudonné [**EGA I**, p. 6] listed the titles of the chapters they planned to write. The last one, Chapter XIII, is entitled, "Cohomologie de Weil." The next-to-last is entitled, "Schémas abéliens et schémas de Picard." Earlier, at the end of his 1958 ICM talk, Grothendieck had listed five open problems; the fifth is to construct the Picard scheme.

In 1950, Weil published a remarkable note on Abelian varieties. For each complete normal variety X of any dimension in any characteristic, he said there ought to be two associated Abelian varieties, the "Picard" variety P and the "Albanese" variety A, with certain properties, discussed just below. He explained he had complete proofs for smooth complex X, and "sketches" in general. Soon all was proved.

Weil's sketches rest on two criteria for linear equivalence, developed in 1906 by Severi and reformulated in the 1950 note by Weil. He announced proofs of them in 1952, and published the details in 1954. For some more information, see [**Za35**, p. 120] and Remark 5.8. In Weil's commentaries on his '54 paper, he wrote: "Ever since 1949, I considered the construction of an algebraic theory of the Picard variety as the task of greatest urgency in abstract algebraic geometry."

The properties are these. First, P parameterizes the linear equivalence classes of divisors on X. And there exists a map $X \to A$ that is "universal" in the sense that every map from X to an Abelian variety factors through it. In his commentaries on the note, Weil explained that P had been introduced and named by Castelnuovo; so historically speaking, it would be justified to name P after him, but it was better not to tamper with common usage. By contrast, Weil chose to name A after Albanese in order to honor his work in 1934 viewing A as a quotient of a symmetric power of X, although A had been introduced and studied in 1913 by Severi.

Second, if X is an Abelian variety, then X is equal to the Picard variety of P, and each of X and P is called the "dual" of the other. In general, A and P are dual Abelian varieties; in fact, the universal map $X \to A$ induces the canonical isomorphism from the Picard variety of A onto P. If X is a curve, then both A and P coincide with the Jacobian of X, and the universal map $X \to A$ is just the Abel map; in other words, the Jacobian is "autodual." This autoduality can be viewed as an algebro-geometric statement of Abel's theorem and its converse for integrals of the first kind. For some more information, see Remarks 5.24–5.26.

In 1951, Matsusaka gave the first algebraic construction of P. However, he had to extend the ground field because he applied Weil's results: one of the equivalence criteria, and the construction of the Jacobian. Both applications involve the "generic curve," which is the section of X by a generic linear space of complementary dimension. In 1952, Matsusaka gave a different construction; it does not require extending the ground field, but requires X to be smooth.

Both constructions are like Castelnuovo's in that they begin by constructing a complete algebraic system of sufficiently positive divisors, and then form the quotient modulo linear equivalence. To parameterize the divisors, Matsusaka used the theory of "Chow coordinates," which was developed by Chow and van der Waerden in 1938 and refined by Chow contemporaneously. In 1952, Matsusaka also used this theory to form the quotient. In the same paper, he gave the first construction of A, again using the Jacobian of the generic curve, but he did not relate A and P.

In 1954, Chow published a construction of the Jacobian similar to Matsusaka's second construction of P. Chow had announced it in 1949, and both Weil and Matsusaka had referred to it in the meantime. In 1955, Chow constructed A and P by a new procedure; he took the "image" and the "trace" of the Jacobian of a generic curve. Moreover, he showed that the universal map $X \to A$ induces an isomorphism from the Picard variety of A onto P.

In a course at the University of Chicago, 1954–55, Weil gave a more complete and elegant treatment, based on the "see-saw principle," which he adapted from Severi, and on his own Theorem of the Square and Theorem of the cube. This treatment became the core of Lang's 1959 book, "Abelian.Varieties." The idea is to construct A first using the generic curve, and then to construct P as a quotient of A modulo a finite subgroup; thus there is no need for Chow coordinates.

In 1959 and 1960, Nishi and Cartier independently established the duality between A and P in full generality.

Between 1952 and 1957, Rosenlicht published a remarkable series of papers, which grew out of his 1950 Harvard thesis. It was supervised by Zariski, who had studied Abelian functions and algebraic geometry with Castelnuovo, Enriques, and Severi in Rome from 1921 to 1927. Notably Rosenlicht generalized to a curve with arbitrary singularities the notions of linear equivalence and differentials of the first kind. Then he constructed a "generalized Jacobian" over $\mathbb{C}$ by integrating and in arbitrary characteristic by patching. It is not an Abelian variety, but an extension of the Jacobian of the normalized curve by an affine algebraic group.

Rosenlicht cited Severi's 1947 monograph, "Funzioni Quasi Abeliane," where the generalized Jacobian was discussed for the first time, but only for curves with double points. In turn, Severi traced the history of the corresponding theory of quasi-Abelian functions back to Klein, Picard, Poincaré, and Lefschetz.

In 1956, Igusa established the compatibility of specializing a curve with specializing its generalized Jacobian in arbitrary characteristic when the general curve is smooth and the special curve has at most one node. Igusa explained that, in 1952, Néron had studied the total space of such a family of Jacobians, but had not explicitly analyzed the special fiber. Igusa's approach is, in spirit, like Castelnuovo's, Chow's, and Matsusaka's before him and Grothendieck's after him. But Grothendieck went considerably further: he proved compatibility with specialization for a family of varieties of arbitrary dimension with arbitrary singularities, both in equicharacteristic and in mixed.

In 1960, Chevalley constructed a Picard variety for any normal variety X using locally principal divisorial cycles. Cartier had already focused on these cycles in his 1958 Paris thesis. But Chevalley said he would call them simply "divisors," and we follow suit, although they are now commonly called "Cartier divisors."

First, Chevalley constructed a "strict" Albanese variety; it is universal for regular maps (morphisms) into Abelian varieties. Then he took its Picard variety to be that of X. He noted his Picard and Albanese varieties need not be equal to those of a desingularization of X. By contrast, Weil's P and A are birational invariants, and his universal map $X \to A$ is a rational map, which is defined wherever X is smooth. In 1962, Seshadri generalized Chevalley's construction to a variety with arbitrary singularities, thus recovering Rosenlicht's generalized Jacobian.

Back in 1924, van der Waerden initiated the project of rebuilding the whole foundation of algebraic geometry on the basis of commutative algebra. His goal was to develop a rigorous theory of Schubert's enumerative geometry, as called for by Hilbert's fifteenth problem. Van der Waerden drew on Elimination Theory, Ideal Theory, and Field Theory as developed in the schools of Kronecker, of Dedekind, and of Hilbert. Van der Waerden originated, notably, the algebraic notion of specialization as a replacement for the topological notion of continuity.

In 1934, as Zariski wrote his book [**Za35**], he lost confidence in the clarity, precision, and completeness of the algebraic geometry of his Italian teachers. He spent a couple of years studying the algebra of E. Noether and Krull, and aimed to reduce singularities rigorously. He introduced three algebraic tools: normalization, valuation theory, and completion. He worked extensively with the rings obtained by localizing affine coordinate domains at arbitrary primes over arbitrary fields. And, in 1944, he put a topology on the set of all valuation rings in a field of algebraic functions, and used the property that any open covering has a finite subcovering.

In 1949, Weil saw that the "Zariski topology" can be put on his abstract varieties, simplifying the old exposition and suggesting the construction of new objects, such as locally trivial fiber spaces. In his paper of 1950 on Abelian varieties, he noted that line bundles correspond to linear equivalence classes of divisors, and predicted that line bundles would play a role in the theory of quasi-Abelian functions.

In 1955, Serre provided abstract algebraic geometry with a very powerful new tool: sheaf cohomology. Given a variety equipped with the Zariski topology, he assembled the local rings into the stalks of a "structure" sheaf. Then he developed a cohomology theory of coherent sheaves, analogous to the one that he and Cartan and Kodaira and Spencer had just developed in complex analytic geometry, and had so successfully applied to establish and to generalize previous work on complex algebraic varieties.

About the same time, a general theory of abstract algebraic geometry was developed by Chevalley. He did not use sheaves and cohomology, but did work with what he called "schemes," obtained by patching "affine schemes"; an affine scheme is the set of rings obtained by localizing a finitely generated domain over an arbitrary field. Nevertheless, he soon returned to a more traditional theory of "varieties" when he worked on the theory of algebraic groups.

In January 1954, Chevalley lectured on schemes at Kyoto University. His lectures inspired Nagata in [**Na56**] to generalize the theory by replacing the coefficient field by a Dedekind domain. But Nagata used Zariski's term "model," not Chevalley's term "scheme." Earlier, at the 1950 ICM, Weil had recalled Kronecker's dream of an algebraic geometry over the integers; however, Nagata did not cite Weil's talk, and likely was not motivated by it.

In the fall of 1955, Chevalley lectured on schemes over fields at the Séminaire Cartan–Chevalley, and Grothendieck was there. By February 1956 (see [**CS01**, p. 32]), he was patching the spectra of arbitrary Noetherian rings, and studying the cohomology of Cartier's "quasi-coherent" sheaves. There is good reason for the added generality: nilpotents allow better handling of higher-order infinitesimal deformations, of inseparability in positive characteristic, and of passage to the completion; quasi-coherent sheaves have the technical convenience of coherent sheaves without their cumbersome finiteness. By October 1958 (see [**CS01**, p. 63]), Grothendieck and Dieudonné had begun the gigantic program of writing EGA — rebuilding once again the foundation of algebraic geometry in order to provide a more flexible framework, more powerful methods, and a more refined theory.

Also in 1958, there appeared two other papers, which discuss objects similar to Chevalley's schemes: Kähler published a 400 page foundational monograph, which introduced general base changes, and Chow and Igusa published a four-page note, which proved the Künneth Formula for coherent sheaves. The two works are mentioned briefly in [**CS01**, p. 101], in [**EGA I**, p. 8] and in [**EGA G**, p. 6]; however, they seem to have had little or no influence on Grothendieck.

Finally, in 1961–1962, Grothendieck constructed the Hilbert scheme and the Picard scheme. The construction is a technical masterpiece, showing the tremendous power of Grothendieck's new tools. In particular, a central role is played by the theory of flatness. It was introduced by Serre in 1956 as a formal device for use in comparing algebraic functions and analytic functions. Then Grothendieck developed the theory extensively, for he recognized that flatness is the technical condition that best expresses continuity across a family.

Grothendieck [**FGA**, p. 221-1] saw Hilbert schemes as "destined to replace" Chow coordinates. However, he [**FGA**, p. 221-7] had to appeal to the theory of Chow coordinates for a key finiteness result: in projective space, the subvarieties of given degree form a bounded family. A few years later, Mumford [**Mm66**, Lects. 14, 15] gave a simple direct proof of this finiteness; his proof introduced an important new tool, now known as "Castelnuovo–Mumford" regularity. In a slightly modified form, this tool plays a central role in the proofs of the finiteness theorems for the Picard scheme, which are addressed in Chapter 6 below.

In spirit, Grothendieck's construction of the Picard scheme is like Castelnuovo's and Matsusaka's. He began with the component Σ of the Hilbert scheme determined by a sufficiently positive divisor. Then he formed the quotient; in fact, he did so twice for diversity. First, he used "quasi-sections"; second, and more elegantly, he used the Hilbert scheme of Σ.

Grothendieck's definition of the two schemes is yet a greater contribution than his construction of them. He defined them by their functors of points. These schemes are universal parameter spaces; so they receive a map from a scheme T just when T parameterizes a family of subschemes or of invertible sheaves, respectively, and this map is unique.

What is a universal family? The answer seems obvious when we use schemes. But Chow coordinates parameterize positive cycles, not subschemes. And even in the analytic theory of the Picard variety P, there was some question about the sense in which P parameterizes divisor classes. Indeed, the American Journal of Math., **74** (1952), contains three papers on P. First, Igusa constructed P, but left universality unsettled. Then Weil and Chow settled it with different arguments.

A functor of points, or "representable functor," is not an arbitrary contravariant functor from schemes to sets. It is determined locally, so is a sheaf. But it suffices to represent a sheaf locally, as the patching is determined. Thus to construct the Hilbert scheme, the first step is to check that the Hilbert functor is, in fact, a sheaf for the Zariski topology, that is, a "Zariski sheaf."

The naive Picard functor is not a Zariski sheaf. So the first step is to localize it, or form the associated sheaf. This time, the Zariski topology is not fine enough. However, a representable functor is an fpqc sheaf by a main theorem of Grothendieck's descent theory. In practice, it is enough to localize for the étale Grothendieck topology or for the fppf, and these localizations are more convenient to work with. The localizations of the Picard functor are discussed in Chapter 2.

The next step is to cover the localized Picard functor by representable Zariski open subfunctors. This step is elementary. But it is technically involved, more so than any other argument in these notes. It is carried out in the proof of the main result, Theorem 4.8. Each subfunctor is represented by a quotient of an open subscheme of the Hilbert scheme. Thus the Picard scheme is constructed.

In sum, Grothendieck's method of representable functors is like Descartes's method of coordinate axes: simple, yet powerful. Here is one hallmark of genius!

In the notes that follow, our primary aim is to develop in detail most of Grothendieck's original theory of the Picard scheme basically by filling in his sketch in [**FGA**]. Our secondary aim is to review in brief much of the rest of the theory developed by Grothendieck and by others. We review the secondary material in a series of scattered remarks. The remarks refer to each other and to the primary discussion, but the primary discussion never refers to the remarks. So the remarks may be safely ignored in a first reading.

Notably, the primary discussion does not develop Grothendieck's method of "relative representability." Indeed, the details would take us too far afield. On the other hand, were we to use the method, we could obtain certain existence theorems and finiteness theorems in greater generality by reducing to the cases that we do handle. Consequently, in Sections 4–6, a number of results just assume the Picard scheme exists, rather than assume hypotheses guaranteeing it does. However, we do discuss the method and its applications in Remark 4.18 and in other remarks.

These notes also contain many exercises, which call for working examples and constructing proofs. Unlike the remarks, these exercises are an integral part of the primary discussion, which not only is enhanced by them, but also is based in part on them. Furthermore, the exercises are designed to foster comprehension. The answers involve no new concepts or techniques. The exercises are meant to be easy; if a part seems to be hard, then some review and reflection may be in order. However, all the answers are worked out in detail in Appendix A.

These notes assume familiarity with the basic algebraic geometry developed in Chapters II and III of Hartshorne's popular textbook [**Ha83**], and assume familiarity, but to a lesser extent, with the foundational material developed in Grothendieck and Dieudonné's monumental reference books [**EGA I**] to [**EGA G**]. In addition, these notes assume familiarity with basic Grothendieck topology, descent theory, and Hilbert-scheme theory, such as that explained on pp. 129–147, 199–201, and 215–221 in Bosch, Lütkebohmert, and Raynaud's welcome survey book [**BLR**]; this material and more was introduced by Grothendieck in three Bourbaki talks, nos. 190, 212, and 221, which were reprinted in [**FGA**] and are still worth reading. Of course, when specialized results are used below, precise references are provided.

Throughout these notes, we work only with locally Noetherian schemes, just as Grothendieck did in [**FGA**]. Shortly afterward, Grothendieck promoted the elimination of this restriction, through a limiting process that reduces the general case to the Noetherian case. Ever since, it has been common to make this reduction. However, the process is elementary and straightforward. Using it here would only be distracting.

Throughout, we work with a **fixed** map of finite type

$$f\colon X \to S.$$

For convenience, when given an S-scheme T, we set

$$X_T := X \times_S T$$

and denote the projection by $f_T\colon X_T \to T$. Also, when given a T-scheme T' and given quasi-coherent sheaves $\mathcal{N}$ on T and $\mathcal{M}$ on X_T, we denote the pullback sheaves by $\mathcal{N}|T'$ or $\mathcal{N}_{T'}$ and by $\mathcal{M}|X_{T'}$ or $\mathcal{M}_{T'}$.

Given an S-scheme P, we call an S-map $T \to P$ a *T-point* of P, and we denote the set of all T-points by $P(T)$. As T varies, the sets $P(T)$ form a contravariant functor on the category of S-schemes, called the *functor of points* of P.

Section 2 introduces and compares the five common relative Picard functors, the likely candidates for the functor of points of the Picard scheme. They are simply the functor $T \mapsto \operatorname{Pic}(X_T)\big/\operatorname{Pic}(T)$ and its associated sheaves in the Zariski topology, the étale topology, the fppf topology, and the fpqc topology. Section 3 treats relative effective (Cartier) divisors on X/S and the relation of linear equivalence. We prove these divisors are parameterized by an open subscheme of the Hilbert scheme of X/S. Furthermore, we consider the subscheme parameterizing the divisors whose fibers are linearly equivalent, and prove it is of the form $\mathbf{P}(\mathcal{Q})$ where $\mathcal{Q}$ is a certain coherent sheaf on T.

Section 4 begins the study of the Picard scheme $\mathbf{Pic}_{X/S}$ itself. Notably, we prove Grothendieck's main theorem: $\mathbf{Pic}_{X/S}$ exists when X/S is projective and flat and its geometric fibers are integral. Then we work out Mumford's example showing the necessity of the integrality hypothesis. Section 5 studies $\mathbf{Pic}^0_{X/S}$, which is the union of the connected components of the identity element of the fibers of $\mathbf{Pic}^0_{X/S}$. In particular, we compute the tangent space at the identity of each fiber. It is remarkable how much we can prove *formally* about $\mathbf{Pic}^0_{X/S}$. Section 6 proves the two deeper finiteness theorems. They concern $\mathbf{Pic}^\tau_{X/S}$, which consists of the points with a multiple in $\mathbf{Pic}^0_{X/S}$, and $\mathbf{Pic}^\phi_{X/S}$, whose points represent the invertible sheaves with a given Hilbert polynomial ϕ.

Finally, there are two appendices. Appendix A contains detailed answers to all the exercises. Appendix B develops basic divisorial intersection theory, which is used freely throughout Section 6. The treatment is short, simple, and elementary.

2. The several Picard functors

Our first job is to identify a likely candidate for the functor of points of the Picard scheme. In fact, there are several reasonable such Picard functors, and each one is more likely to be representable than the preceding. In this section, they all are formally introduced and compared.

Definition 2.1. The *absolute Picard functor* Pic_X is the functor from the category of (locally Noetherian) S-schemes T to the category of abelian groups defined by the formula

$$\mathrm{Pic}_X(T) := \mathrm{Pic}(X_T).$$

The absolute Picard functor is a "prepared presheaf" in this sense: given any family of S-schemes T_i, we have

$$\mathrm{Pic}_X\big(\coprod T_i\big) = \prod \mathrm{Pic}_X(T_i).$$

Hence, given a covering family $\{T_i \to T\}$ in the Zariski topology, the étale topology, or any other Grothendieck topology on the category of S-schemes, there is no harm, when we consider the sheaf associated to Pic_X, in working simply with the single map $T' \to T$ where $T' := \coprod T_i$. And doing so lightens the notation, making for easier reading. Therefore, we do so throughout, calling $T' \to T$ simply a *covering* in the given topology.

The absolute Picard functor Pic_X is never a separated presheaf in the Zariski topology. Indeed, take an S-scheme T that carries an invertible sheaf $\mathcal{N}$ such that $f_T^*\mathcal{N}$ is nontrivial. (For example, take $T := \mathbf{P}^1_X$ and $\mathcal{N} := \mathcal{O}_T(1)$. Then the diagonal map $X \to X \times X$ induces a section g of f_T; that is, $gf_T = 1_T$. Hence $f_T^*\mathcal{N}$ is nontrivial.) Now, there exists a Zariski covering $T' \to T$ such that the pullback $\mathcal{N}|T'$ is trivial; here, T' is simply the disjoint union of the subsets in a suitable ordinary open covering of T. So the pullback $f_{T'}^*\mathcal{N}|X_{T'}$ is trivial too. Thus $\mathrm{Pic}_X(T)$ has a nonzero element whose restriction is zero in $\mathrm{Pic}_X(T')$.

According to descent theory, every representable functor is a sheaf in the Zariski topology (in fact, in the étale, fppf, and fpqc topologies). Therefore, the absolute Picard functor Pic_X is never representable. So, in the hope of obtaining a representable functor that differs as little as possible from Pic_X, we now define a sequence of five successively more promising relative functors.

Definition 2.2. The *relative Picard functor* $\mathrm{Pic}_{X/S}$ is defined by

$$\mathrm{Pic}_{X/S}(T) := \mathrm{Pic}(X_T)\big/\,\mathrm{Pic}(T).$$

Denote its associated sheaves in the Zariski, étale, fppf, and fpqc topologies by

$$\mathrm{Pic}_{(X/S)\,(\mathrm{zar})},\ \mathrm{Pic}_{(X/S)\,(\mathrm{\acute{e}t})},\ \mathrm{Pic}_{(X/S)\,(\mathrm{fppf})},\ \mathrm{Pic}_{(X/S)\,(\mathrm{fpqc})}\,.$$

We now have a sequence of six Picard functors, and each one maps naturally into the next. So each one maps naturally into any of its successors. If the latter functor is one of the four displayed, then it is a sheaf in the indicated topology; in fact, it is the sheaf associated to any one of its predecessors, and the map between them is the natural map from a presheaf to its associated sheaf, as is easy to check. In particular, the four displayed sheaves are the sheaves associated to the absolute Picard functor Pic_X, as well as to the relative Picard functor $\mathrm{Pic}_{X/S}$.

Since $\mathrm{Pic}_{X/S}$ is not a priori a sheaf, it is remarkable that it is representable so often in practice.

Note that, for every S-scheme T, each T-point of $\mathrm{Pic}_{(X/S)\,(\mathrm{fpqc})}$, or element of $\mathrm{Pic}_{(X/S)\,(\mathrm{fpqc})}(T)$, is represented by an invertible sheaf $\mathcal{L}'$ on $X_{T'}$ for some fpqc-covering $T' \to T$. (In Vistoli's form of the definition of fpqc-covering, $T' \to T$ is faithfully flat, and every quasi-compact open subset of T is the image of one of T'.) Moreover, there must be an fpqc-covering $T'' \to T' \times_T T'$ such that the two pullbacks of $\mathcal{L}'$ to $X_{T''}$ are isomorphic (or to put it informally, the restrictions of $\mathcal{L}'$ must agree on a covering of the overlaps).

Furthermore, a second such sheaf $\mathcal{L}_1$ on X_{T_1} represents the same T-point if and only if there is an fpqc-covering $T_1' \to T_1 \times_T T'$ such that the pullbacks of $\mathcal{L}'$ and $\mathcal{L}_1$ to $X_{T_1'}$ are isomorphic. (Technically, this condition includes the preceding one, which concerns the case where $T_1 = T'$ and $\mathcal{L}_1 = \mathcal{L}'$ since $\mathcal{L}'$ must represent the same T-point as itself.) Of course, similar considerations apply to the Picard functors for the Zariski, étale, and fppf topologies as well.

Exercise 2.3. Given an S-scheme T of the form $T = \mathrm{Spec}(A)$ where A is a local ring, show that the natural maps are isomorphisms

$$\mathrm{Pic}_X(A) \xrightarrow{\sim} \mathrm{Pic}_{X/S}(A) \xrightarrow{\sim} \mathrm{Pic}_{(X/S)\,(\mathrm{zar})}(A)$$

where $\mathrm{Pic}_X(A) := \mathrm{Pic}_X(T)$, where $\mathrm{Pic}_{X/S}(A) := \mathrm{Pic}_{X/S}(T)$, and so forth.

Assume A is Artin local with algebraically closed residue field. Show

$$\mathrm{Pic}_X(A) \xrightarrow{\sim} \mathrm{Pic}_{(X/S)\,(\mathrm{\acute{e}t})}(A).$$

Assume A is an algebraically closed field k. Show

$$\mathrm{Pic}_X(k) \xrightarrow{\sim} \mathrm{Pic}_{(X/S)\,(\mathrm{fppf})}(k).$$

Exercise 2.4. Show that the natural map

$$\mathrm{Pic}_{(X/S)\,(\mathrm{zar})} \to \mathrm{Pic}_{(X/S)\,(\mathrm{\acute{e}t})}$$

need not be an isomorphism. Specifically, take X to be the following curve in the real projective plane:

$$X : u^2 + v^2 + 1 = 0 \text{ in } \mathbf{P}^2_{\mathbb{R}}.$$

Then X has no $\mathbb{R}$-points, but over the complex numbers $\mathbb{C}$, there is an isomorphism

$$\varphi\colon X_{\mathbb{C}} \xrightarrow{\sim} \mathbf{P}^1_{\mathbb{C}}.$$

Show that $\varphi^*\mathcal{O}(1)$ defines an element of $\mathrm{Pic}_{(X/\mathbb{R})\,(\mathrm{\acute{e}t})}(\mathbb{R})$, which is not in the image of $\mathrm{Pic}_{(X/\mathbb{R})\,(\mathrm{zar})}(\mathbb{R})$.

The main result of this section is the following comparison theorem. It identifies two useful conditions: the first guarantees that the first four relative functors can be viewed as subfunctors of the fifth; together, the two conditions guarantee that all five functors coincide. The second condition has three successively weaker forms. Before we can prove the theorem, we must develop some theory.

THEOREM 2.5 (Comparison). *Assume $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ holds universally (that is, for any S-scheme T, the comorphism of f_T is an isomorphism, $\mathcal{O}_T \xrightarrow{\sim} f_{T*}\mathcal{O}_{X_T}$).*

1. *Then the natural maps are injections:*

$$\mathrm{Pic}_{X/S} \hookrightarrow \mathrm{Pic}_{(X/S)\,(\mathrm{zar})} \hookrightarrow \mathrm{Pic}_{(X/S)\,(\text{ét})} \hookrightarrow \mathrm{Pic}_{(X/S)\,(\mathrm{fppf})} \hookrightarrow \mathrm{Pic}_{(X/S)\,(\mathrm{fpqc})}\,.$$

2. *All four maps are isomorphisms if also f has a section; the latter three are isomorphisms if also f has a section locally in the Zariski topology; etc.*

EXERCISE 2.6. Assume $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ holds universally. Using Theorem 2.5, show that its five functors have the same geometric points; in other words, for every algebraically closed field k containing the residue class field of a point of S, the k-points of all five functors are, in a natural way, the same. Show, in fact, that these k-points are just the elements of $\mathrm{Pic}(X_k)$.

What if $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ does not necessarily hold universally?

LEMMA 2.7. *Assume $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$. Then the functor $\mathcal{N} \mapsto f^*\mathcal{N}$ is fully-faithful from the category $\mathcal{C}$ of locally free sheaves of finite rank on S to that on X. The essential image is formed by the sheaves $\mathcal{M}$ on X such that* (i) *the image $f_*\mathcal{M}$ is in $\mathcal{C}$ and* (ii) *the natural map $f^*f_*\mathcal{M} \to \mathcal{M}$ is an isomorphism.*

PROOF. (Compare with [**EGA IV**$_4$, 21.13.2].) For any $\mathcal{N}$ in $\mathcal{C}$, there is a string of three natural isomorphisms

$$\mathcal{N} \xrightarrow{\sim} \mathcal{N} \otimes f_*O_X \xrightarrow{\sim} \mathcal{N} \otimes f_*f^*\mathcal{O}_S \xrightarrow{\sim} f_*f^*\mathcal{N}. \tag{2.7.1}$$

The first isomorphism arises by tensor product with the comorphism of f; this comorphism is an isomorphism by hypothesis. The second isomorphism arises from the identification $\mathcal{O}_X = f^*\mathcal{O}_S$. The third arises from the projection formula.

For any $\mathcal{N}'$ in $\mathcal{C}$, also $\mathcal{H}om(\mathcal{N},\, \mathcal{N}')$ is in $\mathcal{C}$. Hence, (2.7.1) yields an isomorphism

$$\mathcal{H}om(\mathcal{N},\, \mathcal{N}') \xrightarrow{\sim} f_*f^*\, \mathcal{H}om(\mathcal{N},\, \mathcal{N}').$$

Now, since $\mathcal{N}$ and $\mathcal{N}'$ are locally free of finite rank, the natural map

$$f^*\, \mathcal{H}om(\mathcal{N},\, \mathcal{N}') \to \mathcal{H}om(f^*\mathcal{N},\, f^*\mathcal{N}')$$

is an isomorphism locally, so globally. Hence there is an isomorphism of groups

$$\mathrm{Hom}(\mathcal{N},\, \mathcal{N}') \xrightarrow{\sim} \mathrm{Hom}(f^*\mathcal{N},\, f^*\mathcal{N}').$$

In other words, $\mathcal{N} \mapsto f^*\mathcal{N}$ is fully-faithful.

Finally, the essential image consists of the sheaves $\mathcal{M}$ that are isomorphic to those of the form $f^*\mathcal{N}$ for some $\mathcal{N}$ in $\mathcal{C}$. Given such an $\mathcal{M}$ and $\mathcal{N}$, there is an isomorphism $f_*\mathcal{M} \simeq \mathcal{N}$ owing to (2.7.1). Hence $f_*\mathcal{M}$ is in $\mathcal{C}$, and $f^*f_*\mathcal{M} \to \mathcal{M}$ is an isomorphism locally, so globally; thus (i) and (ii) hold. Conversely, if (i) and (ii) hold, then $\mathcal{M}$ is, by definition, in the essential image. □

Proof of Part 1 of Theorem 2.5. Given $\lambda \in \mathrm{Pic}_{X/S}(T)$, represent λ by an invertible sheaf $\mathcal{L}$ on X_T. Suppose λ maps to 0 in $\mathrm{Pic}_{(X/S)\,(\mathrm{fpqc})}(T)$. Then there exist an fpqc covering $p\colon T' \to T$ and an isomorphism $p_X^*\mathcal{L} \simeq f_{T'}^*\mathcal{N}'$ for some invertible sheaf $\mathcal{N}'$ on T'. Hence Lemma 2.7 implies that $f_{T'*}p_X^*\mathcal{L} \simeq \mathcal{N}'$. Now, p is flat, so $p^*f_{T*}\mathcal{L} \xrightarrow{\sim} f_{T'*}p_X^*\mathcal{L}$. So $p^*f_{T*}\mathcal{L} \simeq \mathcal{N}'$. Hence $f_{T*}\mathcal{L}$ is invertible and the natural map $f_T^*f_{T*}\mathcal{L} \to \mathcal{L}$ is an isomorphism, as both statements hold after pullback via p, which is faithfully flat. Therefore, $\lambda = 0$. Thus $\mathrm{Pic}_{X/S} \hookrightarrow \mathrm{Pic}_{(X/S)\,(\mathrm{fpqc})}$.

The rest is formal. Indeed, form the associated sheaves in the Zariski topology. This operation is exact by general (Grothendieck) topology, and $\mathrm{Pic}_{(X/S)\,(\mathrm{fpqc})}$ remains the same, as it is already a Zariski sheaf. Thus $\mathrm{Pic}_{(X/S)\,(\mathrm{zar})} \hookrightarrow \mathrm{Pic}_{(X/S)\,(\mathrm{fpqc})}$. Similarly, $\mathrm{Pic}_{(X/S)\,(\text{ét})} \hookrightarrow \mathrm{Pic}_{(X/S)\,(\mathrm{fpqc})}$ and $\mathrm{Pic}_{(X/S)\,(\mathrm{fppf})} \hookrightarrow \mathrm{Pic}_{(X/S)\,(\mathrm{fpqc})}$.

Alternatively, we can avoid the use of Lemma 2.7 by starting from the fact that $\mathrm{Pic}_{(X/S)\,(\mathrm{fpqc})}$ is the sheaf associated to Pic_X, rather than to $\mathrm{Pic}_{X/S}$. This way, we may assume $\mathcal{N}' = \mathcal{O}_{T'}$. Then $f_{T'*}p_X^*\mathcal{L} \simeq \mathcal{N}'$ because $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ holds universally. We now proceed just as before. □

Definition 2.8. Assume f has a section g (so $fg = 1$). Let T be an S-scheme, and $\mathcal{L}$ a sheaf on X_T. Then a *g-rigidification* of $\mathcal{L}$ is the choice of an isomorphism $u\colon \mathcal{O}_T \xrightarrow{\sim} g_T^*\mathcal{L}$, assuming one exists.

Lemma 2.9. *Assume f has a section g, and let T be an S-scheme. Form the group of isomorphism classes of pairs $(\mathcal{L}, u)$ where $\mathcal{L}$ is an invertible sheaf on X_T and u is a g-rigidification of $\mathcal{L}$. Then this group is carried isomorphically onto* $\mathrm{Pic}_{X/S}(T)$ *by the homomorphism ρ defined by $\rho(\mathcal{L}, u) := \mathcal{L}$.*

Proof. Given λ in $\mathrm{Pic}_{X/S}(T)$, represent λ by an invertible sheaf $\mathcal{M}$ on X_T. Set $\mathcal{L} := \mathcal{M} \otimes (f_T^*g_T^*\mathcal{M})^{-1}$. Then $\mathcal{L}$ too represents λ. Also $g_T^*\mathcal{L} = g_T^*\mathcal{M} \otimes g_T^*f_T^*g_T^*\mathcal{M}^{-1}$. Now, $g_T^*f_T^* = 1$ as $fg = 1$. Hence the natural isomorphism $g_T^*\mathcal{M} \otimes (g_T^*\mathcal{M})^{-1} \xrightarrow{\sim} O_T$ induces a g-rigidification of $\mathcal{L}$. Thus ρ is surjective.

To prove ρ is injective, let $(\mathcal{L}, u)$ represent an element of its kernel. Then there exist an invertible sheaf $\mathcal{N}$ on T and an isomorphism $v\colon \mathcal{L} \xrightarrow{\sim} f_T^*\mathcal{N}$. Set $w := g_T^*v \circ u$, so $w\colon \mathcal{O}_T \xrightarrow{\sim} g_T^*\mathcal{L} \xrightarrow{\sim} \mathcal{N}$. Now, a map of pairs is just a map w' of the first components such that g_T^*w' is compatible with the two g-rigidifications. So $v\colon (\mathcal{L}, u) \xrightarrow{\sim} (f_T^*\mathcal{N}, w)$ and $f_T^*w\colon (\mathcal{O}_{X_T}, 1) \xrightarrow{\sim} (f_T^*\mathcal{N}, w)$. Thus ρ is injective. □

Lemma 2.10. *Assume f has a section g, and assume $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ holds universally. Let T be an S-scheme, $\mathcal{L}$ an invertible sheaf on X_T, and u a g-rigidification of $\mathcal{L}$. Then every automorphism of the pair $(\mathcal{L}, u)$ is trivial.*

Proof. An automorphism of $(\mathcal{L}, u)$ is just an automorphism $v\colon \mathcal{L} \xrightarrow{\sim} \mathcal{L}$ such that $g_T^*v \circ u\colon \mathcal{O}_T \xrightarrow{\sim} g_T^*\mathcal{L} \xrightarrow{\sim} g_T^*\mathcal{L}$ is equal to u. But then $g_T^*v = 1$. Now,

$$v \in \mathrm{Hom}(\mathcal{L}, \mathcal{L}) = \mathrm{H}^0(\mathcal{H}om(\mathcal{L}, \mathcal{L})) = \mathrm{H}^0(\mathcal{O}_{X_T}) = \mathrm{H}^0(\mathcal{O}_T);$$

the middle equation holds since the natural map $O_{X_T} \to \mathcal{H}om(\mathcal{L}, \mathcal{L})$ is locally an isomorphism, so globally one, and the last equation holds since $\mathcal{O}_T \xrightarrow{\sim} f_*\mathcal{O}_{X_T}$. But $g_T^*v = 1$. Therefore, $v = 1$. □

Proof of Part 2 of Theorem 2.5. Suppose f has a section g. Owing to Part 1, it suffices to prove that every $\lambda \in \mathrm{Pic}_{(X/S)\,(\mathrm{fpqc})}(T)$ lies in $\mathrm{Pic}_{X/S}(T)$. Represent λ by a $\lambda' \in \mathrm{Pic}_{X/S}(T')$ where $T' \to T$ is an fpqc covering. Then there is an fpqc covering $T'' \to T' \times_T T'$ such that the two pullbacks of λ' to $X_{T''}$ are equal. We may assume $T'' \xrightarrow{\sim} T' \times T'$ because $\mathrm{Pic}_{X/S}$ is separated for the fpqc topology, again owing to Part 1.

Owing to Lemma 2.9, we may represent λ' by a pair $(\mathcal{L}', u')$ where $\mathcal{L}'$ is an invertible sheaf on $X_{T'}$ and u' is a g-rigidification of $\mathcal{L}'$. Furthermore, on $X_{T'\times T'}$, there is an isomorphism v' from the pullback of $(\mathcal{L}', u')$ via the first projection onto the pullback via the second.

Consider the three projections $X_{T'\times T'\times T'} \to X_{T'\times T'}$. Let v'_{ij} denote the pullback of v' via the projection to the ith and jth factors. Then $v'^{-1}_{13}v'_{23}v'_{12}$ is an automorphism of the pullback of $(\mathcal{L}', u')$ via the first projection $X_{T'\times T'\times T'} \to X_{T'}$. So, owing to Lemma 2.10, this automorphism is trivial. Hence $(\mathcal{L}', u')$ descends to a pair $(\mathcal{L}, u)$ on X_T. Therefore, λ lies in $\mathrm{Pic}_{X/S}(T)$.

The rest is formal. Indeed, suppose that there is a Zariski covering $T' \to T$ such that $f_{T'}$ has a section. Then, by the above, $\mathrm{Pic}_{(X/S)\,(\mathrm{zar})}|T' \xrightarrow{\sim} \mathrm{Pic}_{(X/S)\,(\mathrm{fpqc})}|T'$. Hence, by general (Grothendieck) topology, $\mathrm{Pic}_{(X/S)\,(\mathrm{zar})} \xrightarrow{\sim} \mathrm{Pic}_{(X/S)\,(\mathrm{fpqc})}$ since both source and target are sheaves in the Zariski topology. (A map of sheaves is an isomorphism if it is so locally.) Etc. □

Remark 2.11. There is another way to prove Theorem 2.5. This way is more sophisticated, and yields more information, which we won't need. Here is the idea.

Recall [**Ha83**, Ex. III, 4.5, p. 224] that, for any ringed space R, there is a natural isomorphism

$$\mathrm{Pic}(R) = \mathrm{H}^1(R, \mathcal{O}_R^*). \tag{2.11.1}$$

Now, given any S-scheme T, form the presheaf $T' \mapsto \mathrm{H}^1(X_{T'}, \mathcal{O}^*_{X_{T'}})$ on T. Its associated sheaf is [**Ha83**, Prp. III, 8.1, p. 250] simply $\mathrm{R}^1 f_{T*}\mathcal{O}^*_{X_T}$. Therefore,

$$\mathrm{Pic}_{(X/S)\,(\mathrm{zar})}(T) = \mathrm{H}^0(T, \mathrm{R}^1 f_{T*}\mathcal{O}^*_{X_T}). \tag{2.11.2}$$

Consider the Leray spectral sequence [**Gd58**, Thm. II, 4.17.1, p. 201]

$$\mathrm{E}_2^{pq} := \mathrm{H}^p(T, \mathrm{R}^q f_{T*}\mathcal{O}^*_{X_T}) \Longrightarrow \mathrm{H}^{p+q}(X_T, \mathcal{O}^*_{X_T}),$$

and form its exact sequence of terms of low degree [**Gd58**, Thm. I, 4.5.1, p. 82]

$$\begin{aligned} 0 \to \mathrm{H}^1(T, f_{T*}\mathcal{O}^*_{X_T}) \to \mathrm{H}^1(X_T, \mathcal{O}^*_{X_T}) \to \mathrm{H}^0(T, \mathrm{R}^1 f_{T*}\mathcal{O}^*_{X_T}) \\ \to \mathrm{H}^2(T, f_{T*}\mathcal{O}^*_{X_T}) \to \mathrm{H}^2(X_T, \mathcal{O}^*_{X_T}). \end{aligned} \tag{2.11.3}$$

If $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ holds universally, then $\mathrm{H}^1(T, \mathcal{O}_T^*) \xrightarrow{\sim} \mathrm{H}^1(T, f_{T*}\mathcal{O}^*_{X_T})$. Hence the beginning of (2.11.3) becomes

$$0 \to \mathrm{Pic}(T) \to \mathrm{Pic}(X_T) \to \mathrm{Pic}_{(X/S)\,(\mathrm{zar})}(T).$$

Thus $\mathrm{Pic}_{X/S} \hookrightarrow \mathrm{Pic}_{(X/S)\,(\mathrm{zar})}$.

If also f has a section g, then g induces, for each p, a left inverse of the map $\mathrm{H}^p(T, f_{T*}\mathcal{O}^*_{X_T}) \to \mathrm{H}^p(X_T, \mathcal{O}^*_{X_T})$ induced by f. So the latter is injective. Hence, $\mathrm{H}^1(X_T, \mathcal{O}^*_{X_T}) \to \mathrm{H}^0(T, \mathrm{R}^1 f_{T*}\mathcal{O}^*_{X_T})$ is surjective. Thus $\mathrm{Pic}_{X/S} \xrightarrow{\sim} \mathrm{Pic}_{(X/S)\,(\mathrm{zar})}$.

The preceding argument works for the étale, fppf, and fpqc topologies too with little change. First of all, consider the functor

$$\mathbb{G}_{\mathrm{m}}(T) := \mathrm{H}^0(T, \mathcal{O}_T^*).$$

Let u be an indeterminate. Then $\mathbb{G}_{\mathrm{m}}(T)$ is representable by the S-scheme

$$\mathbb{G}_{\mathrm{m}} := \mathrm{Spec}(\mathcal{O}_S[u, u^{-1}]).$$

Indeed, giving an S-map $T \to \mathbb{G}_{\mathrm{m}}$ is the same as giving an $\mathrm{H}^0(S, \mathcal{O}_S)$-homomorphism from $\mathrm{H}^0(S, \mathcal{O}_S)[u, u^{-1}]$ to $\mathrm{H}^0(T, \mathcal{O}_T)$, and giving such a homomorphism is the same as assigning to u a unit in $\mathrm{H}^0(T, \mathcal{O}_T)$. Now, since $\mathbb{G}_\mathrm{m}(T)$ is representable, it is a sheaf for the étale, fppf, and fpqc topologies.

Grothendieck's generalization [**FGA**, p. 190-16] of Hilbert's Theorem 90 asserts the formula

$$\mathrm{Pic}(T) = \mathrm{H}^1(T, \mathbb{G}_\mathrm{m})$$

where the H^1 can be computed in either the étale, fppf, or fpqc topology. The proof is simple, and similar to the proof of (2.11.1). The H^1 can be computed as a Čech group. And, essentially by definition, for a covering $T' \to T$, a Čech cocycle with values in $\mathbb{G}_\mathrm{m}$ amounts to descent data on $\mathcal{O}_{T'}$. The data is effective by descent theory, and the resulting sheaf on T is invertible since the covering is faithfully flat.

In the present context, the exact sequence (2.11.3) becomes

$$0 \to \mathrm{H}^1(T, f_{T*}\mathbb{G}_\mathrm{m}) \to \mathrm{H}^1(X_T, \mathbb{G}_\mathrm{m}) \to \mathrm{H}^0(T, \mathrm{R}^1 f_{T*}\mathbb{G}_\mathrm{m}) \\ \to \mathrm{H}^2(T, f_{T*}\mathbb{G}_\mathrm{m}) \to \mathrm{H}^2(X_T, \mathbb{G}_\mathrm{m}). \quad (2.11.4)$$

Furthermore, the proof of (2.11.2) yields, for example in the fpqc topology,

$$\mathrm{Pic}_{(X/S)\,(\mathrm{fpqc})}(T) = \mathrm{H}^0(T, \mathrm{R}^1 f_{T*}\mathbb{G}_\mathrm{m}).$$

If $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ holds universally, then it follows from the definitions that $f_{T*}\mathbb{G}_\mathrm{m} = \mathbb{G}_\mathrm{m}$. Hence the beginning of (2.11.4) becomes

$$0 \to \mathrm{Pic}(T) \to \mathrm{Pic}(X_T) \to \mathrm{Pic}_{(X/S)\,(\mathrm{fpqc})}(T).$$

Thus $\mathrm{Pic}_{X/S} \hookrightarrow \mathrm{Pic}_{(X/S)\,(\mathrm{fpqc})}$. And $\mathrm{Pic}_{X/S} \xrightarrow{\sim} \mathrm{Pic}_{(X/S)\,(\mathrm{fpqc})}$ if also f has a section. As before, the rest of Theorem 2.5 follows formally.

The étale group $\mathrm{H}^2(T, \mathbb{G}_\mathrm{m})$ was studied extensively by Grothendieck [**Dix**, pp. 46–188]. He showed that it gives one of two significant generalizations of the Brauer group of central simple algebras over a field. The other is the group of Azumaya algebras on T. He denoted the latter by $\mathrm{Br}(T)$ and the former by $\mathrm{Br}'(T)$. Hence [**Dix**, pp. 127–128], if $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ holds universally, then (2.11.4) becomes

$$0 \to \mathrm{Pic}(T) \to \mathrm{Pic}(X_T) \to \mathrm{Pic}_{(X/S)\,(\text{ét})}(T) \to \mathrm{Br}'(T) \to \mathrm{Br}'(X_T);$$

in particular, the obstruction to representing an element of $\mathrm{Pic}_{(X/S)\,(\text{ét})}(T)$ by an invertible sheaf on X_T is given by an element of $\mathrm{Br}'(T)$, which maps to 0 in $\mathrm{Br}'(X_T)$.

Using the smoothness of $\mathbb{G}_\mathrm{m}$ as an S-scheme, Grothendieck [**Dix**, p. 180] proved the natural homomorphisms are isomorphisms from the étale groups $\mathrm{H}^p(T, \mathbb{G}_\mathrm{m})$ to the corresponding fppf groups. Hence, if $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ holds universally, then it follows from (2.11.4) via the Five Lemma that, whether f has a section locally in the étale topology or not,

$$\mathrm{Pic}_{(X/S)\,(\text{ét})} \xrightarrow{\sim} \mathrm{Pic}_{(X/S)\,(\mathrm{fppf})}.$$

This isomorphism also holds if f is proper, according to [**BLR**, p. 203]. Namely, in (2.11.4), the second term and the last term can be computed in either the étale topology or the fppf topology. Indeed, this assertion can be reduced to the case where $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ holds universally by means of the Stein factorization.

In practice, $f\colon X \to S$ is faithfully flat. Then f is an fppf covering. So f has a section locally in the fppf topology, namely, the diagonal map $X \to X \times X$. Hence, if also $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ holds universally, then $\mathrm{Pic}_{(X/S)\,(\mathrm{fppf})} \xrightarrow{\sim} \mathrm{Pic}_{(X/S)\,(\mathrm{fpqc})}$ by Part 2 of Theorem 2.5. Thus, of the five relative Picard functors, two are of particular interest: $\mathrm{Pic}_{X/S}$ if f has a (global) section, and $\mathrm{Pic}_{(X/S)\,(\text{ét})}$ if not.

Nevertheless, when a discussion is set in the greatest possible generality, it is common to work with $\mathrm{Pic}_{(X/S)\,(\mathrm{fppf})}$ and call it *the* Picard functor.

3. Relative effective divisors

Grothendieck constructed the Picard scheme by taking a suitable family of effective divisors and forming the quotient modulo linear equivalence. This section develops the basic theory of these notions.

3.1 (Effective divisors). A closed subscheme $D \subset X$ is called an *effective (Cartier) divisor* if its ideal $\mathcal{I}$ is invertible. Given an $\mathcal{O}_X$-module $\mathcal{F}$ and $n \in \mathbb{Z}$, set

$$\mathcal{F}(nD) := \mathcal{F} \otimes \mathcal{I}^{\otimes -n}.$$

In particular, $\mathcal{O}_X(-D) = \mathcal{I}$. So the inclusion $\mathcal{I} \hookrightarrow \mathcal{O}_X$ yields, via tensor product with $\mathcal{O}_X(D)$, an injection $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(D)$, which, in turn, corresponds to a global section of $\mathcal{O}_X(D)$. This section is not arbitrary since it corresponds to an injection. Sections corresponding to injections are termed *regular*.

Conversely, given an arbitrary invertible sheaf $\mathcal{L}$ on X, let $\mathrm{H}^0(X, \mathcal{L})_{\mathrm{reg}}$ denote the subset of $\mathrm{H}^0(X, \mathcal{L})$ consisting of the regular sections, those corresponding to injections $\mathcal{L}^{-1} \hookrightarrow \mathcal{O}_X$. And let $|\mathcal{L}|$ denote the set of effective divisors D such that $\mathcal{O}_X(D)$ is, in some way, isomorphic to $\mathcal{L}$. For historical reasons, $|\mathcal{L}|$ is called the *complete linear system* associated to $\mathcal{L}$ (but $|\mathcal{L}|$ needn't be a $\mathbf{P}^n$ if X isn't integral).

Exercise 3.2. Under the conditions of (3.1), establish a canonical isomorphism

$$\mathrm{H}^0(X, \mathcal{L})_{\mathrm{reg}} / \mathrm{H}^0(X, \mathcal{O}_X^*) \xrightarrow{\sim} |\mathcal{L}|.$$

Definition 3.3. A *relative effective divisor* on X/S is an effective divisor $D \subset X$ that is S-flat.

Lemma 3.4. *Let $D \subset X$ be a closed subscheme, $x \in D$ a point, and $s \in S$ its image. Then the following statements are equivalent:*

(i) *The subscheme $D \subset X$ is a relative effective divisor at x (that is, in a neighborhood of x).*
(ii) *The schemes X and D are S-flat at x, and the fiber D_s is an effective divisor on X_s at x.*
(iii) *The scheme X is S-flat at x, and the subscheme $D \subset X$ is cut out at x by one element that is regular (a nonzerodivisor) on the fiber X_s.*

Proof. For convenience, set $A := \mathcal{O}_{S,s}$ and denote its residue field by k. In addition, set $B := \mathcal{O}_{X,x}$ and $C := \mathcal{O}_{D,x}$. Then $B \otimes_A k = \mathcal{O}_{X_s,x}$.

Assume (i), and let's prove (ii). By hypothesis, D is an effective divisor at x. So there is a regular element $b \in B$ that generates the ideal of D. Multiplication by b defines a short exact sequence

$$0 \to B \to B \to C \to 0.$$

In turn, this sequence induces the following exact sequence:

$$\mathrm{Tor}_1^A(B, k) \to \mathrm{Tor}_1^A(B, k) \to \mathrm{Tor}_1^A(C, k) \to B \otimes k \to B \otimes k.$$

By hypothesis, D is S-flat at x; hence, $\mathrm{Tor}_1^A(C, k) = 0$. So $B \otimes k \to B \otimes k$ is injective. Its image is the ideal of D_s. Thus D_s is an effective divisor.

Since $\mathrm{Tor}_1^A(C, k) = 0$, the map $\mathrm{Tor}_1^A(B, k) \to \mathrm{Tor}_1^A(B, k)$ is surjective. This map is given by multiplication by b, and b lies in the maximal ideal of B. Also, $\mathrm{Tor}_1^A(B, k)$ is a finitely generated B-module. Hence, $\mathrm{Tor}_1^A(B, k) = 0$ by Nakayama's lemma. Therefore, by the local criterion of flatness [**SGA 1**, Thm. 5.6, p. 98] or [**OB72**, Thm. 6.1, p. 73], B is A-flat; in other words, X is S-flat at x. Thus (ii) holds.

Assume (ii). To prove (iii), denote the ideal of D in B by I, and that of D_s in $B \otimes k$ by I'. Take an element $b \in I$ whose image b' in $B \otimes k$ generates I'; such a b exists because D_s is an effective divisor at x by hypothesis. For the same reason, b' is regular. It remains to prove b generates I.

Consider the short exact sequence

$$0 \to I \to B \to C \to 0.$$

By hypothesis, C is A-flat. Hence the map $I \otimes k \to B \otimes k$ is injective. Its image is I', which is generated by b'. So the image of b in $I \otimes k$ generates it. Hence, by Nakayama's lemma, b generates I. Thus (iii) holds.

Assume (iii). To prove (i), again denote the ideal of D in B by I. By hypothesis, I is generated by an element b whose image b' in $B \otimes k$ is regular. We have to prove b is regular and C is A-flat.

The exact sequence $0 \to I \to B \to C \to 0$ yields this one:

$$\mathrm{Tor}_1^A(B, k) \to \mathrm{Tor}_1^A(C, k) \to I \otimes k \to B \otimes k. \tag{3.4.1}$$

The last map is injective for the following reason. Since $I = Bb$, multiplication by b induces a surjection $B \to I$, so a surjection $B \otimes k \to I \otimes k$. Consider the composition

$$B \otimes k \to I \otimes k \to B \otimes k.$$

It is given by multiplication by b', so is injective because b' is regular. Hence $B \otimes k \xrightarrow{\sim} I \otimes k$. Therefore, $I \otimes k \to B \otimes k$ is injective.

By hypothesis, B is A-flat. So $\mathrm{Tor}_1^A(B, k) = 0$. Hence the exactness of (3.4.1) implies $\mathrm{Tor}_1^A(C, k) = 0$. Therefore, by the local criterion, C is A-flat.

Since B and C are A-flat and $0 \to I \to B \to C \to 0$ is exact, also I is A-flat.

Define K by the exact sequence $0 \to K \to B \to I \to 0$. Then the sequence

$$0 \to K \otimes k \to B \otimes k \to I \otimes k \to 0$$

is exact since I is A-flat. But $B \otimes k \xrightarrow{\sim} I \otimes k$. Hence $K \otimes k = 0$. Therefore, $K = 0$ by Nakayama's lemma. But K is the kernel of multiplication by b on B. So b is regular. Thus (i) holds. □

Exercise 3.5. Let D and E be relative effective divisors on X/S, and $D + E$ their sum, or union. Show $D + E$ is a relative effective divisor too.

Definition 3.6. Define a functor $\mathrm{Div}_{X/S}$ by the formula

$$\mathrm{Div}_{X/S}(T) := \{\, \text{relative effective divisors } D \text{ on } X_T/T \,\}.$$

Note $\mathrm{Div}_{X/S}$ is indeed a functor. Namely, given a relative effective divisor D on X_T/T and an arbitrary S-map $p\colon T' \to T$, we have to see the T'-flat closed subscheme $D_{T'} \subset X_{T'}$ is an effective divisor. So let $\mathcal{I}$ denote the ideal of D. Since D is T-flat, $p_{X_T}^*\mathcal{I}$ is equal to the ideal of $D_{T'}$. But, since $\mathcal{I}$ is invertible, so is $p_{X_T}^*\mathcal{I}$. Thus $D_{T'}$ is a (relative) effective divisor.

Theorem 3.7. *Assume X/S is projective and flat. Then $\mathrm{Div}_{X/S}$ is representable by an open subscheme $\mathbf{Div}_{X/S}$ of the Hilbert scheme $\mathbf{Hilb}_{X/S}$.*

Proof. Set $H := \mathbf{Hilb}_{X/S}$, and let $W \subset X \times H$ be the universal (closed) subscheme, and $q\colon W \to H$ the projection. Let V denote the set of points $w \in W$ at which W is an effective divisor. Plainly V is open in W. Set $Z := q(W - V)$. Then Z is closed because q is proper. Set $U := H - Z$. Then U is open, and $q^{-1}U$ is an effective divisor in $X \times U$. In fact, since q is flat, $q^{-1}U$ is a relative effective divisor in $X \times U/U$.

It remains to show that U represents $\mathrm{Div}_{X/S}$. So let T be an S-scheme, and $D \subset X_T/T$ a relative effective divisor. By the universal property of the pair (H, W), there exists a unique map $g\colon T \to H$ such that $g_X^{-1}W = D$. We have to show that g factors through U.

For each $t \in T$, the fiber D_t is an effective divisor since it is obtained by base change (or owing to Lemma 3.4). But $D_t = W_{g(t)} \otimes k_t$ where k_t is the residue field of t. So $W_{g(t)}$ too is a divisor, as a field extension is faithfully flat. Hence, since $X \times H$ and W are H-flat, W is, by Lemma 3.4, a relative effective divisor along the fiber over $g(t)$. Therefore, $g(t) \in U$. So, since U is open, g factors through U. □

Exercise 3.8. Assume $f\colon X \to S$ is flat and is projective locally over S. Assume its fibers are curves. Given $m \geq 1$, let $\mathrm{Div}^m_{X/S}$ be the functor whose T-points are the relative effective divisors D on X_T/T with fibers D_t of degree m.

Show the $\mathrm{Div}^m_{X/S}$ are representable by open and closed subschemes of finite type $\mathbf{Div}^m_{X/S} \subset \mathbf{Div}_{X/S}$, which are disjoint and cover.

Let $X_0 \subset X$ be the subscheme where X/S is smooth. Show $X_0 = \mathbf{Div}^1_{X/S}$.

Let X_0^m be the m-fold S-fibered product. Show there is a natural S-map

$$\alpha\colon X_0^m \to \mathbf{Div}^m_{X/S},$$

which is given on T-points by $\alpha(\Gamma_1, \dots, \Gamma_m) = \sum \Gamma_i$.

Remark 3.9. Consider the map α of Exercise 3.8. Plainly α is compatible with permuting the factors of X_0^m. Hence α factors through the symmetric product $X_0^{(m)}$. In fact, α induces an isomorphism

$$X_0^{(m)} \xrightarrow{\sim} \mathbf{Div}^m_{X_0/S}\,.$$

This isomorphism is treated in detail in [**De73**, Prp. 6.3.9, p. 437] and in outline in [**BLR**, Prp. 3, p. 254].

3.10 (The module $\mathcal{Q}$). Assume $f\colon X \to S$ is proper, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module flat over S. Recall from [**EGA III**$_2$, 7.7.6] that there exist a coherent $\mathcal{O}_S$-module $\mathcal{Q}$ and an isomorphism of functors in the quasi-coherent $\mathcal{O}_S$-module $\mathcal{N}$:

$$q\colon\ \mathcal{H}om(\mathcal{Q}, \mathcal{N}) \xrightarrow{\sim} f_*(\mathcal{F} \otimes f^*\mathcal{N}). \tag{3.10.1}$$

The pair $(\mathcal{Q}, q)$ is unique, up to unique isomorphism, and by [**EGA III**$_2$, 7.7.9], forming it commutes with changing the base, in particular, with localizing.

Fix $s \in S$ and assume $S = \mathrm{Spec}(\mathcal{O}_{S,s})$. Note that the following conditions are equivalent:

(i) the $\mathcal{O}_S$-module $\mathcal{Q}$ is free (or equivalently, projective);
(ii) the functor $\mathcal{N} \mapsto f_*(\mathcal{F} \otimes f^*\mathcal{N})$ is right exact;
(iii) for all $\mathcal{N}$, the natural map is an isomorphism, $f_*(\mathcal{F}) \otimes \mathcal{N} \xrightarrow{\sim} f_*(\mathcal{F} \otimes f^*\mathcal{N})$;
(iv) the natural map is a surjection, $\mathrm{H}^0(X, \mathcal{F}) \otimes k_s \twoheadrightarrow \mathrm{H}^0(X_s, \mathcal{F}_s)$.

Indeed, the equivalence of (i)–(iii) is elementary and straightforward. Moreover, (iv) is a special case of (iii). Conversely, (iv) implies (iii) by [**EGA III**$_2$, 7.7.10] or [**OB72**, Cor. 5.1 p. 72]; this useful implication is known as the "property of exchange."

In addition, (i)–(iv) are implied by the following condition:

(v) the first cohomology group of the fiber vanishes, $\mathrm{H}^1(X_s, \mathcal{F}_s) = 0$.

Indeed, (v) implies that $\mathrm{R}^1 f_*(\mathcal{F} \otimes f^*\mathcal{N}) = 0$ for all $\mathcal{N}$ by [**EGA III**$_2$, 7.5.3] or [**OB72**, Cor. 2.1 p. 68]; in turn, this vanishing implies (ii) owing to the long exact sequence of higher direct images.

Exercise 3.11. Assume $f \colon X \to S$ is proper and flat, and its geometric fibers are reduced and connected. Show $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ holds universally.

Definition 3.12. Let $\mathcal{L}$ be an invertible sheaf on X. Define a subfunctor $\mathrm{LinSys}_{\mathcal{L}/X/S}$ of $\mathrm{Div}_{X/S}$ by the formula

$$\mathrm{LinSys}_{\mathcal{L}/X/S}(T) := \{\,\text{relative effective divisors } D \text{ on } X_T/T \text{ such that } \mathcal{O}_{X_T}(D) \simeq \mathcal{L}_T \otimes f_T^*\mathcal{N} \text{ for some invertible sheaf } \mathcal{N} \text{ on } T\,\}.$$

Notice the similarity of this definition with that, Definition 2.2, of $\mathrm{Pic}_{X/S}$: both definitions work with isomorphism classes of invertible sheaves on X_T modulo $\mathrm{Pic}(T)$, in the hope of producing a representable functor. Here, this hope is fulfilled under suitable hypotheses on f, according to the next theorem.

Theorem 3.13. *Assume X/S is proper and flat, and its geometric fibers are integral. Let $\mathcal{L}$ be an invertible sheaf on X, and let $\mathcal{Q}$ be the $\mathcal{O}_S$-module associated in Subsection 3.10 to $\mathcal{F} := \mathcal{L}$. Set $L := \mathbf{P}(\mathcal{Q})$. Then L represents* $\mathrm{LinSys}_{\mathcal{L}/X/S}$.

Proof. Let $D \in \mathrm{LinSys}_{\mathcal{L}/X/S}(T)$. Say $\mathcal{O}_{X_T}(D) \simeq \mathcal{L}_T \otimes f_T^*\mathcal{N}$. Then $\mathcal{N}$ is determined up to isomorphism. Indeed, let $\mathcal{N}'$ be a second choice. Then

$$\mathcal{L}_T \otimes f_T^*\mathcal{N} \simeq \mathcal{L}_T \otimes f_T^*\mathcal{N}'.$$

So $f_T^*\mathcal{N} \simeq f_T^*\mathcal{N}'$ since $\mathcal{L}$ is invertible. Hence $\mathcal{N} \simeq \mathcal{N}'$ by Lemma 2.7, which applies to $f_T \colon X_T \to T$ since $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ holds universally by Exercise 3.11.

Say D is defined by $\sigma \in \mathrm{H}^0(X_T, \mathcal{L}_T \otimes f_T^*\mathcal{N})$. Now, forming $\mathcal{Q}$ commutes with changing the base to T, and so (3.10.1) becomes

$$\mathcal{H}om(\mathcal{Q}_T, \mathcal{N}) \xrightarrow{\sim} f_{T*}(\mathcal{L}_T \otimes f_T^*\mathcal{N}). \tag{3.13.1}$$

Hence σ corresponds to a map $u \colon \mathcal{Q}_T \to \mathcal{N}$.

Let $t \in T$. Since D is a relative effective divisor on X_T/T, its fiber D_t is a divisor on X_t by Lemma 3.4. Since D_t is defined by $\sigma_t \in \mathrm{H}^0(X_t, \mathcal{L}|X_t)$, necessarily $\sigma_t \neq 0$. But σ_t corresponds to $u \otimes k_t$, so $u \otimes k_t \neq 0$. Now, $\mathcal{N}$ is invertible, so $\mathcal{N} \otimes k_t$ is a k_t-vector space of dimension 1. So $u \otimes k_t$ is surjective. Hence, by Nakayama's lemma, u is surjective at t. But t is arbitrary. So u is surjective everywhere.

Therefore, $u \colon \mathcal{Q}_T \to \mathcal{N}$ defines an S-map $p \colon T \to L$ by [**EGA II**, 4.2.3]. Since $(\mathcal{N}, u)$ is determined up to isomorphism, a second choice yields the same p.

Plainly, this construction is functorial in T. Thus we obtain a map of functors,

$$\Lambda \colon \mathrm{LinSys}_{\mathcal{L}/X/S}(T) \to L(T).$$

Let us prove Λ is an isomorphism.

Let $p \in L(T)$, so $p \colon T \to L$ is an S-map. Then p arises from a surjection

$u\colon \mathcal{Q}_T \twoheadrightarrow \mathcal{N}$; namely, $u = p^*\alpha$ where $\alpha\colon \mathcal{Q}_L \twoheadrightarrow \mathcal{O}(1)$ is the tautological map. Moreover, there is only one such pair $(\mathcal{N}, u)$ up to isomorphism.

Via the isomorphism in (3.13.1), the surjection u corresponds to a global section $\sigma \in \mathrm{H}^0(X_T, \mathcal{L}_T \otimes f_T^*\mathcal{N})$. Let $t \in T$. Then $u \otimes k_t$ is surjective, so $u \otimes k_t \neq 0$. But $u \otimes k_t$ corresponds to $\sigma_t \in \mathrm{H}^0(X_t, \mathcal{L}|X_t)$, so $\sigma_t \neq 0$. But X_t is integral since the geometric fibers of X/S are integral by hypothesis. Hence σ_t is regular.

The section σ defines a map $(\mathcal{L}_T \otimes f_T^*\mathcal{N})^{-1} \to \mathcal{O}_{X_T}$. Its image is the ideal of a closed subscheme $D \subset X$, which is cut out locally by one element; moreover, on the fiber X_t, this element corresponds to σ_t, so is regular. Hence D is a relative effective divisor on X_T/T by Lemma 3.4. In fact, $D \in \mathrm{LinSys}_{\mathcal{L}/X/S}(T)$. Plainly, D is the only such divisor corresponding to $(\mathcal{N}, u)$, so mapping to p under Λ.

Thus Λ is an isomorphism. In other words, L represents $\mathrm{LinSys}_{\mathcal{L}/X/S}$. □

Exercise 3.14. Under the conditions of Theorem 3.13, show that there exists a natural relative effective divisor W on X_L/L such that

$$\mathcal{O}_{X_L}(W) = \mathcal{L}_L \otimes f_L^*\mathcal{O}_L(1).$$

Furthermore, W possesses the following universal property: given any S-scheme T and any relative effective divisor D on X_T/T such that $\mathcal{O}_{X_T}(D) \simeq \mathcal{L}_T \otimes f_T^*\mathcal{N}$ for some invertible sheaf $\mathcal{N}$ on T, there exist a unique S-map $w\colon T \to L$ such that $(1 \times w)^{-1}W = D$.

4. The Picard scheme

This section proves Grothendieck's main theorem about the Picard scheme, which asserts its existence if X/S is projective and flat and its geometric fibers are integral; in fact, the functor $\mathrm{Pic}_{(X/S)\,(\text{ét})}$ is representable. The proof involves Grothendieck's method of using functors to prescribe patching. The basic theory is developed in [**EGA G**, Ch. 0, Sct. 4.5, pp. 102–107], and is applied in [**EGA G**, Ch. 1, Sct. 9, pp. 354–401] to the construction of Grassmannians and related parameter schemes. The present construction involves the basic theory, but is more sophisticated and more complicated because it works not simply with the Zariski topology, but also with the étale topology.

Definition 4.1. If any of the five relative Picard functors of Definition 2.2 is representable, then the representing scheme is called the *Picard scheme* and denoted by $\mathbf{Pic}_{X/S}$. Moreover, we say simply that the Picard scheme $\mathbf{Pic}_{X/S}$ exists.

Notice that, although there are five relative Picard functors, there is at most one Picard scheme. Of course, if any functor is representable, then the representing scheme is uniquely determined, up to a unique isomorphism that preserves the identification of the given functor with the functor of points of the representing scheme. But here, there is more to the story.

Indeed, for example, say $\mathrm{Pic}_{(X/S)\,(\text{ét})}$ is representable by $\mathbf{Pic}_{X/S}$. Then, by descent theory, $\mathrm{Pic}_{(X/S)\,(\text{ét})}$ is already a sheaf in the fppf and fpqc topologies; so it is equal to its associated sheaves, namely, $\mathrm{Pic}_{(X/S)\,(\text{fppf})}$ and $\mathrm{Pic}_{(X/S)\,(\text{fpqc})}$. Hence, they too are representable by $\mathbf{Pic}_{X/S}$. On the other hand, $\mathrm{Pic}_{X/S}$ may or may not be representable; however, if it is, then it must be representable by $\mathbf{Pic}_{X/S}$.

Exercise 4.2. Assume $\mathbf{Pic}_{X/S}$ exists, and $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ holds universally. Let T be an S-scheme, and $\mathcal{L}$ an invertible sheaf on X_T. Show that there exist a subscheme $N \subset T$ and an invertible sheaf $\mathcal{N}$ on N with these three properties: first, $\mathcal{L}_N \simeq f_N^*\mathcal{N}$; second, given any S-map $t\colon T' \to T$ such that $\mathcal{L}_{T'} \simeq f_{T'}^*\mathcal{N}'$ for some invertible sheaf $\mathcal{N}'$ on T', necessarily t factors through N and $\mathcal{N}' \simeq t^*\mathcal{N}$; and third, N is a closed subscheme if $\mathbf{Pic}_{X/S}$ is separated. Show also that the first two properties determine N uniquely and $\mathcal{N}$ up to isomorphism.

EXERCISE 4.3. Assume $\mathbf{Pic}_{X/S}$ exists. An invertible sheaf $\mathcal{P}$ on $X \times \mathbf{Pic}_{X/S}$ is called a *universal sheaf*, or *Poincaré sheaf*, if $\mathcal{P}$ possesses the following property: given any S-scheme T and any invertible sheaf $\mathcal{L}$ on X_T, there exists a unique S-map $h\colon T \to \mathbf{Pic}_{X/S}$ such that, for some invertible sheaf $\mathcal{N}$ on T,

$$\mathcal{L} \simeq (1 \times h)^*\mathcal{P} \otimes f_T^*\mathcal{N}.$$

Show that a universal sheaf $\mathcal{P}$ exists if and only if $\mathbf{Pic}_{X/S}$ represents $\mathrm{Pic}_{X/S}$.

Assume $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ holds universally. Show that, if $\mathcal{P}$ exists, then it is unique up to tensor product with the pullback of a unique invertible sheaf on $\mathbf{Pic}_{X/S}$.

Show that, if also f has a section, then a universal sheaf $\mathcal{P}$ exists.

Find an example where no universal sheaf $\mathcal{P}$ exists.

EXERCISE 4.4. Assume $\mathbf{Pic}_{X/S}$ exists, and let S' be an S-scheme. Show that $\mathbf{Pic}_{X_{S'}/S'}$ exists too, and in fact, that

$$\mathbf{Pic}_{X_{S'}/S'} = \mathbf{Pic}_{X/S} \times_S S'.$$

Thus forming the Picard scheme commutes with changing the base.

Find an example where $\mathbf{Pic}_{X_{S'}/S'}$ represents $\mathrm{Pic}_{X_{S'}/S'}$, but $\mathbf{Pic}_{X/S}$ does not represent $\mathrm{Pic}_{X/S}$.

EXERCISE 4.5. Assume $\mathbf{Pic}_{X/S}$ exists, and either it represents $\mathrm{Pic}_{(X/S)\,(\mathrm{fppf})}$ or $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ holds universally. Show the scheme points of $\mathbf{Pic}_{X/S}$ correspond, in a natural bijective fashion, to the classes of invertible sheaves $\mathcal{L}$ on the fibers of X/S. A class is, by definition, represented by an $\mathcal{L}$ on an X_k where k is a field containing the residue field k_s of a (scheme) point $s \in S$; an $\mathcal{L}'$ on an $X_{k'}$ represents the same class if and only if there is a third field k'' containing the other two such that $\mathcal{L}|X_{k''} \simeq \mathcal{L}'|X_{k''}$.

DEFINITION 4.6. The *Abel map* is the natural map of functors

$$A_{X/S}(T)\colon\ \mathrm{Div}_{X/S}(T) \to \mathrm{Pic}_{X/S}(T)$$

defined by sending a relative effective divisor D on X_T/T to the sheaf $\mathcal{O}_{X_T}(D)$. The target $\mathrm{Pic}_{X/S}$ may be replaced by any of its associated sheaves. If $\mathbf{Pic}_{X/S}$ exists, then the term *Abel map* may refer to the corresponding map of schemes

$$\mathbf{A}_{X/S}\colon\ \mathbf{Div}_{X/S} \to \mathbf{Pic}_{X/S}\,.$$

EXERCISE 4.7. Assume X/S is proper and flat with integral geometric fibers. Assume $\mathbf{Pic}_{X/S}$ exists, and denote it by P. View $\mathbf{Div}_{X/S}$ as a P-scheme via the Abel map. Assume a universal sheaf $\mathcal{P}$ exists, and let $\mathcal{Q}$ be the sheaf on P associated to $\mathcal{P}$ as in Subsection 3.10. Show $\mathbf{P}(\mathcal{Q}) = \mathbf{Div}_{X/S}$ as P-schemes.

THEOREM 4.8 (Main). *Assume $f\colon X \to S$ is projective locally over S, and is flat with integral geometric fibers.*

(1) *Then $\mathbf{Pic}_{X/S}$ exists, is separated and locally of finite type over S, and represents $\mathrm{Pic}_{(X/S)\,(\text{ét})}$.*

(2) *If also S is Noetherian and X/S is projective, then $\mathbf{Pic}_{X/S}$ is a disjoint*

union of open subschemes, each an increasing union of open quasi-projective S-schemes.

Proof. By [**EGA G**, (0, 4.5.5), p. 106], it is a local matter on S to represent a Zariski sheaf on the category of S-schemes. Moreover, it is also a local matter on S to prove that an S-scheme is separated and locally of finite type. Hence, in order to prove (1), we may assume S is Noetherian and X/S is projective.

Plainly an S-scheme is separated if it is a disjoint union of separated open subschemes, or if it is an increasing union of separated open subschemes. Hence (1) follows from (2).

To prove (2), owing to Yoneda's lemma, we may view the category of schemes as a full subcategory of the category of functors by identifying a scheme T with its functor of points. Denote this functor too by T in order to lighten the notation. And say that the functor is a scheme, as well as that it is representable. Also, set

$$P := \mathrm{Pic}_{(X/S)\,(\text{ét})}\,.$$

Note $P(T) = \mathrm{Hom}(T, P)$.

Given a polynomial $\phi \in \mathbb{Q}[n]$, let $P^\phi \subset P$ be the étale subsheaf associated to the presheaf whose T-points are represented by the invertible sheaves $\mathcal{L}$ on X_T such that we have

$$\chi(X_t, \mathcal{L}_t^{-1}(n)) = \phi(n) \text{ for all } t \in T. \tag{4.8.1}$$

Notice that this presheaf is well defined, because (4.8.1) remains valid after any base change $p\colon T' \to T$; indeed, for any $t' \in T'$, for any i, and for any n, we have

$$\mathrm{H}^i\big(X_{t'}, \mathcal{L}_{t'}^{-1}(n)\big) = \mathrm{H}^i\big(X_{p(t')}, \mathcal{L}_{p(t')}^{-1}(n)\big) \otimes_{k_t} k_{t'}$$

because cohomology commutes with flat base change by [**Ha83**, Prp. III, 9.3, p. 255]. Hence P^ϕ is well defined too.

Fix a map $T \to P$, and represent it by means of an étale covering $p\colon T' \to T$ and an invertible sheaf $\mathcal{L}'$ on $X_{T'}$. Consider the subset $T'^\phi \subset T'$ defined as follows:

$$T'^\phi := \{\, t' \in T' \mid \chi\big(X_{t'}, \mathcal{L}_{t'}^{-1}(n)\big) = \phi(n)\,\}.$$

Then T'^ϕ is open by [**EGA III**$_2$, 7.9.11].

Set $T^\phi := p(T'^\phi)$. Then $T^\phi \subset T$ is open as $T'^\phi \subset T'$ is open and p is étale.

Moreover, $T'^\phi = p^{-1}(T^\phi)$. Indeed, let $t' \in p^{-1}(T^\phi)$. Say $p(t') = p(t'_1)$ where $t'_1 \in T'^\phi$. Now, there is an étale covering $T'' \to T' \times_T T'$ such that the two pullbacks of $\mathcal{L}'$ to $X_{T''}$ are isomorphic. Let $t'' \in T''$ have image $t' \in T'$ under the first map $T'' \to T'$ and have the image $t'_1 \in T'$ under the second map. Then

$$\chi\big(X_{t'}, \mathcal{L}_{t'}^{-1}(n)\big) = \chi\big(X_{t''}, \mathcal{L}_{t''}^{-1}(n)\big) = \chi\big(X_{t'_1}, \mathcal{L}_{t'_1}^{-1}(n)\big) = \phi(n).$$

Hence $t' \in T'^\phi$. Thus $T'^\phi \supset p^{-1}(T^\phi)$. Therefore, $T'^\phi = p^{-1}(T^\phi)$.

Furthermore, T^ϕ is (represents) the fiber product of functors $P^\phi \times_P T$. Indeed, to see they have the same R-points, let $r\colon R \to T$ be a map; form $R' := R \times_T T'$ and $r'\colon R' \to T'$. Suppose r factors through T^ϕ. Then r' factors through T'^ϕ. So $R' \to T' \to P$ factors through P^ϕ essentially by definition. Now, $R' \to R$ is an étale covering. Hence $R \to T \to P$ factors through P^ϕ since P^ϕ is an étale sheaf.

Conversely, suppose $R \to T \to P$ factors through P^ϕ. Then $R \to P$ is defined by means of an étale covering $R'' \to R$ and an invertible sheaf $\mathcal{L}''$ on $X_{R''}$ such that $\chi(X_u, {\mathcal{L}''}_u^{-1}(n)) = \phi(n)$ for all $u \in R''$. Since both $\mathcal{L}''$ and $\mathcal{L}'$ define $R \to P$, there is an étale covering $R''' \to R'' \times_R R'$ such that the pullbacks of $\mathcal{L}''$ and $\mathcal{L}'$ to

$X_{R'''}$ are isomorphic. Hence the image of $r'\colon R' \to T'$ lies in ${T'}^{\phi}$. But the latter is open. Hence r' factors through it. Therefore, $r\colon R \to T$ factors through T^{ϕ}. Thus T^{ϕ} and $P^{\phi} \times_P T$ have the same R-points.

Let ϕ vary. Plainly the ${T'}^{\phi}$ are disjoint and cover T'. So the T^{ϕ} are disjoint and cover T. Hence, by a general result [**EGA G**, (0, 4.5.4), p. 103], if the P^{ϕ} are (representable by) schemes, then P is their disjoint union. Thus it remains to represent each P^{ϕ} by an increasing union of open quasi-projective S-schemes.

Fix ϕ. Given $m \in \mathbb{Z}$, let $P^{\phi}_m \subset P^{\phi}$ be the étale subsheaf associated to the presheaf whose T-points are represented by the $\mathcal{L}$ on X_T such that, in addition to (4.8.1), we have

$$\mathrm{R}^i f_{T*}\mathcal{L}(n) = 0 \text{ for all } i \geq 1 \text{ and } n \geq m. \tag{4.8.2}$$

Notice that this presheaf is well defined, because (4.8.2) remains valid after any base change $p\colon T' \to T$, as is shown next.

First, let's see that (4.8.2) is equivalent to the following condition:

$$\mathrm{H}^i(X_t, \mathcal{L}_t(n)) = 0 \text{ for all } i \geq 1, \text{ all } n \geq m, \text{ and all } t \in T. \tag{4.8.3}$$

Indeed, (4.8.3) implies (4.8.2); in fact, for any given i, t and n, if $\mathrm{H}^i(X_t, \mathcal{L}_t(n)) = 0$, then $\mathrm{R}^i f_{T*}\big(\mathcal{L}(n) \otimes f_T^*\mathcal{N}\big)_t = 0$ for all quasi-coherent $\mathcal{N}$ on T by [**EGA III**$_2$, 7.5.3] or [**OB72**, Cor. 2.1 p. 68].

Conversely, assume (4.8.2). Fix t and n. Let's proceed by descending induction on i to prove $\mathrm{H}^i(X_t, \mathcal{L}_t(n))$ vanishes. It vanishes for $i \gg 1$ by Serre's Theorem [**EGA III**$_1$, 2.2.2]. Suppose it vanishes for some $i \geq 2$. Then $\mathrm{R}^i f_{T*}\big(\mathcal{L}(n) \otimes f_T^*\mathcal{N}\big)_t$ vanishes for all quasi-coherent $\mathcal{N}$ on T, as just noted. So $\mathrm{R}^{i-1} f_{T*}\big(\mathcal{L}(n) \otimes f_T^*\mathcal{N}\big)_t$ is right exact in $\mathcal{N}$ owing to the long exact sequence of higher direct images. Therefore, by general principles, there is a natural isomorphism of functors

$$\mathrm{R}^{i-1} f_{T*}(\mathcal{L}(n))_t \otimes \mathcal{N}_t \xrightarrow{\sim} \mathrm{R}^{i-1} f_{T*}\big(\mathcal{L}(n) \otimes f_T^*\mathcal{N}\big)_t.$$

Since (4.8.2) holds, both source and target vanish. Taking $\mathcal{N} := k_t$ yields the vanishing of $\mathrm{H}^{i-1}(X_t, \mathcal{L}_t(n))$. Thus (4.8.2) implies (4.8.3).

Finally, for any $t' \in T'$, any i, and any n, we have

$$\mathrm{H}^i\big(X_{t'}, \mathcal{L}_{t'}(n)\big) = \mathrm{H}^i\big(X_{p(t')}, \mathcal{L}_{p(t')}(n)\big) \otimes_{k_t} k_{t'}$$

because cohomology commutes with flat base change. So (4.8.3) remains valid after the base change $p\colon T' \to T$; whence, (4.8.2) does too. Thus the presheaf is well defined, and so P^{ϕ}_m is too.

Arguing much as we did for $P^{\phi} \times_P T$, we find, given a map $T \to P^{\phi}$, that, as m varies, the products $P^{\phi}_m \times_{P^{\phi}} T$ form a nested sequence of open subschemes of T, whose union is T. In the argument, the key change is in proving openness. In place of [**EGA III**$_2$, 7.9.11], we use the following part of Serre's Theorem [**EGA III**$_1$, 2.2.2]: given a coherent sheaf $\mathcal{F}$ on a projective scheme over a Noetherian ring A, there are only finitely many $i \geq 1$ and $n \geq m$ such that $H^i(\mathcal{F}(n))$ is nonzero, and all these nonzero A-modules are finitely generated. Hence, if there is a prime $\mathbf{p}$ of A such that $H^i(\mathcal{F}(n))_{\mathbf{p}} = 0$ for all $i \geq 1$ and $n \geq m$, then there is an $a \notin \mathbf{p}$ such that $H^i(\mathcal{F}(n))_a = 0$ for all $i \geq 1$ and $n \geq m$.

Therefore, again by [**EGA G**, (0, 4.5.4), p. 103], it suffices to represent each P^{ϕ}_m by a quasi-projective S-scheme.

Fix ϕ and m. Set $\phi_0(n) := \phi(m+n)$. Then there is an isomorphism of functors $P^{\phi}_m \xrightarrow{\sim} P^{\phi_0}_0$, which is defined as follows. First, define an endomorphism ε of $\mathrm{Pic}_{X/S}$

by sending an invertible sheaf $\mathcal{L}$ on an X_T to $\mathcal{L}(m)$. Plainly ε is an automorphism. So ε induces an automorphism ε^+ of the associated sheaf P. Plainly ε^+ carries P_m^ϕ onto $P_0^{\phi_0}$. Thus it suffices to represent $P_0^{\phi_0}$ by a quasi-projective S-scheme.

The function $s \mapsto \chi(X_s, \mathcal{O}_{X_s}(n))$ is locally constant on S by [**EGA III**$_2$, 7.9.11]. Hence we may assume it is constant by replacing S by an open and closed subset. Set $\psi(n) := \chi(X_s, \mathcal{O}_{X_s}(n))$.

Consider the Abel map $A_{X/S}\colon \mathrm{Div}_{X/S} \to P$. Note that $\mathrm{Div}_{X/S}$ is a scheme, in fact, an open subscheme of the Hilbert scheme $\mathbf{Hilb}_{X/S}$, by Theorem 3.7. Form the product $P_0^{\phi_0} \times_P \mathrm{Div}_{X/S}$. It is a scheme Z, in fact, an open subscheme of $\mathrm{Div}_{X/S}$, by what was proved above. Set $\theta(n) := \psi(n) - \phi_0(n)$. Plainly Z lies in $\mathbf{Hilb}^\theta_{X/S}(n)$, which is projective over S. Hence Z is quasi-projective.

Let's now prove the projection $\alpha\colon Z \to P_0^{\phi_0}$ is a surjection of étale sheaves. In other words, given a T and a $\lambda \in P_0^{\phi_0}(T)$, we have to find an étale covering $T_1 \to T$ and a $\lambda_1 \in Z(T_1)$ such that $\alpha(\lambda_1) \in P_0^{\phi_0}(T_1)$ is equal to the image of λ.

Represent λ by means of an étale covering $p\colon T' \to T$ and an invertible sheaf $\mathcal{L}'$ on $X_{T'}$. Virtually by definition, the product $T' \times_{P_0^{\phi_0}} Z$ is equal to $\mathrm{LinSys}_{\mathcal{L}'/X_{T'}/T'}$. So, by Theorem 3.13, this product is equal to $\mathbf{P}(\mathcal{Q})$ where $\mathcal{Q}$ is the $\mathcal{O}_{T'}$-module associated to $\mathcal{L}'$ as in Subsection 3.10. Now, $m = 0$, so $\mathrm{H}^1(X_t, \mathcal{L}_t) = 0$ for all $t \in T'$ owing to (4.8.3). Since (v) implies (i) in Subsection 3.10, therefore $\mathcal{Q}$ is locally free.

Hence $\mathbf{P}(\mathcal{Q})$ is smooth over T'. So there exist an étale covering $T_1 \to T'$ and a T'-map $T_1 \to \mathbf{P}(\mathcal{Q})$ by [**EGA IV**$_4$, 17.16.3 (ii)]. Then the composition $T_1 \to \mathbf{P}(\mathcal{Q}) \to Z \to P_0^{\phi_0}$ is equal to the composition $T_1 \to T' \to T \to P_0^{\phi_0}$. In other words, the map $T_1 \to Z$ is a $\lambda_1 \in Z(T_1)$ such that $\alpha(\lambda_1) \in P_0^{\phi_0}(T_1)$ is equal to the image of λ. Since the composition $T_1 \to T' \to T$ is an étale covering, α is thus a surjection of étale sheaves.

Plainly, the map $\alpha\colon Z \to P_0^{\phi_0}$ is defined by the invertible sheaf associated to the universal relative effective divisor on X_Z/Z. So taking $T := Z$ and $T' := T$ above, we conclude that the product $Z \times_{P_0^{\phi_0}} Z$ is a smooth projective Z-scheme. Therefore, the theorem now results from the following general lemma. $\square$

LEMMA 4.9. *Let $\alpha\colon Z \to P$ be a map of étale sheaves, and set $R := Z \times_P Z$. Assume α is a surjection, Z is representable by a quasi-projective S-scheme, and R is representable by a smooth and proper Z-scheme. Then P is representable by a quasi-projective S-scheme, and α is representable by a smooth map.*

PROOF. To lighten the notation, let Z and R denote the corresponding schemes as well. Since $Z \to S$ is quasi-projective, it is separated; whence, $Z \times_S Z \to Z$ is too. But $R \to Z$ is proper, and factors naturally through $Z \times_S Z$. Hence $h\colon R \to Z \times_S Z$ is proper. But h is a monomorphism; that is, h is injective on T-points. Therefore, h is a closed embedding by [**EGA IV**$_3$, 8.11.5].

Plainly, for each S-scheme T, the subset $R(T) \subset Z(T) \times_{S(T)} Z(T)$ is the graph of an equivalence relation. Also, the map of schemes $R \to Z$ is flat and proper, and Z is a quasi-projective S-scheme. It follows that there exist a quasi-projective S-scheme Q and a faithfully flat and projective map $Z \to Q$ such that $R = Z \times_Q Z$. (In other words, a flat and proper equivalence relation on a quasi-projective scheme is effective.) In fact, R defines a map from Z to the Hilbert scheme $\mathbf{Hilb}_{Z/S}$, and its graph lies in the universal scheme as a closed subscheme, which descends to $Q \subset \mathbf{Hilb}_{Z/S}$; for more details, see [**AK80**, Thm. (2.9), p. 70].

Since $Z \to Q$ is flat, it is smooth if (and only if) its fibers are smooth. But these fibers are, up to extension of the ground field, the same as those of $R \to Z$. And $R \to Z$ is smooth by hypothesis. Thus $Z \to Q$ is smooth.

It remains to see that Q represents P. First, observe that $Z \to Q$ is (represents) a surjection of étale sheaves. Indeed, given an element of $Q(T)$, set $A := Z \times_Q T$. Then $A \to T$ is smooth. So there exist an étale covering $T' \to T$ and a T-map $T' \to A$ by [**EGA IV**$_4$, 17.16.3 (ii)]. Then $T' \to A \to Z$ is an element of $Z(T')$, which maps to the element of $Q(T')$ induced by the given element of $Q(T)$.

Since $Z \to Q$ is a surjection of étale sheaves, Q is, in the category of étale sheaves, the coequalizer of the pair of maps $R \rightrightarrows Z$ by Exercise 4.10 below, which is an elementary exercise in general Grothendieck topology. By the same exercise, P too is the coequalizer of this pair of maps. But, in any category, the coequalizer is unique up to unique isomorphism. Thus Q represents P. □

Exercise 4.10. Given a map of étale sheaves $F \to G$, show it is a surjection if and only if G is the coequalizer of the pair of maps $F \times_G F \rightrightarrows F$.

Exercise 4.11. Assume X/S is projective and flat, its geometric fibers are integral, and S is Noetherian. Let $Z \subset \mathbf{Pic}_{X/S}$ be a subscheme of finite type. Show Z is quasi-projective.

Exercise 4.12. Assume $f\colon X \to S$ is projective locally over S, and is flat with integral geometric fibers. First, show that, if a universal sheaf $\mathcal{P}$ exists, then the Abel map $\mathbf{A}_{X/S}\colon \mathbf{Div}_{X/S} \to \mathbf{Pic}_{X/S}$ is projective locally over S.

Second, show that, in general, $\mathbf{A}_{X/S}\colon \mathbf{Div}_{X/S} \to \mathbf{Pic}_{X/S}$ is proper. Proceed by reducing this case to the first: just use $f\colon X \to S$ itself to change the base.

Exercise 4.13. Assume $f\colon X \to S$ is flat and projective locally over S. Assume its geometric fibers X_k are integral curves of arithmetic genus $\dim \mathrm{H}^1(\mathcal{O}_{X_k})$ at least 1. Let $X_0 \subset X$ be the open subscheme where X/S is smooth. Show there is a natural closed embedding $A\colon X_0 \hookrightarrow \mathbf{Pic}_{X/S}$.

Example 4.14. In Theorem 4.8, the geometric fibers of f are assumed to be integral. This condition is needed not only for the proof to work, but also for the statement to hold. The matter is clarified by the following example, which is attributed to Mumford and is described in [**FGA**, p. 236-1] and in [**BLR**, p. 210].

Let $\mathbb{R}[\![t]\!]$ be the ring of formal power series, S its spectrum. Let $X \subset \mathbf{P}^2_S$ be the subscheme with inhomogeneous equation $x^2 + y^2 = t$, and $f\colon X \to S$ the structure map. The generic fiber X_σ is a nondegenerate conic. The special fiber X_0 is a pair of conjugate lines; X_0 is irreducible and geometrically connected. So f is flat by the implication (iii)⇒(i) of Lemma 3.4 with $D := X$. And $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ holds universally by Exercise 3.11. However, as we'll see, $\mathbf{Pic}_{X/S}$ does *not* exist!

On the other hand, set $S' := \mathbb{C}[\![t]\!]$ and $X' := X \times S'$. Then $f_{S'}$ has sections: for example, one section g' is defined by setting

$$x := \sqrt{-1} \text{ and } y := \sqrt{1+t} = 1 + (1/2)t - (1/8)t^2 + \cdots .$$

Hence all five relative Picard functors of X'/S' are equal by the Comparison Theorem, Theorem 2.5. Furthermore, as we'll see, $\mathbf{Pic}_{X'/S'}$ exists, but is *not* separated!

In fact, $\mathbf{Pic}_{X'/S'}$ is a disjoint union of isomorphic open nonseparated subschemes S'_n for $n \in \mathbb{Z}$. Each S'_n is obtained from S' by repeating the origin infinitely often; more precisely, to form S'_n, take a copy $S'_{a,b}$ of S' for each pair $a, b \in \mathbb{Z}$ with $a+b = n$, and glue the $S'_{a,b}$ together off their closed points. Each $S'_{a,b}$ parameterizes a different degeneration of the invertible sheaf of degree n on the generic fiber X'_σ; the degenerate sheaf has degree a on the first line, and degree b on the second. Also, complex conjugation interchanges the two lines, so interchanges $S'_{a,b}$ and $S'_{b,a}$.

Suppose $\mathbf{Pic}_{X/S}$ exists. Then $\mathbf{Pic}_{X/S} \times_S S' = \mathbf{Pic}_{X'/S'}$ by Exercise 4.4. Since the closed points of $S'_{a,b}$ and $S'_{b,a}$ are conjugate, they map to the same point of $\mathbf{Pic}_{X/S}$. This point lies in an affine open subscheme U. So the two closed points lie in the preimage U' of U in $\mathbf{Pic}_{X'/S'}$. But U' is affine since U, S and S' are affine. However, if $a \neq b$, then the two closed points are distinct, and so lie in no common affine U'. We have a contradiction. Thus $\mathbf{Pic}_{X/S}$ cannot exist.

Finally, let's prove $\mathrm{Pic}_{X'/S'}$ is representable by $\amalg S'_n$. First, note X' is regular; in fact, in $\mathbf{P}^2_{\mathbb{C}} \times \mathbf{A}^1_{\mathbb{C}}$, the equation $x^2 + y^2 = t$ defines a smooth surface. So on X' every reduced curve is an effective divisor. In particular, consider these three: the line $L : x = \sqrt{-1}y$, $t = 0$, the line $M : x = -\sqrt{-1}y$, $t = 0$, and the image A of the section g of $f_{S'}$ defined above.

Set $\mathcal{P}'_{a,b} := \mathcal{O}_{X'}(bL + nA)$ where $n = a + b$. The restriction to the generic fiber $\mathcal{P}'_{a,b}|X'_\sigma$ has degree n since $L \cap X'_\sigma$ is empty and A is the image of a section. And $\mathcal{P}'_{a,b}|M$ has degree b since $A \cap M$ is empty and $L \cap M$ is a reduced $\mathbb{C}$-rational point. And $\mathcal{P}'_{a,b}|L$ has degree a since, in addition, $\mathcal{O}_{X'}(L) \otimes \mathcal{O}_{X'}(M) \simeq \mathcal{O}_{X'}$ as the ideal of $L + M$ is generated by t. Lastly, every invertible sheaf on S' is trivial; fix a g-rigidification $\mathcal{O}_{S'} \xrightarrow{\sim} g^*\mathcal{P}'$, and use it to identify the two sheaves.

On $X' \times_{S'} \amalg S'_n$, form an invertible sheaf $\mathcal{P}'$ by placing $\mathcal{P}'_{a,b}$ on $S'_{a,b}$; plainly, the $\mathcal{P}'_{a,b}$ patch together. It now suffices to show this: given any S'-scheme T and any invertible sheaf $\mathcal{L}$ on on X'_T, there exist a unique S'-map $q\colon T \to \amalg S'_n$ and some invertible sheaf $\mathcal{N}$ on T such that $(1 \times q)^*\mathcal{P}' \simeq \mathcal{L} \otimes f_T^*\mathcal{N}$.

Replace $\mathcal{L}$ by $\mathcal{L} \otimes f_T^* g_T^* \mathcal{L}^{-1}$. Then $g_T^*\mathcal{L} = \mathcal{O}_T$ since $g_T^* f_T^* = 1$. Hence, if q and $\mathcal{N}$ exist, then necessarily $\mathcal{N} \simeq \mathcal{O}_T$ since $g_T^*(1 \times q)^*\mathcal{P}' = q^* g^* \mathcal{P}'$ and $g^*\mathcal{P}' = \mathcal{O}_{S'}$.

Plainly, we may assume T is connected. Then the function $s \mapsto \chi(X'_t, \mathcal{L}_t)$ is constant on T by [**EGA III**$_2$, 7.9.5]. Set $n := \chi(X'_t, \mathcal{L}_t) - 1$.

Fix a, b with $a + b = n$. Set

$$\mathcal{M} := \mathcal{L}^{-1} \otimes (1 \times \tau)^*\mathcal{P}'_{a,b} \text{ and } \mathcal{N} := f_{T*}\mathcal{M}.$$

Plainly $g_T^*\mathcal{M} = \mathcal{O}_T$. Form the natural map $u\colon f_T^*\mathcal{N} \to \mathcal{M}$.

Let T_σ be the generic fiber of the structure map $\tau\colon T \to S'$. Then $T_\sigma \subset T$ is open. Let $t \in T_\sigma$. Then X'_t is a nondegenerate plane conic with a rational point A_t. So $X'_t \simeq \mathbf{P}^1_{k_t}$. Hence $\mathcal{L}_t \simeq \mathcal{O}_{X'_t}(nA_t)$. So $\mathcal{M}_t \simeq \mathcal{O}_{X'_t}$. Hence $\mathrm{H}^1(X'_t, \mathcal{M}_t) = 0$ and $\mathrm{H}^0(X'_t, \mathcal{M}_t) = k_t$ by Serre's explicit computation [**EGA III**$_1$, 2.1.12].

Therefore, $\mathcal{N}$ is invertible at t, and forming $\mathcal{N}$ commutes with passing to X'_t, owing to the theory in Subsection 3.10. So forming $u\colon f_T^*\mathcal{N} \to \mathcal{M}$ commutes with passing to X'_t too. But u is an isomorphism on X'_t. Hence u is surjective along X'_t by Nakayama's lemma. But both source and target of u are invertible along X'_t. Hence u is an isomorphism along X'_t, so over T_σ as $t \in T_\sigma$ is arbitrary. Now, $g_T^*\mathcal{M} = \mathcal{O}_T$ and $g_T^* f_T^* = 1$. Hence $\mathcal{N}|T_\sigma = \mathcal{O}_{T_\sigma}$. Therefore, $\mathcal{L}|X'_{T_\sigma} = (1 \times \tau)^*\mathcal{P}'_{a,b}|X'_{T_\sigma}$.

Let $T'_{a,b}$ be the set of $t \in T$ such that $\tau(t) \in S'$ is the closed point and $\mathcal{L}|L_t$ has degree a and $\mathcal{L}|M_t$ has degree b. Fix $t \in T'_{a,b}$. Then $\mathcal{M}|L_t$ has degree 0, so $\mathcal{M}|L_t \simeq \mathcal{O}_{L_t}$. Similarly, $\mathcal{M}|M_t \simeq \mathcal{O}_{M_t}$. Consider the natural short exact sequence

$$0 \to \mathcal{O}_{L_t}(-1) \to \mathcal{O}_{X'_t} \to \mathcal{O}_{M_t} \to 0.$$

Twist by $\mathcal{M}$ and take cohomology. Thus $\mathrm{H}^1(X'_t, \mathcal{M}_t) = 0$ and $\mathrm{H}^0(X'_t, \mathcal{M}_t) = k_t$.

Hence, on X'_t, the map u becomes a map $\mathcal{O}_{X'_t} \to \mathcal{M}_t$. This map is surjective as it is surjective after restriction to L_t and to M_t. So this map is an isomorphism.

As above, we conclude u is an isomorphism on an open neighborhood V' of X'_t. Set $W' := f_{T'}(X'_T - V')$. Since f is proper, W' is open. But $f^{-1}W' \subset V'$. So u is an isomorphism over W'. Hence $O_{X'_{t'}} \xrightarrow{\sim} \mathcal{M}_{t'}$ for all $t' \in W'$. So $W' \subset T'_{a,b} \cup T_\sigma$.

Set $T_{a,b} := T'_{a,b} \cup T_\sigma$. Then $T_{a,b} \subset T$ is open as it contains a neighborhood of each of its points. Furthermore, u is an isomorphism over $T_{a,b}$. Hence $\mathcal{N}|T_\sigma = \mathcal{O}_{T_\sigma}$, again since $g_T^*\mathcal{M} = \mathcal{O}_T$ and $g_T^* f_T^* = 1$. Therefore, $\mathcal{L}|X'_{T_{a,b}} = (1 \times \tau)^*\mathcal{P}'_{a,b}|X'_{T_{a,b}}$.

Let $q_{a,b} \colon T_{a,b} \to \amalg S'_n$ be the composition of the structure map $\tau \colon T \to S'$ and the inclusion of S' in S'_n as $S'_{a,b}$. Plainly $(1 \times q_{a,b})^*\mathcal{P}' = \mathcal{L}|X'_{T_{a,b}}$. Plainly, as a and b and n vary, the $q_{a,b}$ patch to a map $q \colon T \to \amalg S'_n$ such that $(1 \times q)^*\mathcal{P}' = \mathcal{L}$. Plainly this map q is the only S'-map q such that $(1 \times q)^*\mathcal{P}' \simeq \mathcal{L} \otimes f_T^*\mathcal{N}$ for some invertible sheaf $\mathcal{N}$ on T. Thus $\amalg S'_n$ represents $\mathrm{Pic}_{X'/S'}$, and $\mathcal{P}'$ is a universal sheaf.

Exercise 4.15. Assume $X = \mathbf{P}(\mathcal{E})$ where $\mathcal{E}$ is a locally free sheaf on S and is everywhere of rank at least 2. Show $\mathbf{Pic}_{X/S}$ exists, and represents $\mathrm{Pic}_{X/S}$; in fact, $\mathbf{Pic}_{X/S} = \mathbb{Z}_S$ where $\mathbb{Z}_S$ stands for the disjoint union of copies of S indexed by $\mathbb{Z}$.

Exercise 4.16. Consider the curve $X/\mathbb{R}$ of Exercise 2.4. Show $\mathbf{Pic}_{X/S} = \mathbb{Z}_\mathbb{R}$.

Proposition 4.17. *If* $\mathbf{Pic}_{X/S}$ *exists and represents* $\mathrm{Pic}_{(X/S)\,(\mathrm{fppf})}$, *then* $\mathbf{Pic}_{X/S}$ *is locally of finite type.*

Proof. Set $P := \mathrm{Pic}_{(X/S)\,(\mathrm{fppf})}$. Owing to [**EGA IV**$_3$, 8.14.2], we just need to check the following condition. For every filtered inverse system of affine S-schemes T_i, the natural map is a bijection:

$$\varinjlim P(T_i) \xrightarrow{\sim} P\big(\varprojlim T_i\big).$$

To check it is injective, set $T := \varprojlim T_i$. Fix i and let $\lambda_i \in P(T_i)$. Represent λ_i by a sheaf $\mathcal{L}'_i$ on $X_{T'_i}$ where $T'_i \to T_i$ is an fppf covering. Set $T' := T'_i \times_{T_i} T$. Set $\mathcal{L}' := \mathcal{L}_i|X_{T'}$. Let λ be the image of λ_i in $P(T)$. Then $\mathcal{L}'$ represents λ.

Suppose $\lambda = 0 \in P(T)$. Then there exists an fppf covering $T'' \to T'$ such that $\mathcal{L}'|X_{T''} \simeq \mathcal{O}_{X_{T''}}$. It follows from [**EGA IV**$_3$, 8.8.2, 8.10.5(vi), 11.2.6, 8.5.2(i), and 8.5.2.4] that there exist a $j \ge i$ and an fppf covering $T''_j \to T'_j$ with $T'_j := T'_i \times_{T_j} T$ such that $\mathcal{L}'|X_{T''_j} \simeq \mathcal{O}_{X_{T''_j}}$. So λ_i maps to $0 \in P(T_j)$. Thus the map is injective.

Surjectivity can be proved similarly. □

Remark 4.18. There are three important existence theorems for $\mathbf{Pic}_{X/S}$, which refine Theorem 4.8. They were proved soon after it, and each involves new ideas.

First, Mumford proved the following generalization of Theorem 4.8, and it fits in nicely with Example 4.14.

Theorem 4.18.1. *Assume X/S is projective and flat, and its geometric fibers are reduced and connected; assume the irreducible components of its ordinary fibers are geometrically irreducible. Then* $\mathbf{Pic}_{X/S}$ *exists.*

Mumford stated this theorem at the bottom of Page viii in [**Mm66**]. He said the proof is a generalization of that [**Mm66**, Lects. 19–21] in the case where S is the spectrum of an algebraically closed field and X is a smooth surface. That proof is based on his theory of independent 0-cycles. This theory is further developed in [**AK79**, pp. 23–28] and used to prove [**AK79**, Thm. (3.1)], which asserts the existence of a natural compactification of $\mathbf{Pic}_{X/S}$ when X/S is flat and locally projective with integral geometric fibers.

On the other hand, Grothendieck [**FGA**, p. 236-1] attributed to Mumford a slightly different theorem, in which neither the geometric fibers nor the ordinary fibers are assumed connected (see [**BLR**, p. 210] also). Grothendieck said the proof is based on a refinement of Mumford's construction of quotients, and referred to the forthcoming notes of a Harvard seminar of Mumford and Tate's, held in the spring of 1962.

Mumford was kind enough, in November 2003, to mail the present author his personal folder of handwritten notes from the seminar; the folder is labeled, "Groth–Mumford–Tate," and contains notes from talks by each of the three, and notes written by each of them. Virtually all the content has appeared elsewhere; Mumford's contributions appeared in Mumford's books [**Mm65**], [**Mm66**], and [**Mm70**].

The notes contain, in Mumford's hand, a precise statement of the theorem and a rough sketch of the proof. This statement too is slightly different from that of Theorem 4.18.1: he crossed out the hypothesis that the geometric fibers are connected; and he made the weaker assumption that the ordinary fibers are connected. The proof is broadly like his proof in [**Mm66**, Lects. 19–21].

Second, Grothendieck proved this theorem of "generic representability."

Theorem 4.18.2. *Assume X/S is proper, and S is integral. Then there exists a nonempty open subset $V \subset S$ such that $\mathbf{Pic}_{X_V/V}$ exists, represents $\mathrm{Pic}_{(X_V/V)\,(\mathrm{fppf})}$, and is a disjoint union of open quasi-projective subschemes.*

A particularly important special case is covered by the following corollary.

Corollary 4.18.3. *Assume S is the spectrum of a field k, and X is complete. Then $\mathbf{Pic}_{X/k}$ exists, represents $\mathrm{Pic}_{(X/k)\,(\mathrm{fppf})}$, and is a disjoint union of open quasi-projective subschemes.*

Before Grothendieck discovered Theorem 4.18.2, he [**FGA**, Cor. 6.6, p. 232-17] proved Corollary 4.18.3 assuming X/k is projective. To do so, he developed a method of "relative representability," by which Theorem 4.8 implies the existence of the Picard scheme in other cases. The method incorporates a "dévissage" of Oort's [**Oo62**, §6]; the latter yields the Picard scheme as an extension of a group subscheme of $\mathbf{Pic}_{X_{\mathrm{red}}/k}$ by an affine group scheme.

In [**FGA**, Rem. 6.6, p. 232-17], Grothendieck said it is "extremely plausible" that Corollary 4.18.3 holds in general, and can be proved by extending the method of relative representability, so that it covers the case of a surjective map $X' \to X$ with X' projective, such as the map provided by Chow's lemma [**EGA II**, Thm. 5.6.1]. Furthermore, he conjectured, in [**FGA**, Rem. 6.6, p. 232-17], that, for any surjective map $X' \to X$ between proper schemes over a field, the induced map on Picard schemes is affine. He said he was led to the conjecture by considerations of "nonflat descent," a version of descent theory where the maps are not required to be flat, but the objects are.

Thanking Grothendieck for help, Murre [**Mr63**, (II.15)] gave the first proof of the heart of Corollary 4.18.3: if X/k is proper, then $\mathbf{Pic}_{X/k}$ exists and is locally of finite type. Murre did not use the method of relative representability. Rather, he identified seven conditions [**Mr63**, (I.2.1)] that are necessary and sufficient for the representability of a functor from schemes over a field to Abelian groups. Then he showed $\mathrm{Pic}_{(X/k)\,(\mathrm{fpqc})}$ satisfies the seven. To handle the last two, he used Chow's lemma and Theorem 4.8.

In the meantime, Grothendieck had proved Theorem 4.18.2, according to Murre [**Mr63**, p. 5]. Later, the proof appeared in two parts. The first part established a key intermediate result, the following theorem.

Theorem 4.18.4. *Assume X/S is proper. Let $\mathcal{F}$ be a coherent sheaf on X, and $S_{\mathcal{F}} \subset S$ the subfunctor of all S-schemes T such that $\mathcal{F}_T$ is T-flat. Then $S_{\mathcal{F}}$ is representable by an unramified S-scheme of finite type.*

Murre sketched Grothendieck's proof of this theorem in [**Mr65**, Cor. 1, p. 294-11]. The proof involves identifying and checking eight conditions that are necessary and sufficient for the representability of a functor by a separated and unramified S-scheme locally of finite type. As in Murre's proof of Corollary 4.18.3, a key step is to show the functor is "pro-representable": there exist certain natural topological rings, and if the functor is representable by Y, then these rings are the completions of the local rings at the points of Y that are closed in their fibers over S.

In the second part of Grothendieck's proof of Theorem 4.18.2, the main result is the following theorem of relative representability.

Theorem 4.18.5. *Assume S is integral. Let $X' \to X$ be a surjective map of proper S-schemes. Then there is a nonempty open subset $V \subset S$ such that the map*

$$\mathrm{Pic}_{(X_V/V)\,(\mathrm{fppf})} \to \mathrm{Pic}_{(X'_V/V)\,(\mathrm{fppf})}$$

is representable by quasi-affine maps of finite type.

In other words, for every S-scheme T and map $T \to \mathrm{Pic}_{(X'_V/V)\,(\mathrm{fppf})}$, the fibered product $\mathrm{Pic}_{(X_V/V)\,(\mathrm{fppf})} \times_{\mathrm{Pic}_{(X'_V/V)\,(\mathrm{fppf})}} T$ is representable and its projection to T is a quasi-affine map of finite type between schemes.

Raynaud [**Ra71**, Thm. 1.1] gave Grothendieck's proof of Theorem 4.18.5; the main ingredients are, indeed, Oort dévissage and nonflat descent. As a first consequence, Raynaud [**Ra71**, Cor. 1.2] derived Theorem 4.18.2. In order to show $\mathbf{Pic}_{X/S}$ is a disjoint union of open quasi-projective subschemes, he used a finiteness theorem for $\mathbf{Pic}_{X/S}$, which Grothendieck had stated under (v) in [**FGA**, p. C-08], and which is proved below as Theorem 6.16. As a second consequence of Theorem 4.18.5, Raynaud [**Ra71**, Cors. 1.5] established Grothendieck's conjecture that, over a field, a surjective map of proper schemes induces an affine map on Picard schemes.

The third important existence theorem for $\mathbf{Pic}_{X/S}$ is due to M. Artin, whose work greatly clarifies the situation. Artin proved $\mathbf{Pic}_{X/S}$ exists when it should, but not as a scheme. Rather, it exists as a more general object, called an "algebraic space," which is closer in nature to a (complex) analytic space.

Grothendieck [**FGA**, Rem. 5.2, p. 232-13] had said: "it is not ruled out that $\mathbf{Pic}_{X/S}$ exists whenever X/S is proper and flat and such that $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ holds universally. At least, this statement is proved in the context of analytic spaces when X/S is, in addition, projective." Mumford's example, Example 4.14, shows the statement is false for schemes; Artin's theorem shows the statement holds for algebraic spaces.

Algebraic spaces were introduced by Artin [**Ar68**, §1], and the theory developed by himself and his student Knutson [**Kn71**]. The spaces are constructed by gluing together schemes along open subsets that are isomorphic locally in the étale topology. Over $\mathbb{C}$, these open sets are locally analytically isomorphic; so an algebraic space is a kind of complex analytic space. In general and more formally, an algebraic space is the quotient in the category of étale sheaves of a scheme divided by an étale equivalence relation.

Artin [**Ar67**, Thm. 3.4, p. 35], proceeding in the spirit of Grothendieck and Murre, identified five conditions on a functor that are necessary and sufficient for it to be a locally separated algebraic space that is locally of finite type over a field or over an excellent Dedekind domain. In the proof of necessity, a key new ingredient is Artin's approximation theorem; it implies that the topological rings given by pro-representability can be algebraized. The resulting schemes are then glued together via an étale equivalence relation.

By checking that the five conditions hold, Artin [**Ar67**, Thm. 7.3, p. 67] proved the following theorem.

Theorem 4.18.6. *Let $f\colon X \to S$ be a flat, proper, and finitely presented map of algebraic spaces. Assume forming $f_*\mathcal{O}_X$ commutes with changing S (or f is "cohomologically flat in dimension zero"). Then $\mathbf{Pic}_{X/S}$ exists as an algebraic space, which is locally of finite presentation over S.*

Since f is finitely presented, the statement can be reduced to the case where S is locally of finite type over a field or over an excellent Dedekind domain. Artin's proof of Theorem 4.18.6 is direct: it involves no reduction whatsoever to Theorem 4.8.

On the other hand, it is not known, in general, whether Theorem 4.18.6 implies Theorem 4.8, which asserts $\mathbf{Pic}_{X/S}$ is a scheme. However, more is known over a *field*. Indeed, given an algebraic space that is locally of finite type over a field, Artin [**Ar67**, Lem. 4.2, p. 43] proved this: if the space is a sheaf of groups, then it is a group scheme. Thus Theorem 4.18.6 implies the heart of Corollary 4.18.3.

Thus, in Theorem 4.18.6, the fibers over S of the algebraic space $\mathbf{Pic}_{X/S}$ are schemes; they are the Picard schemes of the fibers X_s, though the X_s need not be schemes. In particular, if S and X are schemes — so the X_s are too — then the $\mathbf{Pic}_{X_s/k_s}$ are schemes. Furthermore, then they form a family; its total space $\mathbf{Pic}_{X/S}$ is an algebraic space, but not necessarily a scheme.

5. The connected component of the identity

Having treated the existence of the Picard scheme $\mathbf{Pic}_{X/S}$, we now turn to its structure. In this section, we study the union $\mathbf{Pic}^0_{X/S}$ of the connected components of the identity element, $\mathbf{Pic}^0_{X_s/k_s}$, for $s \in S$. We establish a number of basic properties, especially when S is the spectrum of a field.

It is remarkable how much we can prove about $\mathbf{Pic}^0_{X/S}$ formally, or nearly so, from general principles. Notably, we can do without the finiteness theorems proved in the next section. In order to emphasize the formal nature and corresponding generality of the arguments, most of the results are stated with the general hypothesis that $\mathbf{Pic}_{X/S}$ exists instead of with specific hypotheses that imply it exists.

Lemma 5.1. *Let k be a field, and G a group scheme locally of finite type. Let G^0 denote the connected component of the identity element e.*

(1) *Then G is separated.*

(2) *Then G is smooth if it has a geometrically reduced open subscheme.*

(3) *Then G^0 is an open and closed group subscheme of finite type; it is geometrically irreducible; and forming it commutes with extending k.*

Proof. Since e is a k-point, it is closed. Define a map $\alpha\colon G \times G \to G$ on T-points by $\alpha(g, h) := gh^{-1}$. Then $\alpha^{-1}e \subset G \times G$ is a closed subscheme. Its T-points are just the pairs (g, g) for $g \in G(T)$; so $\alpha^{-1}e$ is the diagonal. Thus (1) holds.

Suppose G has a geometrically reduced open subscheme V. To prove G is smooth, we may replace k by its algebraic closure. Then V contains a nonempty smooth open subscheme W. Furthermore, given any two closed points g, $h \in G$, there is an automorphism of G that carries g to h, namely, multiplication by $g^{-1}h$. Taking $g \in W$, we conclude that G is smooth at h. Thus (2) holds.

Consider (3). By definition, G^0 is the largest connected subspace containing e. But the closure of a connected subspace is plainly connected; so G^0 is closed. By [**EGA I**, 6.1.9], in any locally Noetherian topological space, the connected components are open. Thus G^0 is open too.

Since G^0 is connected and has a k-point, G^0 is geometrically connected by [**EGA IV**$_2$, 4.5.14]. Thus forming G^0 commutes with extending k.

Furthermore, $G^0 \times G^0$ is connected by [**EGA IV**$_2$, 4.5.8]. So $\alpha(G^0 \times G^0) \subset G$ is connected, and contains e, so lies in G^0. Thus G^0 is a subgroup.

To prove G^0 is geometrically irreducible and quasi-compact, we may replace k by its algebraic closure. Then $G^0_{\text{red}} \times G^0_{\text{red}}$ is reduced by [**EGA IV**$_2$, 4.6.1]. Hence α induces a map from $G^0_{\text{red}} \times G^0_{\text{red}}$ into G^0_{red}. So G^0_{red} is a subgroup. Thus we may replace G^0 by G^0_{red}.

Since G^0 is reduced and k is algebraically closed, G^0 contains a nonempty smooth affine open subscheme U. Take arbitrary k-points $g \in U$ and $h \in G^0$. Then $hg^{-1}U$ is smooth and open, and contains h. Hence G^0 is irreducible locally at h. But h is arbitrary, and G^0 is connected. So G^0 is irreducible by [**EGA I**, 6.1.10].

Since G^0 is irreducible, its open subschemes U and hU meet. So their intersection contains a k-point g_1 since k is algebraically closed. Then $g_1 = hh_1$ for some a k-point $h_1 \in U$. Then $h = g_1h_1^{-1}$. But $h \in G^0$ is arbitrary. Hence $\alpha(U \times U) = G^0$. Now, $U \times U$ is affine, so quasi-compact. Hence G^0 is quasi-compact. By hypothesis, G is locally of finite type. Hence G^0 is of finite type. Thus (3) holds. □

Remark 5.2. Let G be a group scheme of finite type over a field, G_{red} its reduction. Then G_{red} need not be a group subscheme, because $G_{\text{red}} \times G_{\text{red}}$ need not be reduced. Waterhouse [**Wa79**, p. 53] gives two conditions equivalent to reducedness when G is finite in Exercise 9, and he gives a counterexample in Exercise 10.

Proposition 5.3. *Assume S is the spectrum of a field k. Assume $\mathbf{Pic}_{X/k}$ exists and represents $\text{Pic}_{(X/k)\,(\text{fppf})}$. Then $\mathbf{Pic}_{X/k}$ is separated, and it is smooth if it has a geometrically reduced open subscheme. Furthermore, the connected component of the identity $\mathbf{Pic}^0_{X/k}$ is an open and closed group subscheme of finite type; it is geometrically irreducible; and forming it commutes with extending k.*

Proof. This result follows formally from Proposition 4.17 and Lemma 5.1. □

Theorem 5.4. *Assume S is the spectrum of a field k. Assume X/k is projective, and X is geometrically integral. Then $\mathbf{Pic}^0_{X/k}$ exists and is quasi-projective. If also X is geometrically normal, then $\mathbf{Pic}^0_{X/k}$ is projective.*

Proof. Theorem 4.8 implies $\mathbf{Pic}_{X/k}$ exists and represents $\text{Pic}_{(X/k)\,(\text{ét})}$, so $\text{Pic}_{(X/k)\,(\text{fppf})}$. Hence $\mathbf{Pic}^0_{X/k}$ exists and is of finite type by Proposition 5.3 (in fact, here Proposition 4.17 is logically unnecessary since $\mathbf{Pic}_{X/k}$ is locally of finite type by Theorem 4.8). So $\mathbf{Pic}^0_{X/k}$ is quasi-projective by Exercise 4.11.

Suppose X is also geometrically normal. Since $\mathbf{Pic}^0_{X/k}$ is quasi-projective, to prove it is projective, it suffices to prove it is proper. By Lemma 5.1, forming

$\mathbf{Pic}^0_{X/k}$ commutes with extending k. And by [**EGA IV**$_2$, 2.7.1(vii)], a k-scheme is complete if (and only if) it is after extending k. So we may, and do, assume k is algebraically closed.

Recall the structure theorem of Chevalley and Rosenlicht for algebraic groups, or reduced connected group schemes of finite type over k; see [**Co02**, Thm. 1.1, p. 3]. The theorem says that every algebraic group is an extension of an Abelian variety (or complete algebraic group) by a linear (or affine) algebraic group. Recall also that every solvable linear algebraic k-group is triangularizable (the Lie–Kolchin theorem), so contains a copy of the multiplicative group or of the additive group; see [**Bo69**, (10.5) and (10.2)]. Hence it suffices to show that, if T denotes the affine line minus the origin, then every k-map $t\colon T \to (\mathbf{Pic}^0_{X/k})_{\mathrm{red}}$ is constant.

Since k is algebraically closed, X/k has a section. So t arises from an invertible sheaf $\mathcal{L}$ on $X \times T$ by the Comparison Theorem, Theorem 2.5. Since $X \times T$ is integral, there is a divisor D such that $\mathcal{O}(D) = \mathcal{L}$ by [**Ha83**, Ex. II, 6.15, p. 145].

Form the projection $p\colon X \times T \to X$. Restrict $\mathcal{L}$ to its generic fiber. This restriction is trivial as T is an open subset of the line. So there is a rational function ϕ on $X \times T$ such that $(\phi) + D$ restricts to the trivial divisor. Let $s\colon X \to X \times T$ be a section. Set $E := s^*((\phi) + D)$; then E is a well-defined divisor on X. Plainly, p^*E and $(\phi) + D$ coincide as cycles. Hence they coincide as divisors since $X \times T$ is normal. Therefore, $\mathcal{L} = p^*\mathcal{O}(E)$. Thus $t\colon T \to \mathbf{Pic}_{X/k}$ is constant. □

Corollary 5.5. *Assume S is the spectrum of an algebraically closed field k. Assume X is projective and integral. Set $P := \mathbf{Pic}^0_{X/k}$, and let $\mathcal{P}$ be the restriction to X_P of a Poincaré sheaf. Then a* Poincaré family *W exists; by definition, W is a relative effective divisor on X_P/P such that*

$$\mathcal{O}_{X_P}\big(W - (W_0 \times P)\big) \simeq \mathcal{P} \otimes f_P^*\mathcal{N}$$

where W_0 is the fiber over $0 \in P$ and where $\mathcal{N}$ is an invertible sheaf on P.

Proof. Note that P exists and is quasi-projective by Theorem 5.4 and that $\mathcal{P}$ exists by Exercise 3.11 and Exercise 4.3. Since P is Noetherian, Serre's Theorem [**EGA III**$_1$, 2.2.1] implies there is an N such that $R^i f_{P*}\mathcal{P}(n) = 0$ for all $i > 0$ and $n \geq N$. Recall that (4.8.2) implies (4.8.3); similarly, $\mathrm{H}^i(\mathcal{P}_t(n)) = 0$ for all $t \in P$. Fix an $n \geq N$ such that $\dim \mathrm{H}^0(\mathcal{O}_X(n)) > \dim P$.

Say $\lambda \in \mathbf{Pic}_{X/k}$ represents $\mathcal{O}_X(n)$. Form the automorphism of $\mathbf{Pic}_{X/k}$ of multiplication by λ. Plainly, P is carried onto the connected component, P' say, of λ. Let $q\colon P \xrightarrow{\sim} P'$ be the induced isomorphism. Let $\mathcal{P}'$ be the restriction to $X_{P'}$ of a Poincaré sheaf. Plainly, $(1 \times q)^*\mathcal{P}' \simeq \mathcal{P}(n) \otimes f_P^*\mathcal{N}$ for some invertible sheaf $\mathcal{N}$.

By Exercise 4.7, there is a coherent sheaf $\mathcal{Q}$ on P such that $\mathbf{P}(\mathcal{Q}) = \mathbf{Div}_{X/S}$. Moreover, $\mathcal{Q}|P'$ is locally free of rank $\dim \mathrm{H}^0(\mathcal{O}_X(n))$ owing to Subsection 3.10. So $\mathcal{Q}|P'$ is of rank at least $1 + \dim P$. Now, there is an m such that the sheaf $\mathcal{H}om(\mathcal{Q}|P', \mathcal{O}_P)(m)$ is generated by finitely many global sections; so a general linear combination of them vanishes nowhere by a well-known lemma [**Mm66**, p. 148] attributed to Serre. Hence there is a surjection $\mathcal{Q}|P' \twoheadrightarrow \mathcal{O}_P(m)$. Correspondingly, there is a P'-map $h'\colon P' \to \mathbf{P}(\mathcal{Q}|P')$; in other words, h' is a section of the restriction over P' of the Abel map $\mathbf{A}_{X/S}\colon \mathbf{Div}_{X/S} \to \mathbf{Pic}_{X/S}$.

Let $W' \subset X_{P'}$ be the pullback under $1 \times h'$ of the universal relative effective divisor. Then $\mathcal{O}_{X_{P'}}(W') = \mathcal{P}'$ since h' is a section of $\mathbf{A}_{X/S}|P'$. So, in particular, $\mathcal{O}_X(W'_\lambda) = \mathcal{O}_X(n)$. Set $W := (1 \times q)^{-1}W'$. Plainly, W is a Poincaré family. □

Remark 5.6. More generally, Theorem 5.4 holds whenever X/k is proper, whether X is integral or not. The proof is essential the same, but uses Corollary 4.18.3 in place of Theorem 4.8. In fact, the proof of the quasi-projectivity assertion is easier, and does not require Proposition 5.3, since $\mathbf{Pic}^0_{X/k}$ is given as contained in a quasi-projective scheme. On the other hand, the proof of the projectivity assertion requires an additional step: the reduction, when k is algebraically closed, to the case where X is irreducible. Here is the idea: since X is normal, X is the disjoint union of its irreducible components X_i, and plainly $\mathbf{Pic}_{X/k} = \prod \mathbf{Pic}_{X_i/k}$; hence, if the $\mathbf{Pic}^0_{X_i/k}$ are complete, so is $\mathbf{Pic}^0_{X/k}$.

Chow [**Ch57**, Thm. p.128] proved every algebraic group — indeed, every homogeneous variety — is quasi-projective. Hence, in Lemma 5.1 , if G is reduced, then G^0 is quasi-projective. In characteristic 0, remarkably G is smooth, so reduced; this result is generally attributed to Cartier, and is proved in [**Mm66**, p. 167].

It follows that Theorem 5.4 holds in characteristic 0 whenever X is a proper algebraic k-space. Indeed, the quasi-projectivity assertion holds in view of the preceding discussion and of the discussion at the end of Remark 4.18. The proof of the projectivity assertion has one more complication: it is necessary to work with an étale covering $U \to X$ where U is a scheme and with an invertible sheaf $\mathcal{L}$ on $U \times T$. However, the proof shows that, in arbitrary characteristic, if X is a proper and normal algebraic k-space, then $\mathbf{Pic}^0_{X/k}$ is proper.

Exercise 5.7. Assume X/S is projective and smooth, its geometric fibers are irreducible, and S is Noetherian. Using the Valuative Criterion [**Ha83**, Thm. 4.7, p. 101] rather than the Chevalley–Rosenlicht structure theorem, prove that a closed subscheme $Z \subset \mathbf{Pic}_{X/S}$ is projective over S if it is of finite type.

Remark 5.8. There are three interesting alternative proofs of the second assertion of Theorem 5.4. The first alternative uses Exercise 5.7. It was sketched by Grothendieck [**FGA**, p. 236-12], and runs basically as follows. Proceed by induction on the dimension r of X/k. If $r = 0$, then $\mathbf{Pic}^0_{X/k}$ is S, so trivially projective. If $r = 1$, then $\mathbf{Pic}^0_{X/k}$ is projective by Exercise 5.7.

Suppose $r \geq 2$. As in the proof of the theorem, reduce to the case where k is algebraically closed. Let Y be a general hyperplane section of X. Then Y too is normal [**Se50**, Thm. 7′, p. 376]. Plainly the inclusion $\varphi\colon Y \hookrightarrow X$ induces a map

$$\varphi^* : \mathbf{Pic}^0_{X/k} \to \mathbf{Pic}^0_{Y/k} \,.$$

By induction, $\mathbf{Pic}^0_{Y/k}$ is projective. So $\mathbf{Pic}^0_{X/k}$ is projective too if φ^* is finite.

In order to handle φ^*, Grothendieck suggested using a version of the "known equivalence criteria." In this connection, he [**FGA**, p. 236-02] announced that [**SGA 2**] contains some key preliminary results, which must be combined with the existence theorems for the Picard scheme. In fact, [**SGA 2**, Cor. 3.6, p. 153] does directly imply φ^* is injective for $r \geq 3$. Hence φ^* is generically finite. So φ^* is finite since it is homogeneous.

For any $r \geq 2$, [**Kl73**, Lem. 3.11, p. 639, and Rem. 3.12, p. 640] assert $\ker \varphi^*$ is finite and unipotent, so trivial in characteristic 0; however, the proofs in [**Kl73**] require the compactness of $\mathbf{Pic}^0_{X/k}$. It would be good to have a direct proof of this assertion also when $r = 2$, a proof in the spirit of [**SGA 2**].

In characteristic 0, Mumford [**Mm67**, p. 99] proved as follows $\ker \varphi^*$ vanishes. Suppose not. First of all, $\ker \varphi^*$ is reduced by Cartier's theorem. So $\ker \varphi^*$ contains a point λ of order $n > 1$. And λ defines an unramified Galois cover X'/X with group $\mathbb{Z}/n$. Set $Y' := Y \times_X X'$. Then Y' is a disjoint union of n copies of Y because $\lambda \in \ker \varphi^*$. On the other hand, Y' is ample since Y is; hence, Y' is connected (by Corollary B.29 for example). We have a contradiction. Thus $\ker \varphi^*$ vanishes.

In characteristic 0, the injectivity of φ^* also follows from the Kodair Vanishing Theorem. Indeed, as just noted, $\ker \varphi^*$ is reduced. So φ^* is injective if and only if its differential is zero. Now, this differential is, owing to Theorem 5.11 below, equal to the natural map

$$\mathrm{H}^1(\mathcal{O}_X) \to \mathrm{H}^1(\mathcal{O}_Y).$$

Its kernel is equal to $\mathrm{H}^1(\mathcal{O}_X(-1))$ owing to the long exact sequence of cohomology.

In characteristic 0, if X is smooth, then $\mathrm{H}^1(\mathcal{O}_X(-1))$ vanishes by the Kodaira Vanishing Theorem. If also $r = 2$, then the dual group $\mathrm{H}^1(\Omega^2_X(1))$ vanishes by the theorem on the regularity of the adjoint system, which was proved by Picard in 1906 using Abelian integrals and by Severi in 1908 using algebro-geometric methods. For more information, see [**Za35**, pp. 181, 204–206] and [**Mm67**, pp. 94–97].

For any $r \geq 2$, there are, as Grothendieck [**FGA**, p. 236-12] suggested, finitely many smooth irreducible curves $Y_i \subset X$ such that the induced map is injective:

$$\mathbf{Pic}^0_{X/k} \to \prod \mathbf{Pic}^0_{Y_i/k}.$$

So, again, since the $\mathbf{Pic}^0_{Y_i/k}$ are projective, $\mathbf{Pic}^0_{X/k}$ is projective too.

To find the Y_i, use the final version of the "equivalence criteria" proved by Weil [**We54**, Cor. 2, p. 159]; it says in other words that, if W/T is the family of smooth 1-dimensional linear-space sections of X (or even a nonempty open subfamily), then the induced map is injective:

$$\mathbf{Pic}^0_{X/k}(k) \to \mathbf{Pic}^0_{W/T}(T).$$

For each finite set F of k-points of T, let K_F be the kernel of the map

$$\mathbf{Pic}^0_{X/k} \to \textstyle\prod_{t\in F} \mathbf{Pic}_{W_t/k}.$$

Since K_F is closed, we may assume, by Noetherian induction, that K_F contains no strictly smaller K_G. Suppose K_F has a nonzero k-point. It yields a nonzero T-point of $\mathrm{Pic}_{W/T}$, so a nonzero k-point of $\mathbf{Pic}_{W_t/k}$ for some k-point t of T. Let G be the union of F and $\{t\}$. Then K_G is strictly smaller than K_F. So $K_F = 0$. Take the Y_i to be the W_t for $t \in F$.

In characteristic 0 or if $r = 2$, another way to finish is to take a desingularization $\vartheta\colon X' \to X$. Since X is normal, a divisor D on X is principal if $\vartheta^* D$ is principal; hence, the induced map

$$\vartheta^* : \mathbf{Pic}^0_{X/k} \to \mathbf{Pic}^0_{X'/k}$$

is injective. But $\mathbf{Pic}^0_{X'/k}$ is projective by Exercise 5.7. So $\mathbf{Pic}^0_{X/k}$ is projective too.

The second alternative proof of Theorem 5.4 is similar to the proof in Answer 5.7. (It too may be due to Grothendieck — see [**EGA IV**$_4$, 21.14.4, iv] — but was indicated to the author by Mumford in a private communication in 1974). Again we reduce to the case where k is algebraically closed. Then, using a more refined form of the Valuative Criterion (obtained modifying [**Ha83**, Ex. 4.11, p. 107] slightly), we need only check this statement: given a k-scheme T of the form $T = \mathrm{Spec}(C)$ where C is a complete discrete valuation ring with algebraically closed residue field k_0, and with fraction field K say, and given a divisor D on X_K, its closure D' is a divisor on X_T.

This statement follows from the Ramanujam–Samuel Theorem [**EGA IV**$_4$, 21.14.1], a result in commutative algebra. Apply it taking B to be the local ring of a closed point of X_T, and A to be the local ring of the image point in X_{k_0}. The hypotheses hold because A and B share the residue field k_0. The completion $\widehat{A}$ is a normal domain by a theorem of Zariski's [**ZS60**, Thm. 32, p. 320]. And $A \to B$ is formally smooth because C is a formal power series ring over k_0 by a theorem of Cohen's [**ZS60**, Thm. 32, p. 320].

The third alternative proof is somewhat like the second, but involves some geometry instead of the Ramanujam–Samuel Theorem; see [**AK74**, Thm. 19, p. 138]. Moreover, C need not be complete, just discrete, and k_0 need not be algebraically closed. Here is the idea. Let $E \subset X_{k_0}$ be the closed fiber of D'/T. In $\mathbf{Hilb}_{X_L/L}$, form the sets U and V parameterizing the divisorial cycles linearly equivalent to those of the form D_L+H and E_L+H as H ranges over the divisors whose associated sheaves are algebraically equivalent to $\mathcal{O}_{X_L}(n)$ for a suitably large n.

It can be shown that U and V are dense open subsets of the same irreducible component of $\mathbf{Hilb}_{X_L/L}$. Hence they have a common point. Let I and J be the ideals of D and E. Then there are invertible sheaves $\mathcal{L}$ and $\mathcal{M}$ on X_L such that $I_L \otimes \mathcal{L}$ and $J_L \otimes \mathcal{M}$ are isomorphic. Since I_L is invertible, so is J_L. Hence so is J. Thus E is a divisor, as desired.

More generally, if X is not necessarily projective, but is simply complete and normal, then $\mathbf{Pic}^0_{X/k}$ is still complete, whether X is a scheme or algebraic space. Indeed, the original proof and its second alternative work without essential change. The first and third alternatives require X to be projective. However, it is easy to see as follows that this case implies the general case.

Namely, we may assume k is algebraically closed. By Chow's lemma, there is a projective variety Y and a birational map $\gamma\colon Y \to X$. Since X and Y are normal, a divisor D on X is the divisor of a function h if and only if γ^*D is the divisor of γ^*h. Hence the induced map $\mathbf{Pic}^0_{X/k} \to \mathbf{Pic}^0_{Y/k}$ is injective. It follows, as above, that $\mathbf{Pic}^0_{X/k}$ is complete since $\mathbf{Pic}^0_{Y/k}$ is.

Definition 5.9. Assume S is the spectrum of a field k. Let $\mathcal{L}$ and $\mathcal{N}$ be invertible sheaves on X. Then $\mathcal{L}$ is said to be *algebraically equivalent to* $\mathcal{N}$ if, for some n and all i with $1 \le i \le n$, there exist a connected k-scheme of finite type T_i, geometric points s_i, t_i of T_i with the same field, and an invertible sheaf $\mathcal{M}_i$ on X_{T_i} such that

$$\mathcal{L}_{s_1} \simeq \mathcal{M}_{1,s_1},\ \mathcal{M}_{1,t_1} \simeq \mathcal{M}_{2,s_2},\ \ldots,\ \mathcal{M}_{n-1,t_{n-1}} \simeq \mathcal{M}_{n,s_n},\ \mathcal{M}_{n,t_n} \simeq \mathcal{N}_{t_n}.$$

Proposition 5.10. *Assume S is the spectrum of a field k. Assume $\mathbf{Pic}_{X/k}$ exists and represents $\mathrm{Pic}_{(X/k)\,(\mathrm{fppf})}$. Let $\mathcal{L}$ be an invertible sheaf on X, and $\lambda \in \mathbf{Pic}_{X/k}$ the corresponding point. Then $\mathcal{L}$ is algebraically equivalent to $\mathcal{O}_X$ if and only if $\lambda \in \mathbf{Pic}^0_{X/k}$.*

Proof. Suppose $\mathcal{L}$ is algebraically equivalent to $\mathcal{O}_X$, and use the notation of Definition 5.9. Then $\mathcal{M}_i$ defines a map $\tau_i\colon T_i \to \mathbf{Pic}_{X/k}$. Now, $\mathcal{M}_{n,t_n} \simeq \mathcal{O}_{X_{t_n}}$. So $\tau_n(t_n) \in \mathbf{Pic}^0_{X/k}$. Suppose $\tau_i(t_i) \in \mathbf{Pic}^0_{X/k}$. Then $\tau_i(T_i) \subset \mathbf{Pic}^0_{X/k}$ since T_i is connected. So $\tau_i(s_i) \in \mathbf{Pic}^0_{X/k}$. But $\mathcal{M}_{i,s_i} \simeq \mathcal{M}_{i-1,t_{i-1}}$. So $\tau_{i-1}(t_{i-1}) \in \mathbf{Pic}^0_{X/k}$. Descending induction yields $\tau_1(s_1) \in \mathbf{Pic}^0_{X/k}$. But $\mathcal{M}_{1,s_1} \simeq \mathcal{L}_{s_1}$. Thus $\lambda \in \mathbf{Pic}^0_{X/k}$.

Conversely, suppose $\lambda \in \mathbf{Pic}^0_{X/k}$. The inclusion $\mathbf{Pic}^0_{X/k} \hookrightarrow \mathbf{Pic}_{X/k}$ is defined by an invertible sheaf $\mathcal{M}$ on X_T for some fppf covering $T \to \mathbf{Pic}^0_{X/k}$. Let t_1, t_2 be geometric points of T lying over λ, $0 \in \mathbf{Pic}^0_{X/k}$. Let $T_1, T_2 \subset T$ be irreducible components containing t_1, t_2, and $T'_1, T'_2 \subset \mathbf{Pic}^0_{X/k}$ their images. The latter contain open subsets because an fppf map is open by [**EGA IV**$_2$, 2.4.6]. Since $\mathbf{Pic}^0_{X/k}$ is irreducible by Lemma 5.1 , these open subsets contain a common point. Say it is the image of geometric points t_1, s_2 of T_1, T_2. Then $\mathcal{M}_{1,t_1} \simeq \mathcal{M}_{2,s_2}$ by Exercise 2.6. Set $\mathcal{M}_i := \mathcal{M}|T_i$. Thus $\mathcal{L}$ is algebraically equivalent to $\mathcal{O}_X$. □

THEOREM 5.11. *Assume S is the spectrum of a field k. Assume $\mathbf{Pic}_{X/k}$ exists and represents $\mathrm{Pic}_{(X/k)\,(\text{ét})}$. Let $\mathrm{T}_0\,\mathbf{Pic}_{X/k}$ denote the tangent space at 0. Then*

$$\mathrm{T}_0\,\mathbf{Pic}_{X/k} = \mathrm{H}^1(\mathcal{O}_X).$$

PROOF. Let P be any k-scheme locally of finite type, $e \in P$ a rational point, A its local ring, and $\mathbf{m}$ its maximal ideal. Usually, by the "tangent space" T_eP is meant the Zariski tangent space $\mathrm{Hom}(\mathbf{m}/\mathbf{m}^2, k)$. However, as in differential geometry, T_eP may be viewed as the vector space of k-derivations $\delta\colon A \to k$. Indeed, $\delta(\mathbf{m}^2) = 0$; so δ induces a linear map $\mathbf{m}/\mathbf{m}^2 \to k$. Conversely, every such linear map arises from a δ, and δ is unique because $\delta(1) = 0$.

Let k_ε be the ring of "dual numbers," the ring obtained from k by adjoining an element ε with $\varepsilon^2 = 0$. Then any (fixed) derivation δ induces a local homomorphism of k-algebras $u\colon A \to k_\varepsilon$ by $u(a) := \overline{a} + \delta(a)\varepsilon$ where $\overline{a} \in k$ is the residue of a. Conversely, every such u arises from a unique δ.

On the other hand, to give a u is the same as to give a k-map t_ε from the "free tangent vector" $\mathrm{Spec}(k_\varepsilon)$ to P such that the image of t_ε has support at e. Denote the set of t_ε by $P(k_\varepsilon)_e$. Thus, as sets,

$$T_eP = P(k_\varepsilon)_e. \tag{5.11.1}$$

The vector space structure on T_eP transfers as follows. Given $a \in k$, define a k-algebra homomorphism $\mu_a\colon k_\varepsilon \to k_\varepsilon$ by $\mu_a\varepsilon := a\varepsilon$. Now, let δ be a derivation, and u the corresponding homomorphism. Then $a\delta$ corresponds to $\mu_a u$. Thus multiplication by a transfers as $P(\mu_a)\colon P(k_\varepsilon)_e \to P(k_\varepsilon)_e$.

Let $k_{\varepsilon,\varepsilon'}$ denote the ring obtained from k_ε by adjoining an element ε' with $\varepsilon\varepsilon' = 0$ and $(\varepsilon')^2 = 0$. Define a homomorphism $\sigma_1\colon k_{\varepsilon,\varepsilon'} \to k_\varepsilon$ by $\varepsilon \mapsto \varepsilon$ and $\varepsilon' \mapsto 0$. Define another $\sigma_2\colon k_{\varepsilon,\varepsilon'} \to k_\varepsilon$ by $\varepsilon \mapsto 0$ and $\varepsilon' \mapsto \varepsilon$. The σ_i induce a map of sets

$$\pi\colon P(k_{\varepsilon,\varepsilon'})_e \to P(k_\varepsilon)_e \times P(k_\varepsilon)_e$$

where $P(k_{\varepsilon,\varepsilon'})_e$ is the set of maps with image supported at e. Plainly π is bijective.

Define a third homomorphism $\sigma\colon k_{\varepsilon,\varepsilon'} \to k_\varepsilon$ by $\varepsilon \mapsto \varepsilon$ and $\varepsilon' \mapsto \varepsilon$. Given two derivations δ, δ', define a homomorphism $v\colon A \to k_{\varepsilon,\varepsilon'}$ by $v(a) := \overline{a} + \delta(a)\varepsilon + \delta'(a)\varepsilon'$. Then $\delta + \delta'$ corresponds to σv. Therefore, addition on T_eP transfers as

$$\alpha\colon P(k_\varepsilon)_e \times P(k_\varepsilon)_e \to P(k_\varepsilon)_e \text{ where } \alpha := P(\sigma)\pi^{-1}.$$

Suppose P is a group scheme, $e \in P$ the identity. The natural ring homomorphism $\rho\colon k_\varepsilon \to k$ induces a group homomorphism $P(\rho)\colon P(k_\varepsilon) \to P(k)$. Plainly

$$P(k_\varepsilon)_e = \ker P(\rho). \tag{5.11.2}$$

The left side T_0P is a vector space; the right side is a group. Does addition on the left match multiplication on the right? Yes, indeed! We know α is the addition map. We must show α is also the multiplication map. Let us do so.

Since π and $P(\sigma)$ arise from ring homomorphisms, both are group homomorphisms. Now $\alpha := P(\sigma)\pi^{-1}$. Hence α is a group homomorphism too. So

$$\alpha(m, n) = \alpha(m, e) \cdot \alpha(e, n)$$

where $e \in P(k_\varepsilon)_e$ is the identity. So we have to show $\alpha(m,e) = m$ and $\alpha(e,n) = n$.

Consider the inclusion $\iota\colon k_\varepsilon \to k_{\varepsilon,\varepsilon'}$. Plainly $\sigma_2\iota\colon k_\varepsilon \to k_{\varepsilon'}$ factors through $\rho\colon k_\varepsilon \to k$. Hence $P(\sigma_2)P(\iota)(m) = e$ for any $m \in P(k_\varepsilon)_e$ owing to Formula (5.11.2). On the other hand, $\sigma_1\iota$ is the identity of k_ε. Thus $\pi P(\iota)(m) = (m,e)$

Plainly $\sigma\iota\colon k_\varepsilon \to k_\varepsilon$ is also the identity of k_ε. So $P(\sigma)P(\iota)(m) = m$. Hence

$$\alpha(m,e) = P(\sigma)\pi^{-1}\pi P(\iota)(m) = m.$$

Similarly $\alpha(e,n) = n$. Thus α is the multiplication map.

Take $P := \mathbf{Pic}_{X/k}$, so $e = 0$. Then Formulas (5.11.1) and (5.11.2) yield

$$\mathrm{T}_0\,\mathbf{Pic}_{X/k} = \ker\bigl(\mathbf{Pic}_{X/k}(k_\varepsilon) \to \mathbf{Pic}_{X/k}(k)\bigr). \tag{5.11.3}$$

To compute this kernel, set $X_\varepsilon := X \otimes_k k_\varepsilon$, and form the truncated exponential sequence of sheaves of Abelian groups:

$$0 \to \mathcal{O}_X \to \mathcal{O}^*_{X_\varepsilon} \to \mathcal{O}^*_X \to 1,$$

where the first map takes a local section b to $1 + b\varepsilon$. This sequence is split by the map $\mathcal{O}^*_X \to \mathcal{O}^*_{X_\varepsilon}$ defined by $a \mapsto a + 0\cdot\varepsilon$. Hence taking cohomology yields this split exact sequence of Abelian groups:

$$0 \to \mathrm{H}^1(\mathcal{O}_X) \to \mathrm{H}^1(\mathcal{O}^*_{X_\varepsilon}) \to \mathrm{H}^1(\mathcal{O}^*_X) \to 1.$$

However, $\mathbf{Pic}_{X/k}$ represents $\mathrm{Pic}_{(X/k)\,(\text{ét})}$, which is the sheaf associated to the presheaf $T \mapsto \mathrm{H}^1(\mathcal{O}^*_{X_T})$. So there is a natural commutative square of groups

$$\begin{array}{ccc} \mathrm{H}^1(\mathcal{O}^*_{X_\varepsilon}) & \longrightarrow & \mathrm{H}^1(\mathcal{O}^*_X) \\ \big\downarrow & & \big\downarrow \\ \mathbf{Pic}_{X/k}(k_\varepsilon) & \longrightarrow & \mathbf{Pic}_{X/k}(k). \end{array} \tag{5.11.4}$$

Hence, there is an induced homomorphism between the horizontal kernels. Owing to Formula (5.11.3), this homomorphism is an additive map

$$v\colon \mathrm{H}^1(\mathcal{O}_X) \to \mathrm{T}_0\,\mathbf{Pic}_{X/k}\,.$$

Let $a \in k$. On $T_0\,\mathbf{Pic}_{X/k}$, scalar multiplication by a is, owing to the discussion after Formula (5.11.1), the map induced by $\mu_a\colon k_\varepsilon \to k_\varepsilon$. Now, μ_a induces an endomorphism of the above square. At the top, it arises from the map of sheaves of groups $\mathcal{O}^*_X \to \mathcal{O}^*_X$ defined by $\varepsilon \mapsto a\varepsilon$. So the induced endomorphism of $\mathrm{H}^1(\mathcal{O}_X)$ is scalar multiplication by a. Thus v is a map of k-vector spaces.

Square (5.11.4) maps to the corresponding square obtained by making a field extension K/k. Since the kernels are vector spaces, there is an induced square

$$\begin{array}{ccc} \mathrm{H}^1(\mathcal{O}_X)\otimes_k K & \longrightarrow & \mathrm{H}^1(\mathcal{O}_{X_K}) \\ {\scriptstyle v\otimes_k K}\big\downarrow & & \big\downarrow \\ \mathrm{T}_0\,\mathbf{Pic}_{X/k}\otimes_k K & \longrightarrow & \mathrm{T}_0\,\mathbf{Pic}_{X_K/K}\,. \end{array}$$

The two horizontal maps are isomorphisms. Hence, if the right-hand map is an isomorphism, so is v. Thus we may assume k is algebraically closed.

In Square (5.11.4), the two vertical maps are isomorphisms by Exercise 2.3 since $\mathbf{Pic}_{X/k}$ represents $\mathrm{Pic}_{(X/k)\,(\text{ét})}$ and k is algebraically closed. Therefore, v too is an isomorphism, as desired. □

Remark 5.12. There is a relative version of Theorem 5.11. Namely, assume $\mathbf{Pic}_{X/S}$ exists, represents $\mathrm{Pic}_{(X/S)\,(\text{ét})}$, and is locally of finite type, but let S be arbitrary. Then $\mathrm{R}^1 f_* \mathcal{O}_X$ is equal to the normal sheaf of $\mathbf{Pic}_{X/S}$ along the identity section, its "Lie algebra"; the latter is simply the dual of the restriction to this section of the sheaf of relative differentials. For more information, see the recent treatment [**LLR**, § 1] and the references it cites.

Corollary 5.13. *Assume S is the spectrum of a field k. Assume $\mathbf{Pic}_{X/k}$ exists and represents $\mathrm{Pic}_{(X/k)\,(\text{ét})}$. Then*

$$\dim \mathbf{Pic}_{X/k} \le \dim \mathrm{H}^1(\mathcal{O}_X).$$

Equality holds if and only if $\mathbf{Pic}_{X/k}$ is smooth at 0; if so, then $\mathbf{Pic}_{X/k}$ is smooth of dimension $\dim \mathrm{H}^1(\mathcal{O}_X)$ everywhere.

Proof. Plainly we may assume k is algebraically closed. Then, given any closed point $\lambda \in \mathbf{Pic}_{X/k}$, there is an automorphism of $\mathbf{Pic}_{X/k}$ that carries 0 to λ, namely, "multiplication" by λ. So $\mathbf{Pic}_{X/k}$ has the same dimension at λ as at 0, and $\mathbf{Pic}_{X/k}$ is smooth at λ if and only if it is smooth at 0.

By general principles, $\dim_0 \mathbf{Pic}_{X/k} \le \dim \mathrm{T}_0 \mathbf{Pic}_{X/k}$, and equality holds if and only if $\mathbf{Pic}_{X/k}$ is regular at 0. Moreover, $\mathbf{Pic}_{X/k}$ is regular at 0 if and only if it is smooth at 0 since k is algebraically closed. Therefore, the corollary results from Theorem 5.11. □

Corollary 5.14. *Assume S is the spectrum of a field k. Assume $\mathbf{Pic}_{X/k}$ exists and represents $\mathrm{Pic}_{(X/k)\,(\text{ét})}$. If k is of characteristic 0, then $\mathbf{Pic}_{X/k}$ is smooth of dimension $\dim \mathrm{H}^1(\mathcal{O}_X)$ everywhere.*

Proof. Since k is of characteristic 0, any group scheme locally of finite type over k is smooth by Cartier's theorem [**Mm66**, Thm. 1, p. 167]. So the assertion follows from Corollary 5.13. □

Remark 5.15. Over a field k of positive characteristic, $\mathbf{Pic}_{X/k}$ need not be smooth, even when X is a connected smooth projective surface. Examples were constructed by Igusa [**Ig55**] and Serre [**Sr56**, n° 20].

On the other hand, Mumford [**Mm66**, Lect. 27, pp. 193–198] proved that $\mathbf{Pic}_{X/k}$ is smooth if and only if all of Serre's Bockstein operations β_i vanish; here

$$\beta_1 : \mathrm{H}^1(\mathcal{O}_X) \to \mathrm{H}^2(\mathcal{O}_X) \text{ and } \beta_i \colon \ker \beta_{i-1} \to \operatorname{cok} \beta_{i-1} \text{ for } i \ge 2.$$

In fact, the tangent space to $\mathbf{Pic}_{X_{\mathrm{red}}/k}$ is the subspace of $\mathrm{H}^1(\mathcal{O}_X)$ given by

$$\mathrm{T}_0 \mathbf{Pic}_{X_{\mathrm{red}}/k} = \bigcap \ker \beta_i.$$

Moreover, here X need not be smooth or 2-dimensional.

The examples illustrate further pathologies. Set

$$g := \dim \mathbf{Pic}_{X/k},\ h^{0,1} := \dim \mathrm{H}^1(\mathcal{O}_X),\ \text{and } h^{1,0} := \dim \mathrm{H}^0(\Omega^1_X).$$

In Igusa's example, $g = 1$, $h^{0,1} = 2$, and $h^{1,0} = 2$; in Serre's, $g = 0$, $h^{0,1} = 1$, and $h^{1,0} = 0$. Moreover, Igusa had just proved that, in any event, $g \le h^{1,0}$.

By contrast, in characteristic 0, Serre's Comparison Theorem [**Ha83**, Thm. 2.1, p. 440] says that $h^{0,1}$ and $h^{1,0}$ can be computed by viewing X as a complex analytic manifold. Hence Hodge Theory yields

$$h^{0,1} = h^{1,0} \text{ and } h^{0,1} + h^{1,0} = b$$

where b is the first Betti number; see [**Za35**, p. 200]. Therefore, the following exercise now yields the Fundamental Theorem of Irregular Surfaces (1.4).

EXERCISE 5.16. Assume S is the spectrum of a field k. Assume X is a projective, smooth, and geometrically irreducible surface. According to the original definitions as stated in modern terms, the "geometric genus" of X is the number $p_g := \dim \mathrm{H}^0(\Omega^2_X)$; its "arithmetic genus" is the number $p_a := \phi(0) - 1$ where $\phi(n)$ is the polynomial such that $\phi(n) = \dim \mathrm{H}^0(\Omega^2_X(n))$ for $n \gg 0$; and its "irregularity" q is the difference between the two genera, $q := p_g - p_a$.

Show $\dim \mathbf{Pic}_{X/k} \le q$, with equality in characteristic 0.

EXERCISE 5.17. Assume S is the spectrum of an algebraically closed field k. Assume X is projective and integral. Set $q := \dim \mathrm{H}^1(\mathcal{O}_X)$.

Show $q = 0$ if and only if every algebraic system of curves is "contained completely in a linear system." The latter condition means just that, given any relative effective divisor D on X_T/T where T is a connected k-scheme, there exist invertible sheaves $\mathcal{L}$ on X and $\mathcal{N}$ on T such that $\mathcal{O}_{X_T}(D) \simeq \mathcal{L}_T \otimes f_T^*\mathcal{N}$. The condition may be put more geometrically: in the notation of Exercise 3.14, it means there is a map, necessarily unique, $w\colon T \to L$ such that $(1 \times w)^{-1}W = D$.

In characteristic 0, show $q = 0$ if the condition holds for all smooth such T.

REMARK 5.18. Assume S is the spectrum of a field k. Assume X/k is projective, and X is geometrically integral. Let $D \subset X$ be an effective divisor, and $\mathcal{N}_D$ its normal sheaf. Let $\delta \in \mathbf{Div}_{X/k}$ be the point representing D, and $\lambda \in \mathbf{Pic}_{X/k}$ the point representing $\mathcal{O}_X(D)$.

Then the tangent space at δ is given by the formula

$$\mathrm{T}_\delta \, \mathbf{Div}_{X/k} = \mathrm{H}^0(\mathcal{N}_D), \tag{5.18.1}$$

which respects the vector space structure of each side. This formula can be proved with a simple elementary computation; see [**Mm66**, Cor., p. 154].

Form the fundamental exact sequence of sheaves

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{N}_D \to 0,$$

and consider its associated long exact sequence of cohomology groups

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^0(\mathcal{O}_X) &\longrightarrow \mathrm{H}^0(\mathcal{O}_X(D)) \longrightarrow \mathrm{H}^0(\mathcal{N}_D) \\ \xrightarrow{\partial^0} \mathrm{H}^1(\mathcal{O}_X) &\longrightarrow \mathrm{H}^1(\mathcal{O}_X(D)) \xrightarrow{u} \mathrm{H}^1(\mathcal{N}_D) \xrightarrow{\partial^1} \mathrm{H}^2(\mathcal{O}_X) \end{aligned}$$

Another elementary computation shows that the boundary map ∂^0 is equal to the tangent map of the Abel map, $\mathrm{T}_\delta \, \mathbf{Div}_{X/k} \to T_\lambda \, \mathbf{Pic}_{X/k}$; see [**Mm66**, Prp., p. 165].

By definition, D is said to be "semiregular" if the boundary map ∂^1 is injective. Plainly, it is equivalent that $u = 0$. So it is equivalent that $\dim \mathrm{H}^0(\mathcal{N}_D) = R$ where

$$R := \dim \mathrm{H}^1(\mathcal{O}_X) + \dim \mathrm{H}^0(\mathcal{O}_X(D)) - 1 - \dim \mathrm{H}^1(\mathcal{O}_X(D)).$$

Semiregularity was recognized in 1944 by Severi as precisely the right positivity condition for the old (1904) theorem of the completeness of the characteristic system, although Severi formulated the condition in an equivalent dual manner.

In its modern formulation, the theorem of the completeness of the characteristic system asserts that $\mathbf{Div}_{X/k}$ is smooth of dimension R at δ if and only if D is semiregular, provided the characteristic is 0, or more generally, $\mathbf{Pic}_{X/k}$ is smooth. Indeed, Formula (5.18.1) says that the characteristic system of $\mathbf{Div}_{X/k}$ on D is always equal to the complete linear system of the invertible sheaf $\mathcal{N}_D$. But $\mathbf{Div}_{X/k}$ can have a nilpotent or a singularity at δ. So, in effect, Enriques and Severi had simply sought conditions guaranteeing $\mathbf{Div}_{X/k}$ is smooth of dimension R at δ.

The first purely algebraic discussion of the theorem was made by Grothendieck [**FGA**, Sects. 221-5.4 to 5.6], and he proved it in the two most important cases. Specifically, he noted that, if $\mathrm{H}^1(\mathcal{O}_X(D)) = 0$, then the Abel map is smooth; see the end of the proof of Theorem 4.8. Hence $\mathbf{Div}_{X/k}$ is smooth at δ if and only if $\mathbf{Pic}_{X/k}$ is smooth; furthermore, $\mathbf{Pic}_{X/k}$ is smooth in characteristic zero by Cartier's theorem. Grothendieck pointed out that this case had been treated with transcendental means by Kodaira in 1956.

Grothendieck also observed $\mathrm{H}^1(\mathcal{N}_D)$ houses the obstruction to deforming D in X. Hence, if this group vanishes, then $\mathbf{Div}_{X/k}$ is smooth at δ in any characteristic. Mumford [**Mm66**, pp. 157–159] explicitly worked out the obstruction and its image under ∂^1. If $\partial^1 = 0$, so if D is semiregular, then $\mathbf{Div}_{X/k}$ is smooth if this image vanishes. Inspired by work of Kodaira and Spencer in 1959, Mumford used the exponential in characteristic 0 and proved the image vanishes. Mumford did not use Cartier's theorem; so the latter results from taking a D with $\mathrm{H}^1(\mathcal{O}_X(D)) = 0$.

A purely algebraic proof of the full completeness theorem is given in [**Kl73**, Thm., p. 307]. This proof was inspired by Kempf's (unpublished) thesis. The proof does not use obstruction theory, but only simple formal properties of a scheme of the form $\mathbf{P}(\mathcal{Q})$ where $\mathcal{Q}$ arises from an invertible sheaf $\mathcal{F}$ as in Subsection 3.10. This proof works, more generally, if S is arbitrary and if X/S is projective and flat and has integral geometric fibers. Here D is the divisor on the geometric fiber through δ. Then provided $\mathbf{Pic}_{X/S}$ is smooth, $\mathbf{Div}_{X/S}$ is smooth of relative dimension R at δ if and only if D is semiregular.

There is a celebrated example, valid over an algebraically closed field k of any characteristic, where $\mathbf{Div}_{X/k}$ is nonreduced at δ. The example was discovered by Severi and Zappa in the 1940s, and is explained in [**Mm66**, pp. 155–156]. Here is the idea. Let C be an elliptic curve, and $0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{O}_C \to 0$ the nontrivial extension; set $X := \mathbf{P}(\mathcal{E})$.

Let D be the section of X/C defined by $\mathcal{E} \to \mathcal{O}_C$. Then $\mathcal{N}_D = \mathcal{O}_C$ by [**Ha83**, Prp. 2.8, p. 372]; so $\dim T_\delta\, \mathbf{Div}_{X/k} = 1$. However, δ is an isolated point. Otherwise, the connected component of δ contains a second closed point. And it represents a curve D' algebraically equivalent to D. So $\deg \mathcal{O}_D(D') = \deg \mathcal{N}_D = 0$. Hence D and D' are disjoint. Let F be a fiber. Then $\deg \mathcal{O}_F(D) = \deg \mathcal{O}_F(D') = 1$. Hence D' is a second section. Therefore, $\mathcal{E}$ is decomposable, a contradiction.

Proposition 5.19. *Assume $\mathbf{Pic}_{X/S}$ exists and represents $\mathrm{Pic}_{(X/S)\,(\text{ét})}$. Let $s \in S$ be a point such that $\mathrm{H}^2(\mathcal{O}_{X_s}) = 0$. Then there exists an open neighborhood of s over which $\mathbf{Pic}_{X/S}$ is smooth.*

Proof. By the Semicontinuity Theorem [**EGA III**$_2$, 7.7.5-I)], there exists an open neighborhood U of s such that $\mathrm{H}^2(\mathcal{O}_{X_t}) = 0$ for all $t \in U$. Replace S by U.

By [**EGA III**$_1$, **0**-10.3.1, p. 20], there is a flat local homomorphism from $\mathcal{O}_s$ into a Noetherian local ring B whose residue field is algebraically closed. By [**EGA IV**$_2$, 6.8.3], if f_B is smooth, then $f\colon X \to S$ is smooth along X_s. Replace S by $\mathrm{Spec}(B)$.

By the Infinitesimal Criterion for Smoothness [**SGA 1**, p. 67], it suffices to show this: given any S-scheme T of the form $T = \mathrm{Spec}(A)$ where A is an Artin local ring that is a finite $\mathcal{O}_s$-algebra and given any closed subscheme $R \subset T$ whose ideal $\mathcal{I}$ has square 0, every R-point of $\mathbf{Pic}_{X/S}$ lifts to a T-point.

The residue field of A is a finite extension of k_s, which is algebraically closed; so the two fields are equal. Hence the R-point is defined by an invertible sheaf on X_R by Exercise 2.3. So we want to show invertible sheaves on X_R lift to X_T.

Since $\mathcal{I}^2 = 0$, we can form the truncated exponential sequence

$$0 \to f_T^*\mathcal{I} \to \mathcal{O}_{X_T}^* \to \mathcal{O}_{X_R}^* \to 1.$$

It yields the following exact sequence:

$$\mathrm{H}^1(\mathcal{O}_{X_T}^*) \to \mathrm{H}^1(\mathcal{O}_{X_R}^*) \to \mathrm{H}^2(f_T^*\mathcal{I}).$$

Hence it suffices to show $\mathrm{H}^2(f_T^*\mathcal{I}) = 0$.

Since T is affine, $\mathrm{H}^2(f_T^*\mathcal{I}) = H^0(\mathrm{R}^2 f_{T*}f_T^*\mathcal{I})$ owing to [**Ha83**, Prp. 8.5, p. 251]. But $\mathrm{H}^2(\mathcal{O}_{X_t}) = 0$ for all $t \in S$. Hence $\mathrm{R}^2(f_T^*\mathcal{I}) = 0$ by the Property of Exchange [**EGA III**$_2$, 7.7.5 II and 7.7.10]. Thus $\mathrm{H}^2(f_T^*\mathcal{I}) = 0$, as desired. □

Proposition 5.20. *Assume* $\mathbf{Pic}_{X/S}$ *exists, and represents* $\mathrm{Pic}_{(X/S)\,(\mathrm{fppf})}$. *For* $s \in S$, *assume all the* $\mathbf{Pic}^0_{X_s/k_s}$ *are smooth of the same dimension. Then* $\mathbf{Pic}_{X/S}$ *has an open group subscheme* $\mathbf{Pic}^0_{X/S}$ *of finite type whose fibers are the* $\mathbf{Pic}^0_{X_s/k_s}$. *Furthermore, if* S *is reduced, then* $\mathbf{Pic}^0_{X/S}$ *is smooth over* S. *Moreover, if all the* $\mathbf{Pic}^0_{X_s/k_s}$ *are complete and if* $\mathbf{Pic}_{X/S}$ *is separated over* S, *then* $\mathbf{Pic}^0_{X/S}$ *is closed in* $\mathbf{Pic}_{X/S}$ *and proper over* S.

Proof. First off, $\mathbf{Pic}_{X/S}$ is locally of finite type by Proposition 4.17. Now, for every $s \in S$, the schemes $\mathbf{Pic}_{X/S} \otimes k_s$ and $\mathbf{Pic}_{X_s/k_s}$ coincide by Exercise 4.4. And the $\mathbf{Pic}^0_{X_s/k_s}$ are smooth of the same dimension by hypothesis. So the $\mathbf{Pic}^0_{X_s/k_s}$ form an open subscheme $\mathbf{Pic}^0_{X/S}$ of $\mathbf{Pic}_{X/S}$ and the structure map $\sigma\colon \mathbf{Pic}^0_{X/S} \to S$ is universally open by [**EGA IV**$_3$, 15.6.3 and 15.6.4]. Furthermore, if S is reduced, then σ is flat by [**EGA IV**$_3$, 15.6.7], so smooth by [**EGA IV**$_4$, 17.5.1].

Define a map $\alpha\colon \mathbf{Pic}^0_{X/S} \times_S \mathbf{Pic}^0_{X/S} \to \mathbf{Pic}_{X/S}$ by $\alpha(g,h) := gh^{-1}$. Then α factors through the open subscheme $\mathbf{Pic}^0_{X/S}$ because forming α commutes with passing to the fibers. Hence $\mathbf{Pic}^0_{X/S}$ is a subgroup

To prove $\mathbf{Pic}^0_{X/S}$ is of finite type, we may work locally on S, and so assume S is Noetherian. Since $\mathbf{Pic}_{X/S}$ is locally of finite type, we need only prove $\mathbf{Pic}^0_{X/S}$ is quasi-compact.

Let $V \subset S$ be a nonempty affine open subscheme, and $U \subset \sigma^{-1}V$ another. Then $\sigma U \subset S$ is open since σ is an open map. Set $U' := \sigma^{-1}\sigma U$. Then U' is open, and α restricts to a map $\alpha'\colon U \times_V U \to U'$. In fact, α' is surjective because its geometric fibers are surjective by an argument at the end of the proof of Lemma 5.1. Now, $U \times_V U$ so quasi-compact. Hence U' is quasi-compact.

Set $T := S - \sigma U$. By Noetherian induction, we may assume $\sigma^{-1}T$ is quasi-compact. Therefore, $U' \cup \sigma^{-1}T$ is quasi-compact. But it is equal to $\mathbf{Pic}^0_{X/S}$. Thus $\mathbf{Pic}^0_{X/S}$ is quasi-compact, as desired.

Moreover, if all the $\mathbf{Pic}^0_{X_s/k_s}$ are complete and if $\mathbf{Pic}_{X/S}$ is separated over S, then $\mathbf{Pic}^0_{X/S}$ is proper over S by [**EGA IV**$_3$, 15.7.11]. Finally, consider the inclusion map $\mathbf{Pic}^0_{X/S} \hookrightarrow \mathbf{Pic}_{X/S}$. It is proper as $\mathbf{Pic}^0_{X/S}$ is proper and $\mathbf{Pic}_{X/S}$ is separated; hence, $\mathbf{Pic}^0_{X/S}$ is closed. □

Remark 5.21. Assume the characteristic is 0 and $f\colon X \to S$ is smooth and proper. Then, in the last two propositions, more can be said, as Visoli explained to the author in early May 2004. These additions result from Part (i) of Theorem (5.5) on p. 123 in Deligne's article [**De68**] (which uses Hodge theory on p. 121). Deligne's theorem asserts that, under the present conditions, all the sheaves $\mathrm{R}^q f_*\Omega^p_{X/S}$ are locally free of finite rank, and forming them commutes with changing the base.

In Proposition 5.19, $\mathbf{Pic}_{X/S}$ *is smooth over the connected component* S_0 *of* s. Indeed, Deligne's theorem implies $\mathrm{R}^2 f_*\mathcal{O}_{X/S}|S_0$ vanishes since it is locally free, so of constant rank, and its formation commutes with passage to every fiber, in particular, to that over $s \in S_0$. Hence, $\mathrm{H}^2(\mathcal{O}_{X_t}) = 0$ for every $t \in S_0$. Therefore, Proposition 5.19, as it stands, implies $\mathbf{Pic}_{X/S}$ is smooth over S_0.

In Proposition 5.20, *there is no need to assume all the* $\mathbf{Pic}^0_{X_s/k_s}$ *are smooth of the same dimension.* Indeed, all the $\mathbf{Pic}^0_{X_s/k_s}$ are smooth of dimension $\dim \mathrm{H}^1(\mathcal{O}_{X_s})$ by Corollary 5.13. But $\dim \mathrm{H}^1(\mathcal{O}_{X_s})$ is constant on each connected component of S owing to Deligne's theorem.

Furthermore, if $f\colon X \to S$ *is smooth and projective locally over* S*, then* $\mathbf{Pic}^0_{X/S}$ *is smooth whether or not* S *is reduced.* Indeed, we may assume S is of finite type over $\mathbb{C}$; the reduction is standard, and sketched by Deligne at the beginning of his proof of his Theorem (5.5). By the Infinitesimal Criterion for Smoothness [**SGA 1**, p. 67], it suffices to show this: given any S-scheme T of the form $T = \mathrm{Spec}(A)$ where A is an Artin local ring that is a finite $\mathbb{C}$-algebra and given any closed subscheme $R \subset T$, every R-point of $\mathbf{Pic}^0_{X/S}$ lifts to a T-point. But, $\mathbf{Pic}^0_{X/S}$ is open in $\mathbf{Pic}_{X/S}$ by the above argument. Hence it suffices to show every R-point of $\mathbf{Pic}^0_{X/S}$ lifts to a T-point of $\mathbf{Pic}_{X/S}$.

The residue field of A is a finite extension of $\mathbb{C}$; so the two fields are equal. Hence the R-point is defined by an invertible sheaf on X_R by Exercise 2.3. So we want to show every invertible sheaf $\mathcal{L}$ on X_R lifts to X_T. By Serre's Comparison Theorem [**Ha83**, Thm. 2.1, p. 440], it suffices to lift $\mathcal{L}$ to an analytic invertible sheaf since $f_T\colon X_T \to T$ is projective. So pass now to the analytic category.

Let X_0 denote the closed fiber of X_T, and form the exponential sequence:

$$0 \to \mathbb{Z}_{X_0} \to \mathcal{O}_{X_0} \to \mathcal{O}^*_{X_0} \to 0.$$

Consider the class of $\mathcal{L}_{X_0}$ in $\mathrm{H}^1(\mathcal{O}^*_{X_0})$. It maps to 0 in $\mathrm{H}^2(\mathbb{Z}_{X_0})$ because this group is discrete and $\mathcal{L}$ defines an R-point of $\mathbf{Pic}^0_{X/S}$.

Form the exponential map $\mathcal{O}_{X_R} \to \mathcal{O}^*_{X_R}$, form its kernel $\mathcal{Z}$, and form the natural map $\kappa\colon \mathcal{Z} \to Z_{X_0}$. Then κ is bijective. Indeed, let a be a local section of $\mathcal{Z}$. Then $1 + a + a^2/2 + \cdots = 1$. Set $u := 1 + a/2 + \cdots$. Then $au = 0$. Suppose a maps to the local section 0 of $\mathcal{O}_{X_0}$. Then a is nilpotent. So u is invertible. Hence $a = 0$. Thus κ is injective. But κ is obviously surjective. Thus κ is bijective.

Consider the class λ of $\mathcal{L}$ in $\mathrm{H}^1(\mathcal{O}^*_{X_R})$. It follows that λ maps to 0 in $\mathrm{H}^2(\mathcal{Z})$. Hence λ comes from a class γ in $\mathrm{H}^1(\mathcal{O}_{X_R})$. By Deligne's theorem, γ lifts to a class γ' in $\mathrm{H}^1(\mathcal{O}_{X_T})$. The image of γ' in $\mathrm{H}^1(\mathcal{O}^*_{X_T})$ gives the desired lifting of $\mathcal{L}$ to X_T.

EXAMPLE 5.22. The following example complements Proposition 5.20, and was provided, in early May 2004, by Vistoli. The example shows $\mathbf{Pic}^0_{X/S}$ *can be smooth and proper over* S *and open and closed in* $\mathbf{Pic}_{X/S}$*, although* $\mathbf{Pic}_{X/S}$ *isn't smooth.* In fact, $f\colon X \to S$ is smooth and projective, its geometric fibers are integral, and S is a smooth curve over an algebraically closed field k of arbitrary characteristic. Furthermore, $\mathbf{Pic}_{X/S}$ has a component that is a reduced k-point; so it is not smooth over S, nor even flat.

To construct $f\colon X \to S$, set $P := \mathbb{P}^3_k$ and fix $d \geq 4$. Set $\mathcal{Q} := \mathrm{H}^0(\mathcal{O}_P(d))^*$ where the '$*$' means dual. Set $H := \mathbb{P}(\mathcal{Q})$. Then H represents the functor (on

k-schemes T) whose T-points are the T-flat closed subschemes of P_T whose fibers are surfaces of degree d by Exercise 5.17 and Theorem 3.13. Let $W \subset P \times H$ be the universal subscheme. Its ideal $\mathcal{O}_{P\times H}(-W)$ is equal to the tensor product of the pullbacks of $\mathcal{O}_P(-d)$ and $\mathcal{O}_H(-1)$ by Exercise 3.14. Set $N := \dim H$.

Let G be the Grassmannian of lines in P, and $L \subset P \times G$ the universal line. Let $\pi\colon L \to G$ be the projection. Form the exact sequence of locally free sheaves

$$0 \to \mathcal{K} \to \pi_*\mathcal{O}_{P\times G}(d) \to \pi_*\mathcal{O}_L(d) \to 0,$$

which defines $\mathcal{K}$; the right-hand map is surjective, since forming it commutes with passing to the fibers, and on the fibers, it is plainly surjective. Set $I := \mathbf{P}(\mathcal{K}^*)$. Then I is smooth, irreducible, and of dimension $4 + N - (d + 1)$, or $N - d + 3$.

Note that $\pi_*\mathcal{O}_{P\times G}(d) = \mathcal{Q}_G^*$. So there is a surjection $\mathcal{Q}_G \to \mathcal{K}^*$, and it induces a closed embedding $I \subset G \times H$. Furthermore, given a T-point of $G \times H$, it lies in I if and only if $L_T \subset W_T$. Indeed, the latter means the ideal of $W_T \subset P_T$ maps to 0 in $\mathcal{O}_{L_T}$; in other words, the composition

$$\mathcal{O}_{P_T}(-d) \otimes \mathcal{O}_H(-1)_{P_T} \to O_{P_T} \to \mathcal{O}_{L_T}$$

vanishes. Equivalently, the composition

$$\mathcal{O}_H(-1)_T \to g_{T*}O_{P_T}(d) \to g_{T*}\mathcal{O}_{L_T}(d)$$

vanishes. But the first map is equal to the natural map $\mathcal{O}_H(-1)_T \to \mathcal{Q}_T^*$. So it is equivalent that this map factors through $\mathcal{K}_T$, or that $\mathcal{Q}_T \to \mathcal{O}_H(1)_T$ factors through $\mathcal{K}_T^*$. Equivalently, the T-point of $G \times H$ lies in I. Thus I is the graph of the incidence correspondence.

Let $J \subset H$ be the image of I. Note $J \neq H$ since $\dim I = \dim H - d + 3$ and $d \geq 4$. Consider the open set $U \subset H$ over which $W \to H$ is smooth. Then $U \cap J$ is nonempty. Indeed, choose coordinates w, x, y, z for P. Then $U \cap J$ contains the point representing the surface $\{w^d - x^d = y^d - z^d\}$ if $p \nmid d$ or the surface $\{wx^{d-1} = y^{d-1}\}$ if $p \mid d$, because, in either case, the surface is smooth and contains a line, either $\{w = x,\ y = z\}$ or $\{w = 0,\ y = 0\}$.

Let $s \in U \cap J$ be a simple k-point. Take a line $S \subset H$ through s and transverse to J. Replace S by $S \cap U$. Let $X \subset W$ be the preimage of S, and $f\colon X \to S$ the induced map. Then f is smooth and projective, and its geometric fibers are integral. Hence $\mathbf{Pic}_{X/S}$ exists, is separated, and represents $\mathrm{Pic}_{(X/S)\,(\text{ét})}$ by Theorem 4.8. Moreover, $\mathrm{H}^1(\mathcal{O}_{X_t}) = 0$ for each $t \in S$; hence, Corollary 5.13 implies $\mathbf{Pic}_{X_t/k_t}$ is reduced and discrete. In particular, $\mathbf{Pic}^0_{X_t/k_t}$ is smooth, of dimension 0, and coplete. Therefore, Proposition 5.20 implies $\mathbf{Pic}^0_{X/S}$ is smooth and proper over S and open and closed in $\mathbf{Pic}_{X/S}$

Since $s \in J$, the surface $X_s \subset P$ contains a line M. And X_s is smooth since $s \in U$; hence M is a divisor. Say $\mathcal{O}_{X_s}(M)$ defines the k-point $\mu \in \mathbf{Pic}_{X/S}$. View μ as a reduced closed subscheme. Then μ is a connected component of its fiber $\mathbf{Pic}_{X_s/k}$ because, as just noted, this fiber is reduced and discrete. It remains to prove μ is a connected component of $\mathbf{Pic}_{X/S}$.

Suppose not, and let's find a contradiction. Let Q be the connected component of $\mu \in \mathbf{Pic}_{X/S}$. Let k_ε be the ring of "dual numbers," and set $T := \mathrm{Spec}(k_\varepsilon)$. Then there is a closed embedding $T \hookrightarrow \mathbf{Pic}_{X/S}$ supported at μ. However, T does not embed into the fiber $\mathbf{Pic}_{X_s/k}$ because the latter is reduced and discrete. Hence the structure map $\mathbf{Pic}_{X/S} \to S$ embeds T into S.

The embedding $T \hookrightarrow \mathbf{Pic}_{X/S}$ corresponds to an invertible sheaf $\mathcal{M}$ on X_T by

Exercise 2.3 since k is algebraically closed. Moreover, $\mathcal{M}|X_s \simeq \mathcal{O}_{X_s}(M)$ since the embedding is supported at μ.

Note $\mathrm{H}^1(\mathcal{O}_{X_s}(M)) = 0$. Indeed, by Serre duality [**Ha83**, Cor. 7.7, p. 244, and Cor. 7.12, p. 246], it suffices to show $\mathrm{H}^1(\omega_{X_s}(-M)) = 0$. Form the sequence

$$0 \to \omega_{X_s}(-M) \to \omega_{X_s} \to \omega_{X_s}|M \to 0.$$

Now, $\omega_{X_s} \simeq \mathcal{O}_{X_s}(d-4)$ since $\omega_{X_s} \simeq \omega_P \otimes \mathcal{O}_{X_s}(X_s)$ by [**Ha83**, Prp. 8.20, p. 182] and $\omega_P \simeq \mathcal{O}_P(-4)$ by [**Ha83**, Eg. 8.20.1, p. 182]. But $\mathrm{H}^1(\mathcal{O}_{X_s}(d-4)) = 0$ because $X_s \subset P$ is a hypersurface, and $\mathrm{H}^0(\mathcal{O}_{X_s}(d-4)) \to \mathrm{H}^0(\mathcal{O}_M(d-4))$ is surjective because $\mathrm{H}^0(\mathcal{O}_P(d-4)) \to \mathrm{H}^0(\mathcal{O}_M(d-4))$ is. Thus $\mathrm{H}^1(\mathcal{O}_{X_s}(M)) = 0$.

Therefore, $\mathrm{H}^0(\mathcal{M}) \otimes k \to \mathrm{H}^0(\mathcal{O}_{X_s}(M))$ is surjective by the implication (v)⇒(iv) of Subsection 3.10. So the section of $\mathcal{O}_{X_s}(M)$ defining M extends to a section of $\mathcal{M}$. The extension defines a relative effective divisor on X_T, which restricts to M, owing to the implication (iii)⇒(i) of Lemma 3.4. It follows that the embedding $T \hookrightarrow H$ factors through I, so through J. However, J and S meet transversally at s; whence, T cannot embedd into $J \cap S$. Here is the desired contradiction. The discussion is now complete.

Exercise 5.23. Assume X/S is projective and flat, its geometric fibers are integral curves of arithmetic genus p_a, and S is Noetherian. Show the "generalized Jacobians" $\mathbf{Pic}^0_{X_s/k_s}$ form a smooth quasi-projective family of relative dimension p_a. And show this family is projective if and only if X/S is smooth.

Remark 5.24. Assume X is an Abelian S-scheme of relative dimension g; that is, X is a smooth and proper S-group scheme with geometrically connected fibers of dimension g. Then X needn't be projective locally over S.

Indeed, according to Raynaud [**Ra66**, Rem. 8.c, p. 1315], Grothendieck found two such examples: one where S is reduced and 1-dimensional, and another where S is the spectrum of the ring of dual numbers of a field of characteristic 0. In [**Ra70**, Ch. XII], Raynaud gave detailed constructions of similar examples.

Assume X/S is projective, and S is Noetherian. Then $\mathbf{Pic}^0_{X/S}$ exists, and is also a projective Abelian S-scheme of relative dimension g. Set $X^* := \mathbf{Pic}^0_{X/S}$.

Indeed, $\mathbf{Pic}_{X/S}$ exists, represents $\mathrm{Pic}_{(X/S)\,(\text{ét})}$, and is locally of finite type by the main theorem, Theorem 4.8. For $s \in S$, the $\mathbf{Pic}^0_{X_s/k_s}$ are smooth and proper of dimension g by [**Mm70**, § 13]. Hence X^* exists by Proposition 5.20, and is projective by Exercise 5.7. Finally, X^* is smooth owing to a more sophisticated version of the proof of Proposition 5.19; see [**Mm65**, pp. 117–118].

There exists a universal sheaf $\mathcal{P}$ on $X \times \mathbf{Pic}_{X/S}$ by Exercise 4.3 since f has a section g, namely, the identity section. Normalize $\mathcal{P}$ by tensoring it with $f^*_{X^*} g^*_{X^*} \mathcal{P}$. Then its restriction to $X \times X^*$ defines a map, which is a "duality" isomorphism

$$\pi \colon X \xrightarrow{\sim} X^{**}.$$

Indeed, forming π commutes with changing S, and π's geometric fibers are isomorphisms by [**Mm70**, Cor., p. 132]. But X and X^{**} are proper over S, and X is flat. Therefore, π is an isomorphism by [**EGA III**$_1$, 4.6.7].

Remark 5.25. Assume S is the spectrum of an algebraically closed field k, and X is normal, integral, and projective. Then $\mathbf{Pic}^0_{X/k}$ is irreducible and projective by Proposition 5.3 and Theorem 5.4. Set $P := (\mathbf{Pic}^0_{X/k})_{\mathrm{red}}$ and $A := \mathbf{Pic}^0_{P/k}$. Then P is plainly an Abelian variety; whence, A is an Abelian variety too by Remark 5.24.

Fix a point $x \in X(k)$. Let B be an Abelian variety, and set $B^* := \mathbf{Pic}^0_{B/k}$. Then B^* is an Abelian variety too, and there is a canonical isomorphism $B \xrightarrow{\sim} B^{**}$ by Remark 5.24. Let $\xi\colon X \to X \times B^*$ be the map defined by $0 \in B^*(k)$, and let $\beta\colon B^* \to X \times B^*$ be the map defined by x.

Consider a map $a\colon B^* \to P$ such that $a(0) = 0$. By the Comparison Theorem, Theorem 2.5, a corresponds to an invertible sheaf $\mathcal{L}$ on $X \times B^*$ such that $\xi^*\mathcal{L} \simeq \mathcal{O}_X$. Normalize $\mathcal{L}$ by tensoring it with $(\beta^*\mathcal{L})_X$. Then $\mathcal{L}$ defines a map $b\colon X \to B$ such that $b(x) = 0$.

Reversing the preceding argument, we see that every such b arises from a unique map $a\colon B^* \to \mathbf{Pic}_{X/k}$ such that $a(0) = 0$. Since B^* is integral, a factors through P. Thus the maps $a\colon B^* \to P$ and $b\colon X \to B$ are in bijective correspondence. Plainly, this correspondence is compatible with maps $b'\colon B \to B'$ such that $b'(0) = 0$. In particular, 1_P corresponds to a natural map $u\colon X \to A$ such that $u(x) = 0$, and every map $b\colon X \to B$ factors uniquely through u.

Remark 5.26. Assume X/S is projective and smooth, its geometric fibers are connected curves of genus $g > 0$, and S is Noetherian. Set $J := \mathbf{Pic}^0_{X/S}$; it exists and is a projective Abelian S-scheme by Exercise 5.23. Set $J^* := \mathbf{Pic}^0_{J/S}$; it exists, is a projective Abelian scheme, and is "dual" to J by Remark 5.24.

Suppose X has an invertible sheaf $\mathcal{L}$ whose fibers $\mathcal{L}_s$ are of degree 1. Define an associated "Abel" map

$$A_{\mathcal{L}}\colon X \to J$$

directly on T-points as follows. Given $t\colon T \to X$, its graph subscheme $\Gamma_t \subset X \times T$ is a relative effective divisor; see Answer 4.13. Use $\mathcal{L}_T \otimes \mathcal{O}_{X_T}(-\Gamma)$ to define $A_{\mathcal{L}}(t)$.

Then $A_{\mathcal{L}}$ induces, via pullback, an "auto-duality" isomorphism

$$A^*_{\mathcal{L}}\colon J^* \xrightarrow{\sim} J.$$

This isomorphism is independent of the choice of $\mathcal{L}$; in fact, it exists even if no $\mathcal{L}$ does. These facts are proved in [**EGK**, Thm. 2.1, p. 595]. In fact, a more general autoduality result is proved: it applies to the natural compactification of J, which parameterizes torsion-free sheaves, when the geometric fibers of X are not necessarily smooth, but integral with double points at worst. And the proof starts from scratch, recovering the original case of a single smooth curve over a field.

Remark 5.27. Assume X/S is proper and flat. Assume its geometric fibers are curves, but not necessarily integral. Then there are two remarkable theorems asserting the existence of $\mathbf{Pic}_{X/S}$ as an algebraic space and of $\mathbf{Pic}^0_{X/S}$ as a separated S-scheme. The importance of these theorems lies in the theory of Néron models; so in [**BLR**, Sect. 9.4], their proofs are sketched, and the original papers, cited.

One theorem is due to Raynaud. He assumes, in addition, that S is the spectrum of a discrete valuation ring, that X is normal, and that $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ holds. Furthermore, given any geometric fiber of X/S, he measures the lengths of the local rings at the generic points of its irreducible components, and he assumes their greatest common divisor is 1. Then he proves the above existence assertions.

The other theorem is due to Deligne. Instead, he assumes, in addition, X/S is semi-stable; that is, its geometric fibers are reduced and connected, and have, at worst, ordinary double points. Then he proves, in addition, $\mathbf{Pic}^0_{X/S}$ is smooth and quasi-projective; in fact, it carries a canonical S-ample invertible sheaf.

6. The torsion component of the identity

This section establishes the two main finiteness theorem for $\mathbf{Pic}_{X/S}$, when X/S is projective and its geometric fibers are integral. The first theorem asserts the finiteness of the torsion component $\mathbf{Pic}^{\tau}_{X/S}$, an open and closed group subscheme. By definition, it consists of the points with a multiple in the connected component $\mathbf{Pic}^{0}_{X/S}$, which was studied in the previous section.

The second theorem asserts the finiteness of a larger sort of subset $\mathbf{Pic}^{\phi}_{X/S}$. Its points represent the invertible sheaves with a given Hilbert polynomial ϕ. The section starts by developing numerical characterizations of $\mathbf{Pic}^{\tau}_{X/S}$, or rather of the corresponding invertible sheaves, when S is the spectrum of an algebraically closed field. This development assumes some familiarity with basic intersection theory, which is developed in Appendix B.

Definition 6.1. Assume S is the spectrum of a field. Let $\mathcal{L}$ and $\mathcal{N}$ be invertible sheaves on X. Then $\mathcal{L}$ is said to be *τ-equivalent to $\mathcal{N}$* if, for some nonzero m depending on $\mathcal{L}$ and $\mathcal{N}$, the mth power $\mathcal{L}^{\otimes m}$ is algebraically equivalent to $\mathcal{N}^{\otimes m}$.

In addition, $\mathcal{L}$ is said to be *numerically equivalent to* $\mathcal{N}$ if, for every complete curve $Y \subset X$, the corresponding intersection numbers are equal:

$$\textstyle\int c_1\mathcal{L}\cdot[Y] = \int c_1\mathcal{N}\cdot[Y].$$

It is sufficient, by additivity, to take Y to be complete and integral. It is then equivalent that $\deg\mathcal{L}_Y = \deg\mathcal{N}_Y$ or that $\deg\mathcal{L}_{Y'} = \deg\mathcal{N}_{Y'}$ where Y' is the normalization of Y, because, in any event,

$$\textstyle\int c_1\mathcal{L}\cdot[Y] = \deg\mathcal{L}_Y = \deg\mathcal{L}_{Y'}.$$

Definition 6.2. Assume S is the spectrum of a field. Let Λ be a family of invertible sheaves on X. Then Λ is said to be *bounded* if there exist an S-scheme T of finite type and an invertible sheaf $\mathcal{M}$ on X_T such that, given $\mathcal{L}\in\Lambda$, there exists a geometric point t of T such that $\mathcal{L}_t \simeq \mathcal{M}_t$.

Theorem 6.3. *Assume S is the spectrum of an algebraically closed field, and X is projective. Let $\mathcal{L}$ be an invertible sheaf on X. Then the following conditions are equivalent:*

(a) *The sheaf $\mathcal{L}$ is τ-equivalent to $\mathcal{O}_X$.*
(b) *The sheaf $\mathcal{L}$ is numerically equivalent to $\mathcal{O}_X$.*
(c) *The family $\{\mathcal{L}^{\otimes p} \mid p\in\mathbb{Z}\}$ is bounded.*
(d) *For every coherent sheaf $\mathcal{F}$ on X, we have $\chi(\mathcal{F}\otimes\mathcal{L}) = \chi(\mathcal{F})$.*
(e) *For every closed integral curve $Y\subset X$, we have $\chi(\mathcal{L}_Y) = \chi(\mathcal{O}_Y)$.*
(f) *For every integer p, the sheaf $\mathcal{L}^{\otimes p}(1)$ is ample.*

If X is irreducible, then all the conditions above are equivalent to the following one:

(g) *For every pair of integers p, n, we have $\chi(\mathcal{L}^{\otimes p}(n)) = \chi(\mathcal{O}_X(n))$.*

If X is irreducible of dimension $r\ge 2$, then all the conditions above are equivalent to the following one:

(h) *Setting $\ell := c_1\mathcal{L}$ and $h := c_1\mathcal{O}_X(1)$, we have $\int \ell h^{r-1} = 0$ and $\int \ell^2 h^{r-2} = 0$.*

Proof. Let us proceed by establishing the following implications:

$$\text{(c)} \Longrightarrow \text{(a)} \Longrightarrow \text{(d)} \Longrightarrow \text{(e)} \Longleftrightarrow \text{(b)} \Longrightarrow \text{(c)} \Longrightarrow \text{(f)} \Longrightarrow \text{(b)};$$

$$\text{(d)} \Longrightarrow \text{(g)} \Longrightarrow \text{(h)} \Longrightarrow \text{(b)};\ \text{and (g)} \Longrightarrow \text{(b) if } \dim X = 1.$$

Assume (c). Then, by definition, there exist an S-scheme T of finite type and an invertible sheaf $\mathcal{M}$ on X_T such that, given $p \in \mathbb{Z}$, there exists a geometric point t of T such that $\mathcal{L}_t^{\otimes p} = \mathcal{M}_t$. Apply the Pigeonhole Principle: say $\mathcal{L}^{\otimes p_1}$ and $\mathcal{L}^{\otimes p_2}$ belong to the same connected component of T, but $p_1 \neq p_2$. Set $m := p_1 - p_2$. Then $\mathcal{L}^{\otimes m}$ is algebraically equivalent to $\mathcal{O}_X$. Thus (a) holds.

Furthermore, for each $t \in T$, there exists an n such that $\mathcal{M}_t(n)$ is ample by [**EGA II**, 4.5.8]. So t has a neighborhood U such that $\mathcal{M}_u(n)$ is ample for every $u \in U$ by [**EGA III**$_1$, 4.7.1]. Since T is quasi-compact, T is covered by finitely many of the U. Let N be the product of the corresponding n. Then $\mathcal{M}_t(N)$ is ample for every $t \in T$. In particular, $\mathcal{L}^{\otimes p}(N)$ is ample for every $p \in \mathbb{Z}$. So $\big(\mathcal{L}^{\otimes q}(1)\big)^{\otimes N}$ is ample for every $q \in \mathbb{Z}$. Thus (f) holds.

Assume (a). The function $n \mapsto \chi(\mathcal{F} \otimes \mathcal{L}^{\otimes n})$ is a polynomial. To prove it is constant, we may replace $\mathcal{L}$ by $\mathcal{L}^{\otimes m}$ for any nonzero m. Thus we may assume $\mathcal{L}$ is algebraically equivalent to $\mathcal{O}_X$. So let T be a connected S-scheme, and $\mathcal{M}$ an invertible sheaf on X_T. Then for a fixed n, as $t \in T$ varies, the function $t \mapsto \chi(\mathcal{F} \otimes \mathcal{L}_t^{\otimes n})$ is constant by [**EGA III**$_2$, 7.9.5]. It follows that (d) holds.

Assume (d). Taking $\mathcal{F} := \mathcal{O}_Y$, we get (e). Taking $\mathcal{F} := \mathcal{L}^{\otimes p}(n)$, we get $\chi(\mathcal{L}^{\otimes (p+1)}(n)) = \chi(\mathcal{L}^{\otimes p}(n))$. Thus whether or not X is irreducible, (g) holds,.

Assume (g). Then X is irreducible, say of dimension r. Set $\ell := c_1\mathcal{L}$ and $h := c_1\mathcal{O}_X(1)$. Write

$$\chi(\mathcal{L}^{\otimes p}(n)) = \sum_{0 \leq i,\, j \leq r} a_{ij} \binom{p+i}{i} \binom{n+j}{j}$$

where $a_{ij} = \int \ell^i h^j$ if $i + j = r$. Then (g) implies $a_{ij} = 0$ if $i \geq 1$. If $r \geq 2$, then (h) follows. Suppose $r = 1$. Then $\int \ell = 0$. Now, X_{red} is the only closed integral curve contained in X, and $\int \ell$ is a multiple of $\int \ell \cdot [X_{\text{red}}]$. Thus (b) holds.

Conditions (e) and (b) are equivalent since $\chi(\mathcal{L}_Y) = \deg(\mathcal{L}_Y) + \chi(\mathcal{O}_Y)$ by Riemann's Theorem.

Assume (f). Then for every closed integral curve $Y \subset X$, we have

$$0 \leq \deg(\mathcal{L}_Y^{\otimes p}(1)) = p \deg(\mathcal{L}_Y) + \deg(\mathcal{O}_Y(1))$$

for every integer p. So $\deg(\mathcal{L}_Y) = 0$. Thus (b) holds.

Assume (h). Then X is irreducible of dimension $r \geq 2$. To prove (b), plainly we may replace X by its reduction. We proceed by induction on r. If $r = 2$, then (b) holds by the Hodge Index Theorem.

Suppose $r \geq 3$. Given a complete integral curve $Y \subset X$, take n so that the twisted ideal $\mathcal{I}_{Y,X}(n-1)$ is generated by its global sections. View these sections as sections of $\mathcal{O}_X(n-1)$. Then they define a linear system that is free of base points on $X - Y$. So the global sections of $\mathcal{I}_Y(n)$ define a linear system that is very ample on $X - Y$. In particular, this system maps $X - Y$ onto a variety of dimension at least 2; in other words, the system is not "composite with a pencil."

Hence the generic member H_η is geometrically irreducible by [**Za58**, Thm. I.6.3, p. 30]. In the first instance, we must apply the cited theorem to the induced system on the normalization of X, and we conclude that the preimage of H_η is geometrically irreducible. But then H_η is too. Therefore, by [**EGA IV**$_3$, 9.7.7], a general member H is irreducible.

Set $\ell_1 := c_1\mathcal{L}_H$ and $h_1 := c_1\mathcal{O}_H(1)$. Then, by the Projection Formula,

$$\textstyle\int \ell_1 h_1^{r-2} = n \int \ell h^{r-1} = 0 \text{ and } \int \ell_1^2 h_1^{r-3} = n \int \ell^2 h^{r-2} = 0.$$

So by induction, $\mathcal{L}_H$ is numerically equivalent to $\mathcal{O}_X$ on H. But $Y \subset H$ since H arises from a section of $\mathcal{I}_Y(n)$. Hence, by the Projection Formula,

$$\int \ell \cdot [Y] = \int \ell_1 \cdot [Y] = 0.$$

Thus (b) holds.

Finally, assume (b). By Lemma 6.6 below, there is an m such that, if $\mathcal{N}$ is an invertible sheaf on X numerically equivalent to $\mathcal{O}_X$, then $\mathcal{N}$ is m-regular. So $\mathcal{N}(m)$ is generated by its global sections, and its higher cohomology groups vanish.

Set $\phi(n) := \chi(\mathcal{O}_X(n))$ and $M := \phi(m)$. Then $\dim \mathrm{H}^0(\mathcal{N}(m)) = M$, since $\chi(\mathcal{N}(n)) = \phi(n)$ also by Lemma 6.6 below. Set $\mathcal{F} := \mathcal{O}_X(-m)^{\oplus M}$. Then $\mathcal{N}$ is a quotient of $\mathcal{F}$.

Set $T := \mathbf{Quot}^{\phi}_{\mathcal{F}/X/k}$. Then T is of finite type. Let $\mathcal{M}$ be the universal quotient. Then there exists a k-point $t \in T$ such that $\mathcal{N} = \mathcal{M}_t$. Let $U \subset X \times T$ be the open set on which $\mathcal{M}$ is invertible. Let $R \subset T$ be the image of the complement of U. Then R is closed. Replace T by $T - R$, and $\mathcal{M}$ by its restriction. Then $t \in T$ still. Thus the invertible sheaves on X numerically equivalent to $\mathcal{O}_X$ form a bounded family. In particular, (c) holds. □

Exercise 6.4. Consider the preceding paragraph, the last one in the proof of Theorem 6.3. Using $\mathbf{Div}_{X/k}$ instead of $\mathbf{Quot}^{\phi}_{\mathcal{F}/X/k}$, give another proof that the invertible sheaves $\mathcal{N}$ on X numerically equivalent to $\mathcal{O}_X$ form a bounded family.

Lemma 6.5. *Assume S is the spectrum of an algebraically closed field, and X is projective of dimension r. Let $\mathcal{F}$ be a coherent sheaf on X. Then there is a number $B_{\mathcal{F}}$ such that, if $\mathcal{L}$ is any invertible sheaf on X numerically equivalent to $\mathcal{O}_X$, then*

$$\dim \mathrm{H}^0(\mathcal{L} \otimes \mathcal{F}(n)) \le B_{\mathcal{F}} \tbinom{n+r}{r} \text{ for all } n \ge 0.$$

Proof. Suppose $r = 0$. Then $\mathcal{L} \otimes \mathcal{F}(n) = \mathcal{F}$. So we may take $B_{\mathcal{F}} = \dim \mathrm{H}^0(\mathcal{F})$.

Suppose $r \ge 1$. Given a short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, we have

$$\dim \mathrm{H}^0(\mathcal{L} \otimes \mathcal{F}(n)) \le \dim \mathrm{H}^0(\mathcal{L} \otimes \mathcal{F}'(n)) + \dim \mathrm{H}^0(\mathcal{L} \otimes \mathcal{F}''(n)).$$

So given $B_{\mathcal{F}'}$ and $B_{\mathcal{F}''}$, we can take $B_{\mathcal{F}} := B_{\mathcal{F}'} + B_{\mathcal{F}''}$.

Say $X = \mathrm{Proj}(A)$ and $\mathcal{F} = \widetilde{M}$ where M is a finitely generated graded A-module. Then there is a filtration by graded submodules

$$M =: M_q \supset M_{q-1} \supset \cdots \supset M_1 \supset M_0 := 0$$

such that $M_{i+1}/M_i \simeq (A/P_i)[p_i]$ where P_i is a homogeneous prime for each i. It follows that we may assume X is integral and $\mathcal{F} = \mathcal{O}_X(p)$.

Let $\mathcal{L}$ be numerically equivalent to $\mathcal{O}_X$. Set $\ell := c_1\mathcal{L}$ and $h := c_1\mathcal{O}_X(1)$. Suppose $\mathcal{L}(p)$ has a nonzero section. It defines a divisor D, possibly 0. Hence

$$0 \le \int h^{r-1}[D] = \int h^{r-1}\ell + p \int h^r.$$

But $\int h^{r-1}\ell = 0$ because h^{r-1} is represented by a curve since $r \ge 1$. And $\int h^r > 0$. Hence $p \ge 0$. Thus $\mathrm{H}^0(\mathcal{L}(-1)) = 0$.

Let H be a hyperplane section of X. Then there is an exact sequence

$$0 \to \mathcal{L}(n-1) \to \mathcal{L}(n) \to \mathcal{L}_H(n) \to 0.$$

By induction on r, we may assume there is a number B such that

$$\dim \mathrm{H}^0(\mathcal{L}_H(n)) \le B \tbinom{n+r-1}{r-1} \text{ for all } n \ge 0;$$

moreover, B works for every $\mathcal{L}$. Hence

$$\dim \mathrm{H}^0(\mathcal{L}(n)) - \dim \mathrm{H}^0(\mathcal{L}(n-1)) \le B\binom{n+r-1}{r-1} \text{ for all } n \ge 0.$$

But $\mathrm{H}^0(\mathcal{L}(-1)) = 0$. Since $\binom{n+r-1}{r} + \binom{n+r-1}{r-1} = \binom{n+r}{r}$, induction on n yields

$$\dim \mathrm{H}^0(\mathcal{L}(n)) \le B\binom{n+r}{r} \text{ for all } n \ge 0.$$

Recall $\mathcal{F} = \mathcal{O}_X(p)$. If $p \le 0$, then $\mathcal{F} \subset \mathcal{O}_X$; so we may take $B_{\mathcal{F}} := B$. But if $p \ge 0$, then $\binom{p+n+r}{r} \le \binom{p+r}{r}\binom{n+r}{r}$ since every monomial of degree $p+n$ is the product of one of degree p and one of degree n; so we may take $B_{\mathcal{F}} := B\binom{p+r}{r}$. □

Lemma 6.6. *Assume S is the spectrum of an algebraically closed field, and X is projective. Then there is an integer m such that, if $\mathcal{L}$ is any invertible sheaf on X numerically equivalent to $\mathcal{O}_X$, then $\mathcal{L}$ is m-regular, and*

$$\chi(\mathcal{L}(n)) = \chi(\mathcal{O}_X(n)) \textit{ for all } n.$$

Proof. Set $r := \dim(X)$, and proceed by induction on r. If $r = 0$, then both assertions are trivial. So assume $r \ge 1$.

First, let us establish the asserted equation. Given an $\mathcal{L}$, fix $q \ge 1$ such that $\mathcal{L}(q)$ is very ample. Take effective divisors F and G such that

$$\mathcal{O}_X(F) = \mathcal{O}_X(q) \text{ and } \mathcal{O}_X(G) = \mathcal{L}(q).$$

For every p, plainly $\mathcal{L}_F^{\otimes p}$ and $\mathcal{L}_G^{\otimes p}$ are numerically equivalent to $\mathcal{O}_F$ and $\mathcal{O}_G$.

Form $0 \to \mathcal{O}_X(-F) \to \mathcal{O}_X \to \mathcal{O}_F \to 0$ and $0 \to \mathcal{O}_X(-G) \to \mathcal{O}_X \to \mathcal{O}_G \to 0$. Tensor them with $\mathcal{L}^{\otimes p}(n+q)$. We get

$$0 \to \mathcal{L}^{\otimes p}(n) \to \mathcal{L}^{\otimes p}(n+q) \to \mathcal{L}_F^{\otimes p}(n+q) \to 0, \tag{6.6.1}$$
$$0 \to \mathcal{L}^{\otimes p-1}(n) \to \mathcal{L}^{\otimes p}(n+q) \to \mathcal{L}_G^{\otimes p}(n+q) \to 0.$$

Apply $\chi(\bullet)$ and subtract. We get

$$\chi(\mathcal{L}^{\otimes p}(n)) - \chi(\mathcal{L}^{\otimes p-1})(n)) = \chi(\mathcal{L}_G^{\otimes p}(n+q)) - \chi(\mathcal{L}_F^{\otimes p}(n+q)).$$

By induction, the right hand side varies as a polynomial in n, which is independent of p. Hence there are polynomials $\phi_1(n)$ and $\phi_0(n)$ such that

$$\chi(\mathcal{L}^{\otimes p}(n)) = \phi_1(n)p + \phi_0(n). \tag{6.6.2}$$

Suppose $\phi_1 \ne 0$. Say $\phi_1(n) \ne 0$ for all $n \ge n_1$.

By induction, there is an integer n_2 such that $\mathcal{L}_F^{\otimes p}$ is n_2-regular for every p. So $\mathrm{H}^i(\mathcal{L}_F^{\otimes p}(n)) = 0$ for $i \ge 1$, for $n \ge n_2 - i$, and for every p. Hence, owing to Sequence (6.6.1), there is an isomorphism

$$\mathrm{H}^i(\mathcal{L}^{\otimes p}(n)) \xrightarrow{\sim} \mathrm{H}^i(\mathcal{L}^{\otimes p}(n+q)) \text{ for } i \ge 2, \text{ for } n \ge n_2, \text{ and for every } p.$$

Now, for each $i \ge 2$, each p, and each n, there is a $j \ge 0$ such that

$$\mathrm{H}^i(\mathcal{L}^{\otimes p}(n+jq)) = 0$$

by Serre's Theorem. Therefore,

$$\mathrm{H}^i(\mathcal{L}^{\otimes p}(n)) = 0 \text{ for } i \ge 2, \text{ for } n \ge n_2 \text{ and for every } p.$$

Hence $\mathrm{H}^0(\mathcal{L}^{\otimes p}(n)) \ge \chi(\mathcal{L}^{\otimes p}(n))$ for $n \ge n_2$ and every p. Take $n := \max(n_2, n_1)$. Owing to Equation (6.6.2), then $\mathrm{H}^0(\mathcal{L}^{\otimes p}(n)) \to \infty$ as $p \to \infty$ if $\phi_1(n) > 0$ or as $p \to -\infty$ if $\phi_1(n) < 0$. However, by Lemma (6.5), there is a number B such that $\mathrm{H}^0(\mathcal{L}^{\otimes p}(n)) \leq B$ for any p. This contradiction means $\phi_1 = 0$. Hence Equation (6.6.2) yields $\chi(\mathcal{L}(n)) = \chi(\mathcal{O}_X(n))$ for all n, as desired.

Finally, in order to prove there is an m such that every $\mathcal{L}$ numerically equivalent to $\mathcal{O}_X$ is m-regular, we must modify Mumford's original work [**Mm66**, Lect. 14] because these $\mathcal{L}$ are not ideals. However, as Mumford himself points out [**Mm66**, pp. 102–103], the hypothesis that his sheaf $\mathcal{I}$ is an ideal enters only through the bound $\dim \mathrm{H}^0(\mathcal{I}(n)) \leq \binom{n+r}{r}$. Plainly, this bound can be replaced by the bound $B_{\mathcal{F}}$ with $\mathcal{F} := \mathcal{O}_X$ of Lemma 6.5. Of course, in addition, we must use the fact we just proved, that all the $\mathcal{L}$ have the same Hilbert polynomial. □

Exercise 6.7. Assume S is the spectrum of an algebraically closed field, and X is projective and integral of dimension $r \geq 1$. Set $h := c_1\mathcal{O}_X(1)$. Let $\mathcal{L}$ be an invertible sheaf on X, and set $\ell := c_1\mathcal{L}$. Say

$$\chi(\mathcal{L}(n)) = \textstyle\sum_{0\leq i\leq r} a_i\binom{n+i}{i} \text{ and } a := \int \ell h^{r-1}.$$

Suppose $a < a_r$. Show $\dim \mathrm{H}^0(\mathcal{L}(n)) \leq a_r\binom{n+r}{r}$ for all $n \geq 0$. Furthermore, modifying Mumford's work [**Mm66**, pp. 102–103] slightly, show there is a polynomial Φ_r depending only on r such that $\mathcal{L}$ is m-regular with $m := \Phi_r(a_0, \dots, a_{r-1})$.

In general, show there is a polynomial Ψ_r depending only on r such that $\mathcal{L}$ is m-regular with $m := \Psi_r(a_0, \dots, a_r; a)$.

Definition 6.8. Let G/S be a group scheme. For $n > 0$, let $\varphi_n \colon G \to G$ denote the nth power map. Then G^τ is the set defined by the formula

$$G^\tau := \textstyle\bigcup_{n>0} \varphi_n^{-1} G^0$$

where G^0 is the union of the connected components of the identity G_s^0 for $s \in S$.

Lemma 6.9. *Let k be a field, and G a commutative group scheme locally of finite type. Then G^τ is an open group subscheme, and forming it commutes with extending k. Moreover, if G^τ is quasi-compact, then it is closed and of finite type.*

Proof. By Lemma 5.1, G^0 is an open and closed group subscheme of finite type, and forming it commutes with extending k. Now, the nth power map φ_n is continuous, and forming it commutes with extending k; also, φ_n is a homomorphism since G is commutative. So G^τ is the filtered union of the open and closed group subschemes $\varphi_n^{-1}G^0$, and forming them commutes with extending k. Hence G^τ is an open group subscheme, and forming it commutes with extending k. Moreover, if G^τ is quasi-compact, then G^τ is the union of finitely many of the $\varphi_n^{-1}G^0$, and so G^τ is closed; also, then G^τ is of finite type since G is locally of finite type. □

Exercise 6.10. Let k be a field, and G a commutative group scheme locally of finite type. Let $H \subset G$ be a group subscheme of finite type. Show $H \subset G^\tau$. (Thus, if G^τ is of finite type, then it is the largest group subscheme of finite type.)

Exercise 6.11. Assume S is the spectrum of a field k. Assume $\mathbf{Pic}_{X/k}$ exists and represents $\mathrm{Pic}_{(X/k)\,(\mathrm{fppf})}$. Let $\mathcal{L}$ be an invertible sheaf on X, and $\lambda \in \mathbf{Pic}_{X/k}$ the corresponding point. Show $\mathcal{L}$ is τ-equivalent to $\mathcal{O}_X$ if and only if $\lambda \in \mathbf{Pic}^\tau_{X/k}$.

Proposition 6.12. *Assume S is the spectrum of a field k. Assume $\mathbf{Pic}_{X/k}$ exists and represents $\mathrm{Pic}_{(X/k)\,(\mathrm{fppf})}$. Then $\mathbf{Pic}^\tau_{X/k}$ is an open group subscheme, and forming it commutes with extending k. Moreover, if X is projective, then $\mathbf{Pic}^\tau_{X/k}$ is closed and of finite type.*

PROOF. By Proposition 4.17, $\mathbf{Pic}_{X/S}$ is locally of finite type. So owing to Lemma 6.9, we need only prove $\mathbf{Pic}^{\tau}_{X/k}$ is quasi-compact when X is projective. Since forming $\mathbf{Pic}^{\tau}_{X/k}$ commutes with extending k, we may also assume k is algebraically closed.

At the very end of the proof of Theorem 6.3, we proved that the invertible sheaves $\mathcal{L}$ numerically equivalent to $\mathcal{O}_X$ form a bounded family. In other words, there is a k-scheme T of finite type and an invertible sheaf $\mathcal{M}$ on X_T such that the $\mathcal{L}$ appear among the fibers $\mathcal{M}_t$.

Then $\mathcal{M}$ defines a map $\theta\colon T \to \mathbf{Pic}_{X/k}$. Owing to Theorem 6.3 and to Exercise 6.11, we have $\theta(T) \supset \mathbf{Pic}^{\tau}_{X/k}$. Since T is Noetherian, so is $\theta(T)$; whence, so is any subspace of $\theta(T)$. Thus $\mathbf{Pic}^{\tau}_{X/k}$ is quasi-compact, as needed. □

EXERCISE 6.13. Assume S is the spectrum of a field k. Assume X is projective and geometrically integral. Show $\mathbf{Pic}^{\tau}_{X/k}$ is quasi-projective. If also X is geometrically normal, show $\mathbf{Pic}^{\tau}_{X/k}$ is projective.

REMARK 6.14. In Proposition 6.12, if X is projective, then $\mathbf{Pic}_{X/k}$ does exist and represent $\mathrm{Pic}_{(X/k)\,(\mathrm{fppf})}$ according to Corollary 4.18.3. In fact, this corollary asserts $\mathbf{Pic}_{X/k}$ exists and represents $\mathrm{Pic}_{(X/k)\,(\mathrm{fppf})}$ whenever X is complete; X need not be projective. Furthermore, although we used projective methods to prove Proposition 6.12, we can infer it whenever X is complete, as follows.

Assume X is complete. By Chow's lemma, there is a projective variety X' and a surjective map $\gamma\colon X' \to X$. By Theorem 4.18.5, the induced map

$$\gamma^*\colon\ \mathbf{Pic}_{X/k} \to \mathbf{Pic}_{X'/k}$$

is of finite type. Set $H := (\gamma^*)^{-1}\,\mathbf{Pic}^{\tau}_{X'/k}$. Then H is of finite type since $\mathbf{Pic}^{\tau}_{X'/k}$ is by Proposition 6.12. Now, plainly $\gamma^*\,\mathbf{Pic}^{0}_{X/k} \subset \mathbf{Pic}^{0}_{X'/k}$. So since γ^* is a homomorphism, $\gamma^*\,\mathbf{Pic}^{\tau}_{X/k} \subset \mathbf{Pic}^{\tau}_{X'/k}$; whence, $\mathbf{Pic}^{\tau}_{X/k} \subset H$ (in fact, the two are equal by Exercise 6.10). Since $\mathbf{Pic}^{\tau}_{X/k}$ is open, it is therefore a subscheme of finite type.

Similarly, in Theorem 6.3, Conditions (a)–(e) continue to make sense and to remain equivalent whenever X is complete. Indeed, our proofs of the implications

$$\text{(c)} \Longrightarrow \text{(a)} \Longrightarrow \text{(d)} \Longrightarrow \text{(e)} \Longleftrightarrow \text{(b) and (c)} \Longrightarrow \text{(f)} \Longrightarrow \text{(b)}$$

work without change. However, we used projective methods to prove (b) $\Longrightarrow$ (c). Nevertheless, we can infer this implication whenever X is complete, as follows.

Let $\mathcal{L}$ be numerically equivalent to $\mathcal{O}_X$. Let $\gamma\colon X' \to X$ be as above. Then $\gamma^*\mathcal{L}$ is numerically equivalent to $\mathcal{O}_{X'}$. Indeed, Let $Y' \subset X'$ be a closed curve. Then $\int c_1\gamma^*\mathcal{L}\cdot[Y']$ is a multiple of $\int c_1\mathcal{L}\cdot[\gamma Y']$ by the Projection Formula; hence, the former number vanishes as the latter does.

Let $\lambda \in \mathbf{Pic}_{X/k}$ represent $\mathcal{L}$. Then $\gamma^*\lambda \in \mathbf{Pic}_{X'/k}$ represents $\gamma^*\mathcal{L}$. Now, $\gamma^*\mathcal{L}$ is numerically equivalent to $\mathcal{O}_{X'}$. Hence $\gamma^*\lambda \in \mathbf{Pic}^{\tau}_{X'/k}$ owing to Theorem 6.3 and to Exercise 6.11. So $\lambda \in H := (\gamma^*)^{-1}\,\mathbf{Pic}^{\tau}_{X'/k}$.

The inclusion $H \hookrightarrow \mathbf{Pic}_{X/k}$ is defined by an invertible sheaf $\mathcal{M}$ on X_T for some fppf covering $T \to H$. (Although k is algebraically closed, possibly $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ does not hold universally, so we cannot simply take $T := H$.) Replace T by an open subscheme so that $T \to H$ is of finite type and surjective. Since H is of finite type, so is T.

Let $t \in T$ be a k-point that maps to $\lambda \in H$. Then $\mathcal{M}_t \simeq \mathcal{L}$. Now, for every $p \in \mathbb{Z}$, plainly $\mathcal{L}^{\otimes p}$ is numerically equivalent to $\mathcal{O}_X$. So similarly $\mathcal{L}^{\otimes p} \simeq \mathcal{M}_{t_p}$ for some k-point $t_p \in T$. Thus (c) holds.

Exercise 6.15. Assume $\mathbf{Pic}_{X/S}$ exists and represents $\mathrm{Pic}_{(X/S)\,(\mathrm{fppf})}$. Let Λ be an arbitrary subset of $\mathbf{Pic}_{X/S}$, and L the corresponding family of classes of invertible sheaves on the fibers of X/S in the sense of Exercise 4.5. Show Λ is quasi-compact (with the induced topology) if and only if L is *bounded* in the following sense: there exist an S-scheme T of finite type and an invertible sheaf $\mathcal{M}$ on X_T such that every class in L is represented by a fiber $\mathcal{M}_t$ for some $t \in T$.

Theorem 6.16. *Assume $f\colon X \to S$ is projective locally over S, and flat with integral geometric fibers. Then $\mathbf{Pic}^{\tau}_{X/S}$ is an open and closed group subscheme of finite type, and forming it commutes with changing S. If also X/S is projective, and S is Noetherian, then $\mathbf{Pic}^{\tau}_{X/S}$ is quasi-projective.*

Proof. The second assertion follows from the first and Exercise 4.11. The first assertion is local on S; so, to prove it, we may assume X/S is projective.

Theorem 4.8 asserts $\mathbf{Pic}_{X/S}$ exists and is locally of finite type. So the $\mathbf{Pic}^{\tau}_{X_s/k_s}$ are subgroups of the $\mathbf{Pic}_{X_s/k_s}$, and forming $\mathbf{Pic}^{\tau}_{X_s/k_s}$ commutes with extending k_s by Exercise 4.4 and Lemma 6.9. However, plainly $\mathbf{Pic}^{\tau}_{X/S} = \bigcup_{s\in S} \mathbf{Pic}^{\tau}_{X_s/k_s}$ as sets. Hence, forming $\mathbf{Pic}^{\tau}_{X/S}$ commutes with changing S; moreover, in order to infer $\mathbf{Pic}^{\tau}_{X/S}$ is a group subscheme, we need only prove it is open.

Plainly, a subset A of a topological space B is open or closed if (and only if), for every member B_i of an open covering of B, the intersection $A \cap B_i$ is so in B_i. Hence, in order to infer $\mathbf{Pic}^{\tau}_{X/S}$ is an open and closed group subscheme, we need only prove that, for any affine open subscheme U of $\mathbf{Pic}_{X/k}$, the intersection $U \cap \mathbf{Pic}^{\tau}_{X/S}$ is open and closed in U.

Theorem 4.8 also asserts $\mathbf{Pic}_{X/k}$ represents $\mathrm{Pic}_{(X/S)\,(\text{ét})}$. Thus the inclusion $U \hookrightarrow \mathbf{Pic}_{X/S}$ is defined by an invertible sheaf $\mathcal{M}$ on X_T for some étale covering $T \to U$. Replace T by an open subscheme so that $T \to U$ is of finite type and surjective. Let T^{τ} be the set of $t \in T$ where M_t is τ-equivalent to $\mathcal{O}_{X_t}$. Then T^{τ} is the preimage of $\mathbf{Pic}^{\tau}_{X/S}$ in T owing to Exercise 6.11. So we have to prove T^{τ} is open and closed.

Since U is affine, it is quasi-compact, so of finite type. Hence T is of finite type. So it has only finitely many connected components. But, for every pair p, n, the function $t \mapsto \chi(\mathcal{M}_t^{\otimes p}(n))$ is constant on each connected component of T. Therefore, Theorem 6.3 implies T^{τ} is open and closed, as desired.

It remains to prove $\mathbf{Pic}^{\tau}_{X/S}$ is of finite type. Let L be the corresponding family of classes of invertible sheaves on the fibers of X/S. By Exercise 6.15, we need only prove L is bounded. At the very end of the proof of Theorem 6.3, we proved essentially this statement when S is the spectrum of an algebraically closed field, and X is projective, but not necessarily integral. We can argue similarly here, but must make two important modifications.

First, if a class in L is represented by an invertible sheaf $\mathcal{L}$ on a fiber X_k where k is a field containing the field k_s of a point $s \in S$, then $\chi(\mathcal{L}(n)) = \chi(\mathcal{O}_{X_s}(n))$ for all n, owing to Lemma 6.6. But $\chi(\mathcal{O}_{X_s}(n))$ can vary with s. Nevertheless, it must remain the same on each connected component of S. And the matter in question is local on S. So we may and must assume S is connected.

Second, at the end of the proof of Lemma 6.6, when we modified Mumford's work, we used the bound $B_{\mathcal{F}}$ with $\mathcal{F} := \mathcal{O}_X$ of Lemma 6.5; in fact, in the induction step, we implicitly used the corresponding bounds for various subschemes of X. Unfortunately, it is not clear, in general, how these bounds vary with X. But in place of Lemma 6.6, we can use Exercise 6.7, which provides a uniform m such that $\mathcal{L}$ is m-regular for every $\mathcal{L}$ representing a class in L. □

COROLLARY 6.17. *Assume S is Noetherian. Assume $f\colon X \to S$ is projective locally over S, and is flat with geometrically integral fibers. For each $s \in S$, let k'_s be the algebraic closure of the residue field k_s. Then the torsion group*

$$\mathbf{Pic}^{\tau}_{X_{k'_s}/k'_s}(k'_s) \;/\; \mathbf{Pic}^{0}_{X_{k'_s}/k'_s}(k'_s) \tag{6.17.1}$$

is finite, and its order is bounded.

PROOF. Since $\mathbf{Pic}^{0}_{X_{k'_s}/k'_s}$ is open in $\mathbf{Pic}^{\tau}_{X_{k'_s}/k'_s}$ by Proposition 5.3, the order of their quotient is equal to the number of connected components of $\mathbf{Pic}^{\tau}_{X_{k'_s}/k'_s}$. This number is finite because $\mathbf{Pic}^{\tau}_{X_{k'_s}/k'_s}$ is of finite type by Proposition 6.12.

Moreover, $\mathbf{Pic}^{\tau}_{X_{k'_s}/k'_s}$ is equal to $\mathbf{Pic}^{\tau}_{X_s/k_s} \otimes k'_s$ again by Proposition 6.12, so equal to $\mathbf{Pic}^{\tau}_{X/S} \otimes k'_s$ essentially by Definition 6.8. But, since $\mathbf{Pic}^{\tau}_{X/S}$ is of finite type by Theorem 6.16, the number of connected components of $\mathbf{Pic}^{\tau}_{X/S} \otimes k'_s$ is constant for s in a nonempty open subset of S by [**EGA IV**$_3$, 9.7.9]. Hence the number is bounded by Noetherian induction. □

EXERCISE 6.18. Assume X/S is projective and smooth, its geometric fibers are irreducible, and S is Noetherian. Show that $\mathbf{Pic}^{\tau}_{X/S}$ is projective.

REMARK 6.19. Assume X/S is proper. If $\mathbf{Pic}_{X/S}$ exists and it represents $\mathrm{Pic}_{(X/k)\,(\mathrm{fppf})}$, then $\mathbf{Pic}^{\tau}_{X/S}$ is an open group subscheme of finite type. This fact can be derived from Theorem 6.16 through a series of reduction steps; see [**Kl71**, Thm. 4.7, p. 647].

Assume S is Noetherian in addition. Then, whether or not $\mathbf{Pic}_{X/k}$ exists, the torsion group (6.17.1) is finite and its order is bounded. This fact follows from the preceding one via the proof of Corollary 6.17, since there is an nonempty open subsscheme V of S_{red} such that $\mathbf{Pic}_{X_V/V}$ exists and represents $\mathrm{Pic}_{(X_V/V)\,(\mathrm{fppf})}$ by Grothendieck's Theorem 4.18.2.

Furthermore, the rank of the corresponding "Néron–Severi" group

$$\mathbf{Pic}_{X_{k'_s}/k'_s}(k'_s) \;/\; \mathbf{Pic}^{\tau}_{X_{k'_s}/k'_s}(k'_s)$$

is finite and bounded. This fact is far deeper; see [**Kl71**, Thm. 5.1, p. 650, and Rem. 5.3, p. 652], and see [**Za35**, pp. 121–124]. Moreover, the rank is arithmetic in nature: its value need not be a constructible funtion of $s \in S$; a standard example is discussed in [**BLR**, p. 235].

THEOREM 6.20. *Assume X/S is projective and flat with integral geometric fibers. Given a polynomial $\phi \in \mathbb{Q}[n]$, let $\mathbf{Pic}^{\phi}_{X/S} \subset \mathbf{Pic}_{X/S}$ be the set of points representing invertible sheaves $\mathcal{L}$ such that $\chi(\mathcal{L}(n)) = \phi(n)$ for all n. Then the $\mathbf{Pic}^{\phi}_{X/S}$ are open and closed subschemes of finite type; they are disjoint and cover; and forming them commutes with changing S. If also S is Noetherian, then $\mathbf{Pic}^{\phi}_{X/S}$ is quasi-projective.*

PROOF. Plainly, the $\mathbf{Pic}^{\phi}_{X/S}$ are disjoint and cover, and forming them commutes with changing S. The rest of the proof is similar to the proof of Theorem 6.16. In fact, the present case is simpler because the sheaves in question have the same Hilbert polynomials by hypothesis; there is no need to appeal to Theorem 6.3 nor to Lemma 6.6. □

Exercise 6.21. Assume X/S is locally projective over S and flat, and its geometric fibers are integral curves. Given an integer m, let $\mathbf{Pic}^m_{X/S} \subset \mathbf{Pic}_{X/S}$ be the set of points representing invertible sheaves $\mathcal{L}$ of degree m.

Show the $\mathbf{Pic}^m_{X/S}$ are open and closed subschemes of finite type; show they are disjoint and cover; and show that forming them commutes with changing S.

Show there is no abuse of notation: the fiber of $\mathbf{Pic}^0_{X/S}$ over $s \in S$ is the connected component of $0 \in \mathbf{Pic}_{X_s/k_s}$. Show there is no torsion: $\mathbf{Pic}^0_{X/S} = \mathbf{Pic}^\tau_{X/S}$. Show each $\mathbf{Pic}^m_{X/S}$ is an fppf-torsor under $\mathbf{Pic}^0_{X/S}$; that is, the latter acts naturally on the former, and the two become isomorphic after base change by an fppf-covering.

Show the $\mathbf{Pic}^m_{X/S}$ are quasi-projective if X/S is projective and S is Noetherian.

Remark 6.22. There is another important case where $\mathbf{Pic}^0_{X/S} = \mathbf{Pic}^\tau_{X/S}$, namely, when X is an Abelian S-scheme. Indeed, the equation holds if it does on each geometric fiber of X/S; so we may assume that S is the spectrum of an algebraically closed field. In this case, a modern proof was given by Mumford [**Mm70**, Cor.2, p. 178].

Example 6.23. Theorem 6.20 can fail if a geometric fiber of X/S is reducible. For example, let S be the spectrum of a field k, and let X be the union of two disjoint lines. For each pair $a, b \in \mathbb{Z}$, let $\mathcal{L}_{a,b}$ be the invertible sheaf that restricts to $\mathcal{O}(a)$ on the first line and to $\mathcal{O}(b)$ on the second.

By Riemann's Theorem, $\chi(\mathcal{L}_{a,b}(n)) = (a+b) + 2n + 1$. And it is easy to see that $\mathrm{Pic}_{X/k}$ is the disjoint union of copies of $\mathrm{Spec}(k)$ indexed by $\mathbb{Z} \times \mathbb{Z}$; compare with Exercise 4.15. Moreover, $\mathcal{L}_{a,b}$ is represented by the point with index (a, b). Hence, each set $\mathbf{Pic}^\phi_{X/k}$ is infinite, and so not of finite type.

Remark 6.24. Theorem 6.20 can be modified as follows. Assume S is Noetherian. Assume X/S is projective and flat with integral geometric fibers of dimension r. Then a subset $\Lambda \subset \mathbf{Pic}_{X/S}$ is of finite type if, in the Hilbert polynomials $\sum_{i=0}^r a_i\binom{n+i}{i}$ of the corresponding invertible sheaves, a_{r-1} remains bounded from above and below, and a_{r-2} remains bounded from below alone. Moreover, Λ is of finite type if, instead, $\int \ell h^{r-1}$ remains bounded from above and below, and $\int \ell^2 h^{r-2}$ remains bounded from below alone, where ℓ and h are as usual.

These facts can be derived from Theorem 6.20 by reducing to the case where the fibers are normal and by showing, in this case using simple elementary means, that the given bounds imply bounds on all the a_i; see [**Kl71**, Thm. 3.13, p. 641].

The first fact is essentially equivalent, given Theorems 6.20 and 4.18.5, to the following fact. Assume S is Noetherian, and X/S is projective and flat with fibers whose irreducible components have dimension at least 3. Let Y be a relative effective divisor whose associated sheaf is $\mathcal{O}_X(1)$. Then the induced map of Picard functors is representable by maps of finite type. See [**Kl71**, Thm. 3.8, p. 636].

The latter fact was first proved directly using the "equivalence criterion" mentioned in Remark 5.8 by Grothendieck [**FGA**, p. C-10].

Corollary 6.25. *Assume X/S is projective and flat with integral geometric fibers. Then the connected components of $\mathbf{Pic}_{X/S}$ are open and closed subschemes of finite type.*

Proof. By construction, $\mathbf{Pic}_{X/S}$ is locally Noetherian. Hence, its connected components are open and closed; see the proof of Lemma 5.1. Now, a connected component is always contained in any open and closed set it meets. Hence the connected components of $\mathbf{Pic}_{X/S}$ are of finite type owing to Theorem 6.20. □

Remark 6.26. In Corollary 6.25, X/S must be projective, not simply proper, nor even just projective locally over S. Indeed, Grothendieck [**FGA**, Rem. 3.3, p. 232-07] gave an example where $\mathbf{Pic}_{X/S}$ has a connected component that is not of finite type: here S is a curve with two components that meet in two points, such as the union of a smooth conic and a line in the plane over an algebraically closed field, and X is projective over a neighborhood of each component of S.

Corollary 6.27. *Assume X/S is projective and flat with integral geometric fibers. For $n \neq 0$, the* n*th power map $\varphi_n \colon \mathbf{Pic}_{X/S} \to \mathbf{Pic}_{X/S}$ is of finite type.*

Proof. Owing to Corollary 6.25, we need only prove that, given any connected component U of $\mathbf{Pic}_{X/S}$, the preimage $\varphi_n^{-1}U$ is of finite type too. Since $\mathbf{Pic}_{X/k}$ represents $\mathrm{Pic}_{(X/S)\,(\text{ét})}$ by Theorem 4.8, the inclusion $U \hookrightarrow \mathbf{Pic}_{X/S}$ is defined by an invertible sheaf $\mathcal{M}$ on X_T for some étale covering $T \to U$.

Fix $t \in T$, and set $\psi(p,q) := \chi(\mathcal{M}_t^p(q))$. Fix p, q, and form the set T' of points t' of T such that $\chi(\mathcal{M}_{t'}^p(q)) = \psi(p,q)$. By [**EGA III**$_2$, 7.9.4], the set T' is open, and so is its complement. Hence their images are open in U, and plainly these images are disjoint. But U is connected. Hence $T' = T$.

Let $\lambda \in \varphi_n^{-1}U$. Represent λ by an invertible sheaf $\mathcal{L}$. Set $\theta(m,q) := \chi(\mathcal{L}^m(q))$. Say $\theta(m,q) = \sum a_i(m)\binom{q+i}{i}$ where $a_i(m)$ is a polynomial. Now, $\varphi_n(\lambda) \in U$. So $\theta(mn,q) = \psi(m,q)$. So $a_i(mn)$ is independent of the choice of λ for all m. Hence $a_i(m)$ is too. Set $\phi(n) := \theta(1,q)$. Then $\varphi_n^{-1}U \subset \mathbf{Pic}^{\phi}_{X/S}$. Hence Theorem 6.20 implies $\varphi_n^{-1}U$ is of finite type. □

Remark 6.28. Corollary 6.27 holds in greater generality. Assume X/S is proper, and assume $\mathbf{Pic}_{X/S}$ exists and represents $\mathrm{Pic}_{(X/k)\,(\text{fppf})}$. Then, for $n \neq 0$, the nth power map $\varphi_n \colon \mathbf{Pic}_{X/S} \to \mathbf{Pic}_{X/S}$ is of finite type. This fact can be derived from the corollary through a series of reduction steps similar to those used to generalize Theorem 6.16; see [**Kl71**, Thm. 3.6, p. 635].

Exercise 6.29. Assume S is Noetherian, and X/S is projective and flat. Assume $\mathbf{Pic}_{X/S}$ exists and represents $\mathrm{Pic}_{(X/S)\,(\text{fppf})}$. Let Λ be an arbitrary subset of $\mathbf{Pic}_{X/S}$, and Π the corresponding set of Hilbert polynomials. Show Π is finite if Λ is quasi-compact. Show Π has only one element if Λ is connected.

Appendix A. Answers to all the exercises

The exercises are not meant to be tricky, but are designed to help you check, solidify, and expand your understanding of the ideas and methods. So to promote your own mathematical health, try seriously to do each exercise before you read its answer here. Note that the answer key is the same number as the exercise key.

Answer 2.3. Since A is local, $\mathrm{Pic}(T)$ is trivial; so Definitions 2.1 and 2.2 yield the first isomorphism, $\mathrm{Pic}_X(A) \xrightarrow{\sim} \mathrm{Pic}_{X/S}(A)$.

Let $\mathcal{L}$ be an invertible sheaf on X_A. Suppose the isomorphism class of $\mathcal{L}$ maps to 0 in $\mathrm{Pic}_{(X/S)\,(\text{zar})}(A)$. Then there is a Zariski covering $T' \to T$ and an isomorphism $v' \colon \mathcal{L}_{T'} \xrightarrow{\sim} \mathcal{O}_{X_{T'}}$. Now, T' is a disjoint union of Zariski open subschemes of T. One of them contains the closed point, so is equal to T. Restricting v' yields an isomorphism $\mathcal{L} \xrightarrow{\sim} \mathcal{O}_{X_A}$. Thus $\mathrm{Pic}_X(A) \to \mathrm{Pic}_{(X/S)\,(\text{zar})}(A)$ is injective.

Given $\lambda \in \mathrm{Pic}_{(X/S)\,(\text{zar})}(A)$, represent λ by an invertible sheaf $\mathcal{L}'$ on $X_{T'}$ for a suitable Zariski covering $T' \to T$. Again, T' contains a copy of T as an open and closed subscheme. Restricting $\mathcal{L}'$ yields an invertible sheaf $\mathcal{L}$ on X_A. Since $\mathcal{L}'$ represent λ, there is an isomorphism v'' between the two pullbacks of $\mathcal{L}'$ to $T' \times_T T'$. Restricting v'' to $T' \times_T T$, or T', yields an isomorphism between $\mathcal{L}'$ and $\mathcal{L}_{T'}$. Hence $\mathcal{L}$ too represents λ. Thus $\mathrm{Pic}_X(A) \to \mathrm{Pic}_{(X/S)\,(\mathrm{zar})}(A)$ is surjective too.

Assume A is Artin local with algebraically closed residue field. Then every étale A-algebra B of finite type is a direct product of Artin local algebras, each isomorphic to A, owing to [**EGA IV**$_4$, 17.6.2 and 17.6.3]. So if $T' \to T$ is any étale covering, then T' is a disjoint union of open subschemes, each a copy of T. Hence, reasoning as above, we conclude $\mathrm{Pic}_X(A) \xrightarrow{\sim} \mathrm{Pic}_{(X/S)\,(\text{ét})}(A)$.

Assume $A = k$ where k is an algebraically closed field. Then any fppf covering $T' \to T$ has a section; indeed, at any closed point of T', the local ring is essentially of finite type over k, and so the residue field is equal to k by the Hilbert Nullstellensatz. This point is not necessarily isolated in T'; nevertheless, reasoning essentially as above, we conclude $\mathrm{Pic}_X(k) \xrightarrow{\sim} \mathrm{Pic}_{(X/S)\,(\mathrm{fppf})}(k)$. □

Answer 2.4. The extension $\mathbb{C}/\mathbb{R}$ is étale. So if the two pullbacks of $\varphi^*\mathcal{O}(1)$ to $X_{\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}}$ are isomorphic, then $\varphi^*\mathcal{O}(1)$ defines an element λ in $\mathrm{Pic}_{(X/\mathbb{R})\,(\text{ét})}(\mathbb{R})$.

Take an indeterminate w, and identify $\mathbb{C}$ with $\mathbb{R}[w]/(w^2+1)$. Then, by extension of scalars, $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ becomes indentified with $\mathbb{C}[w]/(w-1)(w+1)$. So, by the Chinese Remander Theorem, $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to the product $\mathbb{C} \times \mathbb{C}$. (Correspondingly, $c \otimes 1$ and $1 \otimes c$ are identified with (c, c) and $(c, \overline{c})$ where $\overline{c}$ is the conjugate of c, but this fact is not needed here.)

Thus $X_{\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}}$ is isomorphic, over $\mathbb{R}$, to the disjoint union of two copies of $X_{\mathbb{C}}$. Now, for any field k, an invertible sheaf $\mathcal{L}$ on $\mathbf{P}^1_k$ is determined, up to isomorphism, by a single integer its Euler characteristic $\chi(\mathcal{L})$ by [**Ha83**, Cor. 6.17, p. 145]. Hence, the two pullbacks of $\varphi^*\mathcal{O}(1)$ to $X_{\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}}$ are isomorphic. Thus $\varphi^*\mathcal{O}(1)$ defines a λ in $\mathrm{Pic}_{(X/\mathbb{R})\,(\text{ét})}(\mathbb{R})$.

(Similarly, the isomorphism class of $\varphi^*\mathcal{O}(1)$ on $X_{\mathbb{C}}$ is independent of the choice of the isomorphism $\varphi\colon X_{\mathbb{C}} \xrightarrow{\sim} \mathbf{P}^1_{\mathbb{C}}$. So λ is independent too. But this fact too is not needed here.)

Finally, we must show λ is not in the image of $\mathrm{Pic}_{(X/\mathbb{R})\,(\mathrm{zar})}(\mathbb{R})$. By way of contradicition, suppose λ is. Then λ arises from an invertible sheaf $\mathcal{L}$ on X. A priori, the pullback $\mathcal{L}|X_{\mathbb{C}}$ need not be isomorphic to $\varphi^*\mathcal{O}(1)$. Rather, these two invertible sheaves need only become isomorphic after they are pulled back to X_A where A is some étale $\mathbb{C}$-algebra.

However, cohomology commutes with flat base change. So

$$\dim_{\mathbb{R}} \mathrm{H}^0(\mathcal{L}) = \mathrm{rank}_A \mathrm{H}^0(\mathcal{L}|X_A) = \dim_{\mathbb{C}} \mathrm{H}^0(\varphi^*\mathcal{O}(1)) = 2.$$

Hence, $\mathcal{L}$ has a nonzero section. It defines an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{L} \to \mathcal{O}_D \to 0.$$

Similarly $\mathrm{H}^1(\mathcal{O}_X) = 0$. Hence $\dim_{\mathbb{R}} \mathrm{H}^0(\mathcal{O}_D) = 1$. Therefore, D is an $\mathbb{R}$-point of X. But X has no $\mathbb{R}$-point. Thus λ is not in the image of $\mathrm{Pic}_{(X/\mathbb{R})\,(\mathrm{zar})}(\mathbb{R})$. □

Answer 2.6. First of all, we have $\mathrm{Pic}_{(X/S)\,(\mathrm{fpqc})}(k) = \mathrm{Pic}_{(X_k/k)\,(\mathrm{fpqc})}(k)$ essentially by definition, because a map $T' \to T$ of k-schemes is an fpqc-covering if and only if it is an fpqc-covering when viewed as a map of S-schemes. And a similar analysis applies to the other four functors. Now, $f_k\colon X_k \to k$ has a section; indeed, f_k is of finite type and k is algebraically closed, and so any closed point of X has residue field k by the Hilbert Nullstellensatz. Hence, by Part 2 of Theorem 2.5, the k-points of all five functors are the same. Finally, $\mathrm{Pic}_{X/S}(k) = \mathrm{Pic}(X_k)$ because $\mathrm{Pic}(T)$ is trivial whenever T has only one closed point.

Whether or not $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ holds universally, the first four functors have the same geometric points by Exercise 2.3; in fact, given an algebraically closed field k, the k-points of these functors are just the elements of $\mathrm{Pic}(X_k)$. □

Answer 3.2. By definition, a section in $\mathrm{H}^0(X, \mathcal{L})_{\mathrm{reg}}$ corresponds to an injection $\mathcal{L}^{-1} \hookrightarrow \mathcal{O}_X$. Its image is an ideal $\mathcal{I}$ such that $\mathcal{L}^{-1} \xrightarrow{\sim} \mathcal{I}$. So $\mathcal{I}$ is the ideal of an effective divisor D. Then $\mathcal{O}_X(-D) = \mathcal{I}$. So $\mathcal{L}^{-1} \xrightarrow{\sim} \mathcal{O}_X(-D)$. Taking inverses yields $\mathcal{O}_X(D) \simeq \mathcal{L}$. So $D \in |\mathcal{L}|$. Thus we have a map $\mathrm{H}^0(X, \mathcal{L})_{\mathrm{reg}} \to |\mathcal{L}|$.

If the section is multiplied by a unit in $\mathrm{H}^0(X, \mathcal{O}_X^*)$, then the injection $\mathcal{L}^{-1} \hookrightarrow \mathcal{O}_X$ is multiplied by the same unit, so has the same image $\mathcal{I}$; so then D is unaltered. Conversely, if D arises from a second section, corresponding to a second isomorphism $\mathcal{L}^{-1} \xrightarrow{\sim} \mathcal{I}$, then these two isomorphism differ by an automorphism of $\mathcal{L}^{-1}$, which is given by multiplication by a unit in $\mathrm{H}^0(X, \mathcal{O}_X^*)$; so then the two sections differ by multiplication by this unit. Thus $\mathrm{H}^0(X, \mathcal{L})_{\mathrm{reg}}/\mathrm{H}^0(X, \mathcal{O}_X^*) \hookrightarrow |\mathcal{L}|$.

Finally, given $D \in |\mathcal{L}|$, by definition there exist an ismorphism $\mathcal{O}_X(D) \simeq \mathcal{L}$. Since $O_X(-D)$ is the ideal $\mathcal{I}$ of D, the inclusion $\mathcal{I} \hookrightarrow \mathcal{O}_X$ yields an injection $\mathcal{L}^{-1} \hookrightarrow \mathcal{O}_X$. The latter corresponds to a section in $\mathrm{H}^0(X, \mathcal{L})_{\mathrm{reg}}$, which yields D via the procedure of the first paragraph. Thus $\mathrm{H}^0(X, \mathcal{L})_{\mathrm{reg}}/\mathrm{H}^0(X, \mathcal{O}_X^*) \xrightarrow{\sim} |\mathcal{L}|$. □

Answer 3.5. Let $x \in D + E$. If $x \notin D \cap E$, then $D + E$ is a relative effective divisor at x, as $D + E$ is equal to D or to E on a neighborhood of x. So suppose $x \in D \cap E$. Then Lemma 3.4 says X is S-flat at x, and each of D and E is cut out at x by one element that is regular on the fiber X_s through x. Form the product of the two elements. Plainly, it cuts out $D + E$ at x, and it too is regular on X_s. Hence $D + E$ is a relative effective divisor at x by Lemma 3.4 again. □

Answer 3.8. Consider a relative effective divisor D on X_T/T. Each fiber D_t is of dimension 0. So its Hilbert polynomial $\chi(\mathcal{O}_{D_t}(n))$ is constant. Its value is $\dim \mathrm{H}^0(\mathcal{O}_{D_t})$, which is just the degree of D_t.

The assertions are local on S; so we may assume X/S is projective. Then $\mathrm{Div}_{X/S}$ is representable by an open subscheme $\mathbf{Div}_{X/S} \subset \mathbf{Hilb}_{X/S}$ by Theorem 3.7. And $\mathbf{Hilb}_{X/S}$ is the disjoint union of open and closed subschemes of finite type $\mathbf{Hilb}^{\phi}_{X/S}$ that parameterize the subschemes with Hilbert polynomial ϕ. Set $\mathbf{Div}^m_{X/S} := \mathbf{Div}_{X/S} \bigcap \mathbf{Hilb}^m_{X/S}$. Then the $\mathbf{Div}^m_{X/S}$ have all the desired properties.

In general, whenever X/S is separated, X represents $\mathrm{Hilb}^1_{X/S}$, and the diagonal subscheme $\Delta \subset X \times X$ is the universal subscheme. Indeed, the projection $\Delta \to X$ is an isomorphism, so $\Delta \in \mathrm{Hilb}^1_{X/S}(X)$. Now, given any S-map $g : T \to X$, note $(1 \times g)^{-1}\Delta = \Gamma_g$ where $\Gamma_g \subset X \times T$ is the graph subscheme of g, because the T'-points of both $(1 \times g)^{-1}\Delta$ and Γ_g are just the pairs (gp, p) where $p \colon T' \to T$. So $\Gamma_g \in \mathrm{Hilb}^1_{X/S}(T)$.

Conversely, let $\Gamma \in \mathrm{Hilb}^1_{X/S}(T)$. So Γ is a closed subscheme of $X \times T$. The projection $\pi \colon \Gamma \to T$ is proper, and its fibers are finite; hence, it is finite by Chevalley's Theorem [**EGA III**$_1$, 4.4.2]. So $\Gamma = \mathrm{Spec}(\pi_*\mathcal{O}_\Gamma)$. Moreover, $\pi_*\mathcal{O}_\Gamma$ is locally free, being flat and finitely generated over $\mathcal{O}_T$. And forming $\pi_*\mathcal{O}_\Gamma$ commutes with passing to the fibers, so its rank is 1. Hence $\mathcal{O}_T \xrightarrow{\sim} \pi_*\mathcal{O}_\Gamma$. Therefore, π is an isomorphism. Hence Γ is the graph of a map $g \colon T \to X$. So, $(1 \times g)^{-1}\Delta = \Gamma$ by the above; also, g is the only map with this property, since a map is determined by its graph. Thus X represents $\mathrm{Hilb}^1_{X/S}$, and $\Delta \subset X \times X$ is the universal subscheme.

In the case at hand, $\mathrm{Div}^1_{X/S}$ is therefore representable by an open subscheme $U \subset X$ by Theorem 3.7. In fact, its proof shows U is formed by the points $x \in X$ where the fiber Δ_x is a divisor on X_x. Now, Δ_x is a k_x-rational point for any $x \in X$; so Δ_x is a divisor if and only if X_x is regular at Δ_x. Since X/S is flat, X_x is regular at Δ_x if and only if $x \in X_0$. Thus $X_0 = \mathbf{Div}^1_{X/S}$.

Finally, set $T := X_0^m$ and let $\Gamma_i \subset X \times T$ be the graph subscheme of the ith projection. By the above analysis, $\Gamma_i \in \mathrm{Div}^1_{X/S}(T)$. Set $\Gamma := \sum \Gamma_i$. Then $\Gamma \in \mathrm{Div}^m_{X/S}(T)$ owing to Exercise 3.5 and to the additivity of degree. Plainly Γ represents the desired T-point of $\mathbf{Div}^m_{X/S}$. □

Answer 3.11. Let $s \in S$. Let K be the algebraically closure of k_s, and set $A := \mathrm{H}^0(X_K, \mathcal{O}_{X_K})$. Since f is proper, A is finite dimensional as a K-vector space; so A is an Artin ring. Since X_K is connected, A is not a product of two nonzero rings by [**EGA III**$_2$, 7.8.6.1]; so A is an Artin local ring. Since X_K is reduced, A is reduced; so A is a field, which is a finite extension of K. Since K is algebraically closed, therefore $A = K$. Since cohomology commutes with flat base change, consequently $k_s \xrightarrow{\sim} \mathrm{H}^0(X, \mathcal{O}_{X_s})$.

The isomorphism $k_s \xrightarrow{\sim} \mathrm{H}^0(X, \mathcal{O}_{X_s})$ factors through $f_*(\mathcal{O}_X) \otimes k_s$:

$$k_s \to f_*(\mathcal{O}_X) \otimes k_s \to \mathrm{H}^0(X_s, \mathcal{O}_{X_s}).$$

So the second map is a surjection. Hence this map is an isomorphism by the implication (iv)⇒(iii) of Subsection 3.10 with $\mathcal{F} := \mathcal{O}_X$ and $\mathcal{N} := k_s$. Therefore, the first map is an isomorphism too.

It follows that $\mathcal{O}_S \to f_*\mathcal{O}_X$ is surjective at s. Indeed, denote its cokernel by $\mathcal{G}$. Since tensor product is right exact and since $k_s \to f_*(\mathcal{O}_X) \otimes k_s$ is an isomorphism, $\mathcal{G} \otimes k_s = 0$. So by Nakayama's lemma, the stalk $\mathcal{G}_s$ vanishes, as claimed

Let $\mathcal{Q}$ be the $\mathcal{O}_S$-module associated to $\mathcal{F} := \mathcal{O}_X$ as in Subsection 3.10. Then $\mathcal{Q}$ is free at s by the implication (iv)⇒(i) of Subsection 3.10. And $\operatorname{rank} \mathcal{Q}_s = 1$ owing to the isomorphism in (3.10.1) with $\mathcal{N} := k_s$. But, with $\mathcal{N} := \mathcal{O}_S$, the isomorphism becomes $\mathcal{H}om(\mathcal{Q}, \mathcal{O}_X) \xrightarrow{\sim} f_*\mathcal{O}_X$. Hence $f_*\mathcal{O}_X$ too is free of rank 1 at s. Therefore, the surjection $\mathcal{O}_S \to f_*\mathcal{O}_X$ is an isomorphism at s. Since s is arbitrary, $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ everywhere.

Finally, let T be an arbitrary S-scheme. Then $f_T \colon X_T \to T$ too is proper and flat, and its geometric fibers are reduced and connected. Hence, by what we just proved, $\mathcal{O}_T \xrightarrow{\sim} f_T * \mathcal{O}_{X_T}$. □

Answer 3.14. By Theorem 3.13, L represents $\mathrm{LinSys}_{\mathcal{L}/X/S}$. So by Yoneda's Lemma [**EGA G**, (0,1.1.4), p. 20], there exists a $W \in \mathrm{LinSys}_{\mathcal{L}/X/S}(L)$ possessing the required universal property. And W corresponds to the identity map $p \colon L \to L$. The proof of Theorem 3.13 now shows $\mathcal{O}_{X_L}(W) = (\mathcal{L}|X_L) \otimes f_L^*\mathcal{O}_L(1)$. □

Answer 4.2. The structure sheaf $\mathcal{O}_X$ defines a section $\sigma \colon S \to \mathbf{Pic}_{X/S}$. Its image is a subscheme, which is closed if $\mathbf{Pic}_{X/S}$ is separated, by [**EGA G**, Cors. (5.1.4), p. 275, and (5.2.4), p. 278]. Let $N \subset T$ be the pullback of this subscheme under the map $\lambda \colon T \to \mathbf{Pic}_{X/S}$ defined by $\mathcal{L}$. Then the third property holds.

Both $\mathcal{L}_N$ and $\mathcal{O}_X$ define the same map $N \to \mathbf{Pic}_{X/S}$. So, since $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ holds universally, the Comparison Theorem, Theorem, Theorem 2.5, implies that there exists an invertible sheaf $\mathcal{N}$ on N such that the first property holds.

Consider the second property. Then $\mathcal{L}_{T'} \simeq f_{T'}^*\mathcal{N}'$. So $\lambda t\colon T' \to \mathbf{Pic}_{X/S}$ is also defined by $\mathcal{O}_{X_{T'}}$; hence, λt factors through $\sigma\colon S \to \mathbf{Pic}_{X/S}$. Therefore, $t\colon T' \to T$ factors through N. So, since the first property holds, $\mathcal{L}_{T'} \simeq f_{T'}^* t^*\mathcal{N}$. Hence $\mathcal{N}' \simeq t^*\mathcal{N}$ by Lemma (2.7). Thus the second property holds.

Finally, suppose the pair $(N_1, \mathcal{N}_1)$ also possesses the first property. Taking t to be the inclusion of N_1 into T, we conclude that $N_1 \subset N$ and $\mathcal{N}_1 \simeq \mathcal{N}|N$. Suppose $(N_1, \mathcal{N}_1)$ possess the second property too. Then, similarly, $N \subset N_1$. Thus $N = N_1$ and $\mathcal{N}_1 \simeq \mathcal{N}$, as desired. □

Answer 4.3. By Yoneda's Lemma [**EGA G**, (0,1.1.4), p. 20], a universal sheaf $\mathcal{P}$ exists if and only if $\mathbf{Pic}_{X/S}$ represents $\mathrm{Pic}_{X/S}$. Set $P := \mathbf{Pic}_{X/S}$.

Assume $\mathcal{P}$ exists. Then, for any invertible sheaf $\mathcal{N}$ on P, plainly $\mathcal{P} \otimes f_P^*\mathcal{N}$ is also a universal sheaf. Moreover, if $\mathcal{P}'$ is also a universal sheaf, then $\mathcal{P}' \simeq \mathcal{P} \otimes f_P^*\mathcal{N}$ for some invertible sheaf $\mathcal{N}$ on P by the definition with $h := 1_P$.

Assume $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ holds universally. If $\mathcal{P} \otimes f_P^*\mathcal{N} \simeq \mathcal{P} \otimes f_P^*\mathcal{N}'$ for some invertible sheaves $\mathcal{N}$ and $\mathcal{N}'$ on P, then $\mathcal{N} \simeq \mathcal{N}'$ by Lemma 2.7.

By Part 2 of Theorem 2.5, if also f has a section, then $\mathbf{Pic}_{X/S}$ does represent $\mathrm{Pic}_{X/S}$; so then $\mathcal{P}$ exists. Furthermore, the curve $X/\mathbb{R}$ of Exercise 2.4 provides an example where no $\mathcal{P}$ exists, because $\mathrm{Pic}_{(X/\mathbb{R})\,(\text{ét})}$ is representable by Theorem 4.8, but $\mathrm{Pic}_{X/\mathbb{R}}$ is not since the two functors differ. □

Answer 4.4. Say $\mathbf{Pic}_{X/S}$ represents $\mathrm{Pic}_{(X/S)\,(\text{ét})}$. Now, for any S'-scheme T,

$$\mathrm{Pic}_{(X_{S'}/S')\,(\text{ét})}(T) = \mathrm{Pic}_{(X/S)\,(\text{ét})}(T),$$

which holds essentially by definition, since a map of S'-schemes is an étale-covering if and only if it is an étale-covering when viewed as a map of S-schemes. However,

$$(\mathbf{Pic}_{X/S} \times_S S')(T) = \mathbf{Pic}_{X/S}(T)$$

because the structure map $T \to S'$ is fixed. Since the right-hand sides of the two displayed equations are equal, so are their left-hand sides. Thus $\mathbf{Pic}_{X/S} \times_S S'$ represents $\mathrm{Pic}_{(X_{S'}/S')\,(\text{ét})}$. Of course, a similar analysis applies when $\mathbf{Pic}_{X/S}$ represents one of the other relative Picard functors.

An example is provided by the curve $X \subset \mathbf{P}^2_{\mathbb{R}}$ of Exercise 2.4. Indeed, since the functors $\mathrm{Pic}_{X/\mathbb{R}}$ and $\mathrm{Pic}_{(X/\mathbb{R})\,(\text{ét})}$ differ, $\mathrm{Pic}_{X/\mathbb{R}}$ is not representable. But $\mathrm{Pic}_{(X/\mathbb{R})\,(\text{ét})}$ is representable by the Main Theorem, 4.8. Finally, since $X_{\mathbb{C}}$ has a $\mathbb{C}$-point, all its relative Picard functors are equal by the Comparision Theorem, 2.5. □

Answer 4.5. An $\mathcal{L}$ on an X_k defines a map $\mathrm{Spec}(k) \to \mathbf{Pic}_{X/S}$; assign its image to $\mathcal{L}$. Then, given any field k'' containing k, the pullback $\mathcal{L}|X_{k''}$ is assigned the same scheme point of $\mathbf{Pic}_{X/S}$.

Consider an $\mathcal{L}'$ on an $X_{k'}$. If $\mathcal{L}$ and $\mathcal{L}'$ represent the same class, then there is a k'' containing both k and k' such that $\mathcal{L}|X_{k''} \simeq \mathcal{L}'|X_{k''}$; hence, then both $\mathcal{L}$ and $\mathcal{L}'$ are assigned the same scheme point of $\mathbf{Pic}_{X/S}$. Conversely, if $\mathcal{L}$ and $\mathcal{L}'$ are assigned the same point, take k'' to be any algebraically closed field containing both k and k'. Then $\mathcal{L}|X_{k''}$ and $\mathcal{L}'|X_{k''}$ define the same map $\mathrm{Spec}(k'') \to \mathbf{Pic}_{X/S}$. Hence $\mathcal{L}|X_{k''} \simeq \mathcal{L}'|X_{k''}$ by Exercise 2.3 or 2.6.

Finally, given any scheme point of $\mathbf{Pic}_{X/S}$, let k be the algebraic closure of its residue field. Then $\mathrm{Spec}(k) \to \mathbf{Pic}_{X/S}$ is defined by an $\mathcal{L}$ on X_k by Exercise 2.3 or 2.6. So the given point is assigned to $\mathcal{L}$. Thus the classes of invertible sheaves on the fibers of X/S correspond bijectively to the scheme points of $\mathbf{Pic}_{X/S}$. □

ANSWER 4.7. An S-map $h\colon T \to \mathbf{Div}_{X/S}$ corresponds to a relative effective divisor D on X_T. So the composition $\mathbf{A}_{X/S}h\colon T \to P$ corresponds to the invertible sheaf $\mathcal{O}_{X_T}(D)$. Hence $O_{X_T}(D) \simeq (1 \times \mathbf{A}_{X/S}h)^*\mathcal{P} \otimes f_P^*\mathcal{N}$ for some invertible sheaf $\mathcal{N}$ on T. Therefore, if T is viewed as a P-scheme via $\mathbf{A}_{X/S}h$, then D defines a T-point η of $\mathrm{LinSys}_{\mathcal{P}/X\times P/P}$. Plainly, the assigment $h \mapsto \eta$ is functorial in T. Thus if $\mathbf{Div}_{X/S}$ is viewed as a P-scheme via $\mathbf{A}_{X/S}$, then there is a natural map Λ from its functor of points to $\mathrm{LinSys}_{\mathcal{P}/X\times P/P}$.

Furthermore, Λ is an isomorphism. Indeed, let T be a P-scheme. A T-point η of $\mathrm{LinSys}_{\mathcal{P}/X\times P/P}$ is given by a relative effective divisor D on X_T such that $\mathcal{O}_{X_T}(D) \simeq \mathcal{P}_T \otimes f_T^*\mathcal{N}$ for some invertible sheaf $\mathcal{N}$ on T. Then $\mathcal{O}_{X_T}(D)$ and $\mathcal{P}_T$ define the same S-map $T \to P$. But $\mathcal{P}_T$ defines the structure map. And $\mathcal{O}_{X_T}(D)$ defines the composition $\mathbf{A}_{X/S}h$ where $h\colon T \to \mathbf{Div}_{X/S}$ is the map defined by D. Thus $\eta = \Lambda(h)$, and h is determined by η; hence, Λ is an isomorphism.

In other words, $\mathbf{Div}_{X/S}$ represents $\mathrm{LinSys}_{\mathcal{P}/X\times P/P}$. But $\mathbf{P}(\mathcal{Q})$ too represents $\mathrm{LinSys}_{\mathcal{P}/X\times P/P}$ by Theorem 3.13. Therefore, $\mathbf{P}(\mathcal{Q}) = \mathbf{Div}_{X/S}$ as P-schemes. □

ANSWER 4.10. First, suppose $F \to G$ is a surjection. Given a map of étale sheaves $\varphi\colon F \to H$ such that the two maps $F \times_G F \to H$ are equal, we must show there is one and only one map $G \to H$ such that $F \to G \to H$ is equal to φ.

Let $\eta \in G(T)$. By hypothesis, there exist an étale covering $T' \to T$ and an element $\zeta' \in F(T')$ such that ζ' and η have the same image in $G(T')$. Set $T'' := T' \times_T T'$. Then the two images of ζ' in $F(T'')$ define an element ζ'' of $(F \times_G F)(T'')$. Since the two maps $F \times_G F \to H$ are equal, the two images of ζ'' in $H(T'')$ are equal. But these two images are equal to those of $\varphi(\zeta') \in H(T')$. Since H is a sheaf, therefore $\varphi(\zeta')$ is the image of a unique element $\theta \in H(T)$.

Note $\theta \in H(T)$ is independent of the choice of T' and $\zeta' \in F(T')$. Indeed, let $\zeta'_1 \in F(T'_1)$ be a second choice. Arguing as above, we find $\varphi(\zeta'_1) \in H(T'_1)$ and $\varphi(\zeta') \in H(T')$ have the same image in $H(T'_1 \times_T T')$. So ζ'_1 also leads to θ.

Define a map $G(T) \to H(T)$ by $\eta \mapsto \theta$. Plainly this map behaves functorially in T. Thus there is a map of sheaves $G \to H$. Plainly, $F \to G \to H$ is equal to $\varphi\colon F \to H$. Finally, $G \to H$ is the only such map, since the image of η in $H(T)$ is determined by the image of η in $G(T')$, and the latter must map to $\varphi(\zeta') \in H(T')$. Thus G is the coequalizer of $F \times_G F \rightrightarrows F$.

Conversely, suppose G is the coequalizer of $F \times_G F \rightrightarrows F$. Form the étale subsheaf $H \subset G$ associated to the presheaf whose T-points are the images in $G(T)$ of the elements of $F(T)$. Then the map $F \to G$ factors through H. So the two maps $F \times_G F \to H$ are equal. Since G is the coequalizer, there is a map $G \to H$ so that $F \to G \to H$ is equal to $F \to H$. Hence $F \to G \to H \hookrightarrow G$ is equal to $F \to G$. So $G \to H \hookrightarrow G$ is equal to 1_G by uniqueness. Therefore, $H = G$. Thus $F \to G$ is a surjection. □

ANSWER 4.11. Theorem 4.8 implies each connected component Z' of Z lies in an increasing union of open quasi-projective subschemes of $\mathbf{Pic}_{X/S}$. So Z' lies in one of them since Z' is quasi-compact. So Z' is quasi-projective. But Z has only finitely many components Z'. Therefore, Z is quasi-projective. □

ANSWER 4.12. Set $P := \mathbf{Pic}_{X/S}$, which exists by Theorem 4.8. If $\mathcal{P}$ exists, then $\mathbf{A}_{X/S}$ is, by Exercise 4.7, the structure map of the bundle $\mathbf{P}(\mathcal{Q})$ where $\mathcal{Q}$ denotes the coherent sheaf on $\mathbf{Pic}_{X/S}$ associated to $\mathcal{P}$ as in Subsection 3.10. In particular, $\mathbf{A}_{X/S}$ is projective locally over S.

In general, forming P commutes with extending S by Exercise 4.4. Similarly, forming $\mathbf{A}_{X/S}$ does too. But a map is proper if it is after an fpqc base extension by [**EGA IV**$_2$, 2.7.1(vii)].

However, $f\colon X \to S$ is fpqc. Moreover, $f_X\colon X \times X \to X$ has a section, namely, the diagonal. So use f as a base extension. Then, by Exercises 3.11 and 4.3, a universal sheaf $\mathcal{P}$ exists. Therefore, $\mathbf{A}_{X/S}$ is proper by the first case. □

Answer 4.13. Let's use the ideas and notation of Answer 4.12. Now, X_0 represents $\mathrm{Div}^1_{X/S}$ by Exercise 3.8. Hence the Abel map $\mathbf{A}_{X/S}$ induces a natural map $A\colon X_0 \to P$, and forming A commutes with extending S. But a map is a closed embedding if it is after an fpqc base extension by [**EGA IV**$_2$, 2.7.1(xii)]. So we may assume $\mathbf{P}(\mathcal{Q}) = \mathbf{Div}_{X/S}$.

The function $\lambda \mapsto \deg \mathcal{P}_\lambda$ is locally constant. Let $W \subset P$ be the open and closed subset where the function's value is 1. Plainly $\mathbf{P}(\mathcal{Q}_W) = \mathbf{Div}^1_{X/S}$ owing to the above. Therefore, $X_0 = \mathbf{P}(\mathcal{Q}_W)$, and $A\colon X_0 \to P$ is equal to the structure map of $\mathbf{P}(\mathcal{Q}_W)$. So it remains to show that this structure map is a closed embedding.

Fix $\lambda \in W$. Then $\dim_{k_\lambda}(\mathcal{Q} \otimes k_\lambda) = \dim_{k_\lambda} \mathrm{H}^0(X_\lambda, \mathcal{P}_\lambda)$. Suppose P_λ has two independent global sections. Each defines an effective divisor of degree 1, which is a k_λ-rational point x_i. Since neither section is a multiple of the other, the x_i are distinct. Hence the sections generate P_λ. So they define a map $h\colon X_\lambda \to \mathbf{P}^1_{k_\lambda}$ by [**EGA II**, 4.2.3] or [**Ha83**, Thm. II, 7.1, p. 150]. Then h is birational since each x_i is the scheme-theoretic inverse image of a k_λ-rational point of $\mathbf{P}^1_{k_\lambda}$. Hence h is an isomorphism. But, by hypothesis, X_λ is of arithmetic genus at least 1. So there is a contradiction. Therefore, $\dim_{k_\lambda}(\mathcal{Q} \otimes k_\lambda) \le 1$.

By Nakayama's lemma, $\mathcal{Q}$ can be generated by a single element on a neighborhood $V \subset W$ of λ. So there is a surjection $\mathcal{O}_V \twoheadrightarrow \mathcal{Q}_V$. It defines a closed embedding $\mathbf{P}(\mathcal{Q}_V) \hookrightarrow \mathbf{P}(\mathcal{O}_V)$. But the structure map $\mathbf{P}(\mathcal{O}_V) \to V$ is an isomorphism. Hence $\mathbf{P}(\mathcal{Q}_V) \to V$ is a closed embedding. But $\lambda \in W$ is arbitrary. So $\mathbf{P}(\mathcal{Q}_W) \to W$ is indeed a closed embedding. □

Answer 4.15. Representing $\mathrm{Pic}_{X/S}$ is similar to representing $\mathrm{Pic}_{X'/S'}$ in Example 4.14, but simpler. Indeed, On $X \times_S \mathbb{Z}_S$, form an invertible sheaf $\mathcal{P}$ by placing $\mathcal{O}_X(n)$ on the nth copy of X. Then it suffices to show this: given any S-scheme T and any invertible sheaf $\mathcal{L}$ on X_T, there exist a unique S-map $q\colon T \to \mathbb{Z}_S$ and some invertible sheaf $\mathcal{N}$ on T such that $(1 \times q)^*\mathcal{P} = \mathcal{L} \otimes f_T^*\mathcal{N}$.

Plainly, we may assume T is connected. Then the function $s \mapsto \chi(X_t, \mathcal{L}_t)$ is constant on T by [**EGA III**$_2$, 7.9.11]. Now, X_t is a projective space of dimension at least 1 over the residue field k_t; so $\mathcal{L}_t \simeq \mathcal{O}_{X_t}(n)$ for some n by [**Ha83**, Prp. 6.4, p. 132, and Cors. 6.16 and 6.17, p. 145]. Hence n is independent of t.

Set $\mathcal{M} := \mathcal{L}^{-1}(n)$. Then $\mathcal{M}_t \simeq \mathcal{O}_{X_t}$ for all $t \in T$. Hence $\mathrm{H}^1(X_t, \mathcal{M}_t) = 0$ and $\mathrm{H}^0(X_t, \mathcal{M}_t) = k_t$ by Serre's explicit computation [**EGA III**$_1$, 2.1.12]. Hence $f_{T*}\mathcal{M}$ is invertible, and forming it commutes with changing the base T, owing to the theory in Subsection 3.10.

Set $\mathcal{N} := f_{T*}\mathcal{M}$. Consider the natural map $u\colon f_T^*\mathcal{N} \to \mathcal{M}$. Forming u commutes with changing T, since forming $\mathcal{N}$ does. But u is an isomorphism on the fiber over each $t \in T$. So $u \otimes k_t$ is an isomorphism. Hence u is surjective by Nakayama's lemma. But both source and target of u are invertible; so u is an isomorphism. Hence $\mathcal{L} \otimes f_T^*\mathcal{N} = \mathcal{O}_{X_T}(n)$.

Let $q\colon T \to \mathbb{Z}_S$ be the composition of the structure map $T \to S$ and the nth inclusion $S \hookrightarrow \mathbb{Z}_S$. Plainly $(1 \times q)^*\mathcal{P} = \mathcal{O}_{X_T}(n)$, and q is the only such S-map. Thus $\mathbb{Z}_S$ represents $\mathrm{Pic}_{X/S}$, and $\mathcal{P}$ is a universal sheaf. □

ANSWER 4.16. First of all, $\mathbf{Pic}_{X/\mathbb{R}}$ exists by Theorem 4.8. Now, $X_{\mathbb{C}} \simeq \mathbf{P}^1_{\mathbb{C}}$. Hence $\mathbf{Pic}_{X/\mathbb{R}} \times_{\mathbb{R}} \mathbb{C} \simeq \mathbb{Z}_{\mathbb{C}}$ by Exercises 4.4 and 4.15. The induced automorphism of $\mathbb{Z}_{\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}}$ is the identity; indeed, a point of this scheme corresponds to an invertible sheaf on $\mathbf{P}^1_{\mathbb{C}}$, and every such sheaf is isomorphic to its pullback under any $\mathbb{R}$-automorphism of $\mathbf{P}^1_{\mathbb{C}}$. Hence, by descent theory, $\mathbf{Pic}_{X/\mathbb{R}} = \mathbb{Z}_{\mathbb{R}}$.

The above reasoning leads to a second proof $\mathrm{Pic}_{(X/\mathbb{R})\,(\text{ét})}$ is representable. Indeed, set $P := \mathrm{Pic}_{(X/\mathbb{R})\,(\text{ét})}$. By the above, the pair $(P \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) \rightrightarrows P \otimes_{\mathbb{R}} \mathbb{C}$ is representable by the pair $\mathbb{Z}_{\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}} \rightrightarrows \mathbb{Z}_{\mathbb{C}}$, whose coequalizer is $\mathbb{Z}_{\mathbb{R}}$. On the other hand, in the category of étale sheaves, the coequalizer is P owing to Exercise 4.10.

Notice in passing that $\mathbf{Pic}_{X/\mathbb{R}} = \mathbf{Pic}_{\mathbf{P}^1_{\mathbb{R}}/\mathbb{R}}$. However, $\mathrm{Pic}_{X/\mathbb{R}}$ is not representable owing to Exercise 2.4, where as $\mathrm{Pic}_{\mathbf{P}^1_{\mathbb{R}}/\mathbb{R}}$ is representable owing to Exercise 4.15. □

ANSWER 5.7. Exercise 4.11 implies Z is quasi-projective. Hence Z is projective if Z is proper. By [**EGA IV**$_2$, 2.7.1], an S-scheme is proper if it is so after an fpqc base change, such as $f\colon X \to S$. But $f_X\colon X \times X \to X$ has a section, namely, the diagonal. Thus we may assume f has a section.

Using the Valuative Criterion for Properness [**Ha83**, Thm. 4.7, p. 101], we need only check this statement: given an S-scheme T of the form $T = \mathrm{Spec}(A)$ where A is a valuation ring, say with fraction field K, every S-map $u\colon \mathrm{Spec}(K) \to Z$ extends to an S-map $T \to Z$. We do not need to check the extension is unique if it exists; indeed, this uniqueness holds by the Valuative Criterion for Separatedness [**Ha83**, Thm. 4.3, p. 97] since Z is quasi-projective, so separated.

Since f has a section, u arises from an invertible sheaf $\mathcal{L}$ on X_K by Theorem 2.5. We have to extend $\mathcal{L}$ over X_T. Indeed, this extension defines a map $t\colon T \to \mathbf{Pic}_{X/S}$ extending u, and t factors through Z because Z is closed and T is integral.

Plainly it suffices to extend $\mathcal{L}(n)$ for any $n \gg 0$. So replacing $\mathcal{L}$ if need be, we may assume $\mathcal{L}$ has a nonzero section. It is regular since X_K is integral. So X_K has a divisor D such that $\mathcal{O}(D) = \mathcal{L}$.

Let $D' \subset X_T$ be the closure of D. Now, X/S is smooth and T is regular, so X_T is regular by [**EGA IV**$_2$, 6.5.2], so factorial by [**EGA IV**$_2$, 21.11.1]. Hence D' is a divisor. And $\mathcal{O}(D')$ extends $\mathcal{L}$. □

ANSWER 5.16. Serre's Theorem [**Ha83**, Thm. 5.2, p. 228] yields $\mathrm{H}^i(\Omega^2_X(n)) = 0$ for $i > 0$ and $n \gg 0$. So $\phi(n) = \chi(\Omega^2_X(n))$. Hence

$$q = \mathrm{H}^1(\Omega^2_X) - \mathrm{H}^2(\Omega^2_X) + 1.$$

Serre duality [**Ha83**, Cor. 7.13, p. 247] yields $\dim \mathrm{H}^i(\Omega^i_X) = \dim \mathrm{H}^{2-i}(\mathcal{O}_X)$ for all i. And $\dim \mathrm{H}^0(\mathcal{O}_X) = 1$ since X is projective and geometrically integral. So

$$q = \dim \mathrm{H}^1(\mathcal{O}_X).$$

Hence Corollary 5.14 yields $\dim \mathbf{Pic}_{X/S} \leq q$, with equality in characteristic 0. □

ANSWER 5.17. Set $P := \mathbf{Pic}_{X/S}$, which exists by Theorem 4.8. By Exercises 3.11 and 4.3, there exists a universal sheaf $\mathcal{P}$ on $X \times P$

Suppose $q = 0$. Then P is smooth of dimension 0 everywhere by Corollary 5.13. Let D be a relative effective divisor on X_T/T where T is a connected S-scheme.

Then $\mathcal{O}_{X_T}(D)$ defines a map $\tau : T \to P$, and

$$\mathcal{O}_{X_T}(D) \simeq (1 \times \tau)^* \mathcal{P} \otimes f_T^* \mathcal{N}$$

for some invertible sheaf $\mathcal{N}$ on T. Now, T is connected and P is discrete and reduced; so τ is constant. Set $\lambda := \tau T$, and view $\mathcal{P}_\lambda$ as an invertible sheaf $\mathcal{L}$ on X. Then $\mathcal{L}_T = (1 \times \tau)^* \mathcal{P}$. So $\mathcal{O}_{X_T}(D) \simeq \mathcal{L}_T \otimes f_T^* \mathcal{N}$, as required.

Consider the converse. Again by Exercise 4.3, there is a coherent sheaf $\mathcal{Q}$ on P such that $\mathbf{P}(\mathcal{Q}) = \mathbf{Div}_{X/S}$. Furthermore, $\mathcal{Q}$ is nonzero and locally free at any closed point λ representing an invertible sheaf $\mathcal{L}$ on X such that $\mathrm{H}^1(\mathcal{L}) = 0$ by Subsection 3.10; for example, take $\mathcal{L} := \mathcal{O}_X(n)$ for $n \gg 0$.

Let $U \subset P$ be a connected open neighborhood of λ on which Q is free. Let $T \subset \mathbf{P}(\mathcal{Q})$ be the preimage of U, and let D be the universal relative effective divisor on X_T/T. Then the natural map $A\colon T \to U$ is smooth with irreducible fibers. So T is connected. Moreover, A is the map defined by $\mathcal{O}_{X_T}(D)$.

Suppose $\mathcal{O}_{X_T}(D) \simeq \mathcal{M}_T \otimes f_T^* \mathcal{N}$ for some invertible sheaves $\mathcal{M}$ on X and $\mathcal{N}$ on T. Then $A\colon T \to U$ is also defined by $\mathcal{M}_T$. Say $\mu \in P$ represents $\mathcal{M}$. Then A factors through the inclusion of the closed point μ. Hence $\mu = \lambda$; moreover, since A is smooth and surjective, its image, the open set U, is just the reduced closed point λ. Now, there is an automorphism of P that carries 0 to λ, namely, "multiplication" by λ. So P is smooth of dimension 0 at 0. Therefore, $q = 0$ by Corollary 5.13.

In characteristic 0, a priori P is smooth by Corollary 5.14. Now, $A\colon T \to U$ is smooth. Hence, T is smooth too. But the preceding argument shows that, if the condition holds for this T, then $q = 0$, as required. □

Answer 5.23. By hypothesis, $\dim X_s = 1$ for $s \in S$; so $\mathrm{H}^2(\mathcal{O}_{X_s}) = 0$. Hence the $\mathbf{Pic}^0_{X_s/k_s}$ are smooth by Proposition 5.19, so of dimension p_a by Proposition 5.13. Hence, by Proposition 5.20, the $\mathbf{Pic}^0_{X_s/k_s}$ form a family of finite type, whose total space is the open subscheme $\mathbf{Pic}^0_{X/S}$ of $\mathbf{Pic}_{X/S}$. And $\mathbf{Pic}_{X/S}$ is smooth over S again by Proposition 5.19.

Hence $\mathbf{Pic}^0_{X/S}$ is quasi-projective by Exercise 4.11.

If X/S is smooth, then $\mathbf{Pic}^0_{X/S}$ is projective over S by Exercise 5.7. Alternatively, use Theorem 5.4 and Proposition 5.20 again to conclude $\mathbf{Pic}^0_{X/S}$ is proper, so projective since it is quasi-projective.

Conversely, assume $\mathbf{Pic}^0_{X/S}$ is proper, and let us prove X/S is smooth, Since X/S is flat, we need only prove each X_s is smooth. So we may replace S by the spectrum of the algebraic closure of k_s. If $p_a = 0$, then X is smooth, indeed $X = \mathbf{P}^1$, by [**Ha83**, Ex. 1.8(b), p. 298].

Suppose $p_a > 0$. Let X_0 be the open subscheme where X is smooth. Then there is a closed embedding $A\colon X_0 \hookrightarrow \mathbf{Pic}_{X/S}$ by Exercise 4.13. Its image consists of points λ representing invertible sheaves of degree 1. Fix a rational point λ, and define an automorphism β of $\mathbf{Pic}_{X/S}$ by $\beta(\kappa) := \kappa\lambda^{-1}$. Then βA is a closed embedding of X_0 in $\mathbf{Pic}^0_{X/S}$.

By assumption, $\mathbf{Pic}^0_{X/S}$ is proper. So X_0 is proper. Hence $X_0 \hookrightarrow X$ is proper since X is separated. Hence X_0 is closed in X. But X_0 is dense in X since X is integral and the ground field is algebraically closed. Hence $X_0 = X$; in other words, X is smooth. □

Answer 6.4. As before, by Lemma 6.6, there is an m such that every $\mathcal{N}(m)$

is generated by its global sections. So there is a section that does not vanish at any given associated point of X; since these points are finite in number, if σ is a general linear combination of the corresponding sections, then σ vanishes at no associated point. So σ is regular, whence defines an effective divisor D such that $\mathcal{O}_X(-D) = \mathcal{N}^{-1}(-m)$.

Plainly $\mathcal{N}^{-1}$ is numerically equivalent to $\mathcal{O}_X$ too. So $\chi(\mathcal{N}^{-1}(n)) = \chi(\mathcal{O}_X(n))$ by Lemma 6.6. Hence the sequence $0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$ yields

$$\chi(\mathcal{O}_D(n)) = \psi(n) \text{ where } \psi(n) := \chi(\mathcal{O}_X(n)) - \chi(\mathcal{O}_X(n-m)).$$

Let $T \subset \mathbf{Div}_{X/k}$ be the open and closed subscheme parameterizing the effective divisors with Hilbert polynomial $\psi(n)$. Then T is a k-scheme of finite type. Let $\mathcal{M}'$ be the invertible sheaf on X_T associated to the universal divisor; set $\mathcal{M} := \mathcal{M}'(-n)$. Then there exists a rational point $t \in T$ such that $\mathcal{N} = \mathcal{M}_t$. Thus the $\mathcal{N}$ numerically equivalent to $\mathcal{O}_X$ form a bounded family. □

Answer 6.7. Suppose $a < a_r$. Suppose $\mathcal{L}(-1)$ has a nonzero section. It defines an effective divisor D, possibly 0. Hence

$$0 \le \int h^{r-1}[D] = \int h^{r-1}\ell - \int h^r = a - a_r < 0,$$

which is absurd. Thus $\mathrm{H}^0(\mathcal{L}(-1)) = 0$.

Let H be a hyperplane section of X. Then there is an exact sequence

$$0 \to \mathcal{L}(n-1) \to \mathcal{L}(n) \to \mathcal{L}_H(n) \to 0.$$

It yields the following bound:

$$\dim \mathrm{H}^0(\mathcal{L}(n)) - \dim \mathrm{H}^0(\mathcal{L}(n-1)) \le \dim \mathrm{H}^0(\mathcal{L}_H(n)). \tag{A.6.7.1}$$

Since $\binom{n+i}{i} - \binom{n-1+i}{i} = \binom{n+i-1}{i-1}$, the sequence also yields the following formula:

$$\chi(\mathcal{L}_H(n)) = \textstyle\sum_{0 \le i \le r-1} a_{i+1}\binom{n+i}{i}.$$

Suppose $r = 1$. Then $\dim \mathrm{H}^0(\mathcal{L}_H(n)) = \chi(\mathcal{L}_H(n)) = a_1$. Therefore, owing to Equation (A.6.7.1), induction on n yields $\dim \mathrm{H}^0(\mathcal{L}(n)) \le a_1(n+1)$, as desired.

Furthermore, $\mathcal{L}_H$ is 0-regular. Set $m := \dim \mathrm{H}^1(\mathcal{L}(-1))$. Then $\mathcal{L}$ is m-regular by Mumford's conclusion at the bottom of [**Mm66**, p. 102]. But

$$m = \dim \mathrm{H}^0(\mathcal{L}(-1)) - \chi(\mathcal{L}(-1)) = 0 - a_1(-1+1) - a_0 = -a_0.$$

Thus we may take $\Phi_1(u_0) := -u_0$ where u_0 is an indeterminate.

Suppose $r \ge 2$. Then we may take H irreducible by Bertini's Theorem [**Se50**, Thm. 12, p. 374] or [**Jo79**, Cor. 6.7, p. 80]. Set $h_1 := c_1 O_H(1)$ and $\ell_1 := c_1\mathcal{L}_H$. Then $\int \ell_1 h_1^{r-2} = \int \ell h^{r-2}[H] = a < a_r$. So by induction on r, we may assume

$$\dim \mathrm{H}^0(\mathcal{L}_H(n)) \le a_r\binom{n+r-1}{r-1}.$$

Therefore, owing to Equation (A.6.7.1), induction on n yields the desired bound.

Furthermore, we may assume $\mathcal{L}_H$ is m_1-regular where $m_1 := \Phi_{r-1}(a_1 \dots, a_{r-1})$. Set $m := m_1 + \dim \mathrm{H}^1(\mathcal{L}(m_1 - 1))$. By Mumford's same work, $\mathcal{L}$ is m-regular. But

$$\begin{aligned} m &= m_1 + \dim \mathrm{H}^0(\mathcal{L}(m_1-1)) - \chi(\mathcal{L}(m_1-1)) \\ &\le m_1 + a_r\binom{m_1-1+r}{r} - \textstyle\sum_{0\le i\le r} a_i\binom{m_1-1+i}{i}. \end{aligned}$$

The latter expression is a polynomial in $a_0, \dots, a_{r-1}$ and m_1. So it is a polynomial Φ_r in $a_0, \dots, a_{r-1}$ alone, as desired.

In general, consider $\mathcal{N} := \mathcal{L}(-a)$. Then

$$\chi(\mathcal{N}(n)) = \textstyle\sum_{0\le i\le r} b_i\binom{n+i}{i} \text{ where } b_i := \sum_{j=0}^{r-i} a_{i+j}(-1)^j\binom{a-i-j}{j}.$$

Set $\nu := c_1\mathcal{N}$ and $b := \int \nu h^{r-1}$. Then

$$b = \textstyle\int \ell h^{r-1} - a\int h^r = a - aa_r \le 0 < a_r.$$

Hence $\mathcal{N}$ is m-regular where $m := \Phi_r(b_0, \dots, b_{r-1})$. But the b_i are polynomials in $a_0, \dots, a_r$ and a. Hence there is a polynomial Ψ_r depending only on r such that $m := \Psi_r(a_0, \dots, a_r; a)$, as desired. □

Answer 6.10. Let k' be the algebraic closure of k. If $H \otimes k' \subset G^\tau \otimes k'$, then $H \subset G^\tau$. But $G^\tau \otimes k' = (G \otimes k')^\tau$ by Lemma 6.10. Thus we may assume $k = k'$.

Then $H \subset \bigcup_{h\in H(k)} hG^0$. But G^0 is open, so hG^0 is too. And H is quasi-compact. So H lies in finitely many hG^0. So $G^0(k)$ has finite index in $H(k)G^0(k)$, say n. Then $h^n \in G^0(k)$ for every $h \in H(k)$. So $\varphi_n(H) \subset G^0$. Thus $H \subset G^\tau$. □

Answer 6.11. For any n, plainly $\mathcal{L}^{\otimes n}$ corresponds to $\varphi_n\lambda$. And $\mathcal{L}^{\otimes n}$ is algebraically equivalent to $\mathcal{O}_X$ if and only if $\varphi_n\lambda \in \mathbf{Pic}^0_{X/k}$ by Proposition 5.10. So $\mathcal{L}$ is τ-equivalent to $\mathcal{O}_X$ if and only if $\lambda \in \mathbf{Pic}^\tau_{X/k}$ by Definitions 6.1 and 6.8. □

Answer 6.13. Theorem 4.8 implies $\mathbf{Pic}_{X/k}$ exists and represents $\mathrm{Pic}_{(X/S)\,(\text{ét})}$. So $\mathbf{Pic}^\tau_{X/k}$ is of finite type by Proposition 6.12. Hence $\mathbf{Pic}^\tau_{X/k}$ is quasi-projective by Exercise 4.11.

Suppose X is also geometrically normal. Since $\mathbf{Pic}^\tau_{X/k}$ is quasi-projective, to prove it is projective, it suffices to prove it is complete. By Proposition 6.12, forming $\mathbf{Pic}^0_{X/k}$ commutes with extending k. And by [**EGA IV**$_2$, 2.7.1(vii)], a k-scheme is complete if (and only if) it is after extending k. So assume k is algebraically closed.

As λ ranges over the k-points of $\mathbf{Pic}^\tau_{X/k}$, the cosets $\lambda\,\mathbf{Pic}^0_{X/k}$ cover $\mathbf{Pic}^\tau_{X/k}$. So finitely many cosets cover, since $\mathbf{Pic}^0_{X/k}$ is an open by Proposition 5.3 and since $\mathbf{Pic}^\tau_{X/k}$ is quasi-compact, Now, $\mathbf{Pic}^0_{X/k}$ is projective by Theorem 5.4, so complete, And $\mathbf{Pic}^\tau_{X/k}$ is closed again by Proposition 6.12. Therefore, $\mathbf{Pic}^\tau_{X/k}$ is complete. □

Answer 6.15. Suppose L is bounded. Then $\mathcal{M}$ defines a map $\theta\colon T \to \mathbf{Pic}_{X/S}$, and $\theta(T) \supset \Lambda$. Since T is Noetherian, plainly so is $\theta(T)$; whence, plainly so is any subspace of $\theta(T)$. Thus Λ is quasi-compact.

Conversely, suppose Λ is quasi-compact. Since $\mathbf{Pic}_{X/S}$ is locally of finite type by Proposition 4.17, there is an open subscheme of finite type containing any given point of Λ. So finitely many of the subschemes cover Λ. Denote their union by U.

The inclusion $U \hookrightarrow \mathbf{Pic}_{X/S}$ is defined by an invertible sheaf $\mathcal{M}$ on X_T for some fppf covering $T \to U$. Replace T be an open subscheme so that $T \to U$ is of finite type and surjective. Since U is of finite type, so is T. Given $\lambda \in \Lambda$, let $t \in T$ map to λ. Then λ corresponds to the class of $\mathcal{M}_t$. □

Answer 6.18. Theorem 6.16 asserts $\mathbf{Pic}^\tau_{X/S}$ is of finite type. So it is projective by Exercise 5.7. □

Answer 6.21. Plainly, replacing S by an open subset, we may assume X/S is projective and S is connected. Given $s \in S$, set $\psi(n) := \chi(\mathcal{O}_{X_s}(n))$. Then $\psi(n)$ is independent of s. Given m, set $\phi(n) := m + \psi(n)$.

Let $\lambda \in \mathbf{Pic}_{X/S}$. Then $\lambda \in \mathbf{Pic}^m_{X/S}$ if and only if λ represents an invertible sheaf $\mathcal{L}$ of degree m. And $\lambda \in \mathbf{Pic}^{\phi}_{X/S}$ if and only if $\chi(\mathcal{L}(n)) = \phi(n)$. But,

$$\chi(\mathcal{L}(n)) = \deg(\mathcal{L}(n)) + \psi(0) = \deg(\mathcal{L}) + \psi(n)$$

by Riemann's Theorem and the additivity of $\deg(\bullet)$. Hence $\mathbf{Pic}^{m}_{X/S} = \mathbf{Pic}^{\phi}_{X/S}$. So Theorem 6.20 yields all the assertions, except for the two middle about $\mathbf{Pic}^{0}_{X/S}$.

To show $\mathbf{Pic}^{0}_{X/S} = \mathbf{Pic}^{\tau}_{X/S}$, similarly we need only show $\deg \mathcal{L} = 0$ if and only if $\mathcal{L}$ is τ-equivalent to $\mathcal{O}_X$, for, by Exercise 6.11, the latter holds if and only if $\lambda \in \mathbf{Pic}^{\tau}_{X/S}$. Plainly, we may assume $\mathcal{L}$ lives on a geometric fiber of X/S. Then the two conditions on $\mathcal{L}$ are equivalent by Theorem 6.3.

Since deg is additive, multiplication carries $\mathbf{Pic}^{0}_{X/S} \times \mathbf{Pic}^{m}_{X/S}$ set-theoretically into $\mathbf{Pic}^{m}_{X/S}$. So $\mathbf{Pic}^{0}_{X/S}$ acts on $\mathbf{Pic}^{m}_{X/S}$ since these two sets are open in $\mathbf{Pic}_{X/S}$.

Since X/S is flat with integral geometric fibers, its smooth locus X_0 provides an fppf covering of S. Temporarily, make the base change $X_0 \to S$. After it, the new map $X_0 \to S$ has a section. Its image is a relative effective divisor D, and tensoring with $\mathcal{O}_X(mD)$ defines the desired isomorphism from $\mathbf{Pic}^{0}_{X/S}$ to $\mathbf{Pic}^{m}_{X/S}$.

Finally, to show there is no abuse of notation, we must show the fiber $(\mathbf{Pic}^{0}_{X/S})_s$ is connected. To do so, we instead make the base change to the spectrum of an algebraically closed field $k \supset k_s$. Then X_0 has a k-rational point D, and again tensoring with $\mathcal{O}_X(mD)$ defines an isomorphism from $\mathbf{Pic}^{0}_{X/k}$ to $\mathbf{Pic}^{m}_{X/k}$. So it suffices to show $\mathbf{Pic}^{m}_{X/k}$ is connected for some $m \geq 1$.

Let $\beta\colon X_0^m \to \mathbf{Div}^{m}_{X/k} \to \mathbf{Pic}^{m}_{X/S}$ be composition of the map α of Exercise 3.8 and the Abel map. Since X is integral, so is the m-fold product X_0^m. Hence it suffices to show β is surjective for some $m \geq 1$.

By Exercise 6.7, there is an $m_0 \geq 1$ such that every invertible sheaf on X of degree 0 is m_0-regular. Set $m := \deg(\mathcal{O}_X(m_0))$. Then every invertible sheaf $\mathcal{L}$ on X of degree m is 0-regular, so generated by its global sections.

In particular, for each singular point of X, there is a global section that does not vanish at it. So, since k is infinite, a general linear combination of these sections vanishes at no singular point. This combination defines an effective divisor E such that $\mathcal{O}_X(E) = \mathcal{L}$. It follows that β is surjective, as desired. □

Answer 6.29. Suppose Λ is quasi-compact. Then, owing to Exercise 6.15, there exist an S-scheme T of finite type and an invertible sheaf $\mathcal{M}$ on X_T such that every polynomial $\phi \in \Pi$ is of the form $\phi(n) = \chi(\mathcal{M}_t(n))$ for some $t \in T$. Hence, by [**EGA III**$_2$, 7.9.4], the number of ϕ is at most the number of connected components of T. Thus Π is finite.

Suppose Λ is connected. Then its closure is too. So we may assume Λ is closed. Give Λ its reduced subscheme structure. Then the inclusion $\Lambda \hookrightarrow \mathbf{Pic}_{X/S}$ is defined by an invertible sheaf $\mathcal{M}$ on X_T for some fppf covering $T \to \Lambda$. Fix $t \in T$ and set $\phi(n) := \chi(\mathcal{M}_t(n))$. Fix n, and form the set T' of points t' of T such that $\chi(\mathcal{M}_{t'}(n)) = \phi(n)$. By [**EGA III**$_2$, 7.9.4], the set T' is open, and so is its complement. Hence their images are open in Λ, and plainly these images are disjoint. But Λ is connected. Hence $T' = T$. Thus $\Pi = \{\phi\}$. □

Appendix B. Basic intersection theory

This appendix contains an elementary treatment of basic intersection theory, which is more than sufficient for many purposes, including the needs of Section 6. The approach was originated in 1959–60 by Snapper. His results were generalized and his proofs were simplified immediately afterward by Cartier [**Ca60**]. Their work was developed further in fits and starts by the author.

The index theorem was proved by Hodge in 1937. Immediately afterward, B. Segre [**Se37**, § 1] gave an algebraic proof for surfaces, and this proof was rediscovered by Grothendieck in 1958. Their work was generalized a tad in [**Kl71**, p. 662], and a variation appears below in Theorem B.27. From the index theorem, Segre [**Se37**, §6] derived a connectedness statement like Corollary B.29 for surfaces, and the proof below is basically his.

Definition B.1. Let $\mathbf{F}(X/S)$ or $\mathbf{F}$ denote the Abelian category of coherent sheaves $\mathcal{F}$ on X whose support $\operatorname{Supp}\mathcal{F}$ is proper over an Artin subscheme of S. For each $r \geq 0$, let $\mathbf{F}_r$ denote the full subcategory of those $\mathcal{F}$ such that $\dim \operatorname{Supp}\mathcal{F} \leq r$.

Let $\mathbf{K}(X/S)$ or $\mathbf{K}$ denote the "Grothendieck group" of $\mathbf{F}$, namely, the free Abelian group on the $\mathcal{F}$, modulo short exact sequences. Abusing notation, let $\mathcal{F}$ also denote its class. And if $\mathcal{F} = \mathcal{O}_Y$ where $Y \subset X$ is a subscheme, then let $[Y]$ also denote the class. Let $\mathbf{K}_r$ denote the subgroup generated by $\mathbf{F}_r$.

Let $\chi\colon \mathbf{K} \to \mathbb{Z}$ denote the homomorphism induced by the Euler characteristic.

Given $\mathcal{L} \in \operatorname{Pic}(X)$, let $c_1(\mathcal{L})$ denote the endomorphism of $\mathbf{K}$ defined by the following formula:

$$c_1(\mathcal{L})\mathcal{F} := \mathcal{F} - \mathcal{L}^{-1} \otimes \mathcal{F}.$$

Note that $c_1(\mathcal{L})$ is well defined since tensoring with $\mathcal{L}^{-1}$ preserves exact sequences.

Lemma B.2. *Let $\mathcal{L} \in \operatorname{Pic}(X)$. Let $Y \subset X$ be a closed subscheme with $\mathcal{O}_Y \in \mathbf{F}$. Let $D \subset Y$ be an effective divisor such that $\mathcal{O}_Y(D) \simeq \mathcal{L}_Y$. Then*

$$c_1(\mathcal{L}) \cdot [Y] = [D].$$

Proof. The left side is defined since $\mathcal{O}_Y \in \mathbf{F}$. The equation results from the sequence $0 \to \mathcal{O}_Y(-D) \to \mathcal{O}_Y \to \mathcal{O}_D \to 0$ since $\mathcal{O}_Y(-D) \simeq \mathcal{L}^{-1} \otimes \mathcal{O}_Y$. □

Lemma B.3. *Let $\mathcal{L}$, $\mathcal{M} \in \operatorname{Pic}(X)$. Then the following relations hold:*

$$c_1(\mathcal{L})c_1(\mathcal{M}) = c_1(\mathcal{L}) + c_1(\mathcal{M}) - c_1(\mathcal{L} \otimes \mathcal{M}),$$
$$c_1(\mathcal{L})c_1(\mathcal{L}^{-1}) = c_1(\mathcal{L}) + c_1(\mathcal{L}^{-1}),$$
$$c_1(\mathcal{O}_X) = 0.$$

Proof. Let $\mathcal{F} \in \mathbf{K}$. By definition, $c_1(\mathcal{O}_X)\mathcal{F} = 0$; thus the third relation holds. Plainly, each side of the first relation carries $\mathcal{F}$ into

$$\mathcal{F} - \mathcal{L}^{-1} \otimes \mathcal{F} - \mathcal{M}^{-1} \otimes \mathcal{F} + \mathcal{L}^{-1} \otimes \mathcal{M}^{-1} \otimes \mathcal{F}.$$

Thus the first relation holds. And it and the third relation imply the second. □

Lemma B.4. *Given $\mathcal{F} \in \mathbf{F}_r$, let $Y_1, \ldots, Y_s$ be the r-dimensional irreducible components of* $\operatorname{Supp}\mathcal{F}$ *equipped with their induced reduced structure, and let l_i be the length of the stalk of $\mathcal{F}$ at the generic point of Y_i. Then, in $\mathbf{K}_r$,*

$$\mathcal{F} \equiv \sum l_i \cdot [Y_i] \bmod \mathbf{K}_{r-1}.$$

Proof. The assertion holds if it does after we replace S by a neighorhood of the image of $\operatorname{Supp}\mathcal{F}$. So we may assume S is Noetherian.

Let $\mathbf{F}' \subset \mathbf{F}_r$ denote the family of $\mathcal{F}$ for which the assertion holds. Since length($\bullet$) is an additive function, $\mathbf{F}'$ is "exact" in the following sense: for any short eact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ such that two of the $\mathcal{F}$s belong to $\mathbf{F}'$, then the third does too. Trivially, $\mathcal{O}_Y \in \mathbf{F}'$ for any closed integral subscheme $Y \subset X$. Hence $\mathbf{F}' = \mathbf{F}_r$ by the "Lemma of Dévissage," [**EGA III**$_1$, Thm. 3.1.2]. □

Lemma B.5. *Let* $\mathcal{L} \in \operatorname{Pic}(X)$. *Then* $c_1(\mathcal{L})\mathbf{K}_r \subset \mathbf{K}_{r-1}$ *for all* r.

Proof. Let $\mathcal{F} \in \mathbf{F}_r$. Then $\mathcal{F}$ and $\mathcal{L}^{-1} \otimes \mathcal{F}$ are isomorphic at the generic point of each component of $\operatorname{Supp}\mathcal{F}$. So Lemma B.4 implies $c_1(\mathcal{L})\mathcal{F} \in \mathbf{K}_{r-1}$. □

Lemma B.6. *Let* $\mathcal{L} \in \operatorname{Pic}(X)$, *let* $\mathcal{F} \in \mathbf{K}_r$, *and let* $m \in \mathbb{Z}$. *Then*

$$\mathcal{L}^{\otimes m} \otimes \mathcal{F} = \textstyle\sum_{i=0}^{r} \binom{m+i-1}{i} c_1(\mathcal{L})^i \mathcal{F}.$$

Proof. Let x be an indeterminate, and consider the formal identity

$$(1-x)^n = \textstyle\sum_{i \geq 0} (-1)^i \binom{n}{i} x^i.$$

Replace x by $1 - y^{-1}$, set $n := -m$, and use the familiar identity

$$(-1)^i \tbinom{n}{i} = \tbinom{m+i-1}{i},$$

to obtain the formal identity

$$y^m = \textstyle\sum \binom{m+i-1}{i} (1-y^{-1})^i.$$

It yields the assertion, because $c_1(L)^i \mathcal{F} = 0$ for $i > r$ owing to Lemma B.5. □

Theorem B.7 (Snapper). *Let* $\mathcal{L}_1, \ldots, \mathcal{L}_n \in \operatorname{Pic}(X)$, *let* $m_1, \ldots, m_n \in \mathbb{Z}$, *and let* $\mathcal{F} \in \mathbf{K}_r$. *Then the Euler characteristic* $\chi\big(\mathcal{L}_1^{\otimes m_1} \otimes \cdots \otimes \mathcal{L}_n^{\otimes m_n} \otimes \mathcal{F}\big)$ *is given by a polynomial in the* m_i *of degree at most* r. *In fact,*

$$\chi\big(\mathcal{L}_1^{\otimes m_1} \otimes \cdots \otimes \mathcal{L}_n^{\otimes m_n} \otimes \mathcal{F}\big) = \textstyle\sum a(i_1, \ldots, i_n) \binom{m_1+i_1-1}{i_1} \cdots \binom{m_n+i_n-1}{i_n}$$

where $i_j \geq 0$ *and* $\sum i_j \leq r$ *and where* $a(i_1, \ldots, i_n) := \chi\big(c_1(\mathcal{L}_1)^{i_1} \cdots c_1(\mathcal{L}_n)^{i_n} \mathcal{F}\big)$.

Proof. The theorem follows from Lemmas B.6 and B.5. □

Definition B.8. Let $\mathcal{L}_1, \ldots, \mathcal{L}_r \in \operatorname{Pic}(X)$, repetitions allowed. Let $\mathcal{F} \in \mathbf{K}_r$. Define the *intersection number* or *intersection symbol* by the formula

$$\textstyle\int c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_r)\mathcal{F} := \chi\big(c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_r)\mathcal{F}\big) \in \mathbb{Z}.$$

If $\mathcal{F} = \mathcal{O}_X$, then also write $\int c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_r)$ for the number. If $\mathcal{L}_j = \mathcal{O}_X(D_j)$ for a divisor D_j, then also write $(D_1 \cdots D_r \cdot \mathcal{F})$, or just $(D_1 \cdots D_r)$ if $\mathcal{F} = \mathcal{O}_X$.

Theorem B.9. *Let* $\mathcal{L}_1, \ldots, \mathcal{L}_r \in \operatorname{Pic}(X)$ *and* $\mathcal{F} \in \mathbf{K}_r$.

(1) *If* $\mathcal{F} \in \mathbf{F}_{r-1}$, *then* $\int c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_r)\mathcal{F} = 0$.

(2) (symmetry and additivity) *The symbol* $\int c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_r)\mathcal{F}$ *is symmetric in the* $\mathcal{L}_j$. *Furthermore, it is a homomorphism separately in each* $\mathcal{L}_j$ *and in* $\mathcal{F}$.

(3) *Set* $\mathcal{E} := \mathcal{L}_1^{-1} \oplus \cdots \oplus \mathcal{L}_r^{-1}$. *Then*

$$\textstyle\int c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_r)\mathcal{F} = \sum_{i=0}^{r} (-1)^i \chi\big(\big(\bigwedge^i \mathcal{E}\big) \otimes \mathcal{F}\big).$$

Proof. Part (1) results from Lemma B.5. So the symbol is a homomorphism in $\mathcal{L}_j$ owing to Lemma B.3. The remaining assertion result from the definitions. □

Corollary B.10. *Let* $\mathcal{L}_1$, $\mathcal{L}_2 \in \operatorname{Pic}(X)$ *and* $\mathcal{F} \in \mathbf{K}_2$. *Then*

$$\textstyle\int c_1(\mathcal{L}_1)c_1(\mathcal{L}_2)\mathcal{F} = \chi(\mathcal{F}) - \chi(\mathcal{L}_1^{-1} \otimes \mathcal{F}) - \chi(\mathcal{L}_2^{-1} \otimes \mathcal{F}) + \chi(\mathcal{L}_1^{-1} \otimes \mathcal{L}_2^{-1} \otimes \mathcal{F}).$$

Proof. The assertion is a special case of Part (3) of Proposition B.9. □

Corollary B.11. *Let* $D_1, \ldots, D_r$ *be effective divisors on* X, *and* $\mathcal{F} \in \mathbf{F}_r$. *Set* $Z := D_1 \cap \cdots \cap D_r$. *Suppose* $Z \cap \operatorname{Supp} F$ *is finite, and at each of its points,* F *is Cohen–Macaulay. Then*

$$(D_1 \cdots D_r \cdot \mathcal{F}) = \dim \mathrm{H}^0(\mathcal{F}_Z) \ \text{ where } \mathcal{F}_Z := \mathcal{F} \otimes \mathcal{O}_Z.$$

PROOF. For each j, set $\mathcal{L}_j := \mathcal{O}_X(D_j)$ and let $\sigma_j \in \mathrm{H}^0(\mathcal{L}_j)$ be the section defining D_j. Set $\mathcal{E} := \mathcal{L}_1^{-1} \oplus \cdots \oplus \mathcal{L}_r^{-1}$. Form the corresponding Koszul complex $(\bigwedge^\bullet \mathcal{E}) \otimes \mathcal{F}$ and its cohomology sheaves $\mathcal{H}^i\big((\bigwedge^\bullet \mathcal{E}) \otimes \mathcal{F}\big)$. Then

$$\textstyle\int c_1(\mathcal{L}_1)\cdots c_1(\mathcal{L}_r)\mathcal{F} = \sum_{i=0}^r (-1)^i \chi\big(\mathcal{H}^i\big((\bigwedge^\bullet \mathcal{E}) \otimes \mathcal{F}\big)\big).$$

owing to Part (3) of Proposition B.9 and to the additivity of χ. Furthermore, essentially by definition, $\mathcal{H}^0\big((\bigwedge^\bullet \mathcal{E}) \otimes \mathcal{F}\big) = \mathcal{F}_Z$. And by standard local algebra, the higher $\mathcal{H}^i$ vanish. Thus the assertion holds. □

LEMMA B.12. *Let $\mathcal{L}_1, \dots, \mathcal{L}_r \in \mathrm{Pic}(X)$ and $\mathcal{F} \in \mathbf{F}_r$. Let $Y_1, \dots, Y_s$ be the r-dimensional irreducible components of* $\mathrm{Supp}\,\mathcal{F}$ *given their induced reduced structure, and let l_i be the length of the stalk of $\mathcal{F}$ at the generic point of Y_i. Then*

$$\textstyle\int c_1(\mathcal{L}_1)\cdots c_1(\mathcal{L}_r)\mathcal{F} = \sum_i l_i \int c_1(\mathcal{L}_1)\cdots c_1(\mathcal{L}_r)[Y_i].$$

PROOF. Apply Lemma B.4 and Parts (1) and (2) of Proposition B.9. □

LEMMA B.13. *Let $\mathcal{L}_1, \dots, \mathcal{L}_r \in \mathrm{Pic}(X)$ and $Y \subset X$ a closed subscheme with $\mathcal{O}_Y \in \mathbf{F}$. Let $D \subset Y$ be an effective divisor such that $\mathcal{O}_Y(D) \simeq \mathcal{L}_r|Y$. Then*

$$\textstyle\int c_1(\mathcal{L}_1)\cdots c_1(\mathcal{L}_r)[Y] = \int c_1(\mathcal{L}_1)\cdots c_1(\mathcal{L}_{r-1})[D].$$

PROOF. Apply Lemma B.2. □

PROPOSITION B.14. *Let $\mathcal{L}_1, \dots, \mathcal{L}_r \in \mathrm{Pic}(X)$ and $\mathcal{F} \in \mathbf{F}_r$. If all the $\mathcal{L}_j$ are relatively ample and if $\mathcal{F} \notin \mathbf{K}_{r-1}$, then*

$$\textstyle\int c_1(\mathcal{L}_1)\cdots c_1(\mathcal{L}_r)\mathcal{F} > 0.$$

PROOF. Proceed by induction on r. If $r = 0$, then $\int \mathcal{F} = \dim \mathrm{H}^0(\mathcal{F})$ essentially by definition, and $H^0(\mathcal{F}) \neq 0$ since $\mathcal{F} \notin \mathbf{K}_{r-1}$ by hypothesis.

Suppose $r \geq 1$. Owing to Proposition B.12, we may assume $\mathcal{F} = \mathcal{O}_Y$ where Y is integral. Owing to Part (2) of Theorem B.9, we may replace $\mathcal{L}_r$ by a multiple, and so assume it is very ample. Then, for the corresponding embedding of Y, a hyperplane section D is a nonempty effective divisor such that $\mathcal{O}_Y(D) \simeq \mathcal{L}_r|Y$. Hence the assertion results from Proposition B.13 and the induction hypothesis. □

LEMMA B.15. *Let $g\colon X' \to X$ be an S-map. Let $\mathcal{L}_1, \dots, \mathcal{L}_r \in \mathrm{Pic}(X)$ and let $\mathcal{F} \in \mathbf{F}_r(X'/S)$. Then*

$$\textstyle\int c_1(g^*\mathcal{L}_1)\cdots c_1(g^*\mathcal{L}_r)\mathcal{F} = \int c_1(\mathcal{L}_1)\cdots c_1(\mathcal{L}_r)g_*\mathcal{F}.$$

PROOF. Define a map $\mathbf{F}_r(X'/S) \to \mathbf{K}_r(X/S)$ by $\mathcal{G} \mapsto \sum_{i=0}^r (-1)^i \mathrm{R}^i g_* \mathcal{G}$. It induces a homomorphism $\mathrm{R}f_*\colon \mathbf{K}_r(X'/S) \to \mathbf{K}_r(X/S)$. And $\chi(Rg_*(\mathcal{G})) = \chi(\mathcal{G})$ owing to the Leray Spectral Sequence [**EGA III**$_1$, **0**-12.2.4] and to the additivity of χ [**EGA III**$_1$, **0**-11.10.3]. Furthermore, $\mathcal{L} \otimes \mathrm{R}^i g_*(\mathcal{G}) \xrightarrow{\sim} \mathrm{R}^i g_*(g^*\mathcal{L} \otimes \mathcal{G})$ for any $\mathcal{L} \in \mathrm{Pic}(X)$ by [**EGA III**$_1$, **0**-12.2.3.1]. Hence $c_1(\mathcal{L})Rg_*(\mathcal{G}) = Rg_*(c_1(g^*\mathcal{L})\mathcal{G})$. Therefore,

$$\textstyle\int c_1(g^*\mathcal{L}_1)\cdots c_1(g^*\mathcal{L}_r)\mathcal{F} = \int c_1(\mathcal{L}_1)\cdots c_1(\mathcal{L}_r)\mathrm{R}g_*\mathcal{F}.$$

Finally, $\mathrm{R}g_*\mathcal{F} \equiv \mathrm{g}_*\mathcal{F} \bmod \mathbf{K}_{r-1}(X/S)$, because $\mathrm{R}^i g_*\mathcal{F} \in \mathbf{F}_{r-1}$ for $i \geq 1$ since, if $W \subset X'$ is the locus where $\mathrm{Supp}\,\mathcal{F} \to X$ has fibers of dimension at least 1, then $\dim g(W) \leq r - 1$. So Part (1) of Theorem B.9 yields the asserted formula. □

PROPOSITION B.16 (Projection Formula). *Let $g\colon X' \to X$ be an S-map. Let $\mathcal{L}_1, \dots, \mathcal{L}_r \in \operatorname{Pic}(X)$. Let $Y' \subset X'$ be an integral subscheme with $\mathcal{O}_{Y'} \in \mathbf{F}_r(X'/S)$. Set $Y := gY' \subset X$, give Y its induced reduced structure, and let* $\deg(Y'/Y)$ *be the degree of the function field extension if finite and be* 0 *if not. Then*

$$\textstyle\int c_1(g^*\mathcal{L}_1)\cdots c_1(g^*\mathcal{L}_r)[Y'] = \deg(Y'/Y)\int c_1(\mathcal{L}_1)\cdots c_1(\mathcal{L}_r)[Y].$$

PROOF. Lemma B.4 yields $g_*\mathcal{O}_{Y'} \equiv \deg(Y'/Y)[Y] \bmod \mathbf{K}_{r-1}(X/S)$. So the assertion results from Lemma B.15 and from Part (1) of Theorem B.9. □

PROPOSITION B.17. *Assume S is the spectrum of a field, and let T be the spectrum of an extension field. Let $\mathcal{L}_1, \dots, \mathcal{L}_r \in \operatorname{Pic}(X)$ and $\mathcal{F} \in \mathbf{F}_r(X/S)$. Then*

$$\textstyle\int c_1(\mathcal{L}_{1,T})\cdots c_1(\mathcal{L}_{r,T})\mathcal{F}_T = \int c_1(\mathcal{L}_1)\cdots c_1(\mathcal{L}_r)\mathcal{F}.$$

PROOF. The base change $T \to S$ preserves short exact sequences. So it induces a homomorphism $\kappa\colon \mathbf{K}_r(X/S) \to \mathbf{K}_r(X_T/T)$. Plainly κ preserves the Euler characteristic. The assertion now follows. □

PROPOSITION B.18. *Let $\mathcal{L}_1, \dots, \mathcal{L}_r \in \operatorname{Pic}(X)$. Let $\mathcal{F}$ be a flat coherent sheaf on X. Assume* $\operatorname{Supp}\mathcal{F}$ *is proper and of relative dimension r. Then the function*

$$\textstyle y \mapsto \int c_1(\mathcal{L}_1)\cdots c_1(\mathcal{L}_r)\mathcal{F}_y$$

is locally constant.

PROOF. The assertion results from Definition B.8 and [**EGA III**$_2$, 7.9.11]. □

DEFINITION B.19. Let $\mathcal{L}, \mathcal{N} \in \operatorname{Pic}(X)$. Call them *numerically equivalent* if $\int c_1(\mathcal{L})[Y] = \int c_1(\mathcal{N})[Y]$ for all closed integral curves $Y \subset X$ with $\mathcal{O}_Y \in \mathbf{F}_1$.

PROPOSITION B.20. *Let $\mathcal{L}_1, \dots, \mathcal{L}_r; \mathcal{N}_1, \dots, \mathcal{N}_r \in \operatorname{Pic}(X)$ and $\mathcal{F} \in \mathbf{K}_r$. If $\mathcal{L}_j$ snd $\mathcal{N}_j$ are numerically equivalent for each j, then*

$$\textstyle\int c_1(\mathcal{L}_1)\cdots c_1(\mathcal{L}_r)\mathcal{F} = \int c_1(\mathcal{N}_1)\cdots c_1(\mathcal{N}_r)\mathcal{F}.$$

PROOF. If $r = 1$, then the assertion results from Lemma B.12. Suppose $r \geq 2$. Then $c_1(\mathcal{L}_2)\cdots c_1(\mathcal{L}_r)\mathcal{F} \in \mathbf{K}_1$ by Lemma B.5. Hence

$$\textstyle\int c_1(\mathcal{L}_1)c_1(\mathcal{L}_2)\cdots c_1(\mathcal{L}_r)\mathcal{F} = \int c_1(\mathcal{N}_1)c_1(\mathcal{L}_2)\cdots c_1(\mathcal{L}_r)\mathcal{F}.$$

Similarly, $c_1(\mathcal{N}_1)c_1(\mathcal{L}_3)\cdots c_1(\mathcal{L}_r)\mathcal{F} \in \mathbf{K}_1$, and so

$$\textstyle\int c_1(\mathcal{N}_1)c_1(\mathcal{L}_2)c_1(\mathcal{L}_3)\cdots c_1(\mathcal{L}_r)\mathcal{F} = \int c_1(\mathcal{N}_1)c_1(\mathcal{N}_2)c_1(\mathcal{L}_3)\cdots c_1(\mathcal{L}_r)\mathcal{F}.$$

Continuing in this fashion yields the assertion. □

PROPOSITION B.21. *Let $g\colon X' \to X$ be an S-map. Let $\mathcal{L}, \mathcal{N} \in \operatorname{Pic}(X)$.*

(1) *If $\mathcal{L}$ and $\mathcal{N}$ are numerically equivalent, then so are $g^*\mathcal{L}$ and $g^*\mathcal{N}$.*

(2) *Conversely, when g is surjective, if $g^*\mathcal{L}$ and $g^*\mathcal{N}$ are numerically equivalent, then so are $\mathcal{L}$ and $\mathcal{N}$.*

PROOF. Let $Y' \subset X'$ be a closed integral curve with $\mathcal{O}_{Y'} \in \mathbf{F}_1(X'/S)$. Set $Y := g(Y')$ and give Y its induced reduced structure. Then Proposition B.16 yields

$$\textstyle\int c_1(g^*\mathcal{L})[Y'] = \deg(Y'/Y)\int c_1(\mathcal{L})[Y] \text{ and}$$
$$\textstyle\int c_1(g^*\mathcal{N})[Y'] = \deg(Y'/Y)\int c_1(\mathcal{N})[Y].$$

Part (1) follows.

Conversely, suppose g is surjective. Let $Y \subset X$ be a closed integral curve with

$\mathcal{O}_Y \in \mathbf{F}_1(X/S)$. let $y \in Y$ be the generic point, $y' \in g^{-1}Y$ a closed point, and Y' the closure of y'. Give Y' its induced reduced structure. Plainly $\mathcal{O}_{Y'} \in \mathbf{F}_1(X'/S)$ and $\deg(Y'/Y \neq 0$. The two equations displayed above now yield Part (2). □

Definition B.22. Assume S is Artin, and X a proper curve. Let $\mathcal{L} \in \operatorname{Pic}(X)$. Define its *degree* $\deg(\mathcal{L})$ by the formula

$$\deg(\mathcal{L}) := \textstyle\int c_1(\mathcal{L}).$$

Let D be a divisor on X. Define its *degree* $\deg(D)$ by $\deg(D) := \deg(\mathcal{O}_X(D))$.

Proposition B.23. *Assume S is Artin, and X a proper curve.*
(1) *The map* $\deg\colon \operatorname{Pic}(X) \to \mathbb{Z}$ *is a homomorphism.*
(2) *Let $D \subset X$ be an effective divisor. Then*

$$\deg(D) = \dim \mathrm{H}^0(\mathcal{O}_D).$$

(3) (Riemann's Theorem) *Let $\mathcal{L} \in \operatorname{Pic}(X)$. Then*

$$\chi(\mathcal{L}) = \deg(\mathcal{L}) + \chi(\mathcal{O}_X).$$

(4) *Suppose X is integral, and let $g\colon X' \to X$ be the normalization map. Then*

$$\deg(\mathcal{L}) = \deg(g^*\mathcal{L}).$$

Proof. Part (1) results from Theorem B.9 (2). And Part (2) results from Lemma B.13. As to Part (3), note $\deg(\mathcal{L}^{-1}) = -\deg(\mathcal{L})$ by Part (1). And the definitions yield $\deg(\mathcal{L}^{-1}) = \chi(\mathcal{O}_X) - \chi(\mathcal{L})$. Thus Part (3) holds. Finally, Part (3) results from the definition and the Projection Formula. □

Definition B.24. Assume S is Artin, and X a proper surface. Given a divisor D on X, set

$$p_a(D) := 1 - \chi\big(c_1(\mathcal{O}_X(D))\,\mathcal{O}_X\big).$$

Proposition B.25. *Assume S is Artin, and X a proper surface. Let D and E be divisors on X. Then*

$$p_a(D+E) = p_a(D) + p_a(E) + (D\cdot E) - 1.$$

Furthermore, if D is effective, then

$$p_a(D) = 1 - \chi(\mathcal{O}_D);$$

in other words, $p_a(D)$ is equal to the arithmetic genus *of D.*

Proof. The assertions result from Lemmas B.3 and B.2. □

Proposition B.26 (Riemann–Roch for surfaces). *Assume S is the spectrum of a field, and X is a projective, equidimensional, Cohen–Macaulay surface. Let ω be a dualizing sheaf, and set $\mathcal{K} := \omega - \mathcal{O}_X \in \mathbf{K}_1$. Let D be a divisor on X. Then*

$$p_a(D) = \frac{(D^2) + (D\cdot\mathcal{K})}{2} + 1 \quad \textit{and} \quad \chi(\mathcal{O}_X(D)) = \frac{(D^2) - (D\cdot\mathcal{K})}{2} + \chi(\mathcal{O}_X).$$

If X/S is Gorenstein, that is, $\omega = \mathcal{O}_X(K)$ for some "canonical" divisor K, then

$$p_a(D) = \frac{\big(D\cdot(D+K)\big)}{2} + 1 \quad \textit{and} \quad \chi(\mathcal{O}_X(D)) = \frac{\big(D\cdot(D-K)\big)}{2} + \chi(\mathcal{O}_X).$$

Proof. Set $\mathcal{L} := \mathcal{O}_X(D)$. Then $(D^2) := \int c_1(\mathcal{L})^2 = -\int c_1(\mathcal{L})c_1(\mathcal{L}^{-1})$ by Parts (1) and (2) of Theorem B.9. Now, the definitions yield

$$c_1(\mathcal{L})(-c_1(\mathcal{L}^{-1})\mathcal{O}_X + \mathcal{K}) = c_1(\mathcal{L})(\mathcal{L} - 2\mathcal{O}_X + \omega)$$
$$= \mathcal{L} + \omega - 3\mathcal{O}_X + 2\mathcal{L}^{-1} - \mathcal{L}^{-1} \otimes \omega \quad \text{and}$$
$$c_1(\mathcal{L})(-c_1(\mathcal{L}^{-1})\mathcal{O}_X - \mathcal{K}) = c_1(\mathcal{L})(\mathcal{L} - \omega) = \mathcal{L} - \omega - \mathcal{O}_X + \mathcal{L}^{-1} \otimes \omega.$$

But, $\mathrm{H}^i(\mathcal{L})$ is dual to $\mathrm{H}^{2-i}(\mathcal{L}^{-1} \otimes \omega)$ by duality theory; see [**Ha83**, Cor. 7.7, p. 244], where k needn't be taken algebraically closed. So $\chi(\mathcal{L}) = \chi(\mathcal{L}^{-1} \otimes \omega)$. Similarly, $\chi(\mathcal{O}_X) = \chi(\omega)$. Therefore,

$$(D^2) + (D \cdot \mathcal{K}) = 2(\chi(\mathcal{L}^{-1}) - \chi(\mathcal{O}_X)) \quad \text{and} \quad (D^2) - (D \cdot \mathcal{K}) = 2(\chi(\mathcal{L}) - \chi(\mathcal{O}_X)).$$

Now, $-c_1(\mathcal{O}_X(D))\,\mathcal{O}_X = \mathcal{L}^{-1} - \mathcal{O}_X$. The first assertion follows.

Suppose ω is invertible. Then $-c_1(\omega^{-1})\mathcal{O}_X = \mathcal{K}$ owing to the definitions. And $-\int c_1(\mathcal{L})c_1(\omega^{-1}) = \int c_1(\mathcal{L})c_1(\omega)$ by Part (2) of Theorem B.9. Therefore, $(D \cdot \mathcal{K}) = (D \cdot K)$. Hence Part (2) of Theorem B.9 yields the second assertion. □

Theorem B.27 (Hodge Index). *Assume S is the spectrum of a field, and X is a geometrically irreducible complete surface. Assume there is an $\mathcal{H} \in \mathrm{Pic}(X)$ such that $\int c_1(\mathcal{H})^2 > 0$. Let $\mathcal{L} \in \mathrm{Pic}(X)$. Assume $\int c_1(\mathcal{L})c_1(\mathcal{H}) = 0$ and $\int c_1(\mathcal{L})^2 \geq 0$. Then $\mathcal{L}$ is numerically equivalent to $\mathcal{O}_X$.*

Proof. We may extend the ground field to its algebraic closure owing to Proposition B.17. Furthermore, we may replace X by its reduction; indeed, the hypotheses are preserved owing to Lemma B.12, and the conclusion is preserved owing to Definition B.19.

By Chow's Lemma, there is a surjective map $g\colon X' \to X$ where X' is an integral projective surface. Furthermore, we may replace X' by its normalization. Now, we may replace X by X' and $\mathcal{H}$ and $\mathcal{L}$ by $g^*\mathcal{H}$ and $g^*\mathcal{L}$. Indeed, the hypotheses are preserved owing to the Projection Formula, Proposition B.16. And the conclusion is preserved owing to Part (2) of Proposition B.21.

By way of contradiction, assume that there exists a closed integral subscheme $Y \subset X$ such that $\int c_1(\mathcal{L})\mathcal{O}_Y \neq 0$. Let $g\colon X' \to X$ be the blowing-up along Y, and $E := g^{-1}Y \subset X$ the exceptional divisor. Let $E_1, \dots, E_s$ be the irreducible components of E, and give them their induced reduced structure.

Since X is normal, it has only finitely may singular points. Off them, Y is a divisor, and g is an isomorphism. Hence one of the E_i, say E_1 maps onto Y, and the remaining E_i map onto points. Therefore, $\int c_1(g^*\mathcal{L})[E_1] = \int c_1(g\mathcal{L})[Y]$ and $\int c_1(g^*\mathcal{L})[E_i] = 0$ for $i \geq 2$ by the Projection Formula. Hence Lemma B.12 yields $\int c_1(g^*\mathcal{L})[E] = \int c_1(\mathcal{L})[Y]$. The latter is nonzero by the new assumption, and the former is equal to $\int c_1(g^*\mathcal{L})c_1(\mathcal{O}_{X'}(E))$ by Lemma B.13.

Set $\mathcal{M} := \mathcal{O}_{X'}(E)$. Then $\int c_1(g^*\mathcal{L})c_1(\mathcal{M}) \neq 0$. Moreover, by the Projection Formula, $\int c_1(g^*\mathcal{H}) > 0$ and $\int c_1(g^*\mathcal{L})c_1(g^*\mathcal{H}) = 0$ and $\int c_1(g^*\mathcal{L})^2 \geq 0$. Let's prove this situation is absurd. First, replace X by X' and $\mathcal{H}$ and $\mathcal{L}$ by $g^*\mathcal{H}$ and $g^*\mathcal{L}$.

Let $\mathcal{G}$ be an ample invertible sheaf on X. Set $\mathcal{H}_1 := \mathcal{G}^{\otimes m} \otimes \mathcal{M}$. Then

$$\int c_1(\mathcal{L})c_1(\mathcal{H}_1) = m \int c_1(\mathcal{L})c_1(\mathcal{G}) + \int c_1(\mathcal{L})c_1(\mathcal{M})$$

by additivity (see Part (2) of Theorem B.9). Now, $\int c_1(\mathcal{L})c_1(\mathcal{M}) \neq 0$. Hence there is an $m > 0$ so that $\int c_1(\mathcal{L})c_1(\mathcal{H}_1) \neq 0$ and so that $\mathcal{H}_1$ is ample.

Set $\mathcal{L}_1 := \mathcal{L}^{\otimes p} \otimes \mathcal{H}^{\otimes q}$. Since $\int c_1(\mathcal{L})c_1(\mathcal{H}) = 0$, additivity yields

$$\int c_1(\mathcal{L}_1)^2 = p^2 \int c_1(\mathcal{L})^2 + q^2 \int c_1(\mathcal{H})^2, \text{ and}$$
$$\int c_1(\mathcal{L}_1)c_1(\mathcal{H}_1) = p \int c_1(\mathcal{L})c_1(\mathcal{H}_1) + q \int c_1(\mathcal{H})c_1(\mathcal{H}_1).$$

Since $\int c_1(\mathcal{L})c_1(\mathcal{H}_1) \neq 0$, there are p, q with $q \neq 0$ so that $\int c_1(\mathcal{L}_1)c_1(\mathcal{H}_1) = 0$. Then $\int c_1(\mathcal{L}_1)^2 > 0$ since $\int c_1(\mathcal{L})^2 \geq 0$ and $\int c_1(\mathcal{H})^2 > 0$. Replace $\mathcal{L}$ by $\mathcal{L}_1$ and $\mathcal{H}$ by $\mathcal{H}_1$. Then $\mathcal{H}$ is ample, $\int c_1(\mathcal{L})c_1(\mathcal{H}) = 0$ and $\int c_1(\mathcal{L})^2 > 0$.

Set $\mathcal{N} := \mathcal{L}^{\otimes n} \otimes \mathcal{H}^{-1}$ and $\mathcal{H}_1 := \mathcal{L} \otimes \mathcal{H}^a$. Take $a > 0$ so that $\mathcal{H}_1$ is ample. By additivity,

$$\int c_1(\mathcal{N})c_1(\mathcal{H}_1) = n \int c_1(\mathcal{L})^2 - a \int c_1(\mathcal{H})^2.$$

Take $n > 0$ so that $\int c_1(\mathcal{N})c_1(\mathcal{H}_1) > 0$. Then additivity and Proposition B.14 yield

$$\int c_1(\mathcal{N})c_1(\mathcal{H}) = -\int c_1(\mathcal{H})^2 < 0,$$
$$\int c_1(\mathcal{N})^2 = n^2 \int c_1(\mathcal{L})^2 + \int c_1(\mathcal{H})^2 > 0.$$

But this situation stands in contradiction to the next lemma. □

LEMMA B.28. *Assume S is the spectrum of a field, and X is an integral surface. Let $\mathcal{N} \in \operatorname{Pic}(X)$, and assume $\int c_1(\mathcal{N})^2 > 0$. Then these conditions are equivalent:*

(i) *For every ample sheaf $\mathcal{H}$, we have $\int c_1(\mathcal{N})c_1(\mathcal{H}) > 0$.*
(i′) *For some ample sheaf $\mathcal{H}$, we have $\int c_1(\mathcal{N})c_1(\mathcal{H}) > 0$.*
(ii) *For some $n > 0$, we have $\mathrm{H}^0(\mathcal{N}^{\otimes n}) \neq 0$.*

PROOF. Suppose (ii) holds. Then there exists an effective divisor D such that $\mathcal{N}^{\otimes n} \simeq \mathcal{O}_X(D)$. And $D \neq 0$ since $\int c_1(\mathcal{N})^2 > 0$. Hence (i) results as follows:

$$\int c_1(\mathcal{N})c_1(\mathcal{H}) = \int c_1(\mathcal{H})c_1(\mathcal{N}) = \int c_1(\mathcal{H})[D] > 0$$

by symmetry, by Lemma B.13, and by Proposition B.14.

Trivially, (i) implies (i′). Finally, assume (i′), and let's prove (ii). Let ω be a dualizing sheaf for X; then ω is torsion free of rank 1, and $\mathrm{H}^2(\mathcal{L})$ is dual to $\operatorname{Hom}(\mathcal{L}, \omega)$ for any coherent sheaf $\mathcal{F}$ on X; see [**FGA**, p. 149-17], [**AK70**, (1.3), p. 5, and (2.8), p. 8], and [**Ha83**, Prp, 7.2, p. 241]. Set $\mathcal{K} := \omega - \mathcal{O}_X \in \mathbf{K}_1$.

Suppose $\mathcal{L}$ is invertible and $\mathrm{H}^2(\mathcal{L})$ is nonzero. Then there is a nonzero map $\mathcal{L} \to \omega$, and it is injective since X is integral. Let $\mathcal{F}$ be its cokernel. Then

$$\mathcal{K} = \mathcal{F} - c_1(\mathcal{L}^{-1})\mathcal{O}_X \text{ in } \mathbf{K}_1.$$

Hence Proposition B.14, symmetry, and additivity yield

$$\int c_1(\mathcal{H})\mathcal{K} \geq \int c_1(\mathcal{L})c_1(\mathcal{H}).$$

Take $\mathcal{L} := \mathcal{N}^{\otimes n}$. Then $\int c_1(\mathcal{H})c_1(\mathcal{L}) = n \int c_1(\mathcal{H})c_1(\mathcal{N})$ by additivity. But $\int c_1(\mathcal{N})c_1(\mathcal{H}) > 0$ by hypothesis. Hence $\mathrm{H}^2(\mathcal{N}^{\otimes n})$ vanishes for $n \gg 0$. Now,

$$\chi(\mathcal{N}^{\otimes n}) = \int c_1(\mathcal{N})^2 \binom{n+1}{2} + a_1 n + a_0$$

for some a_1, a_0 by Snapper's Theorem, Theorem B.7. But $\int c_1(\mathcal{N})^2 > 0$ by hypthesis. Therefore, (ii) holds. □

COROLLARY B.29. *Assume S is the spectrum of a field, and X a geometrically irreducible projective r-fold with $r \geq 2$. Let D, E be effective divisors, with E possibly trivial. Assume D is ample. Then $D + E$ is connected.*

Proof. Plainly we may assume the ground field is algebraically closed and X is reduced. Fix $n > 0$ so that $nD + E$ is ample; plainly we may replace D and E by $nD + E$ and 0. Proceeding by way of contradiction, assume D is the disjoint union of two closed subschemes D_1 and D_2. Plainly D_1 and D_2 are divisors; so $D = D_1 + D_2$.

Proceed by induction on r. Suppose $r = 2$. Then, since D_1 and D_2 are disjoint, $(D_1 \cdot D_2) = 0$ by Lemma B.13. Now, D is ample. Therefore, Proposition B.14 yields

$$(D_1^2) = (D.D_1) > 0 \text{ and } (D_2^2) = (D.D_2) > 0.$$

These conclusions contradict Theorem B.27 with $\mathcal{H} := \mathcal{O}_X(D_1)$ and $\mathcal{L} := \mathcal{O}_X(D_2)$.

Finally, suppose $r \geq 3$. Let H be a general hyperplane section of X. Then H is integral by Bertini's Theorem [**Se50**, Thm. 12, p. 374]. And H is not a component of D. Set $D' := D \cap H$ and $D'_i := D_i \cap H$. Plainly D'_1 and D'_2 are disjoint, and $D' = D'_1 + D'_2$; also, D' is ample. So induction yields the desired contradiction. □

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