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Integral Method for Estimating Soil Hydraulic Properties

Mingan Shao and Robert Horton*

ABSTRACT

Soil hydraulic properties are required to fully understand and predict soil water distribution. Soil hydraulic properties include the soil water characteristic curve and hydraulic conductivity. We used an integral method to solve the problem of water absorption into a horizontal soil column. The integral solutions to the problem were used to estimate the parameters α and *n* in the van Genuchten model for soil water characteristic curves. The two parameters, α and *n*, in the characteristic curve model were estimated by the length of the wetted zone, sorptivity, and saturated hydraulic conductivity. This new integral method uses both Richards' equation and the closed-form equations of soil hydraulic properties. Six soils ranging from sandy loam to clay loam were used to test the method. Soil water characteristic curves estimated by the infiltration method are in good agreement with measured characteristic curves. The integral method provides a transient water flow approach to estimate the soil water characteristic curve instead of the usual equilibrium method. This is a new and simple means to determine soil hydraulic properties.

INCREASING EVIDENCE shows that the quality of soil and water resources on the Earth is being adversely affected by the release of a variety of agricultural and industrial pollutants into the environment (van Genuchten, 1992). Water is the most important carrier of the pollutants into our soils. Rates of soil water movement in various soil water flow processes (e.g., infiltration, redistribution, root uptake, and drainage) are important for making practical soil management decisions to minimize potential groundwater contamination and degradation of soil quality from land-applied chemicals. Numerical solutions of the flow and transport problems in the vadose zone are the most important approaches to predict quantitatively the dynamic behavior of the system. Unsaturated flow and transport modeling usually requires accurate and complete information about the unsaturated hydraulic properties for the model to function properly. Soil hydraulic properties include a soil water characteristic curve (the relation between volumetric water content [θ] and pressure head [h]) $\theta(h)$, hydraulic conductivity (K), and water diffusivity (D). Because the three hydraulic properties are related by $K = D\,\mathrm{d}\theta/\mathrm{d}h$, only two of them are independent. Usually the hydraulic conductivity and the soil water characteristic curve are considered to be two of the most important hydraulic properties.

The methods of determining unsaturated hydraulic properties are conveniently divided into two groups, direct methods and indirect methods (Neuman, 1973; van Genuchten, 1992). For the direct group, most methods for measuring soil hydraulic properties, i.e., the soil water characteristic curve and hydraulic conductivity, both in the laboratory and in situ, have been described by Green et al. (1986) and by Klute and Dirksen (1986). Although direct methods are relatively clear in concept, they have some limitations that restrict their use in practice (van Genuchten, 1992). Time consumption and uncertainty in the estimated hydraulic parameters are the common limitations for most direct methods, especially for field methods.

Because the direct determination of hydraulic properties is relatively time consuming and expensive, various efforts have been made to relate hydraulic conductivity and the soil water characteristic curve to easily determined soil physical properties. This approach results in indirect methods. For example, soil texture data were successfully used (Bouma and van Lanen, 1987; Puckett et al., 1985; Dane and Puckett, 1992; Tyler and Wheatcraft, 1989, 1990; Wösten et al., 1995) to predict soil water characteristic curves, which could subsequently be used to estimate hydraulic conductivity based on the models of Brooks and Corey (1964), Mualem (1976), and van Genuchten (1980). Recent application of indirect methods (Kool et al., 1987; Kool and Parker, 1988; Russo et al., 1991; Sisson and van Genuchten, 1991; Arya and Dierolf, 1992; Wu and Vomocil, 1992) have shown several advantages compared with the direct techniques (van Genuchten, 1992; van Dam et al., 1992). Complete hydraulic property estimation across a wide range of soil water content and information on parameter uncertainty are major advantages of indirect methods. However, a number of problems, such as convergence and parameter uniqueness, related to indirect methods still remains to be solved (van Genuchten, 1992).

To remove some limitations from both direct methods and indirect methods, we developed an integral method for estimating soil hydraulic properties. The integral method is theoretically based on the Richard equation of water flow in soils and is practical, easy, and convenient in terms of the required measurements. The integral method gives approximate solutions to nonlinear partial differential equations (PDE). The essential idea of the integral method is to approximate the solution to the PDE with a simple function that contains adjustable parameters, and then determine the values of these parameters by requiring the solution to satisfy both the PDE and initial and boundary conditions in an integral sense. The integral method was first used to solve diffusion problems by Landahl (1953). There have been applications of this method in flow problems of porous media (Prasad and Romkens, 1982; Zimmerman and Bodvarsson, 1989; Zimmerman et al., 1990). We used the integral approach to solve the highly nonlinear hori-

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M. Shao, Natl. Lab. of Soil Erosion and Dryland Agriculture, Inst. of Soil and Water Conservation, Chinese Academy of Sciences and Ministry of Water Resources, Yangling, Shaanxi, China 712100; and R. Horton, Dep. of Agronomy, Iowa State Univ., Ames, IA 50011-1010. Journal paper no. 16953 of the Iowa Agriculture and Home Economics Exp. Stn., Ames, Projects no. 3262 and 3287. Received 15 July 1996. *Corresponding author (rhorton@iastate.edu).

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Abbreviations: ODE, ordinary differential equation; PDE, partial differential equations; RMSE, root mean square error.

zontal water flow equation in soils and to estimate the parameter values of the van Genuchten (1980) soil hydraulic property models. We also compared the predicted and measured soil water characteristic curves for six soils ranging from sandy loam to clay loam.

THEORY

The equation describing one-dimensional horizontal unsaturated flow of water in soils is

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left[K(h)\frac{\partial h}{\partial x}\right] \qquad [1]$$

where θ is the volumetric soil water content (m^3/m^3), h is the water pressure head (m), $K(h)$ is the unsaturated hydraulic conductivity (m/s), x is the horizontal distance (m), and t is the time (s).

The initial and boundary conditions are similar to those of the Bruce and Klute (1956) water absorption problem. The only difference is that the variable to describe the initial and boundary conditions in Bruce and Klute (1956) water absorption is water content and the variable in current application is pressure head. Mathematically, these conditions are described as follows:

$$h(x,0) = h_i \qquad [2]$$

$$h(0,t) = 0 \qquad [3]$$

$$h(\infty,t) = h_i \qquad [4]$$

where h_i is the initial pressure head. Without loss of generality, a zero pressure head at the inlet boundary is assumed because the solution for a non-zero pressure boundary is related in a simple way (Philip, 1957) to the solution of zero inlet boundary. The Boltzmann transformation variable, $\lambda = x/t^{0.5}$, is used to convert the PDE, Eq. [1], into an ordinary differential equation (ODE). With this transformation variable, Eq. [1] is transformed into

$$\frac{d}{d\lambda}\left[K(h)\frac{dh}{d\lambda}\right] + \frac{\lambda}{2}\frac{d\theta}{d\lambda} = 0 \qquad [5]$$

The initial and boundary conditions (Eq. [2]–[4]) are converted to

$$h(0) = 0 \qquad [6]$$

$$h(\infty) = h_i \qquad [7]$$

By performing the transformation, the mixed problem of PDE (Eq. [1]–[4]) is reduced to a two-point ODE boundary value problem, given by Eq. [5] through [7]. In Eq. [5], there are two variables, h and θ; an additional equation (the soil water characteristic curve) that relates the two variables is needed to solve the two-point ODE boundary value problem.

The most commonly used closed-form equations for characterizing the soil water characteristic curve and hydraulic conductivity in soil physics are those of van Genuchten (1980) and Mualem (1976). The equations are

$$\theta = \theta_r + (\theta_s - \theta_r)[1 + (\alpha|h|)^n]^{-m} \qquad [8]$$

$$K(h) = \frac{K_s\,[1 - (\alpha|h|)^{n-1}\,(1 + (\alpha|h|)^n)^{-m}]^2}{[1 + (\alpha|h|)^n]^{m/2}} \qquad [9]$$

where $\theta(h)$ is volumetric water content as a function of water pressure head, θ_s is saturated water content, θ_r is the residual water content, α is a scaling parameter that is inversely proportional to the mean pore diameter, $1/\alpha$ is similar to air-entry pressure in the model of Brooks and Corey (1964, 1966), n is the soil water characteristic curve index (shape parameter of the curve) or the pore-size distribution index, K_s is the saturated hydraulic conductivity, and $m = 1 - 1/n$.

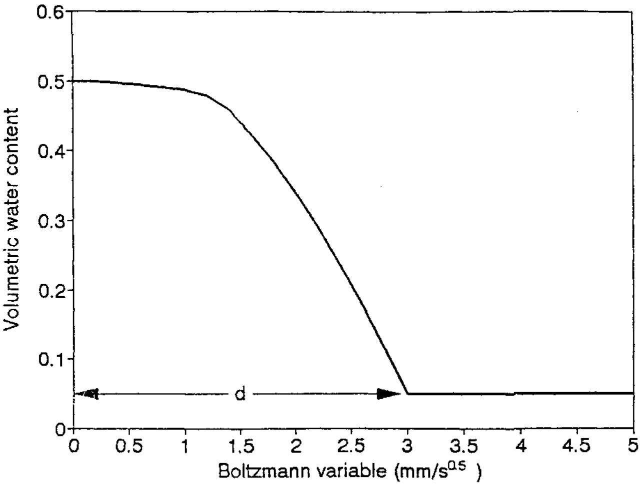

Fig. 1. An assumed water content profile of horizontal infiltration.

An appropriate water content profile that can be described by a simple form of function, $\theta(\lambda)$, may be obtained by the following reasoning. The flux of water infiltrating into the soil is finite. That means $dh/d\lambda$ must be finite at $\lambda = 0$. It is convenient to express the relationship, $h(\lambda)$, in terms of MacLaurin's series, i.e.

$$h = a_0 + a_1\lambda + a_2\lambda^2 + \ldots + \ldots \qquad [10]$$

Because $h(0) = 0$, this means $a_0 = 0$, then $h = a_1\lambda + \ldots + \ldots$, a_1 is a negative constant. Again h is usually negative. For convenience, let $b_1 = -a_1$; then b_1 is a positive constant. Substitution of this into Eq. [8] to make the first-order approximation provides a simple representative form of the water distribution as

$$\frac{\theta - \theta_r}{\theta_s - \theta_r} = 1 - m(\alpha b_1\lambda)^n \qquad [11]$$

The length of wetted zone is denoted by d. In order to find b_1, we use the condition at $\lambda = d$, $\theta(d) = \theta_i$, so the term mb_1^n is given by

$$mb_1^n = \frac{\theta_s - \theta_i}{(\theta_s - \theta_r)\,(\alpha d)^n} \qquad [12]$$

Combining Eq. [11] and [12] gives the appropriate water content profile (also see Fig. 1):

$$\theta(\lambda) = \theta_s - (\theta_s - \theta_i)(\lambda/d)^n \qquad 0 < \lambda < d \qquad [13]$$

$$\theta(\lambda) = \theta_i \qquad d \le \lambda < \infty \qquad [14]$$

Equation [13] describes an absorption profile the same as the exact solution described by Philip (Philip, 1960, Table 1, no. 2):

$$\lambda(\Theta) = \varepsilon(1 - \Theta)^p \qquad p > 0 \qquad [15]$$

where Θ is normalized volumetric water content, $(\theta - \theta_i)/(\theta_s - \theta_i)$, ε is the maximum value of λ, the same as d in Eq. [13], and p is the slope factor. If $p = 1/n$, one can verify that Eq. [13] and [15] are equivalent. This substantiates the water content profile described by Eq. [13] and [14]. Equation [13] will also be verified by experimental evidence.

The characteristic wetting length, d (referred to here simply as *wetting length*), can be related to the parameters of van

Table 1. Some physical properties of the five soils.

Soil	Specific surface	Particle density	Saturated water content	Residual water content
	10^{-3} m^2/kg	Mg/m^3	m^3/m^3	
Sandy loam	–	–	0.440	0.000
Silt loam	41	2.67	0.502	0.118
Loam	40	2.69	0.503	0.130
Sandy clay loam	58	2.64	0.542	0.115
Silty clay loam	79	2.67	0.562	0.163
Clay loam	141	2.57	0.569	0.182

Genuchten's (1980) model by integrating Eq. [5] from $\lambda = 0$ to $\lambda = \infty$ with Eq. [13] and [14] substituted for $\theta(\lambda)$. The first term in Eq. [5] is

$$\int_0^\infty \frac{d}{d\lambda}\left[K(h)\frac{dh}{d\lambda}\right] d\lambda = \left[K(h)\frac{dh}{d\lambda}\right]\Big|_0^\infty = b_1 K_s \qquad [16]$$

which is derived based on the fact that at $\lambda = \infty$, $dh/d\lambda = 0$, and at $\lambda = 0$, $dh/d\lambda = -b_1$. From Eq. [12], b_1 is expressed as

$$b_1 = \frac{1}{\alpha d}\left[\frac{(\theta_s - \theta_i)}{m(\theta_s - \theta_r)}\right]^{1/n} \qquad [17]$$

The second term in Eq. [5] is

$$\int_0^\infty \frac{\lambda}{2}\frac{d\theta}{d\lambda} d\lambda = \int_0^d \frac{\lambda}{2}\frac{d\theta}{d\lambda} d\lambda = \frac{-n(\theta_s - \theta_i)d}{2(n + 1)} \qquad [18]$$

Equation [18] is obtained by the fact that the integral is zero in the interval (d,∞) since $d\theta/d\lambda$ is zero in this interval.

Combining Eq. [16], [17], and [18] with the integral of Eq. [5] gives

$$\alpha = \frac{2(n + 1)\,K_s}{n(\theta_s - \theta_i)\,d^2}\left[\frac{1}{m}\left(\frac{\theta_s - \theta_i}{\theta_s - \theta_r}\right)\right]^{1/n} \qquad [19]$$

From Eq. [19], the scaling parameter, α, is related not only to d but also to K_s and n. In order to estimate both α and n, one more relation is needed if K_s is available (usually K_s is measured). This is obtained by applying Darcy's flux equation to the horizontal absorption. At $x = 0$ (the inlet boundary), the water flux is expressed as

$$q = -\left[K(h)\frac{\partial h}{\partial x}\right]\Big|_{x=0} \qquad [20]$$

and

$$\frac{\partial h}{\partial x}\Big|_{x=0} = \left(\frac{dh}{d\lambda}\right)\Big|_{\lambda=0}\left(\frac{\partial \lambda}{\partial x}\right)\Big|_{x=0} \qquad [21]$$

From Eq. [10], one can get

$$\frac{\partial h}{\partial \lambda}\Big|_{\lambda=0} = a_1 = -b_1 \qquad [22]$$

Considering $K(h) = K_s$ at $x = 0$ and the definition of the Boltzmann variable, and combining Eq. [20], [21], and [22] gives

$$q = K_s b_1 t^{-0.5} \qquad [23]$$

For horizontal infiltration, if one uses Philip's two-term equation, the infiltration rate or flux density is given by

$$q = \frac{S}{2t^{0.5}} \qquad [24]$$

where S is sorptivity that can be relatively easily obtained by analyzing the infiltration rate with time (a simple regression will find S by using Eq. [24]). Combining Eq. [17], [23], and [24] gives

$$\alpha = \frac{2K_s}{Sd}\left[\frac{1}{m}\left(\frac{\theta_s - \theta_i}{\theta_s - \theta_r}\right)\right]^{1/n} \qquad [25]$$

Comparing Eq. [19] with [25] gives an estimation of n as

$$n = \frac{S}{d(\theta_s - \theta_i) - S} \qquad [26]$$

Equations [19] and [26] complete the parameter estimation for the van Genuchten (1980) model of hydraulic properties of a soil. First, n is obtained by measuring the characteristic wetting length and sorptivity. Then, with a K_s measurement, the scaling parameter, α, is found by using Eq. [19]. Experimentally, if one records both infiltration and wetting front with time in a horizontal absorption experiment and measures the saturated hydraulic conductivity of the column after absorption, the parameter estimation for the van Genuchten (1980) model can be completed because saturated and residual water content are easy to measure or estimate.

MATERIALS AND METHODS

Six soils were used to test the integral approach in this study. The first five soils are a silt loam obtained from land mapped as Flagler series (a coarse-loamy, mixed, mesic Typic Hapludoll, 11.4% sand, 70.0% silt, and 18.6% clay), Nicollet loam (a fine-loamy, mixed, superactive, mesic Aquic Hapludoll, 50.9% sand, 32.6% silt, and 16.5% clay), Keswick sandy clay loam (a fine, smectitic, mesic Aquertic Chromic Hapludalf, 67.7% sand, 11.3% silt, and 21.0% clay), Monona silty clay loam (a fine-silty, mixed, superactive, mesic Typic Hapludoll, 2.4% sand, 69.5% silt, and 28.1% clay), Webster clay loam (a fine-loamy, mixed, superactive, mesic, Typic Endoaquoll, 32.1% sand, 39.2% silt, and 28.7% clay). The sixth soil is a Manawatu fine sandy loam (a Dystric Fluventic Eutrochrept). Information on the soil water characteristic curve of the sixth soil was obtained from Clothier and Scotter (1982).

Some basic physical properties of the first five soils were measured. The specific surface areas were measured by using the ethylene glycol monoethyl ether (EGME) technique (Cihacek and Bremner, 1979; Carter et al., 1986). Particle densities were determined by using the pycnometer method (Blake and Hartge, 1986b). Bulk densities were determined by the clod method (Blake and Hartge, 1986a). The saturated water contents of the first five soils were obtained by measuring both mass water contents and their bulk densities at saturation. The residual water contents were estimated as the water contents at −1.5 MPa matric pressure (van Genuchten, 1980).

The soil water characteristic curves for the first five soils were measured by pressure plate technique. Additionally, conventional horizontal-infiltration experiments of the Bruce and Klute (1956) type were performed for the determination of the characteristic depth (d) and sorptivity (S). Air-dried soil was packed into sectioned Plexiglas tubes 0.15 m long (15 sections) and 0.038 m in diameter with a controlled bulk density of 1.30 Mg/m^3. During infiltration, water was supplied to one end of the soil column through a ceramic plate. During the horizontal infiltration (absorption), the advance of the wetting front with time and the amount of water infiltrated into the soil column were recorded. The horizontal absorption experiment was ended when the wetting front reached about half length of the column. The saturated hydraulic conductivities of the first five soils were measured by a constant-head technique (Klute and Dirksen, 1986).

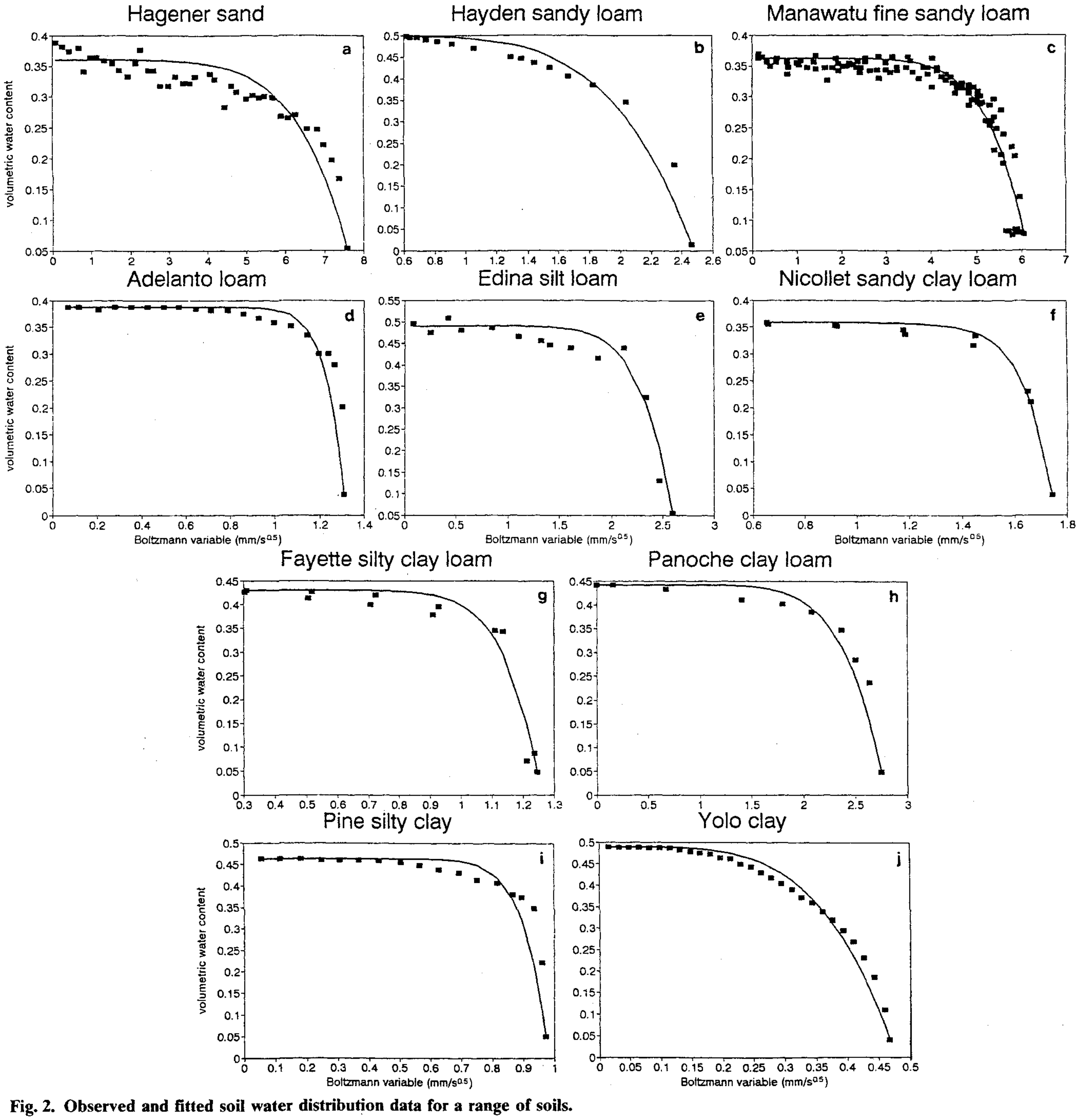

Fig. 2. Observed and fitted soil water distribution data for a range of soils.

DISCUSSION

Equations [19] and [26], representing the parameter estimation of the van Genuchten (1980) model of soil hydraulic properties, depend on six parameters, K_s, S, d, θ_s, θ_r, and θ_i. The two water contents, θ_s and θ_i, are easy to measure as the infiltration boundary and initial condition. The value of θ_r needs to be estimated (for example taking the water content at -1.5 MPa matric pressure as θ_r). The characteristic length (d) of the wetted zone is easy to observe visually during infiltration. The value of S is also relatively easy to determine from infiltration data. The only parameter left to determine is K_s, which can be conveniently measured by using the same soil column after the absorption experiment.

Equation [13] is used to derive Eq. [19] and [26]. As mentioned above, Eq. [13] is approximate. Mathematically, Eq. [13] and [15] are the same. Equation [15] should cover most absorption profiles of soil water content (Clothier et al., 1983). However, testing of Eq. [13] by measured data of soil water content distribution profiles should be performed. To test Eq. [13], measured data for 10 soils ranging from sand to clay were taken from the literature. The 10 soils are: Hagener sand (Selim et al., 1970), Hayden sandy loam (Whisler et al., 1968), Manawatu fine sandy loam (Clothier et al., 1983),

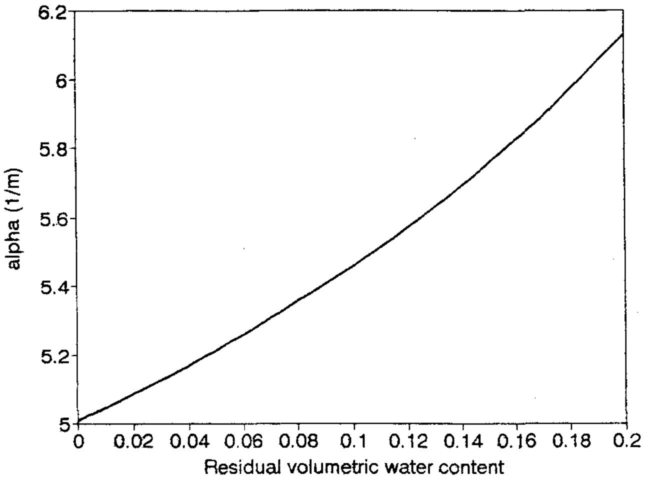

Fig. 3. The sensitivity of the scaling factor (α) to residual water content (θ_r).

Adelanto loam (Jackson, 1963), Edina silt loam (Selim et al., 1970), Nicollet sandy clay loam (McBride and Horton, 1985), Fayette silty clay loam (McBride and Horton, 1985), Panoche clay loam (Reichardt et al., 1972), Pine silty clay (Jackson, 1963), and Yolo clay (Nofziger, 1978). Figure 2a to 2j present the fit of Eq. [13] to soil water distribution data for these 10 soils. From Fig. 2, one can see that Eq. [13] is appropriate for describing the soil water distribution of a horizontal absorption experiment. In the derivation of Eq. [19] and [26], Eq. [13] is the only approximate expression. Both theoretical and experimental verifications of Eq. [13] give confidence that Eq. [19] and [26] should be appropriate for parameter estimations of the van Genuchten (1980) model of a soil water characteristic curve.

Equation [19] should be examined carefully. The scaling parameter, α, is directly proportional to K_s for a given n. Larger values of K_s generally correspond to coarser textured soils, therefore to larger α values. The air-entry pressure is inversely proportional to α. This implies that the coarser textured soils have smaller air-entry values than do finer textured soils. This conclusion is expected.

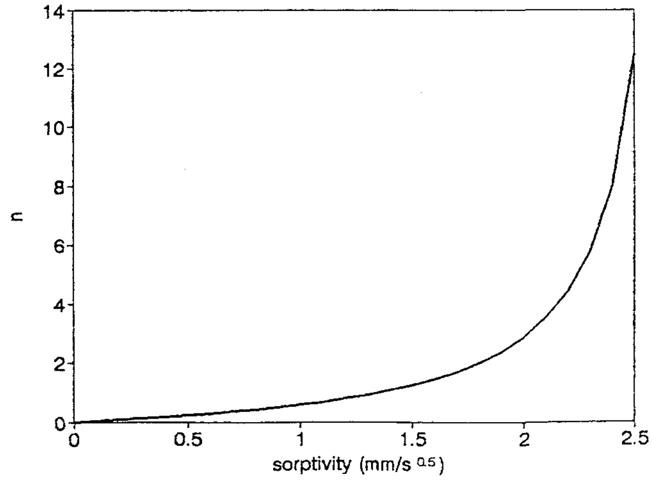

Fig. 4. The relationship between the shape parameter (n) and sorptivity (S).

Table 2. Hydraulic parameter values for the six soils.

Soil	Sorptivity	Saturated hydraulic conductivity	Length of wetted zone
	10^{-4} m/s$^{0.5}$	10^{-7} m/s	10^{-3} m/s$^{0.5}$
Sandy loam	14.70	112.00	6.05
Silt loam	9.87	32.80	3.50
Loam	9.84	30.90	3.44
Sandy clay loam	7.56	5.30	2.00
Silty clay loam	5.30	1.11	1.53
Clay loam	5.16	2.21	1.55

It is necessary to make a sensitivity analysis of α to θ_r because θ_r is approximated by the volumetric water content of -1.5 MPa water pressure of the soil. Assume that $n = 3$, $S = 1.5$ mm/s$^{0.5}$, $K_s = 0.01$ mm/s, $d = 6$ mm/s$^{0.5}$, $\theta_s = 0.44$, and $\theta_i = 0.02$ (the soil may correspond to sandy loam). The sensitivity of α to θ_r is shown in Fig. 3. From Fig. 3 it can be concluded that the effect of θ_r on α is small. For example, an increase in θ_r from 0.01 to 0.2 corresponds with an increase in α from 2.52 to 3.07 m^{-1}. That implies that a change in θ_r of a factor of 20 only makes a 20% change in α. Therefore an approximation of θ_r should not produce a large error in the scaling parameter.

The shape parameter, n, in the van Genuchten (1980) model is related both to sorptivity (S) and the characteristic length (d) of the wetted zone. Clothier and Scotter (1982) provided experimental evidence that, for a given soil, the relationship $\theta(\lambda)$ is unique regardless of the duration of the sorption experiment. This means that d is a constant for a given soil. Therefore d is an indicator of hydraulic properties of a soil, and different soils have different values of d (also see McBride and Horton, 1985). For a given d (here, $d = 6$ mm/s$^{0.5}$, $\theta_s = 0.5$, and $\theta_i = 0.05$), n is proportional to S (also see Fig. 4). An overestimation of S will lead to an overestimation of n. An error in estimating n will also result in an error in estimating α. The accuracy of estimating both α and n depends mainly on the accuracy of sorptivity estimation for a given soil because the determination of K_s, θ_s, θ_r, and θ_i should not produce large errors. Sorptivity estimation by fitting Eq. [24] to observed infiltration values is straightforward and hopefully does not allow a large error either. Therefore this integral method for estimating soil hydraulic properties should be accurate. Below, comparisons between soil water characteristic curves estimated by integral method and measured soil water characteristic curves are made to provide experimental confidence for the method.

To give more information about the soils, the particle

Table 3. Values of the scaling factor (α), shape parameter (n), and the saturated (θ_s) and residual (θ_r) water contents for the six soils.

	Integral method				Curve fitting			
Soil	α	n	θ_s	θ_r	α	n	θ_s	θ_r
Sandy loam	2.65	3.15	0.44	0.00	2.77	2.89	0.44	0.00
Silt loam	4.97	1.45	0.50	0.12	10.92	1.18	0.50	0.00
Loam	5.13	1.43	0.50	0.13	17.81	1.16	0.50	0.00
Sandy clay loam	1.51	1.59	0.54	0.12	0.58	1.59	0.54	0.09
Silty clay loam	0.51	1.79	0.56	0.16	1.36	1.24	0.56	0.00
Clay loam	1.12	1.71	0.56	0.18	1.25	2.38	0.56	0.07

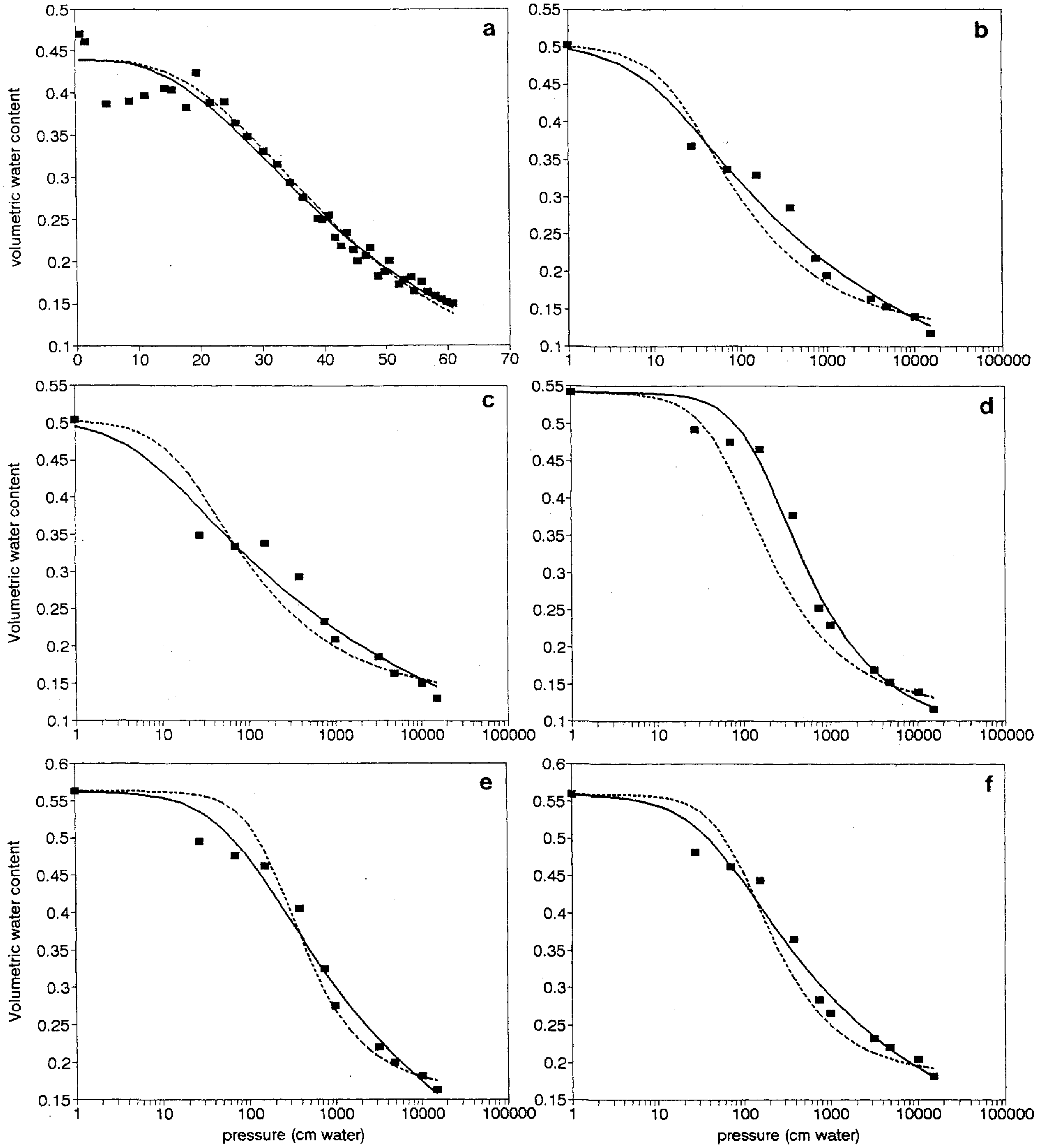

Fig. 5. Comparison of water characteristic curves obtained by the integral method (dashed curve) and by curve-fitting (solid curve) with observed data (filled square) for (a) fine sandy loam, (b) silt loam, (c) loam, (d) sandy clay loam, (e) silty clay loam, and (f) clay loam.

densities (ρ_s) and specific surfaces (SS) of the first five soils, together with θ_s and θ_r of all six soils are listed in Table 1.

The θ_s for fine sandy loam was obtained by averaging the first three measured water contents near saturation from the data of Clothier and Scotter (1982, their Fig. 1) because the water content near saturation seemed to be irregular. The θ_r was assumed to be zero for the fine sandy loam. This assumption may be safe for such a coarse-textured soil, and by using the van Genuchten (1980) model, the regression results of the water characteristic curve also gave a zero θ_r.

The three important parameters for estimating α and n are S, K_s, and d (values of the Boltzmann variable at wetting fronts). Here they are referred to as *hydraulic parameters*. The values of the hydraulic parameters for the six soils are shown in Table 2. The parameters of the sandy loam were taken from the literature (Clothier and Scotter, 1982; Clothier and Wooding, 1983).

An example of calculating n and α may be helpful. For sandy loam (also see Table 2), the value of S is found to be 14.7×10^{-4} m/s$^{0.5}$. The S values are obtained by curve-fitting Eq. [24] to the observed infiltration data; the value of d is 6.05×10^{-3} m/s$^{0.5}$ (the value of the Boltzmann variable obtained from the wetting front at the end of the infiltration experiment); the value of K_s is 112×10^{-7} m/s (the K_s values for the other soils are measured by the constant-head technique [Klute and Dirksen, 1986]); the values of θ_s, θ_r, and θ_i are 0.44, 0.00, and 0.08, respectively. Using Eq. [26] and the above values of d, S, θ_s, and θ_i, n is found to be 3.15; using Eq. [19] and the above values of K_s, d, θ_s, θ_r, and θ_i, α is found to be 2.65 m^{-1}.

Table 2 shows that both S and d tend to decrease when soils become finer in texture. With measured values of the parameters K_s, S, d, θ_s, θ_i, and estimated θ_r, parameters α and n can be determined from Eq. [19] and [26].

The calculated values of α and n from the integral method for all six soils, together with those determined by curve-fitting the actual water characteristic curve data with the van Genuchten (1980) model, are listed in Table 3. The measured θ_s and estimated θ_r along with those obtained by curve-fitting are also included in Table 3. The α and n values from both the integral method and from curve-fitting (van Genuchten, 1980) show similar trends of decreasing from sandy loam (coarser texture) to clay loam (finer texture). In general, curve-fit values of θ_r are consistently lower than the estimated values of θ_r for each soil. The fitted residual water contents do not have a clear relationship to soil textures.

The soil water characteristic curves of the first five soils estimated from the integral method are compared with those measured by pressure plate technique (Fig. 5b–5f). Comparison data (Fig. 5a) for the sixth soil (sandy loam) are taken from the literature (Clothier and Scotter, 1982). The fitted characteristic curves for all six soils, obtained by fitting the closed-form equation of van Genuchten (1980) to the observed data, are also shown in Fig. 5a to 5f. Generally, the soil characteristic curves estimated by the integral method are in good agreement with the observed data for all six soils. The estimated characteristic curves for the first five soils tend to overestimate water contents in the range of 0 to −1 m in pressure head and underestimate water contents for the range of −1 to −10 m in pressure head. When pressure head is less than −10 m, the estimated characteristic curves compare well with the measured ones. The estimated curves cross with the fitted curves somewhere between −10 and −1000 m pressure head.

A statistical comparison of the results was carried out to find out the accuracy of the integral method. An objective and quantitative measure, root mean square error (RMSE) (Willmott et al., 1985), was used to estimate the accuracy. The RMSE values of the integral method along with those of curve fitting by using the van Genuchten (1980) model are listed in Table 4. The results show that the mean value of RMSE for the integral method is 0.034 while that for the curve fitting is 0.019. This result is expected because the curve-fitting method uses the measured data directly.

Table 4. Root mean square errors (RMSE) of the integral method and curve-fitting technique.

Soil	Integral method	Curve fitting
Sandy loam	**0.018**	**0.017**
Silt loam	**0.031**	**0.018**
Loam	**0.033**	**0.021**
Sandy clay loam	**0.052**	**0.020**
Silty clay loam	**0.030**	**0.020**
Clay loam	**0.029**	**0.019**
Mean	**0.034**	**0.019**

CONCLUSION

The integral method has been used to give closed-form approximate solutions to the problem of horizontal absorption. Solutions are used to estimate the parameters of hydraulic property models of Mualem (1976) and van Genuchten (1980). With a horizontal absorption experiment, K_s is the only parameter needed to give complete information on hydraulic properties of a soil. The curve index, n, is estimated by d and S, which can be measured in the horizontal absorption procedure. The scaling parameter, α, is estimated by K_s, d, and S. The approximate solutions presented here show in theory how to estimate soil water characteristic curves and unsaturated hydraulic conductivity from simple horizontal infiltration experiments.

The experimental evidence provided in this study also shows that the integral method can be used to estimate soil hydraulic properties. The soil water characteristic curves estimated by the integral method are in good agreement with those observed for all six soils. Several weeks are needed to measure the water characteristic curves of six soils by using pressure plate equipment, whereas estimates can be accomplished with the integral method in several days by using very simple equipment (a horizontal infiltration device). The infiltration method can simultaneously estimate both the soil characteristic curve and unsaturated hydraulic conductivity from a horizontal infiltration experiment. Therefore, the integral method does not need specialized and expensive equipment and does not require substantial special operation skills either. The new method provides an attractive approach for estimating soil hydraulic properties.

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TECHNICAL ARTICLES

AN APPROXIMATE SOLUTION TO THE CONVECTION-DISPERSION EQUATION OF SOLUTE TRANSPORT IN SOIL[1]

M. Shao[2], R. Horton[3], and R. K. Miller[4]

This paper uses a boundary-layer method to solve the convection-dispersion equation (CDE) to predict solute transport in soil. The boundary-layer solution describing chemical transport for a semi-infinite soil column or field soil profile has the advantages over the corresponding exact solution of simplicity in expression and flexibility in manipulation. Comparisons of the boundary-layer solution to the exact solution are conducted for a range of parameter values. Results show that the boundary layer solution is in good agreement with the exact solution. An important application of the boundary-layer solution is the estimation of transport parameters of solute movement through soils under both laboratory and field conditions. This leads to a new method for estimating parameters of solute transport in soils. The new method requires observation of the advance of the depth of boundary layer (solute front) with time. This can be done visually by using a tracer solution with dye in it. The new method is simple, saves time, and is applicable to both laboratory soil columns and field soils. (Soil Science 1998;163:339–345)

Key words: Pollution, chemical movement, solute transport, solute parameter estimation.

TRANSPORT phenomena of agrichemicals through soils are significant processes in both crop production and groundwater quality control. Concern about the transport behavior of various chemicals in soils has resulted in the development of a number of theoretical models describing the basic mechanisms of chemical transport in soils (Parker and van Genuchten 1984). With the development of increasingly sophisticated models, effort has been focused for several decades on estimating various model parameters (e.g., Buchter et al. 1995; Elprince and Day 1977; Kool et al. 1987; Rifai et al. 1956). Two of the most common transport parameters are dispersion coefficient and retardation factor because most models of solute transport contain these two parameters.

The methods of estimating transport parameters are divided into statistical methods and deterministic methods. Statistical parameter estimation techniques such as least-squares methods, maximum-likelihood procedures, and method of moments have proved to be useful (e.g., Bresler and Naor 1987; Elprince and Day 1977; Jury and Sposito 1985; Parker and van Genuchten 1984). Some problems, however, such as parameter uncertainty and uniqueness are still unsolved in statistical methods. Moreover, in practice there is a time assignment bias in estimating transport parameters by curve-fitting (least-squares method)

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[2] Institute of Soil and Water Conservation, Chinese Academy of Sciences, Yangling, Shaanxi, China, 712100.

[3] Dept. of Agronomy, Iowa State Univ., Ames, IA 50011. Dr. Horton is corresponding author. E-mail: rhorton@iastate.edu

[4] Dept. of Mathematics, Iowa State Univ., Ames, IA 50011.

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transport models to breakthrough curve data (Schnabel and Richie 1987). On the other hand, deterministic methods have the advantages of clear concept and uniqueness of parameter estimation. The current deterministic methods, however, can be used only for some limited instances of solute transport. For example, the method presented by Rifai et al. (1956) can be used only to estimate dispersion coefficient of convection-dispersion equation (CDE) for a first-type (concentration-type) inlet boundary condition. However, the most appropriate inlet boundary for most or all solute displacement experiments is a third-type (flux-type) condition (van Genuchten and Parker 1984, 1994) instead of a first-type inlet boundary condition. The Rifai et al. (1956) method is exact but difficult to apply in practice because of limited use of breakthrough data (Yamaguchi et al. 1989). Although the Rifai et al. (1956) method is modified by Yamaguchi et al. (1989) using additional solute breakthrough data, it is still restricted to the first-type inlet boundary condition for estimating transport parameters of CDE.

In this paper, a new method is proposed to estimate both dispersion coefficient and retardation factor simultaneously by using boundary-layer theory. Boundary-layer theory, an integral method, has been used previously to solve heat and mass transfer problems (e.g., Kumar and Narang 1967; Gupta 1974). In this application of boundary-layer theory to solute transfer problems for semi-infinite columns, it is assumed that a chemical boundary layer analogous to the thermal boundary layer in heat transfer and to the velocity boundary layer in mass transfer exists whose thickness increases with time. The thickness of the boundary layer is specified by the distance from the surface down to the interface where conditions of zero solute flux and equality of resident concentration to its initial value are satisfied (depth of solute front). The solute front as a function of time can be used to determine dispersion coefficient and retardation factor.

THEORY

One-dimensional transient solute transport through a homogeneous medium during steady-water flow is traditionally described by the following partial differential equation (CDE)

$$R\frac{\partial C_r}{\partial t} = D\frac{\partial^2 C_r}{\partial x^2} - v\frac{\partial C_r}{\partial x} \quad (1)$$

where C_r is the volume-average (resident) solution concentration (g/cm^3 of soil solution), D is the dispersion coefficient (cm^2/min), v is the average pore-water velocity (cm/min), R is the retardation factor, x is position (cm), and t is time (min). The initial condition for a displacement experiment semi-infinite space is

$$C_r(x, 0) = 0 \quad (2)$$

Eq. (2) represents a soil column that is initially free of any solute; however, our analysis is easily extended to the instance of a uniform initial concentration (C_i) by a simple variable substitution, i.e., $C_r' = C_r - C_i$. The most appropriate boundary conditions for solute displacement experiments (van Genuchten and Parker 1984, 1994) are

$$\left(-D\frac{\partial C_r}{\partial x} + vC_r\right)\Big|_{x=0^+} = vC_0 \quad (3)$$

$$\left(\frac{\partial C_r}{\partial x}\right)\Big|_{x\to\infty} = 0 \quad (4)$$

Eq. (3) is valid for a system in which the entrance reservoir is not connected physically to the column and for systems in which the column is connected directly to the entrance reservoir as long as diffusion across the inlet boundary is small relative to convective transport by water flow (van Genuchten and Parker 1984, 1994).

We assume that there is a boundary layer (see Fig. 1), d(t), where d(t) is the depth of the solute front as a function of time. Then

$$C_r(d(t), t) = \frac{\partial C_r(d(t), t)}{\partial x} = \frac{\partial^2 C_r(d(t), t)}{\partial x^2} = 0 \quad (5)$$

If $I_s(t)$ denotes the cumulative solute entering the soil column across the inlet boundary, then

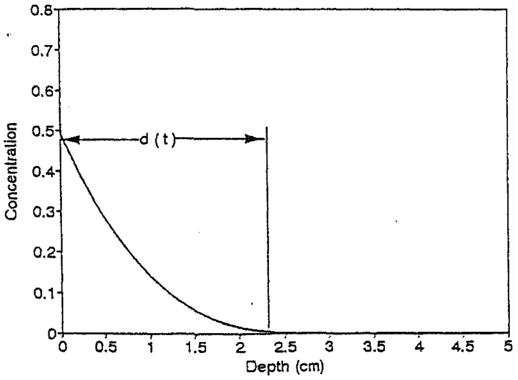

Fig. 1. A schematic diagram of solute boundary layer.

$$I_s(t) = \int_0^{d(t)} C_r(x, t)\, dx \qquad (6)$$

Now integrating Eq. (1) from 0 to d(t), the left hand side (LHS) is

$$R\int_0^{d(t)} \frac{\partial C_r}{\partial t} dx = R\frac{dI_s(t)}{dt} \qquad (7)$$

Eq. (7) is obtained by using the boundary layer condition, Eq. (5).

In a similar way, the right hand side (RHS) of Eq. (1) becomes

$$\int_0^{d(t)} \left(D\frac{\partial^2 C_r}{\partial x^2} - v\frac{\partial C_r}{\partial x}\right) dx = vC_0 \qquad (8)$$

Hence Eq. (1), initial condition, and boundary conditions (inlet boundary and boundary layer) imply that

$$R\frac{dI_s(t)}{dt} = vC_0 \qquad (9)$$

or

$$I_s(t) = \frac{vC_0}{R}\, t \qquad (10)$$

One can assume a parabolic or a cubic polynomial approximate concentration profile for a boundary-layer solution to the problem. For the cubic polynomial concentration profile, resident concentration is written as

$$C_r(x, t) = a_0(t) + a_1(t)\, x + a_2(t)\, x^2 + a_3(t)\, x^3 \qquad (11)$$

By using boundary-layer conditions, Eq. (5), the four time coefficients in Eq. (11) are reduced to a single coefficient, i.e.:

$$C_r(x, t) = a_0(t)\left(1 - \frac{x}{d(t)}\right)^3 \qquad (12)$$

Eq. (12) is valid for $0 < x < d(t)$. When $x > d(t)$, $C_r(x, t) = 0$. Now $a_0(t)$ can be found by using the inlet boundary condition, Eq. (3), then

$$a_0(t) = \frac{vd(t)\, C_0}{vd(t) + 3D} \qquad (13)$$

Therefore a boundary-layer solution to the problem is

$$C_r(x, t) = \frac{vd(t)\, C_0}{vd(t) + 3D}\left(1 - \frac{x}{d(t)}\right)^3 \qquad (14)$$

Combining Eq. (14) with Eq. (6) and integrating yields

$$I_s(t) = \frac{vd(t)^2 C_0}{4\,(vd(t) + 3D)} \qquad (15)$$

Combining Eq. (15) with Eq. (10), we obtain

$$d(t) = \frac{2vt}{R} + \sqrt{\left(\frac{2vt}{R}\right)^2 + \frac{12Dt}{R}} \qquad (16)$$

Eq. (16) is obtained by finding the positive root of a parabolic polynomial equation with unknown d(t). Physically d(t) cannot be negative.

Eq. (14) and Eq. (16) complete the boundary-layer solution to the problem for the instance of cubic polynomial concentration profile. The boundary-layer solution is obtained similarly for a parabolic polynomial concentration profile. The corresponding C_r and d(t) are

$$C_r(x, t) = \frac{vd(t)\, C_0}{vd(t) + 2D}\left(1 - \frac{x}{d(t)}\right)^2 \qquad (17)$$

$$d(t) = \frac{3vt}{2R} + \sqrt{\left(\frac{3vt}{2R}\right)^2 + \frac{6Dt}{R}} \qquad (18)$$

Both Eq. (16) and Eq. (18) contain three parameters, v, R, and D. Usually v can be determined accurately from a solute displacement experiment. Curve-fitting Eq. (18) to observations of boundary layer thickness as a function of time by least squares regression can provide values of R and D. R and D can be estimated if the change of the boundary layer with time is observed. The boundary layer in this instance is physically the depth from the soil surface to the solute front. It is experimentally possible to observe the solute front if a dye tracer solution is used. Brilliant blue has proven to be a safe and useful dye for making such a solution (Flury and Fluhler 1994 a and b). The boundary-layer solution to the CDE can be compared with the corresponding exact solution.

The exact solution to the problem (Lindstrom et al. 1967) is

$$\frac{C_r(x, t)}{C_0} = \frac{1}{2}\, erfc\left[\frac{Rx - vt}{2\,(DRt)^{0.5}}\right] + \left(\frac{v^2 t}{\pi DR}\right)^{\frac{1}{2}} \exp\left[\frac{-\,(Rx - vt)^2}{4DRt}\right] - f \qquad (19)$$

$$f(x, t) = \frac{1}{2}\left(1 + \frac{vx}{D} + \frac{v^2 t}{DR}\right) \exp\left(\frac{vx}{D}\right) erfc\left[\frac{Rx + vt}{2\,(DRt)^{0.5}}\right] \qquad (20)$$

Comparing the exact solution with the boundary-layer solution, it is obvious that the boundary-layer solution to the problem is mathematically much simpler than the exact solution. The boundary-layer solution is an approximate solution because it is based on an integral method. In the following section of this paper, the boundary-layer solution is compared with the exact solution for a range of values of solute transport parameters, D, R, and v.

DISCUSSION

The Change of Boundary Layer with Time

Because Eq. (16) and Eq. (18) have a similar relationship between boundary-layer depth and time, our discussion is focused on Eq. (16). The conclusions of our discussion on Eq. (16) apply as well to Eq. (18). First a simple instance of R = 1 is considered. This implies that the solutes are nonreactive. Then Eq. (16) is reduced to

$$d(t) = 2vt + \sqrt{(2vt)^2 + 12Dt} \qquad (21)$$

From Eq. (21), we can see that the depth of boundary layer for nonreactive solute transport is described by dispersion (D) and convection (v). The sensitivities of d(t) to D and v are shown in Fig. 2 (at a given v) and Fig. 3 (at a given D). The ratio of D/v is dispersivity. Typical values of dispersivity are 0.5 to 2 cm in packed laboratory columns and 5 to 20 cm in the field, and they can be considerably larger in regional groundwater transport (Fried 1975; Jury et al. 1991). In Fig. 2, v = 0.003 cm/min, and dispersivity ranges from 0.5 to 40 cm. At a given average pore-water velocity, the depth of boundary layer (penetration depth of solute) increases with increase of dispersivity. This is expected because an increase of dispersion in this instance will enhance the advance of the solute front. Thus, the increase of the depth from 10 to 40 cm of dispersivity is greater than that from 0.5 to 10 cm of dispersivity most of the time. In Fig. 3, D = 0.03 cm²/min, and the range of dispersivity is the same as in Fig. 2. When dispersivity increases from 0.5 to 10 cm there is little increase of the depth; however, when dispersivity changes from 10 to 40 cm, the increase of the depth is much greater than that in the range of 0.5 to 10 cm. This implies that convection has an important effect on solute transport after v reaches greater values. Combining Fig. 2 and Fig. 3 shows that dispersion has more uniform effects on solute transport than convection.

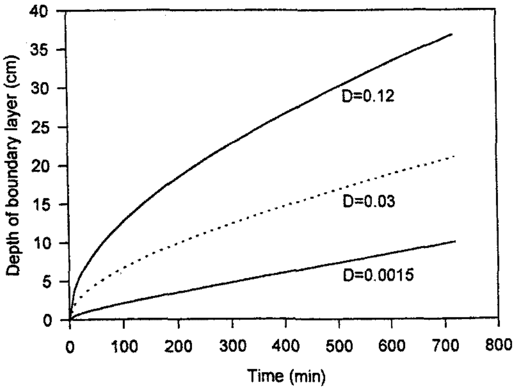

Fig. 2. Effect of dispersion coefficient on the relation of boundary layer with time (v = 0.003 cm/min).

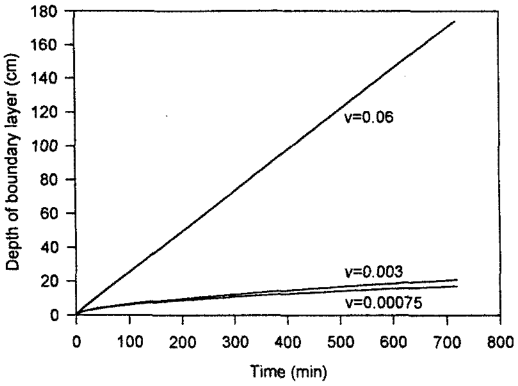

Fig. 3. Effect of pore water velocity on the relation of boundary layer with time (D = 0.03 cm²/min).

Comparisons of Boundary Layer Solution to Exact Solution

In this section, C_0 is assumed to be one. The first instance again is the nonreactive solute transport (R = 1). At a given pore-water velocity (v = 0.001 cm/min), comparisons of the boundary-layer solutions to the corresponding exact solutions for dispersivity ranging from 1 to 40 cm are shown in Figs 3–6. The global error in solute concentration is described as the absolute value of difference between exact solution and boundary-layer solution. For dispersivity equal to 1 cm, both boundary-layer solutions (cubic-polynomial and parabolic-polynomial concentration profiles) are in close agreement with exact solution (Fig. 4). The maximum error of cubic polynomial solution (ME3) is 0.025, and that of parabolic polynomial solution (ME2) is 0.012. Figure 5 shows the comparison for dispersivity of 10 cm. The cubic polynomial solution almost overlaps the exact solution with an ME3 of 0.0010; however, ME2 is still small at 0.0024. As shown in Fig. 6, for a dispersivity of 40 cm, the cubic polynomial solution again almost overlaps the exact solution with an ME3 of 0.0015 and an ME2 of 0.0070. We conclude from the comparisons that the boundary-layer solutions (both cubic and polynomial) are in close agreement with exact solutions and that cubic-

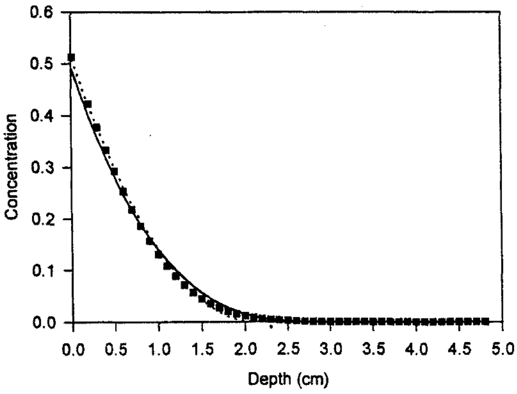

Fig. 4. The comparison of concentration profiles for nonreactive solute transport with dispersivity of 1 cm: filled square = exact solution; solid curve = cubic boundary-layer solution; dashed curve = parabolic boundary-layer solution (t = 300 min).

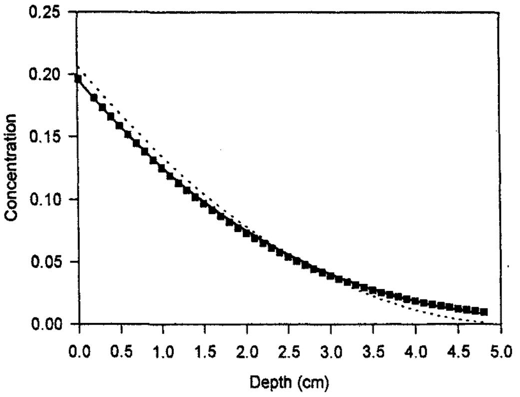

Fig. 5. The comparison of concentration profiles for nonreactive solute transport with dispersivity of 10 cm: filled square = exact solution, solid curve = cubic boundary-layer solution; dashed curve = parabolic boundary-layer solution (t = 300 min).

polynomial solutions predict concentration profile better than parabolic-polynomial solutions. The second instance for comparison is reactive solute transport. At a given dispersivity (10 cm), two values of retardation factor, 0.5 and 2, are used. For these values of R, both boundary-layer solutions predict solute concentration well (Figs. 7 and 8). The cubic-polynomial solutions for both values of R match the exact solutions better than do the parabolic solutions. ME3 is 0.0034 for R = 0.5 and 0.0019 for R = 2.0. ME2 is 0.011 for R = 0.5 and 0.0065 for R = 2.0.

The last instance for comparison is again nonreactive solute transport (Fig. 9). In this instance $R = 1$, dispersivity is fixed (10 cm), and pore-water velocity is increased by one order of magnitude (from 0.001 to 0.01 cm/min). For these values, the parabolic-polynomial solution is better in concentration prediction than a cubic-polynomial solution. The cubic-polynomial in this instance underestimates the concentration profile, and the ME3 is 0.025; however, concentration prediction by the parabolic-polynomial solution almost overlaps the exact solution with an ME2 of 0.0058.

Transport Parameter Estimation

The comparisons above indicate that the boundary-layer solution can predict concentration

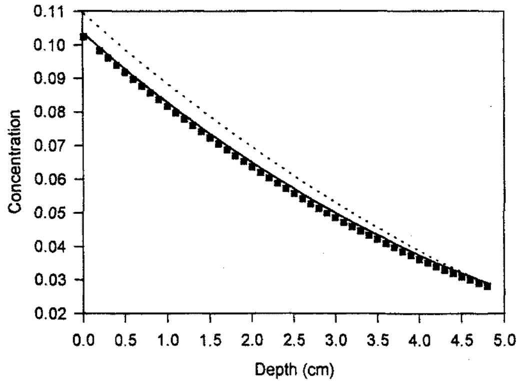

Fig. 6. The comparison of concentration profiles for nonreactive solute transport with dispersivity of 40 cm: filled square = exact solution; solid curve = cubic boundary-layer solution; dashed curve = parabolic boundary-layer solution (t = 300 min).

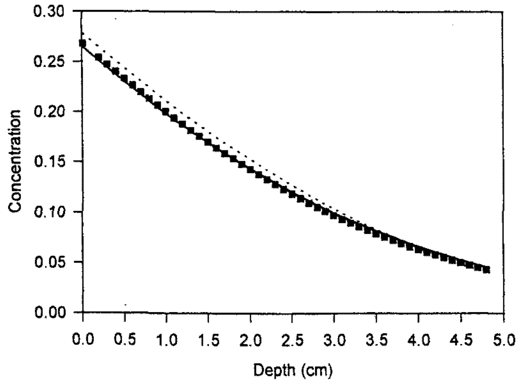

Fig. 7. The comparison of concentration profiles for reactive solute transport (R = 0.5) with dispersivity of 10 cm: filled square = exact solution; solid curve = cubic boundary-layer solution; dashed curve = parabolic boundary-layer solution (t = 300 min).

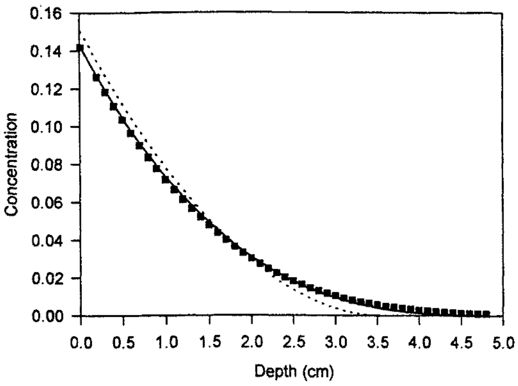

Fig. 8. The comparison of concentration profiles for reactive solute transport (R = 2.0) with dispersivity of 10 cm: filled square = exact solution; solid curve = cubic boundary-layer solution; dashed curve = parabolic boundary-layer solution (t = 300 min).

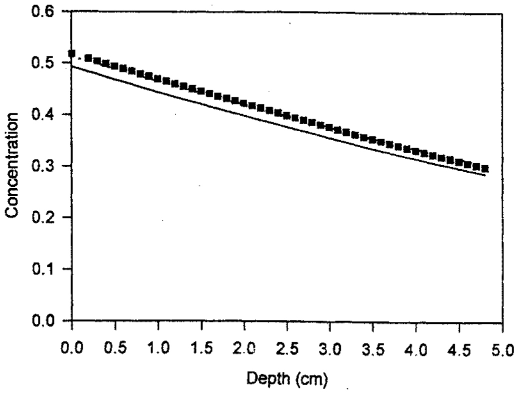

Fig. 9. The comparison of concentration profiles for nonreactive solute transport with dispersivity of 10 cm and pore water velocity of 0.01 cm/min: filled square = exact solution; solid curve = cubic boundary-layer solution; dashed curve = parabolic boundary-layer solution (t = 300 min).

profiles well. Because the boundary-layer solutions are much simpler mathematically than the corresponding exact solution, they may prove to be useful for describing solute transport in soil. One application of the boundary-layer solution is to estimate solute transport parameters. For this purpose, either Eq. (16) or Eq. (18) can be used to fit the observation of the advance of solute front with time. Usually pore-water velocity (v) is easy to determine from infiltration data. Thus, the fitting of the solute advance with time can be used to estimate dispersion coefficient and retardation factor. Mathematically, using the boundary-layer solution to fit data is much easier than using the exact solution to fit data. Physically, in many instances, measuring the advance of solute front is simpler, less time consuming, and requires much less technical equipment than does measuring a complete breakthrough curve or resident concentration profile. The advance of solute front with time can be observed visually both in the laboratory and in the field if the tracer solution has a dye. The new method does not require concentration data, which is often either time consuming (resident and flux concentration in laboratory) or difficult to measure (resident concentration in field conditions). An evaluation of the new method proposed in this paper, both under laboratory and field conditions, will be performed in the near future.

CONCLUSIONS

This study shows that solute transport in a semi-infinite soil column or field profile for flux-type inlet boundary condition can be approximated by boundary-layer theory. The boundary-layer solution is simpler than the corresponding exact solution. This simplicity is not accidental but results from very close approximation of the specific integral method—boundary-layer theory. Generally, the cubic-polynomial solution is better in concentration prediction than the parabolic-polynomial solution. Both are very similar to the exact solution. An important application of the boundary-layer solution is to estimate transport parameters of solute movement through soils both under laboratory and field conditions. This leads to a new method for estimating parameters of solute transport in soils. The observed advance of the solute front with time can be analyzed to determine R and D. The new method provides simplicity, saves time, and overcomes some difficulties in applying the CDE under field conditions.

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TECHNICAL ARTICLES

SOIL WATER DIFFUSIVITY DETERMINATION BY GENERAL SIMILARITY THEORY[1]

Mingan Shao[2] and Robert Horton[3]

A new and simple method is presented for estimating soil water diffusivity. This method utilizes general similarity theory rather than the Boltzmann transformation to evaluate horizontal water infiltration-redistribution processes. It uses the Brooks and Corey function of water diffusivity and requires only measuring the wetting front advance with time. General similarity diffusivities for five soils were compared with those obtained by the Boltzmann transformation and with a third method that used a fitting function to approximate the water distribution data in the Boltzmann transformation method. The comparisons showed that soil water diffusivities for the three methods were in good agreement for the intermediate range of water contents. However, at low water contents, the similarity water diffusivities differed from the other water diffusivities for the five soils. The new method has several advantages over the other methods. It allows the inlet boundary water content to vary in time and initial water content distribution to vary with distance, which is more general than constant water content. The new method also does not require soil water diffusivity to be zero at the initial water content. This represents an improvement over the earlier methods, which give a zero diffusivity at initial water content no matter how high this initial water content is.

SOIL water infiltration rates and subsequent water redistribution are important concerns for development of soil management practices to minimize potential groundwater contamination from land-applied chemicals. Numerical solutions of the flow and transport problems in the vadose zone are the most important approaches to predict quantitatively the dynamic behavior of the system. Unsaturated flow and transport modeling usually requires accurate and complete information about the unsaturated hydraulic properties for the model to function properly. There are three basic hydraulic parameters: hydraulic conductivity (K), water diffusivity (D), and specific water capacity (C). Among the three parameters, only two of them are independent because of the relationship K = CD.

In recent years, there have been increased efforts to determine water diffusivities of unsaturated soils. Usually, horizontal infiltration experiments have been used to relate the soil water diffusivity to the volumetric water content by the method of Bruce and Klute (1956). This method is based on the Boltzmann transformation and constant initial condition, flux upper boundary, and constant water content lower boundary conditions. The water content distribution along the column needs to be measured in order to estimate the water diffusivity. The most common way to find D was shown by Kirkham and Powers (1972, p. 256). However, determining the slope of the water distribution curve can be difficult. Cassel et al. (1968) presented a method for estimating soil water diffusivity values based on time-dependent soil water content distribution in the horizontal redistribution process. Their

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[2] Institute of Soil and Water Conservation, Chinese Academy of Sciences, Yangling, Shaanxi, China 712100.

[3] Department of Agronomy, Iowa State University, Ames, IA 50011-1010. Dr. Horton is corresponding author. E-mail: rhorton@iastate.edu

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method requires measuring water content distribution with time and also involves both relatively intensive calculation and running of the experiments for long time periods. Clothier et al. (1983) presented a fitting function chosen from those presented by Philip (1960) to approximate the water distribution curve in the Bruce-Klute method (1956). This makes possible a simple analytical expression of the water diffusivity by avoiding finding the slopes of the soil water distribution curve. However, the method of Clothier et al. (1983) needs precise measurements for parameter estimation of their analytical expression of water diffusivity; for example, a small error in determining the water content of the inlet boundary with their formula may produce a negative P value (Eq. (15) in their paper), which makes no physical sense. McBride and Horton (1985), based on the Bruce-Klute method (1956), used various aspects of this approach to develop a method of determining the water diffusivity from horizontal infiltration experiments.

In particular, they introduced an empirical function, fitted by least squares regression, to water distribution data. Their approach provides another way to determine the water diffusivity. However, the McBride and Horton approach involves intensive calculations. Warrick (1994) gave a detailed review on soil water diffusivity estimation for fixed water content at the inlet boundary. To our knowledge, little information in the literature deals with variable water content of the inlet boundary in the horizontal infiltration experiments for the purpose of diffusivity determination. However, Anderson and Jeppson (1984) developed a nonlinear diffusion model for semiconductors. They found that at high concentration, impurity diffusion in semiconductors tends to be governed by nonlinear diffusion processes. They used a general similarity approach to deal with nonlinear diffusion processes. Their idea may be used in determining the soil water diffusivity; general similarity theory, rather than the Boltzmann transformation, may be used to give an analytical solution of the horizontal infiltration and the redistribution to get an analytical expression of the water diffusivity for variable water-content of the inlet boundary. One factor associated with water redistribution is capillary hysteresis. A complete analysis of water redistribution in soil should take capillary hysteresis effects into account. Currently, only numerical techniques can actually incorporate hysteresis effects into the water flow model. However, analytical solution of nonhysteretic flow may still have certain applications to soil water redistribution. There is some evidence (Watson and Sardana 1987) that the size of the hysteresis loop decreases for fine-textured soils. Moreover, both theoretical analysis and experimental evidence show that hysteresis has much less effect on hydraulic properties if they are expressed in water content rather than pressure head (Mualem and Dagan 1975; Mualem 1976). The hysteresis phenomenon primarily affects hydraulic properties of soils in the range of capillarity, i.e., in the wet end of hydraulic properties. Therefore nonhysteretic solutions still have applications to water redistribution for certain soil water conditions. Such nonhysteric solutions should be applicable to certain intermediate and low ranges of soil water content.

This paper presents a method for estimating water diffusivities of unsaturated soils by using a nonhysteretic analytical solution to horizontal redistribution, based on general similarity theory. This new method allows the water content of the inlet boundary to be variable with time and allows the initial water content distribution to be variable with distance. It needs only information on the advance of the wetting front with time to obtain water diffusivities of unsaturated soils. The analytical expression of the water diffusivity will be compared with D(θ) data for five soils derived from one-dimensional horizontal absorption experiments by the method of Bruce and Klute (1956) and by the method of Clothier et al. (1983).

THEORY

For one-dimensional horizontal flow, the flow equation is given by (Klute 1952)

$$\partial\theta/\partial t = \partial\,(D\,(\theta)\,\partial\theta/\partial x)\,/\partial x \qquad (1)$$

where $\theta(x,t)$ is the volumetric water content (m^3/m^3), t is the time (s), x is the distance (m), and D is the soil water diffusivity (m^2/s).

The problem to be solved is a two-step problem. First, water infiltrates into the soil, then the water supply is stopped, and the soil water is allowed to redistribute. Our interest is focused on the second step, i.e., the redistribution process. The initial and boundary conditions for the problem of horizontal redistribution are

$$\theta\,(x, 0) = f(x) \qquad x \leq x_0 \qquad (2a)$$

$$\theta\,(x, 0) = \theta_i \qquad x > x_0 \qquad (2b)$$

$$q(0,\ t) = -\,D(\theta)\ \partial\theta\,(0,\ t)\,/\partial x = 0 \qquad (3)$$

$$\theta\,(x_f, t) = \theta_i \qquad (4)$$

in which x_0 is the initial wetting distance from the inlet end of the soil column to where the water content profile intersects the x axis, $f(x)$ is the water content distribution of the infiltrated water, θ_i is the initial water content in the dry zone, $q(0, t)$ is the flux density at the zero-position boundary (in general, flux density $q = q(x, t)$), and x_f is the position of the wetting front.

Simplifying assumptions are needed to solve the nonlinear flow equation analytically for this particular flow problem. First of all, we adopt the Brooks and Corey water diffusivity (1964), a power function of the water diffusivity that has been used by others (Parlange et al. 1980; Parlange and Fleming 1984; Hogarth et al. 1989; Ross and Parlange 1994 a and b), i.e.:

$$D\,(\theta) = D_0\,\theta^\gamma \tag{5}$$

where D_0 and γ are constants.

For the sake of simplicity, let θ_i = constant, and specifically let $\theta_i = 0$ (water redistribution into an oven-dry soil). This assumption was also used by Parlange et al. (1980) to solve the diffusivity equation. With these assumptions, the problem can be reduced as follows:

$$\begin{gathered}\partial\theta/\partial t = \partial\,(D_0\theta^\gamma\,\partial\theta/\partial x)\,/\,\partial x \\ 0 < t\,, 0 < x < \infty\end{gathered} \tag{6}$$

$$\theta\,(x\,, 0) = 0 \qquad x_0 < x < \infty \tag{7}$$

$$\partial\theta\,(0, t)\,/\partial x = 0 \qquad 0 < t \tag{8}$$

$$\theta\,(x_f, t) = 0 \qquad 0 < t \tag{9}$$

By introducing $\tau = D_0\,t$, (6) is then transformed as

$$\partial\theta/\partial\tau \;=\; \partial(\theta^\gamma\,\partial\theta/\partial x)\,/\partial x \tag{10}$$

By using similarity methods, the solution to (10) is written as

$$\theta \;=\; \tau^\alpha\,\phi\,(\xi) \tag{11}$$

$$\xi \;= x/\tau^\beta \tag{12}$$

Inserting (11) and (12) into (10), the LHS of (10) is

$$\partial\theta/\partial\tau = \alpha\tau^{\alpha-1}\,\phi(\xi)\,\text{-}\beta\tau^{\alpha-1}\,\xi\,d\phi\,(\xi)\,/d\xi \tag{13}$$

The RHS of (10) is

$$\begin{gathered}\partial\,(\theta^\gamma\,\partial\theta/\partial x)\,/\partial x = \tau^{\alpha\gamma+\alpha-2\beta} \\ d(\phi^\gamma d\phi\,(\xi)\;\,/d\xi)\,/d\xi\end{gathered} \tag{14}$$

Combining (13) with (14) and dividing both sides by $\tau^{\alpha-1}$ yields

$$\begin{gathered}\alpha\phi - \beta\xi\;\,d\phi/d\xi \;= \tau^{\alpha\gamma-2\beta+1} \\ d(\phi^\gamma\,d\phi/d\xi)\;\,/d\xi\end{gathered} \tag{15}$$

In order to remove τ as an explicit variable in (15), the power of τ should be zero; then

$$\alpha = (2\beta - 1)\,/\gamma \tag{16}$$

and the resulting equation for $\phi(\xi)$ is then

$$\alpha\phi - \beta\xi\;\,d\phi/d\xi = d(\phi^\gamma\,d\phi/d\xi)\,/d\xi \tag{17}$$

Equation (17) is equation (13) of Hogarth et al. (1989) when the gravity term that is considered in that paper, as well, is removed. Hogarth et al. (1989) solved the ordinary differential equation (Eq. (13) in their paper) numerically by a shooting procedure. A particular interest of this paper is to solve equation (17) analytically. Boundary conditions are needed to solve (17). The boundary conditions described by Eqs. (8) and (9) are transformed as

$$d\phi(0)\,/\,d\xi{=}0 \tag{18}$$

$$\phi\,(\xi_f)\,{=}0 \tag{19}$$

By performing the general similarity transformation, the mixed problem of PDE (Eqs (6)–(9)) is reduced to a two-point ODE boundary value problem, given by (17)–(19).

By using the initial condition (7) and mass conservation condition, i.e.:

$$\int_0^\infty \theta\,(x, \tau)\,dx = \tau^{\alpha+\beta}\int_0^\infty\phi(\xi)\,d\xi = H_0 \tag{20}$$

where H_0 is the water applied in the infiltration process (m); of course, H_0 is a constant because the total quantity of water within the redistribution remains unchanged during the process of water redistribution. H_0 is also the total quantity of water (H) in the soil column because of zero initial water content. This statement is also true for non-zero initial condition, but the total quantity of water should consist of applied water infiltrated into the column (H_0) and residual water in the soil column (H_r, i.e., $H = H_0 + H_r$). Therefore H_0 cannot depend explicitly on τ. To remove any explicit τ dependence, we must have $\alpha = -\beta$, which, together with Eq. (16), determines α and β to be

$$\alpha = -\beta = -1/\,(2 + \gamma) \tag{21}$$

With α and β given by (21), Eq. (17) with the boundary conditions of (18) and (19) can be integrated explicitly to yield

$$\phi(\xi) = \phi_0\,(1 - \xi^2/\xi_f^2)^{1/\gamma} \tag{22}$$

In the first integration of Eq. (17), (18) is used to give a zero integration constant. In the second integration, (19) is used to determine the integration constant. In Eq. (22), the general similarity variable at the wetting front, ξ_f, is related to the integration constant, ϕ_0, by

$$\xi_f^2 = 2\ \phi_0^\gamma\ (2 + \gamma)\ /\gamma \tag{23}$$

Furthermore, the integration constant, ϕ_0, is found by inserting Eq. (22) into the second integral of (20), i.e.:

$$\phi_0 = (H_0^2\ \gamma/\ (2\ (2 + \gamma)\ I_\gamma^2)\ )^{1/(2+\gamma)} \tag{24}$$

where I_γ is a definite integral expressed as

$$I_\gamma = \int_0^1 (1-x^2)^{1/\gamma}\ dx = B\ (1/2, 1+1/\gamma)\ /2 \tag{25}$$

where $B(1/2, 1+1/\gamma)$ is a Beta function, and the substitution of $t = x^2$ is used to convert the definite integral to the formal Beta function. The numerical evaluation of the Beta function can be found in Abramowitz and Stegun (1972).

Combining (23) and (24), ξ_f is obtained as

$$\xi_f = (2H_0^\gamma\ (2 + \gamma)\ /\ (\gamma I_\gamma^\gamma)\ )^{1/(2+\gamma)} \tag{26}$$

and the wetting front, x_f, is written as

$$x_f(\tau) = (2\ (2 + \gamma) H_0^\gamma\ \tau/\ (\gamma I_\gamma^\gamma)\ )^{(1/2+\gamma)} \tag{27}$$

It is obvious that (27) gives $x_f\ (0) = 0$. That means the given amount of water (H_0) is concentrated at $x_0 = 0$ with an infinite θ. This is clearly unphysical. We assume that $x_f\ (0) = x_0$ (a finite distance) over which H_0 is distributed. Then, the solution to the original wetting front, $x_f(t)$, is

$$x_f\ (t) = x_0 + (2\ (2 + \gamma)\ H_0^\gamma\ D_0\ t/(\gamma I_\gamma^\gamma))^{1/(2+\gamma)} \tag{28}$$

Equation (28) is physical now. However, experimental data will be used to verify Eq. (28). It is obvious that Eq. (28) can be expressed by

$$x_f\ (t) = x_o + a\ t^b \tag{29}$$

The two constants, a and b, in Eq. (29) can be obtained experimentally by fitting (29) to the wetting front with time observed in the horizontal infiltration-redistribution experiment.

With a and b, then, γ and D_0 are obtained as follows

$$\gamma = 1/b - 2 \tag{30}$$

$$D_0 = a^{\gamma+2}\ \gamma I_\gamma^\gamma/\ (2\ (2+\gamma)\ H_0^\gamma) \tag{31}$$

So far, the relationship of $D(\theta) = D_0\theta^\gamma$ is determined by Eqs. (30) and (31).

EXPERIMENTS

Five soils were used to test the approach in this study: a silt loam obtained from land mapped as Flagler series (Coarse-loamy, mixed, mesic Typic Hapludoll, 0.114 sand, 0.700 silt, and 0.186 clay mass fractions), Nicollet loam (Fine-loamy, mixed, mesic Aquic Hapludoll, 0.509 sand, 0.326 silt, and 0.165 clay), Keswick sandy clay loam (Fine, Montmorillontic, mesic Aquic Hapludoll, 0.677 sand, 0.113 silt, and 0.210 clay), Monona silty clay loam (Fine-silty, mixed, mesic Typic Hapludoll; 0.024 sand, 0.695 silt, and 0.281 clay), and Webster clay loam (Fine-loamy, mixed, mesic, Typic Endoaquoll, 0.321 sand, 0.392 silt, and 0.287 clay). The specific surface areas were measured by using EGME technique (Chihacek and Bremner 1979; Carter et al. 1986). Particle densities were determined by using the pycnometer method (Blake and Hartge 1986). The inlet boundary water contents of −0.03 m water potential, used in the horizontal infiltration and distribution experiments, are listed in Table 1. These water contents are needed to calculate diffusivity by the method of Clothier et al. (1983).

Two types of experiments were carried out to obtain soil water diffusivity data. One was the traditional horizontal-infiltration experiment of the Bruce-Klute method. Air-dried soil was packed into sectioned plexiglas tubes 0.15 m long (15 sections) and 0.038 m in diameter with the controlled bulk density of 1.30 Mg/m^3. A −0.03-m water tension was applied to the inlet boundary of the soil column. This tension was used to reduce water movement along the tube wall. The second was a horizontal experiment of infiltration and redistribution to obtain the two coefficients of the power function of the water diffusivity based on the general similarity theory in this paper. In the second experiment, the oven-dried soil was packed into plexiglas tubes 0.3 m long and 0.038 m in diameter. The bulk density in the second experiment was the same as that in the Bruce-Klute method. A given amount of water (H_0) was applied to each soil column. For example, the amount of water for silt loam soil was 0.003 m. Two methods were used to apply H_0. The first approach was to have H_0 water infiltrate into the soil vertically. After infiltration, the column was placed horizontally while the redistribution of the infiltrated water

TABLE 1

Some physical properties of the five soils

Soil	Specific surface (10^{-3} m^2/kg)	Particle density (Mg/m^3)	Inlet water content (m^3/m^3)
Silt loam	41	2.67	0.454
Loam	40	2.69	0.460
Sandy clay loam	58	2.64	0.447
Silty clay loam	79	2.67	0.462
Clay loam	141	2.57	0.469

occurred. The second approach was to have H_0 water applied to a separate end section of the soil column and then to connect the end section with the water to a dry soil column. During redistribution, the advance of the wetting front with time was recorded. It was easier to observe the wetting front with time for the second manner of water application than for the first method.

RESULTS

Parameter Estimation from the Advance of Wetting Front

As mentioned above, the advance of the wetting front with time is measured in the experiment. This advance can be approximated theoretically by Eq. (29). The a and b parameters are related to D_0 and γ in Eq. (5), which describes the water diffusivity by the power function. The a and b parameters are estimated by least squares regression of fitting the observed data of the advance of wetting front with time by a power function. As an example, the analysis of the silt loam is listed in Table 2. D_0 and γ in Table 2 are calculated by using Eqs. (30) and (31). The fitting of the advance of the wetting front with time by a power function with observed data is shown in Fig. 1. Coefficient of determination, R^2, of the fitting is 0.95. Root Mean Square Error, RMSE, (Willmott et al. 1985), is 0.6 mm. Therefore, Eq. (28) describes the wetting front with time well.

The Water Diffusivity of the Five Soils

The water diffusivities of the general similarity theory for the five soils are obtained by finding the two parameters, γ and D_0, using the observed data of the advance of wetting front with time in the same way as in the above paragraph. The γ and D_0 values estimated by this method are listed in Table 3.

With the parameters in Table 3, the water diffusivity functions of the five soils can be obtained from Eq. (5).

TABLE 2

The parameter values for three samples of silt loam soil

Column	a (10^{-3}m/s^b)	b	γ	D_0 (10^{-5}m^2/s)
1	6.49	0.191	3.236	9.86
2	6.57	0.195	3.134	10.03
3	6.62	0.194	3.164	10.22
mean	6.56	0.193	3.178	10.04

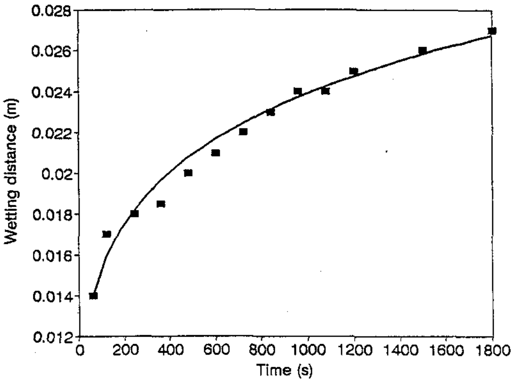

Fig. 1. An example of the wetting front advance with time.

DISCUSSION

The soil water diffusivity of general similarity theory is determined by Eqs. (30) and (31). Because the values of γ and D_0 are dependent on a and b constants in Eq. (29), it may be interesting to look at the sensitivities of the diffusivity to a and b coefficients in Eq. (29). For a given b, i.e., a fixed γ (here b is assumed to be 0.2, then γ is 3), from Eq. (31) it is clear that D_0 will increase 32 times when a is doubled. This is also shown in Fig. 2. Therefore, the a-constant affects only the magnitude of the diffusivity function. However, for a given a-constant (assumed to be 1 cm/min^b), the value of the b-constant affects both the shape and the magnitude of the diffusivity function (Fig. 3). The major effect of the b-constant on the water diffusivity function is the shape of the curve. The smaller the value of b, the steeper the curve. The smaller the value of b, the smaller the value of the diffusivity at lower water content, and the larger the value of the diffusivity at higher water content. Parameter a largely influences the magnitude of the diffusivity, and b largely influences the shape of the diffusivity function.

The water diffusivities determined by the general similarity theory are compared with

TABLE 3

Water diffusivity parameters of the five soils

Soil	γ (dimensionless)	D_0 (10^{-5} m^2/s)
Silt loam	3.18	10.0
Loam	3.47	11.2
Sandy clay loam	2.85	1.12
Silty clay loam	3.34	2.66
Clay loam	3.14	1.69

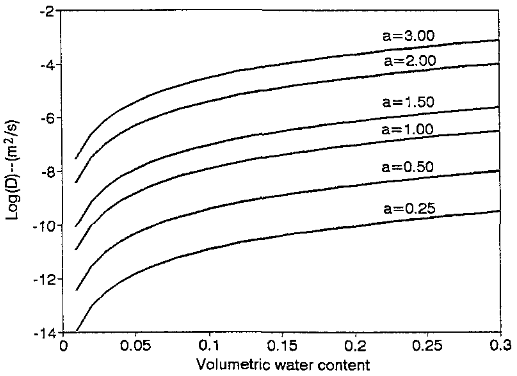

Fig. 2. Sensitivity of diffusivity to parameter a (b = 0.2).

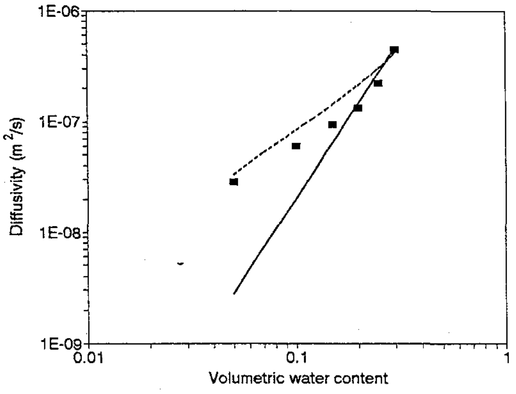

Fig. 4a. Water diffusivities of sandy clay loam: filled square—by Bruce-Klute method; dashed curve—by Clothier et al. method; solid curve—by general similarity.

those obtained by the Bruce-Klute method (1956) and the method of Clothier et al. (1983). The results of the comparisons are shown in Fig. 4. The Bruce-Klute method (1956) provides a relatively accurate determination of soil water diffusivity only in the intermediate range of water contents (the shoulder of the corresponding water content distribution) because it has difficulty obtaining the slopes at small and large water contents. Therefore, particular attention should be given to the water diffusivities at intermediate water contents. The similarity water-diffusivities and the water diffusivities by the Bruce-Klute method for all five soils in the intermediate range of water contents (0.15–0.30 of water contents) are in very good agreement. The Clothier et al. method (1983) tends to overestimate water diffusivities for sandy clay loam and underestimates water diffusivities for silt loam at intermediate water contents. At lower water contents, similarity water diffusivities are consistently lower than the other two water diffusivities for all five soils. The differences for all five soils are within one order of magnitude. However the two water diffusivities determined using the Bruce-Klute method (1956) and the Clothier et al. method (1983) do depend on the initial water content of the soil because both methods give zero water diffusivity at the initial water content. It is not difficult to imagine a jump in water diffusivity near initial water content; the greater the initial water content, the bigger the jump. There is uncertainty in water diffusivities by the two methods (Bruce-Klute and Clothier et al. methods) because of the assumption of zero water diffusivity at initial water content and the inaccurate estimations of the slopes for lower water contents. Therefore the water diffusivity differences for small water con-

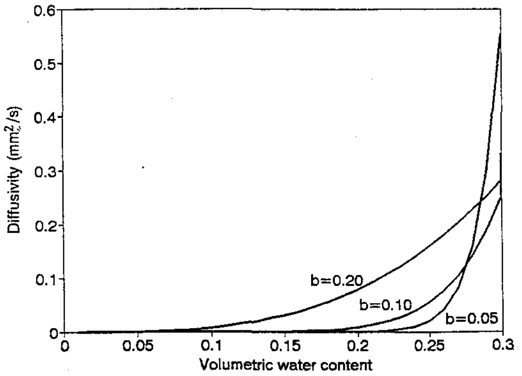

Fig. 3. Sensitivity of diffusivity to parameter b (a = 1).

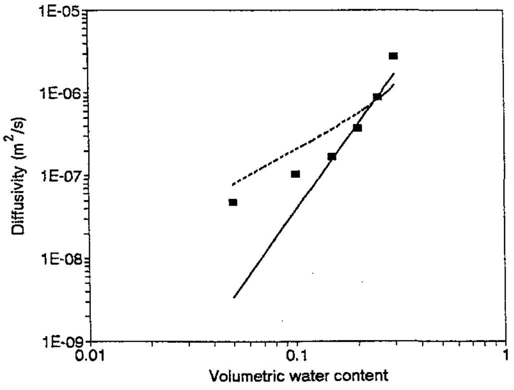

Fig. 4b. Water diffusivities of loam: filled square—by Bruce-Klute method; dashed curve—by Clothier et al. method; solid curve—by general similarity.

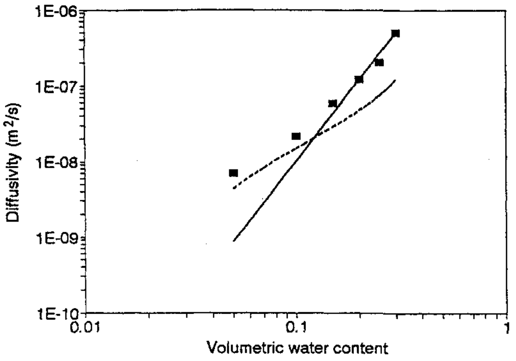

Fig. 4c. Water diffusivities of silt loam: filled square—by Bruce-Klute method; dashed curve—by Clothier et al. method; solid curve—by general similarity.

tents between the similarity theory method and the other two methods are expected. The water diffusivities from the Bruce-Klute method and the Clothier et al. method are similar for all five soils. These results are expected because both methods are based on the same Boltzmann transformation theory.

In the redistribution process, the transformation variable is $\xi = x/t^{\beta}$ (here β ranged from 0.18 to 0.21 for the five soils in this study) rather than the Boltzmann variable ($x/t^{0.5}$) used in the Bruce-Klute method (1956). β is not fixed to the value 0.5, but it can be changed to a specific value for a specific flow problem. It is obvious that the theory is more general than that of the Boltzmann transformation, which is the basic foundation of the Bruce-Klute method. However, it is easy to show that general similarity theory reduces to the special case of the Boltzmann transformation when the water content of the inlet boundary is constant. The Bruce-Klute method (1956) is only a special case of the general similarity method. Thus, the general similarity method is more flexible than the Bruce-Klute method for describing soil water redistribution.

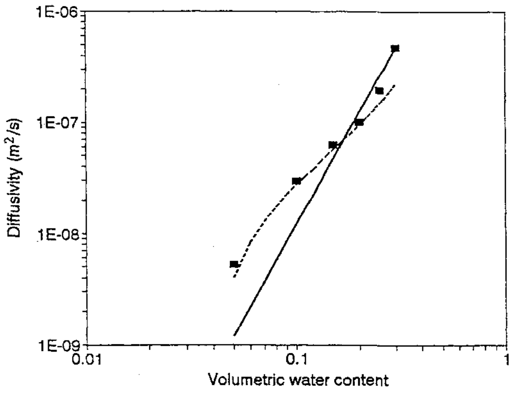

Fig. 4d. Water diffusivities of silty clay loam: filled square—by Bruce-Klute method; dashed curve—by Clothier et al. method, solid curve—by general similarity.

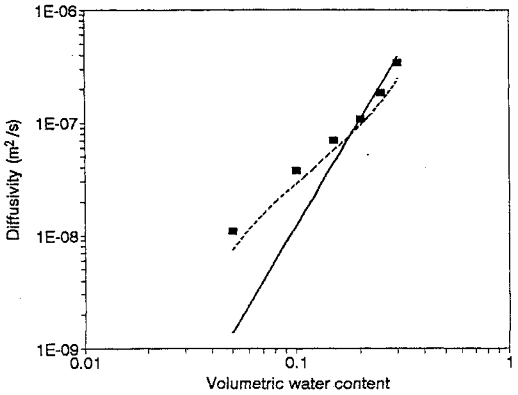

Fig. 4e. Water diffusivities of clay loam: filled square—by Bruce-Klute method; dashed curve—by Clothier et al. method; solid curve—by general similarity.

The results of this investigation suggest that general similarity theory allows a simple method of determining the water diffusivity function that may be more convenient and useful in the laboratory than any other method available currently. The general similarity method allows the inlet boundary to be variable in water content and allows the initial water content to be variable with distance. These conditions are more flexible and general than other constant boundary methods. A limitation for both the Bruce-Klute method (1956) and the Clothier et al. method (1983) is that the diffusivity value associated with the initial water content is zero no matter how high the initial water content. However, the similarity method does not give a zero diffusivity unless the water content is zero. The general similarity method is not only simpler than the current methods (e.g., Bruce-Klute method), it also removes the limitation of zero water diffusivity at the initial water content.

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EXACT SOLUTION FOR HORIZONTAL WATER REDISTRIBUTION BY GENERAL SIMILARITY

MINGAN SHAO AND ROBERT HORTON*

Abstract

This paper presents an exact solution to horizontal water redistribution by using general similarity theory. A power function of soil water diffusivity is used to derive the exact solution. The similarity solution contains initial wetted length, amount of water present, and coefficients of water diffusivity. A similarity solution for a step function initial condition is compared with a corresponding numerical solution. Error analysis indicates that the maximum global error in water content is within 2%. The general similarity theory provides an approach that exactly solves horizontal water redistribution with a variable first-type boundary of a specific form of time dependence and initial conditions; the Boltzmann transformation case is restricted to a horizontal infiltration problem with constant first-type boundary and initial conditions.

THE UNDERSTANDING AND PREDICTION OF THE REDISTRIBUTION OF WATER that has infiltrated soil is just as important as the infiltration process itself (Philip, 1991). Redistribution determines the quantity of water stored in the root zone of crops or natural vegetation and how long this water remains available for uptake by plant roots (Sander et al., 1991). Knowledge of water redistribution is also needed to determine whether or not water or solutes penetrate the root zone, which is useful for agricultural chemical management.

This paper includes the derivation of an exact solution for nonlinear, nonhysteretic redistribution of water in a horizontal soil column by using general similarity theory. The nonlinear water diffusivity is a power function that has been used for more than two decades by a number of soil physicists (Parlange et al., 1980; Parlange and Fleming, 1984; Ross and Parlange, 1994a, 1994b). Philip (1991) gave an analytical solution to the redistribution of water in a horizontal column of infinite dimension. A key for his solution was the similarity character of a horizontal column with two parts, $x < 0$ and $x > 0$, at uniform large and small moisture contents. He used the Boltzmann transformation and assumed power law flux-concentration relations to solve the problem. Philip's solution is an implicit integral that requires iterative numerical integrations to have sorptivity be equal to desorptivity. In our analysis, the column does not necessarily need to have equal distance between the wet and dry parts (i.e., the length of the initially wet soil can be arbitrary).

Shao and Horton (1996) showed that the Boltzmann transformation method is a specific form of the general similarity theory. Parlange and Hogarth (1997) extended the work of Shao and Horton (1996) to provide approximate solutions, including other forms of the diffusivity function. Shao and Horton (1996, 1997) used general similarity to show that measurement of the advance of the wetting front with time for redistributing water can be used to determine soil water diffusivity.

The purpose of this paper is to improve the existing Boltzmann transformation method by presenting an exact solution to horizontal water redistribution using general similarity theory. It will be shown that general similarity theory can be used to describe soil water content distributions during redistribution of water. The theory is quite flexible for much more general initial conditions as compared to the Boltzmann transformation case. An example is presented that compares the exact solution by general similarity with a numerical solution for the special case of a step function initial condition. The step function initial condition is used to represent the physical experiment of joining together a wet soil sample and a dry soil sample.

Theory

The equation for one-dimensional horizontal flow is given by Bruce and Klute (1956)

$$\frac{\partial\theta}{\partial t} = \partial[D(\theta)\partial\theta/\partial x]/\partial x \qquad [1]$$

where θ is volumetric water content ($m^3\ m^{-3}$), t is time (s), x is distance (m), and D is soil water diffusivity ($m^2\ s^{-1}$).

The initial and boundary conditions for the horizontal redistribution are

$$\theta(x,0) = f(x) \qquad x \leq x_0 \qquad [2a]$$

$$\theta(x,0) = \theta_i \qquad x > x_0 \qquad [2b]$$

$$q(0,t) = 0 \qquad [3]$$

$$\theta(x_f,t) = \theta_i \qquad [4]$$

where x_0 is the length of the wet part of the horizontal flow system, $f(x)$ is the water content distribution of the wet part (if the water content is uniform then $f(x)$ is a constant), θ_i is the initial water content in the dry part. $q(0,t)$ is the flux density at the zero-position boundary, and x_f is the position for the leading edge of the wetting front. Water is redistributed from the wet part to the dry part and no water flows in or out of the system.

Shao and Horton (1996) used simplifying assumptions to

M. Shao, National Lab. of Soil Erosion and Dryland Agriculture, Inst. of Soil and Water Conservation, Chinese Academy of Sciences and Ministry of Water Resources, Yangling, Shaanxi, China 712100; R. Horton, Dep. of Agronomy, Iowa State Univ., Ames, IA 50011. Journal Paper No. 17032 of the Iowa Agric. and Home Econ. Exp. Stn., Ames, IA. Project No. 3262 and 3287, and supported in part by Hatch Act and State of Iowa funds, and in part by Project kz 951-B1-211 of resources, ecological, and environmental research at the Chinese Academy of Sciences. Received 21 Dec. 1998. *Corresponding author (rhorton@iastate.edu).

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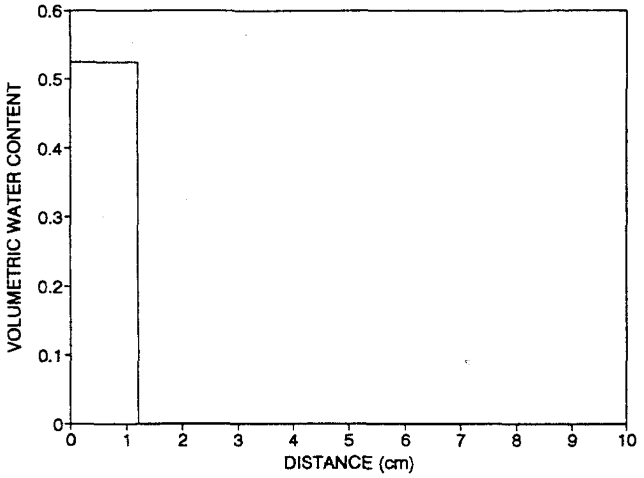

Fig. 1. Initial condition of water redistribution within a two-part column.

solve the nonlinear flow equation analytically for this particular flow problem. First they assumed a power function of the water diffusivity:

$$D(\theta) = D_0\,\theta^{\gamma} \qquad [5]$$

where D_0 and γ are constants.

Shao and Horton (1996) gave the similarity solutions for the case $\theta_i = 0$ as

$$\theta = \tau^{\alpha}\phi(\xi) \qquad [6]$$

$$\xi = x/\tau^{\beta} \qquad [7]$$

where $\tau = D_0\, t$.

Inserting Eq. [5], [6], and [7] into Eq. [1] shows that the power of τ matches only if α and β are related by

$$\alpha = (2\beta - 1)/\gamma \qquad [8]$$

and the resulting equation for $\phi(\xi)$ is then

$$\alpha\phi - \beta\xi\, d\phi/d\xi = d/d\xi\,(\phi^{\gamma}\, d\phi/d\xi) \qquad [9]$$

According to the mass balance:

$$\int_0^{\infty}\theta(x,\tau)dx = \tau^{\alpha+\beta}\int_0^{\infty}\phi(\xi)\,d\xi = H_0 = \int_0^{\infty}f(x)\,dx \qquad [10]$$

where H_0 (a constant) is the amount of water in the wet part (m).

Equation [10] obviously requires $\alpha = -\beta$, which, together with Eq. [8], determines α and β to be

$$\alpha = -\beta = -1/(2 + \gamma) \qquad [11]$$

With α and β given by Eq. [11], Eq. [9] can be integrated explicitly to yield

$$\phi(\xi) = \phi_0\,(1 - \xi^2/\xi_f^2)^{1/\gamma} \qquad [12]$$

in which the characteristic wetting depth, ξ_f, is related to the integration constant, ϕ_0, by

$$\xi_f^2 = 2\phi_0^{\gamma}\,(2 + \gamma)/\gamma \qquad [13]$$

Furthermore, the integration constant, ϕ_0, is determined by

$$\phi_0 = \{H_0^2\gamma/[2(2 + \gamma)I_{\gamma}^2]\}^{1/(2+\gamma)} \qquad [14]$$

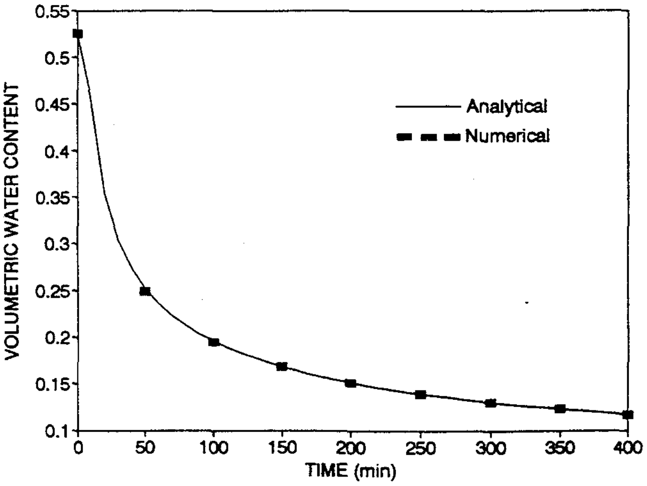

Fig. 2. Comparison of analytical and numerical predictions of the decaying maximum water contents at $x = 0$.

where I_{γ} is a constant [i.e., $I_{\gamma} = \int_0^1 (1 - x^2)^{1/\gamma}\,dx = B(1/2,1 + 1/\gamma)/2$ and $B(1/2,1 + 1/\gamma)$ is a beta function whose value is found in Abramowitz and Stegun (1972)].

By combining Eq. [13] and [14] ξ_f is obtained as

$$\xi_f = [2H_0^{\gamma}(2 + \gamma)/(\gamma I_{\gamma}^{\gamma})]^{1/(2+\gamma)} \qquad [15]$$

and the wetting front, x_f, is written as

$$x_f(\tau) = [2(2 + \gamma)H_0^{\gamma}\,\tau/(\gamma I_{\gamma}^{\gamma})]^{1/(2+\gamma)} \qquad [16]$$

The solution to the original wetting front, $x_f(t)$, is then

$$x_f(t) = [2(2 + \gamma)H_0^{\gamma}\,D_0 t/(\gamma I_{\gamma}^{\gamma})]^{1/(2+\gamma)} \qquad [17]$$

From Eq. [17] it is obvious that D_0 and γ can be obtained by fitting Eq. [17] to observed wetting front data. With D_0 and γ soil water diffusivity can be estimated using Eq. [5].

Collecting our results, we find that $\theta(x,t)$ can be written as

$$\theta(x,\tau) = \theta_0(\tau)[1 - (x/x_f)^2]^{1/\gamma} \qquad [18]$$

$$\theta_0(\tau) = \{(\gamma H_0^2)/[2(2 + \gamma)I_{\gamma}^2\tau]\}^{1/(\gamma+2)} \qquad [19]$$

where $\theta_0(t)$ is a decaying maximum water content at $x = 0$. Equation [16], [18], and [19] complete the analytical solution to this problem. The only step remaining is to incorporate the finite length of the wet part of the soil column. This can be done by relating the initial wetted length of the column with an arbitrarily constant time (τ_0). Then a more general similarity solution that incorporates the initial wetted length is obtained through arbitrary time translations of the previous solution because Eq. [1], with Eq. [5], [6], and [7], is invariant under such time translations:

$$\theta(x,\tau) = \theta_0(\tau + \tau_0)[1 - x^2/x_f^2(\tau + \tau_0)]^{1/\gamma} \qquad [20]$$

From Eq. [7], the arbitrary time constant, τ_0, can be related to the length of the wet part, x_0, initial water, H_0, and water diffusivity coefficient, γ. When $\tau = 0$, $\xi_0 = \xi_f$, and $\xi_0 = x_0/\tau^{1/(\gamma+2)}$

$$\tau_0 = (x_0/\xi_f)^{\gamma+2} = \gamma/[2(\gamma + 2)]\,(I_{\gamma}/H_0)^{\gamma}\,x_0^{\gamma+2} \qquad [21]$$

With Eq. [16], [20], and [21] the general similarity solution to the redistribution problem of soil water is complete. In the following part of this paper, the general similarity solution is compared with a numerical solution for a step function initial

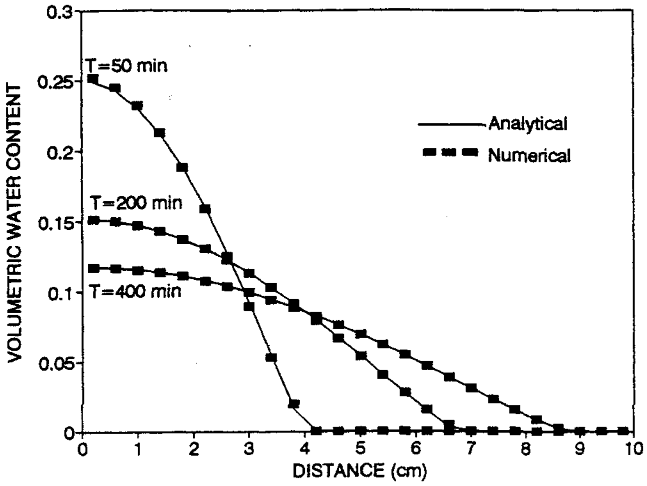

Fig. 3. Comparison of analytical and numerical solutions for soil water content profiles at selected times for a two-part soil column.

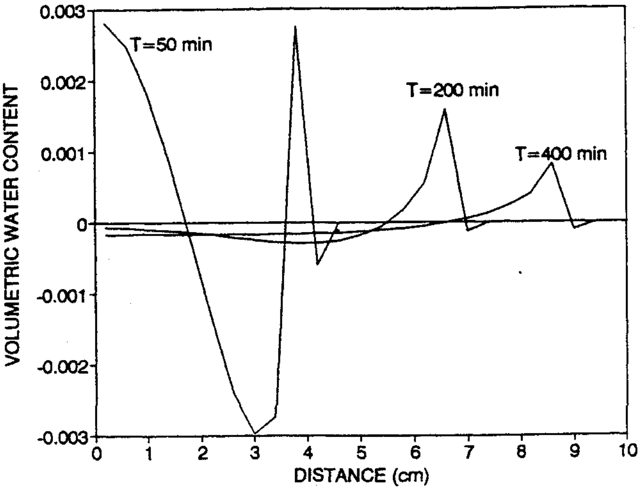

Fig. 4. Global error distribution for water redistribution in a two-part soil column.

condition of soil water distribution. The numerical solution is obtained by using CSMP (continuous system modeling program), a specially designed language that allows users to simulate all types of physical systems with a minimum of programming effort (Speckhart and Green, 1976). In this numerical modeling method the governing partial differential equation (Eq. [1]) is approximated by a set of ordinary differential equations, which describe spatial flow distribution at a given time. The flow rates are integrated numerically over a time step to calculate a new water content distribution. The time step is 0.1 min and the grid spacing is 0.4 cm. The integration method follows the trapezoidal rule.

Results and Discussion

A step function initial condition is selected to demonstrate the use of general similarity theory for describing redistribution of soil water. This example can represent the physical experiment for the redistribution of soil water in a two-part soil column. One part of the soil column is uniformly wet and the other part is uniformly dry. Physically, one column is wetted and then connected with a dry soil column. This allows us to consider the case of water redistribution in the combined column. This initial condition is shown in Fig. 1.

For an example calculation the coefficients of water diffusivity were taken to be $D_0 = 0.12$ cm^2 min^{-1} and $\gamma = 0.71$. The comparison of analytical and numerical decaying maximum water contents ($\theta_0[t]$) at $x = 0$ cm is shown in Fig. 2. The general similarity (referred to as analytical) solution $\theta_0(t)$ is in very close agreement with the numerical values. This indicates the left boundary water content (wetted boundary at $x = 0$) can be well-described by the analytical solution. The comparison of soil water profiles obtained from general similarity theory and the numerical solution is shown in Fig. 3. It can be seen that at the times indicated the water content profiles obtained by general similarity theory and by the numerical solution are almost the same. The area under each curve is also very similar (i.e., both analytical and numerical solutions behave well in accordance with mass conservation by maintaining the initial amount of water [0.63 cm] during the redistribution process).

Figure 4 illustrates the global error in water content, described as the difference between the numerical and similarity solutions. The error is small and decreases with time. This indicates the general similarity solution has the ability to predict water redistribution not only for short times (not less than 20 min in this example), but especially for long times. Long-time prediction of water redistribution by numerical solution can require substantial computing time. The general similarity solution overcomes this limitation of numerical solutions. The maximum error for the time concerned is within 0.003. Maximum error for a specific time happens at the wetting front. The wetting front zone of redistribution is the most difficult part to predict.

Conclusions

A general similarity solution for redistribution of soil water with certain restrictive boundary conditions, but much more general initial-condition and soil-flow properties (D_0 and γ can be chosen arbitrarily), has been presented. The general similarity solution is closed form and explicit. The similarity solution of water redistribution for a step function initial condition compares well with the corresponding numerical solution. Not only can the similarity solution be used to predict soil water distribution, but Shao and Horton (1996) have shown that the similarity solution itself provides a method of estimating soil water diffusivity. This general similarity solution is useful for checking numerical procedures and can be used to analyze the physical experiment of soil water redistribution.

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