

## Plan of the lectures & motivation

1. CS theory: basic ingredients & non-perturbative computations.
  2.  $1/N$  expansion in CS.
  3. CS as an open topological string
  4. CS as a closed topological string & the manifold transition
  5. More general geometries
- 

### 1. CS theory

$M$  3-manifold,

$$S = \frac{k}{4\pi} \int_M \text{Tr} (A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

$A$   $U(N)$  or  $SU(N)$   
connection

\*  $k$  integer,  $M = \mathbb{S}^3$  most of the time

(other groups  $G$   
possible)

\* first remark: this action is topological

→ TFT

\* warning: this is just at the classical level  
it may happen that at the quantum level  
the topological symmetry is anomalous  
will see an example later

(gauge-invariant)

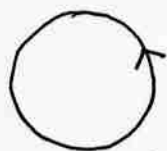
operators: we want to maintain metric independence  
so things like  $\text{Tr}(F^2)$  are not "allowed".

we will ... then use WILSON LOOP operators

$W_R^k(A)$  labeled by  $R$  irrep of  $G$   
 $k$  oriented path in  $M$

notice that classification of oriented paths in  $M = \mathbb{S}^3$  is highly  
nontrivial! this is the subject of KNOT THEORY

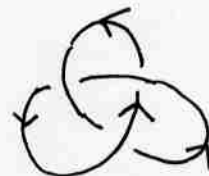
Examples:



unknot  
or trivial knot



trefoil knot



another trefoil ... the mirror  
of the above

and much more! decide for example a knot table

what we do now is to consider the holonomy of the connection  $A$  along  $K$ , called  $U_K$

$$U_K = \text{Pexp} \oint_K A \quad \text{path-ordered exponential}$$

$U_K$  is not gauge invariant, but  $\text{Tr}_R U_K$  is. This is our gauge-invariant, metric independent operator

So we got action + operators.

Go to the quantum theory:

$$Z(M) = \int DA e^{iS(A)}$$

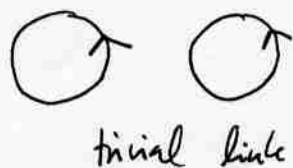
partition function  
→ topological invariant  
of  $M$

$$\langle W_{R_1}^{R_1} \dots W_{R_L}^{R_L} \rangle = W_{R_1 \dots R_L}(L) = \frac{1}{Z(M)} \int DA W_{R_1}^{R_1} \dots W_{R_L}^{R_L} e^{iS(A)}$$

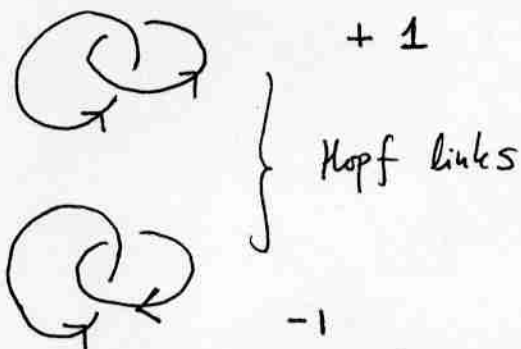
I denote by  $L$  the set of knots  $K_1, \dots, K_L$

this is in general a link of  $L$  components

it can be trivial:

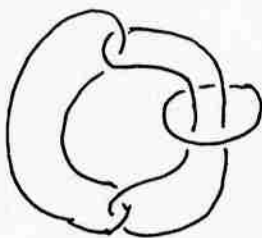


but it can be "linked"



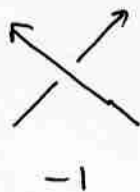
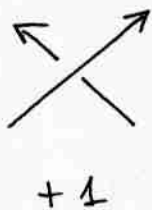
So we also have the problem of classifying links with a given number of components. Notice that out of trivial knots we can get nontrivial links, as shown above.

Another example: the Borromean link! (from the Borromeo family, Italy)

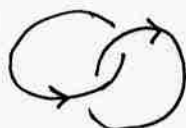


How do we distinguish the above Hopf links? The simplest link invariant is the linking number

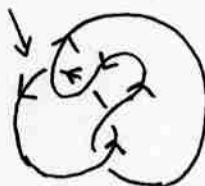
$$lk(K_1, K_2) = \frac{1}{2} \sum_{\substack{\text{crossings} \\ p}} \epsilon(p)$$



notice the topological invariance:



here they cancel



back to the quantum theory:

\* perturbation theory:

1 find the "classical solutions to the EOMs of motion

in CS these are  $F=0$  flat connections on  $M$

$S^3$   $A=0$  trivial flat connections

in general maps  $\pi_1(M) \rightarrow G$

so for example

$S^3/\mathbb{Z}_p = L(p,1)$  lens space

has  $\mathbb{Z}_p \rightarrow U(N)$  nontrivial flat connect

2 expand around a flat connection and compute Feynman diagrams.

a) labels flat connection

$Z^{(c)}(M)$   
a series in  $1/k$

$Z = \sum_{(c)} Z^{(c)}(M)$

series in  $1/k$

$$Z^{(c)}(M) = Z_{1\text{-loop}}^{(c)}(M) e^{\sum_{l=1}^{\infty} S_l^{(c)} x^l}$$

$$x = \frac{2\pi i}{k+N} \quad \text{effective parameter}$$

$$l = \# \text{ number of vertices} / 2 = \# \text{ loops} - 1$$

$$\sum_{l=1}^{\infty} S_l^{(c)} x^l$$

sum over connected diagrams (since they are exponential)  
"vacuum bubbles"

$$l=1 \quad \text{---} \bigcirc \text{---}$$

$$l=2 \quad \text{---} \bigcirc \text{---} \quad \dots$$

$$S_l^{(c)} = \text{Feynman integral} \times \text{group theory factor}$$

$$Z_{1\text{-loop}}^{(c)} = \text{trivial} \propto \frac{1}{\text{vol}(G)}$$

This is a very interesting approach, but we want to do things in a "non-perturbative" way.

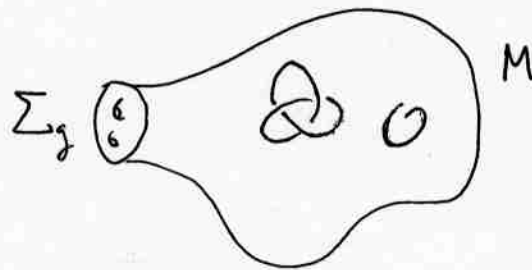
Warning: in CS theory there are no, strictly speaking, nonperturbative effects like instantons, so

by non-pert we mean essentially a way to sum up the full perturbative series

# canonical quantization:

a general fact in QFT

take  $M$  a manifold whose boundary is  $\partial M = \Sigma$



in our case

$\Sigma_g =$  Riemann surface

wavefunctionals

$$\Psi_{M, \mathcal{O}}(\Lambda) = \int_{A|_{\Sigma} = \Lambda} DA e^{iS(A)} \mathcal{O} = \langle \Lambda | \Psi_{M, \mathcal{O}} \rangle$$

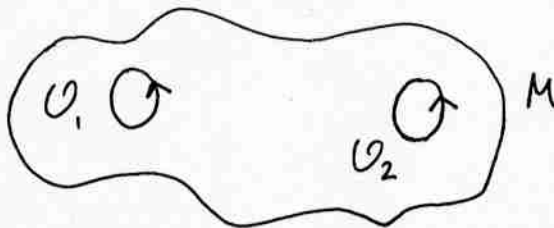
$\mathcal{O}$  local or nonlocal, in our case these are Wilson loop

these wavefunctions live in a Hilbert space  $\mathcal{H}(\Sigma_g)$

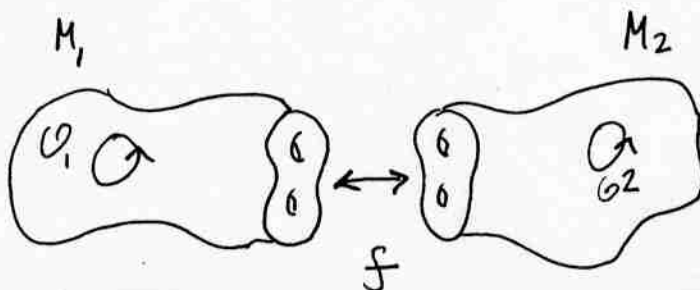
There is also a beautiful relation between inner products and path integrals:

$$\mathcal{Z}(M, \mathcal{O}_1, \mathcal{O}_2) =$$

$$\int DA e^{iS(A)} \mathcal{O}_1 \mathcal{O}_2$$



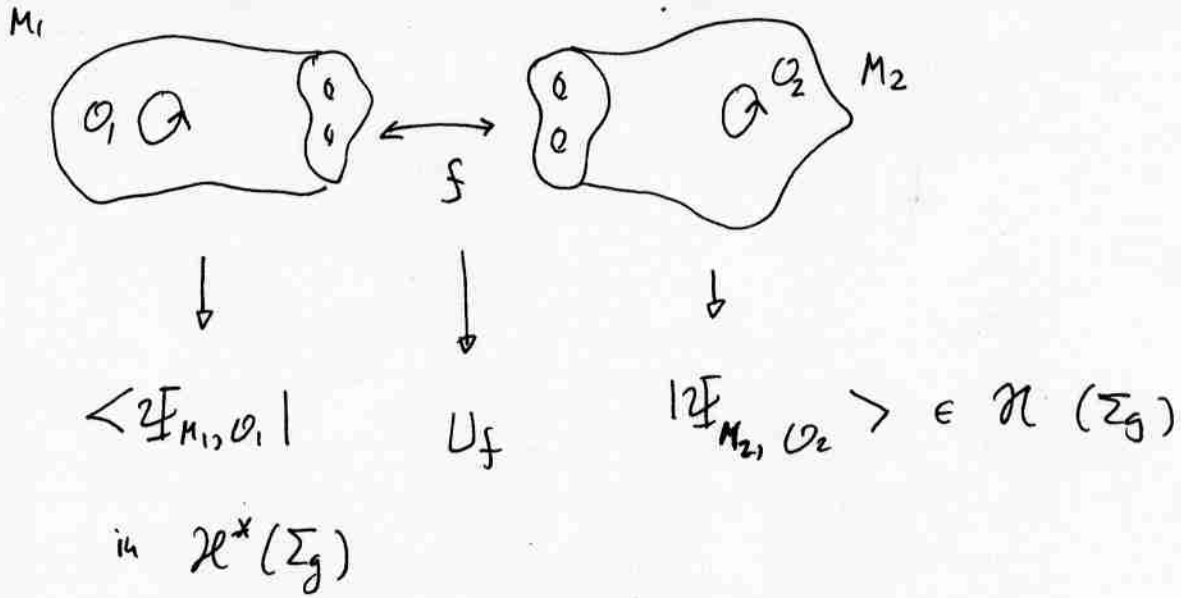
↓ cut it



$$\partial M_2 = \Sigma_g$$

$$\partial M_1 = \overline{\Sigma_g} \quad (\text{with opposite orientation})$$

$$f: \Sigma_g \rightarrow \Sigma_g \quad \text{"gluing instructions"}$$



$$U_f: \mathcal{H}(\Sigma_g) \rightarrow \mathcal{H}(\Sigma_g)$$

$$\text{so } \boxed{Z(M, U_1, U_2) = \langle \Psi_{M_1, U_1} | U_f | \Psi_{M_2, U_2} \rangle}$$

Great result of Witten: determining  $\mathcal{H}(\Sigma_g)$  for CS theory!

fact:  $\mathcal{H}(\Sigma_g) =$  space of conformal blocks of  
WZW on  $\Sigma_g$  with gauge group  $G$   
at level  $k$ .

if you don't know  
this, don't worry!

being more concrete:

$$g=0 \quad \mathcal{H}(\mathbb{S}^2) = \text{one single element}$$

$$g=1 \quad \mathcal{H}(\mathbb{T}^2) = \text{integrable reps of affine lie algebra } \hat{G} \text{ at level } k.$$

labeled by reps  $R$  of  $G$  together with some constraints

Example: for  $SU(2)$  at level  $k$ ,

we have  $j = 0, 1, \dots, k$

this truncates  
the admissible  
reps

we will not worry about this since we are going to work  
with CS  $U(N)$  or  $SU(N)$  with  $k, N$  "very large".

In practical terms

$$\mathcal{H}(\pi^2) = \{ |R\rangle \}$$

$R$  irrep of  $U(N)$

(plus some  
constraint)

$R=0$  included

orthonormal

$$\langle R' | R \rangle = \delta_{RR'}$$

$R$ : Young tableau  
highest weight  
 $\Lambda$

another ingredient we will need are a special class of  
gluing functions

$$f: \pi^2 \rightarrow \pi^2$$

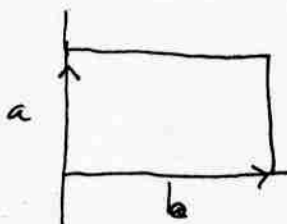
given by  $SL(2, \mathbb{Z})$  transformations

$SL(2, \mathbb{Z})$  generated by  $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

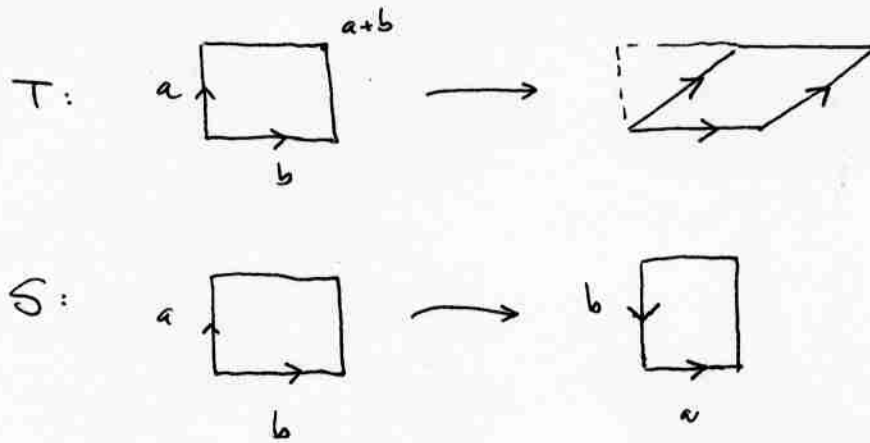
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix}$$

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

where we regard them as matrices acting on  
the one-cycles of a torus



$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ b \end{pmatrix}$$



we know how these homeomorphisms "lift" to  $\mathcal{H}(\pi^2)$ :

$$T^* U_T |R\rangle = e^{2\pi i(h_R - c/24)} |R\rangle$$

$$S^* U_S |R\rangle = \sum_{R'} S_{R'R} |R'\rangle$$

What is  $|R\rangle$  from the point of view of wavefunctions that we used before?

$$|Z_{M,0}\rangle = |R\rangle \quad \text{for some } M$$

$\partial M = \pi^2$   
and some  $\mathcal{O}$

answer: take a solid torus

$$M = \text{torus} = D \times S^1$$

only one noncontractible cycle

$$\mathcal{O} = \text{Tr}_R U_R$$

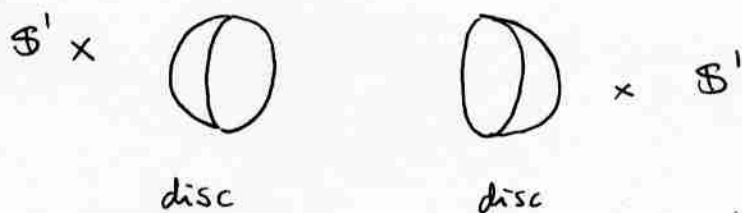
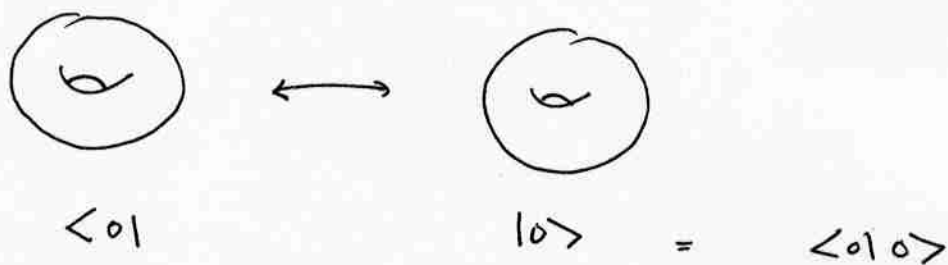
$$|R\rangle = |Z_{D \times S^1, \text{Tr}_R U_R}\rangle$$

no insertion

$$\text{torus} \rightarrow |0\rangle \quad \text{"vacuum"}$$

applications of all this:

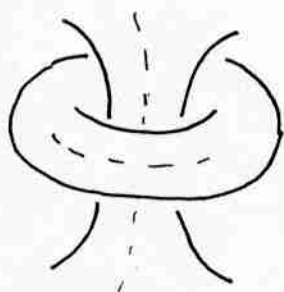
- 1) Consider two solid tori and glue them with  $f=1$  identity  
(with no insertions)



$$= S^1 \times \text{disc} = S^1 \times S^2$$

$$\Rightarrow Z(S^1 \times S^2) = \langle 010 \rangle = 1$$

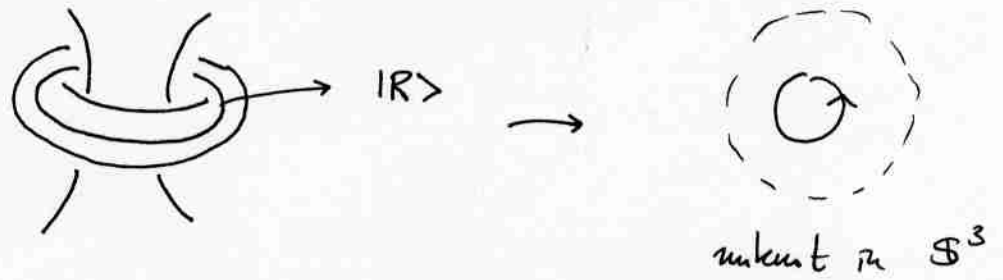
- 2) Same but  $f=S$   $\langle 01S10 \rangle = S_{00}$



$\rightarrow$  gluing them gives  $S^3$

the complement of a solid torus on  $S^3$   
is another solid torus

3)



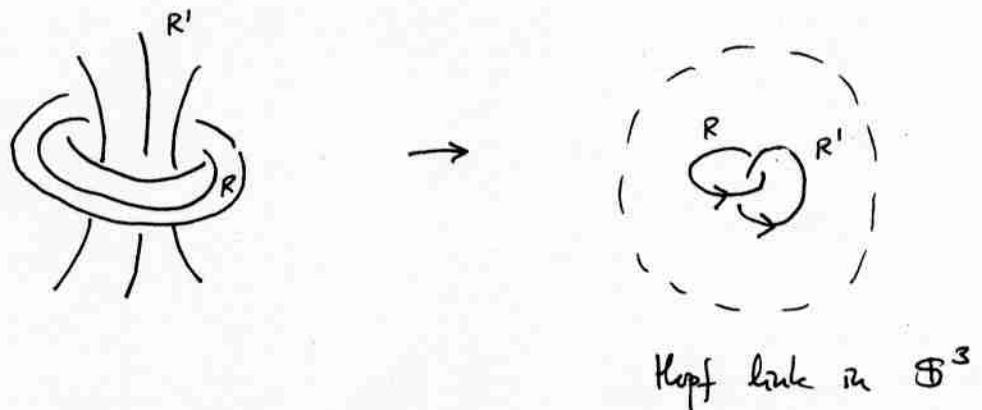
$$\langle 0 | S | R \rangle = S_{0R}$$

$$W_R(\text{unknot}) = \frac{S_{0R}}{S_{00}} = \text{quantum dimension of } R$$

$$q = e^{\frac{2\pi i}{k+N}} \quad \lambda = q^N$$

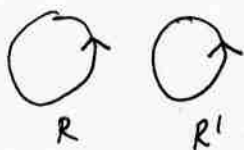
$$R = \square \quad \frac{S_{0R}}{S_{00}} = \frac{\lambda^{\frac{1}{2}} - \lambda^{-\frac{1}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$$

4)



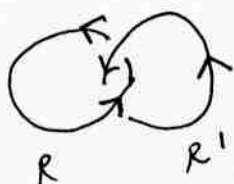
$$\langle R | S | R' \rangle = S_{RR'}$$

show that the theory distinguishes karys:



$$\frac{S_{0R}}{S_{00}} \frac{S_{0R'}}{S_{00}} \xrightarrow{R=R'=D} \left( \frac{\lambda^{1/2} - \lambda^{-1/2}}{q^{1/2} - q^{-1/2}} \right)^2$$

$$\langle W_R^{R_1} W_{R'}^{R_2} \rangle = \langle W_R^{R_1} \rangle \langle W_{R'}^{R_2} \rangle$$



$$\frac{S_{RR'}}{S_{00}} \xrightarrow{R=R'=D} \left( \frac{\lambda^{1/2} - \lambda^{-1/2}}{q^{1/2} - q^{-1/2}} \right)^2 + \lambda^{-1} - 1$$

$$\left. \begin{aligned} W_{RR'}(\text{Hopf}^{-1}) &= \frac{S_{RR'}}{S_{00}} \\ W_{RR'}(\text{Hopf}^{+1}) &= \frac{S_{RR'}^{-1}}{S_{00}} \end{aligned} \right\} \begin{array}{l} \text{are also} \\ \text{in } q^{\pm 1/2}, \lambda^{\pm 1/2} \end{array} \text{ rational functions}$$

$$T_{R_R} U_{R^{-1}} = T_{\bar{R}} U_R \quad \text{and} \quad \underline{S_{\bar{R}R'}} = S_{RR'}^{-1}$$

$\overline{R}$  complex analysis

in general:  $W_{R_1 - R_L}(L)$  is  
for  $G = SU(2)$   $R_1 = \dots = R_L = \mathbb{I}$ : Jones  
 $= SU(N)$  " : HOMFLY

So far we have ignored an important subtlety — the framing of the theory!

To understand what is forming, it is useful to look at U(1) CS, which is a Gaussian theory:

$$S = \frac{k}{4\pi} \int_M A \wedge dA$$

Since  $A \wedge A \wedge A = A_i A_j A_k \epsilon^{ijk} = 0$  if there are no gauge indices.

$$\langle e^{i \oint_{K_1} A} e^{i \oint_{K_2} A} \rangle_{\text{Abelian CS}} \quad K_1 \cup K_2$$

$$= \sum_{m=0}^{\infty} \left\langle \frac{(\oint_{K_1} A)^m}{m!} \frac{(\oint_{K_2} A)^n}{n!} \right\rangle =$$

$$= \frac{1}{2} \langle \oint_{K_1} A \oint_{K_1} A \rangle + \frac{1}{2} \langle \oint_{K_2} A \oint_{K_2} A \rangle + \langle \oint_{K_1} A \oint_{K_2} A \rangle$$

$$\langle \oint_{K_1} A \oint_{K_2} A \rangle \propto \phi(K) = \frac{1}{4\pi} \oint_{K_1} dx^\mu \oint_{K_2} dy^\nu \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3} \quad (*)$$

$$\langle \oint_{K_1} A \oint_{K_2} A \rangle \propto \frac{1}{4\pi} \oint_{K_1} dx^\mu \oint_{K_2} dy^\nu \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3} \quad (**)$$

(\*\*) is a topological invariant of the link made out of  $K_1, K_2$  : it is the linking number of  $K_1, K_2$ ,  $lk(K_1, K_2)$

but (\*) is not a topological invariant : it is the "self-linking" of  $K_1$ , also called the writhe or cotorsion of  $K_1$  :



has cotorsion -1

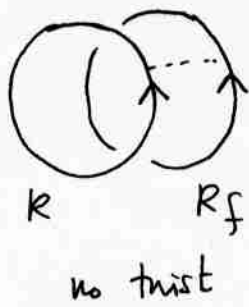


has cotorsion 0

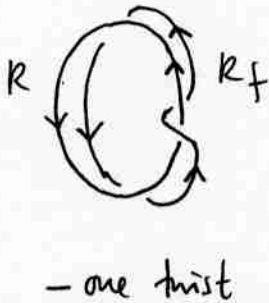
so if we deform the knot we change the writhe

This is the kind of "anomaly" we were mentioning before, and in principle ruins the topological invariance of the theory!

How do we cure that?



instead of doing the integral twice around  $K$ , take a "copy" of  $K$  which runs parallel to  $K$  but can "twist"  $p$  times around it,  $K_f$



now, instead of leaving

$\phi(K)$  we will have  $lk(K, K_f) = p$

The knot  $K$  together with its "copy"  $K_f$  is called a **FRAMED** knot. The framing is specified by an integer  $p$  which gives the number of twists.

MAIN POINT: we don't have a topological theory of knots, but we do have a " " of framed knots!

another way of understanding this from a QFT point of view is that the composite operator  $(\oint_K A)^2$  is ambiguous, and by doing the framing we are choosing a definition of this operator through some kind of point-splitting.

What saves the day is that the change of rep's under change of framing is perfectly under control.

The precise rule in the nonabelian theory is:

$$\boxed{W_R(k) = \langle W_R^k(A) \rangle \rightarrow e^{2\pi i p h_R} W_R(k)} \quad (*)$$

$p$  change of framing,  $h_R = \frac{C_R}{2(k+N)}$

notice:  
conformal weight  
of a primary  $R$   
in WZW

$C_R = \text{Tr}_R(T_a T_a)$  quadratic Casimir

NOTE: this is easy to understand in perturbation theory, since contractions now involve  $A_\mu = \sum_a A_\mu^a T_a$ ,  $T_a$  basis of Lie algebra

$$\overline{A_\mu^a A_\nu^b} \sim \delta^{ab}$$

and  $\text{Tr}_R(A_\mu A_\nu) \rightarrow \text{Tr}_R(T_a T_a)$

For knots in  $\mathbb{S}^3$ , there is something called the CANONICAL or STANDARD framing (where we declare the self-linking to be zero), and any other framing is related to this one by the rule (\*)

## 2. $1/N$ expansion & the string - gauge theory correspondence

back to the perturbative expansion of CS in  $U(N)$

$$F(M) = \log Z(M) = \log Z_{1\text{-loop}} + \sum_{l=1}^{\infty} S_l \cdot x^l$$

$S_l$  depends on  $N$

$$S_l = S_l(N) \quad \text{due to group factors}$$

$$l=1 \quad \bigcirc \rightarrow N(N^2-1)$$

$$l=2 \quad \bigoplus \rightarrow N^2(N^2-1) \quad \text{and so on}$$

note: each usual Feynman diagram has different power of  $N$  associated to it

We want to isolate the  $N$ -dependence in a more precise way. To do that, we introduce the

DOUBLE LINE notation:

$A_{ij}$  has two matrix indices (remember adjoint  $U(N)$  =  $N \times \bar{N}$ )

$$\text{oooo} \rightarrow \begin{matrix} i \\ \longrightarrow \\ j \\ \longleftarrow \end{matrix}$$

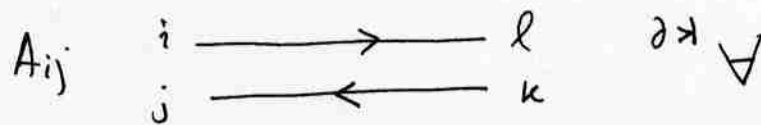
what is the index structure of the propagator?

look at the kinetic term

$$X = \frac{2\pi i}{k+N}$$

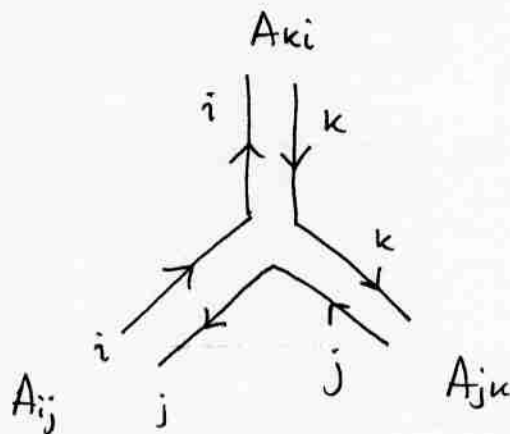
$$\begin{aligned} \frac{1}{X} \text{Tr} (A \wedge dA) &\rightarrow A_{ij} \wedge dA_{ji} \\ &= A_{ij} \wedge dA_{kl} \delta_{il} \delta_{jk} \end{aligned}$$

i.e.  $\langle A_{ij} A_{kl} \rangle \propto \delta_{il} \delta_{jk}$

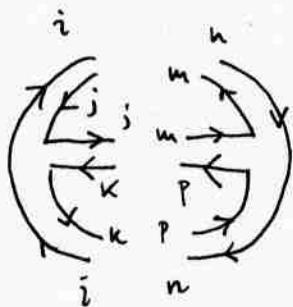


Cubic interaction:

$$\frac{1}{X} \text{Tr}(A \wedge A \wedge A) = \frac{1}{X} A_{ij} \wedge A_{jk} \wedge A_{ki}$$



result



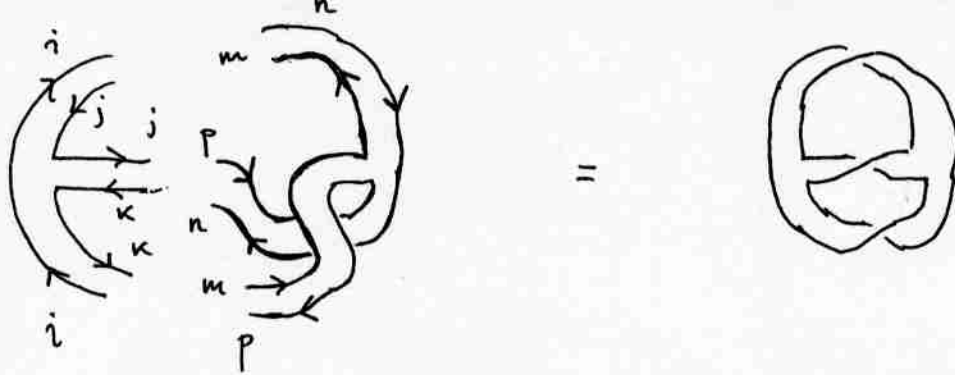
=



$$\sum_{ijklmnp} A_{ij} A_{jk} A_{kl} A_{lm} A_{mn} A_{np}$$

$$\propto \sum_{ijklmnp} \delta_{in} \delta_{jm} \delta_{kl} \delta_{lp} \delta_{pn} \delta_{ij} = N^3$$

3 free indices = 3 closed loops in the diagram



$$\sum_{ijkmp} \overbrace{A_{ij} A_{jk} A_{ki} A_{mn} A_{np} A_{pm}}$$

$$\propto \sum_{ijkmp} \delta_{in} \delta_{jm} \delta_{jp} \delta_{kn} \delta_{km} \delta_{ip} = N$$

one free index = one single closed loop in the diagram

therefore, given a diagram with  $V$  vertices,  $E$  propagators and  $h$  closed loops, we have a factor

$$N^{E-V-h}$$

we now look at the "fat graphs" as Riemann surfaces, and we have

$$2g-2 = E-V-h$$



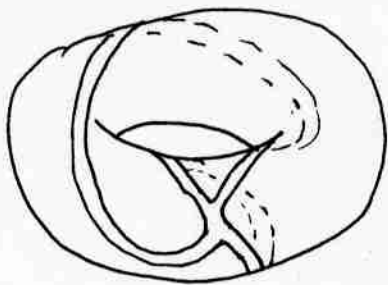
$$E=3, V=2, h=3 \rightarrow g=0$$

a sphere with three holes



$$E=3, V=2, h=1 \rightarrow g=1$$

a torus with one hole



$g=0$  are called  
PLANAR diagrams

$g \geq 1$  are called  
NON-PLANAR diagrams

$$x^{E-V} N^h = x^{2g-2+h} N^h$$

Therefore, the part of the free energy which comes from vacuum bubbles can be written as:

$$\text{vacuum bubbles} = \sum_{g=0}^{\infty} \sum_{h=1}^{\infty} \underbrace{F_{g,h}}_{\text{numerical coefficients}} x^{2g-2+h} N^h \quad (*)$$

numerical coefficients  
(they involve Feynman integrals,  
symmetry factors, etc.)

This structure is of course typical of any field theory  
with gauge group  $U(N)$  (i.e., from matrix models to QCD)

Notice that in the resulting series we have now two parameters  
( $x, N$ ). It is also useful to introduce the so-called  
t Hooft parameter:

$$t = xN$$

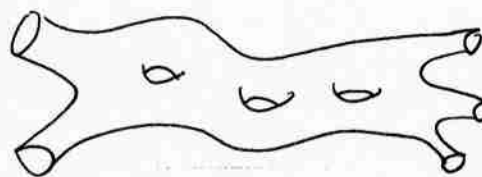
and in terms of  $t$  we have

$$\sum_{g=0}^{\infty} \sum_{h=1}^{\infty} F_{g,h} x^{2g-2} t^h \quad (**)$$

The expansion (\*) looks very much like an OPEN STRING theory expansion with  $U(N)$  Chan-Paton factors,

where

$x \rightarrow g_s$ , open string coupling constant



$$\rightarrow g_s^{2g-2+h} N^h$$

temptation: interpret  $F_{g,h}$  as an amplitude of an open string theory on a Riemann surface  $\Sigma_{g,h}$  (say, the partition function)

How do we compute  $F_{g,h}$ ?

In CS theory on  $\mathbb{S}^3$  there is an obvious way:

Since we know  $Z(\mathbb{S}^3)$  exactly, we can just take the log and expand!

I will give some indications on this:

$$\mathcal{Z} = \frac{1}{(k+N)^{N/2}} \prod_{\alpha > 0} 2 \sin \left( \frac{\pi (\alpha \cdot \rho)}{k+N} \right)$$

$$\alpha \text{ positive roots} = e_i - e_j$$

$$\rho \text{ Weyl vector} = \frac{1}{2} \sum_k (N-2k+1) e_k$$

$$= \frac{1}{(k+N)^{N/2}} \prod_{j=1}^{N-1} \left( 2 \sin \left( \frac{\pi j}{k+N} \right) \right)^{N-j}$$

$$F(\mathbb{S}^3) = -\frac{N}{2} \log(k+N) + \sum_{j=1}^{N-1} \log \left[ 2 \sin \frac{\pi j}{k+N} \right]$$

$$= F^{\text{1-loop}} + F^{\text{vacuum bubbles}}$$

$$\text{use} \quad \sin \pi z = \pi z \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2} \right)$$

$$F^{\text{1-loop}} = -\frac{N^2}{2} \log(k+N) + \frac{1}{2} N(N-1) \log 2\pi + \sum_{j=1}^{N-1} (N-j) \log j$$

$$= \log \frac{(2\pi g_s)^{\frac{1}{2} N^2}}{\text{vol}(U(N))}$$

$$g_s = x = \frac{2n}{k+N}$$

$$= \log \mathcal{Z}_{\text{1-loop}}$$

$$F^{\text{bubbles}} = \sum_{j=1}^{N-1} (N-j) \cdot \sum_{n=1}^{\infty} \log \left( 1 - \frac{j^2 g_s^2}{4n^2 n^2} \right)$$

$$= \sum_{g=0}^{\infty} \sum_{h=2}^{\infty} g_s^{2g-2+h} N^h F_{g,h}$$

$$F_{0,h} = \frac{B_{h-2}}{(h-2)h!} \quad h \geq 4$$

$$F_{1,h} = \frac{1}{12} \frac{B_h}{h \cdot h!}$$

$$F_{g,h} = 2 \frac{3(2g-2+h)}{(2g)^{2g-2+h}} \binom{2g-3+h}{h} \frac{B_{2g}}{2g(2g-2)} \quad g \geq 2$$

From open strings to closed strings:

we can rewrite (\*\*\*) as:

$$\sum_{g=0}^{\infty} \sum_{h=1}^{\infty} F_{g,h} t^h x^{2g-2} = \sum_{g=0}^{\infty} F_g(t) x^{2g-2}$$

$$F_g(t) = \sum_{h=1}^{\infty} F_{g,h} t^h \quad \text{a function of } t$$

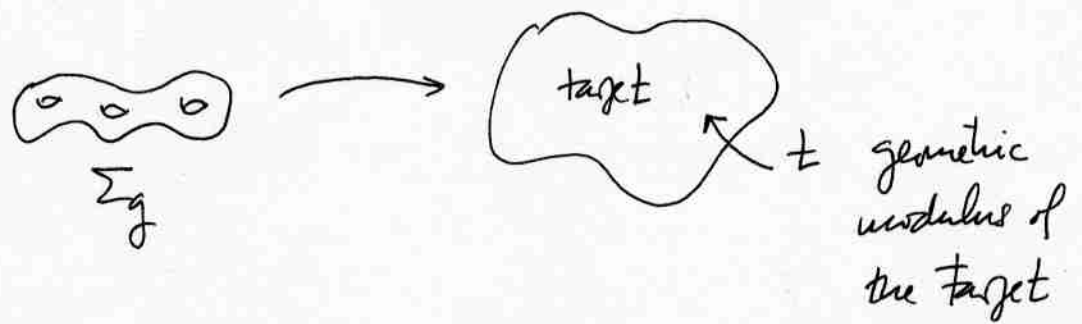
now this looks like a closed string expansion where

$x = g_s$  closed string coupling constant

$F_g(t)$  amplitude for a closed string  
at genus  $g$

$t$  should have, in this context, the interpretation as

a modulus of the target string geometry



notice that in this context we sum over all open string diagrams with a fixed genus and all possible  $\#$  of holes to get  $F_g(t)$ :

$$g=0: \quad \text{[sphere with equator]} t^2 + \text{[sphere with 2 vertical lines]} t^3 + \text{[sphere with 3 horizontal lines]} t^4 + \dots$$

$$= \text{[sphere with many lines]} \text{ sphere } \boxed{\text{planar}} \text{ piece}$$

$$g=1: \quad \text{[torus with 1 hole]} t + \text{[torus with 2 holes]} t^2 + \dots$$

$$= \text{[torus with many holes]} \text{ torus and so on}$$

so by summing over the holes we fill up or close the holes.

$F_0(t)$  comes from planar diagrams

$$x = t/N$$

notice that if we write

$$\begin{aligned}
 & \sum_{g=0}^{\infty} x^{2g-2} F_g(t) \\
 &= \sum_{g=0}^{\infty} N^{2-2g} \left[ t^{2g-2} F_g(t) \right]
 \end{aligned}$$

we have that at fixed  $t$ , large  $N$ ,

$F_0(t)$  is the "leading" piece (goes like  $N^2$ )

$F_g(t)$  are subleading (go like  $O(N^0)$ )

### 3. CS theory as an open string theory Witten hep-th/9207094

We want now to give an interpretation of CS theory  
(realization)  
as an open string theory.

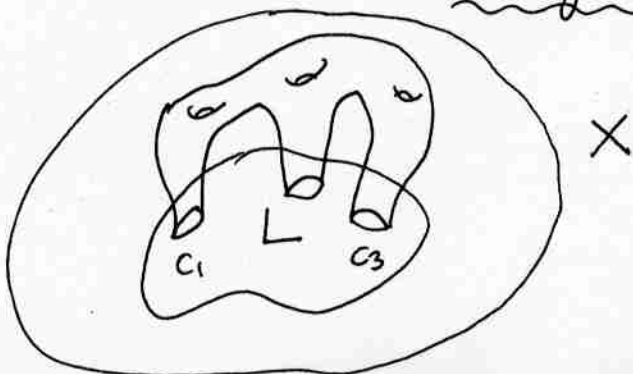
Since CS is a TFT, it seems reasonable to look for an  
open topological string theory. The answer (Witten, 1992) is that  
CS on  $M$  can be realized as a topological string theory on  $T^*M$ .

#### \* open topological strings

remind that a closed topological string is a theory  
of holomorphic maps  $f: \Sigma_g \rightarrow X$   $X \subset 4$  3-fold

open:  $f: \Sigma_{g,h} \rightarrow X$

but we need boundary conditions:  $\begin{cases} \bigcup_i C_i = \partial \Sigma \\ f(C_i) \subset L \end{cases}$

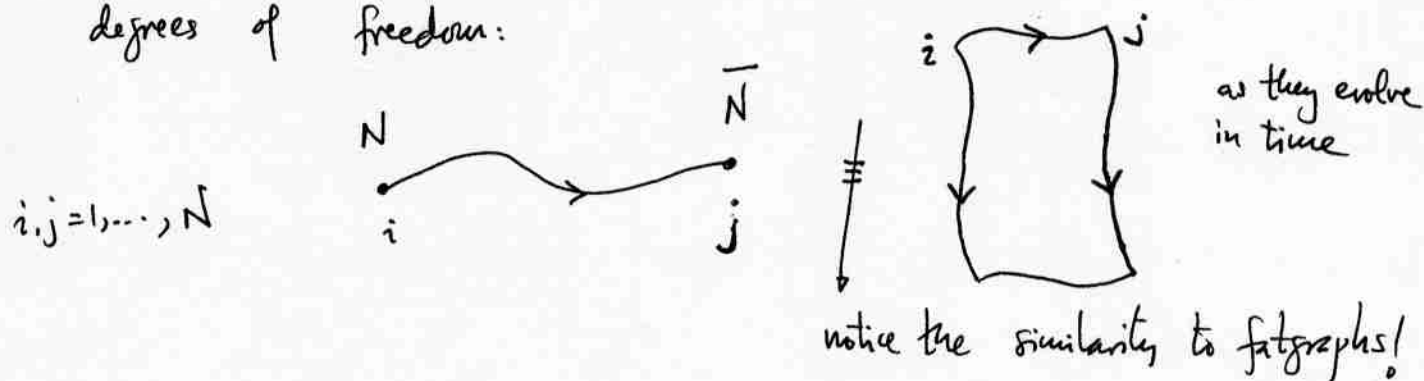


$L$  Lagrangian submanifold  
of  $X$

$$J|_L = 0$$

NOTE: This is like in the analysis of b.c. for  $N=2$  SCFT  
(Ooguri, Oz, Yin), but since we do not worry about spacetime  
SUSY we just get Lagrangian.

We also want to add Chan-Paton degrees of freedom to the  
open strings, so that we get in the spectrum nonabelian  
degrees of freedom:



Reminder: in the usual open string, this gives nonabelian gauge fields

$$\alpha_{-1}^{\mu} |0\rangle \longleftrightarrow A^{\mu}$$

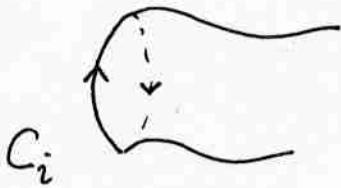
$$\alpha_{-1}^{\mu} |i; j\rangle \longleftrightarrow A_{ij}^{\mu} \quad U(N) \text{ gauge field}$$

Having open strings ending on a Lagrangian submanifold  $L$   
with  $U(N)$  Chan-Paton degrees of freedom is what is usually  
called having open strings in the presence of  $N$  D-branes  
wrapping  $L \subset X$ . The D-branes are called topological in the  
context of topological strings, since we do not require them to be  
spacetime SUSY and they are just Lagrangian.

One thing we will need: coupling of Chan-Paton to background fields

$U(N)$   
A gauge connection on  $L$

$$\rightarrow e^{-L_{\text{top string}}} \prod_{i=1}^h \text{Tr} P_{\text{exp}} \oint_{C_i} f^*(A)$$



\*  $T^*M$

$M$  3-manifold,  $T^*M$

symplectic  $J = \sum_{a=1}^3 dp_a \wedge dq_a$   $q_a$  local coords  $M$   
 $p_a$  " cotangent  
 $\downarrow$   
 find a complex structure s.t.  $J$  is a Kähler form  
 it is in fact a CY  
 non-compact one!

see this more clearly with our main interest case:  $T^*\mathbb{S}^3$

which is nothing but

$$\sum_{\mu=1}^4 \eta_{\mu}^2 = a$$

$$\eta_{\mu} = x_{\mu} + i v_{\mu}$$

$$\sum_{\mu=1}^4 (x_{\mu}^2 - v_{\mu}^2) = a$$

$$\sum_{\mu=1}^4 x_{\mu} v_{\mu} = 0$$

$\mathbb{S}^3$   
 $x_{\mu}$  sphere coords

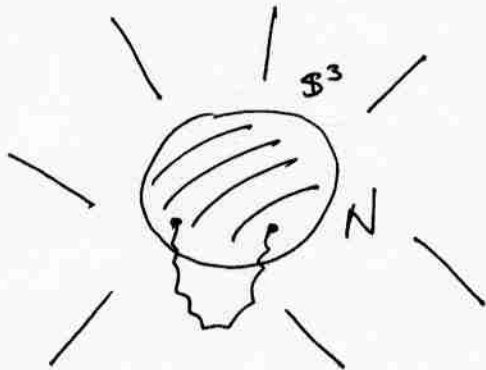
$v_{\mu}$  bundle coordinates

$\delta \quad T^*\mathbb{S}^3 = \text{DEFORMED CONIFOLD}$

Finally:  $M \subset T^*M$  is <sup>obviously</sup> Lagrangian since  $J|_M = 0$

Therefore we can consider open topological strings with U(1) Chan-Paton factors on  $T^*M$  and b.c. specified by  $M$

equivalently,  $N$  top D-branes wrapping  $M$  inside  $T^*M$



We want to argue that the theory of open strings here reduces to CS theory. Equivalently, we want to argue that the <sup>analog of</sup> BI Lagrangian for  $N$  D-branes on  $M$  reduces to CS theory on  $M$ .

Shortcut: use open string field theory for the usual bosonic string! Why? Because the algebraic structure of bosonic strings and topological strings is quite similar.

The basic ingredient is the existence of a BRST operator  $Q_{BRST}$  with  $Q_{BRST}^2 = 0$  and

$$T_{\mu\nu} = \{Q_{BRST}, b_{\mu\nu}\}$$

Witten showed for the bosonic string that the spacetime action is

just:

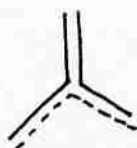
$$S = \frac{1}{g_s} \text{Tr} \int \frac{1}{2} \Psi * \mathcal{Q}_{\text{BRST}} \Psi + \frac{1}{3} \Psi * \Psi * \Psi$$

which describes the spacetime dynamics of  $N$  D-25 branes.

Here,

$$\begin{aligned} \Psi = \text{string field} &= T |p\rangle + A_\mu \alpha_{-1}^\mu |p\rangle + \dots \\ &= \text{functional of spacetime position} \\ &\quad \text{of midpoint of the string} \\ &\quad + \text{all } \underline{\text{oscillator}} \text{ modes} \end{aligned}$$

$*$  star product

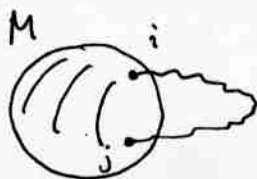


$$\int : \quad \text{---} \quad \text{integration of string functionals}$$

Since open topological strings have the same structure, we should work out the meaning of  $\Psi$ ,  $*$ ,  $\int$  for them, and then the spacetime action should be the same as above.

First question: what is the string field for open top strings on  $T^*M$ ?

it turns out that here strings reduce to point particles!



stretched strings decouple from the spectrum in the same way that in computing  $\text{Tr} (-1)^F e^{-\beta H}$  only states with  $H \sim 0$  are not suppressed as  $\beta \rightarrow \infty$

so we have



and oscillator modes (which are the responsible for stretching the string) shouldn't be included

⇒ downsizing of  $\mathcal{Z}$ !

$\mathcal{Z}$  is just a functional of a point in  $M$

more precisely,  $\mathcal{Z} = A_{\mu}^{ij} dx^{\mu}$  ij Chan-Paton  
a  $U(N)$  connection

NOTE for cosmocondi: g.h.  $\mathcal{Z} = 1$  = degree of the form  
and this is the usual restriction on g.h.  
for string fields.

$\star \rightarrow \wedge$  wedge product (since Witten's star product reduces to the usual product in the limits of point-particle)

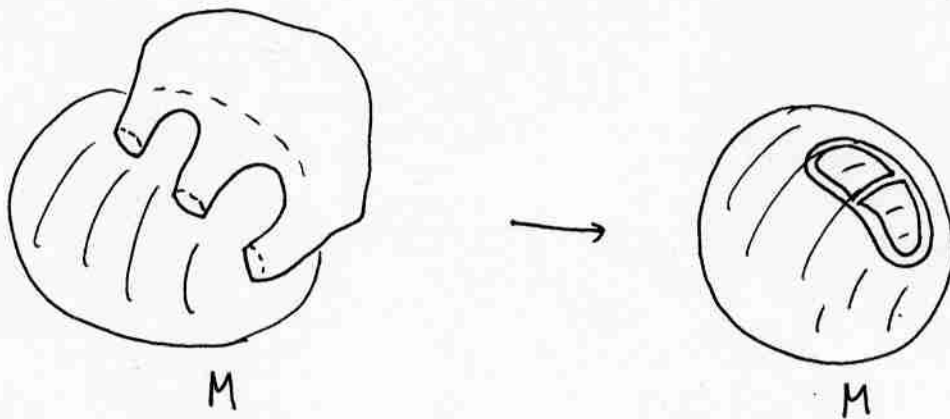
$$\int \rightarrow \int_M$$

$$S \rightarrow \frac{1}{2g_s} \int_M (A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

$$g_s = \frac{2\pi}{k+N}$$

But you must think I'm crazy, since after all a theory of open topological strings is a theory of worldsheet instantons. What are the worldsheet instantons here?

It turns out that, in the limit of point-particles, the worldsheet instantons reduce to fatgraphs of CS theory!



this is a  
Kuranishi degeneration  
limit of open  
string theory...

#### 4. CS as a closed string theory and the conifold transition

We want to know if there is a closed string description in the sense we explained before, i.e.

$$F = \sum_{g=0}^{\infty} \sum_{h=1}^{\infty} F_{g,h} x^{2g-2+h} N^h$$

$$= \sum_{g=0}^{\infty} F_g(t) x^{2g-2}$$

$$x = \frac{2n}{k+n} = g_s$$

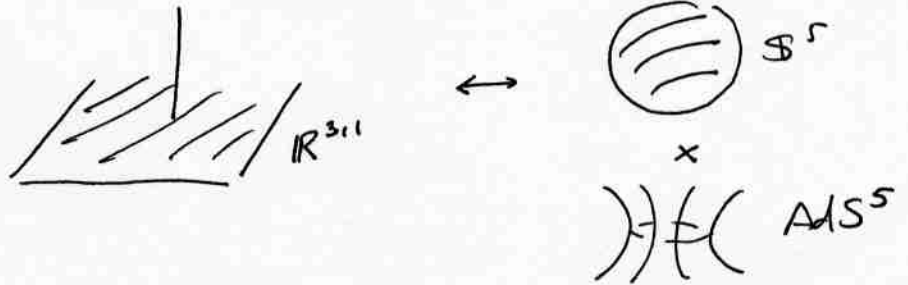
$$F_g(t) = \sum_{h=1}^{\infty} F_{g,h} t^h$$

$$t = Nx$$

for  $M = \mathbb{S}^3$

hint:  $AdS/CFT \quad IIB/R^{9,1} = IIB/AdS_5 \times S^5$

$N$  D-branes on  $R^{3,1}$  no D-branes



take the  $N$ -branes for their effect on the geometry

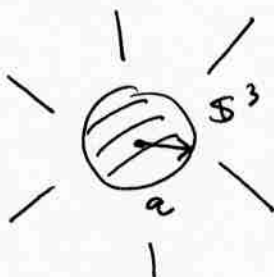
so we should expect

$$\begin{array}{l} T^*S^3 \\ N \text{ D-branes} \\ \text{on } S^3 \end{array} \longleftrightarrow \begin{array}{l} \times \\ \text{a different} \\ \text{background} \\ \text{geometry} \\ \text{with } \underline{\text{no}} \text{ D-bran} \end{array}$$

$\Rightarrow$  GEOMETRIC TRANSITION  
of the background geometry

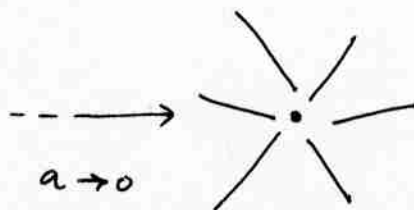
but for  $T^*S^3 =$  deformed conifold there is a well-known transition of the geometry: the CONIFOLD transition.

brief description:



$a$  parameter

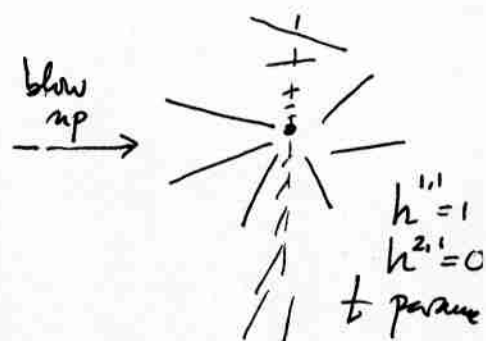
$$h^{2,1}=1, h^{1,1}=0$$



conifold

a singular CY threefold

$$\sum_{\mu=1}^4 \eta_{\mu}^2 = 0$$



$$\begin{array}{l} h^{1,1}=1 \\ h^{2,1}=0 \end{array}$$

$t$  parameter

$$O(-1) \oplus O(-1) \rightarrow \mathbb{P}^1$$

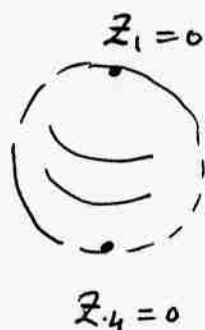
resolved conifold

algebraic description:

$$xy = uv + a \quad \text{def. manifold}$$

$$xy = uv \quad \text{manifold}$$

$$x = \lambda u, \quad u = \lambda y \quad \lambda \text{ inhomogeneous coordinate on } \mathbb{P}^1$$



$$\lambda = \frac{z_1}{z_4}$$

$$(\lambda, x, u) \in \mathbb{C} \times \mathbb{C}^2$$

$$\updownarrow$$

$$(\lambda^{-1}, v, y) \in \mathbb{C} \times \mathbb{C}^2$$



gluing this gives  $\mathcal{O}(-1)^2 \rightarrow \mathbb{P}^1$

\* another algebraic description of  $\mathcal{O}(-1)^2 \rightarrow \mathbb{P}^1$ :

check it with  
Aspinwall's  
lectures

consider the coordinates  $z_1, z_2, z_3, z_4 \in \mathbb{C}$

and the  $U(1)$  action

$$z_1, z_2, z_3, z_4 \rightarrow e^{i\theta} z_1, e^{-i\theta} z_2, e^{-i\theta} z_3, e^{i\theta} z_4$$

Define  $\mathcal{M} = \{ |z_1|^2 + |z_4|^2 - |z_3|^2 - |z_2|^2 = t \} / U(1)$

$$= \mathcal{O}(-1)^2 \rightarrow \mathbb{P}^1$$

notice  $z_2 = z_3 = 0$

$$\mathcal{M} = \{ |z_1|^2 + |z_4|^2 = t \} / U(1)$$

$$= \mathbb{P}^1 \text{ with radius } t^{1/2}$$

and area  $\propto t$

$z_2, z_3$  coordinates for the fibers

$$x = z_1 z_3, \quad y = x_2 x_4, \quad u = x_1 x_2, \quad v = x_3 x_4 \longrightarrow \lambda = x/v = u/y = x_1/x_4$$

# pictorial description:

rewrite  $xy = uv + a$  as follows:

\* we introduce  $z = uv$ , so we write  $uv = z$   
 $xy = a + z$

\* notice the  $\pi^2 = U(1)^2$  symmetry of this last equation  
as:

$$x, y \rightarrow x e^{i\theta_a}, y e^{-i\theta_a} \quad U(1)_a \times U(1)_b$$

$$u, v \rightarrow u e^{i\theta_b}, v e^{-i\theta_b}$$

we have then a fibration

$$\begin{array}{ccc} \pi^2 \times \mathbb{R} & \rightarrow & \mathcal{M}_{\text{conf}}^{\text{def}} \\ & & \downarrow \\ & & \mathbb{R}^3 \end{array}$$

if we fix  $z, \theta_a, \theta_b$ ,

$$u|v| = |z|$$

$$\theta_u + \theta_v = \theta_z$$

$$|x||y| = |z+a|$$

$$\theta_x + \theta_y = \theta_{z+a}$$

with  $\pi^2 \times \mathbb{R}$  corresponding to  
 $(\theta_a, \theta_b, \text{Im } z)$

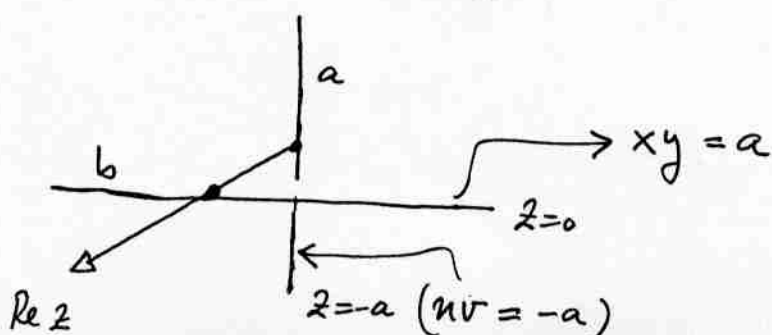
and  $\mathbb{R}^3$  to

$(\text{Re } z, |x|, |u|)$

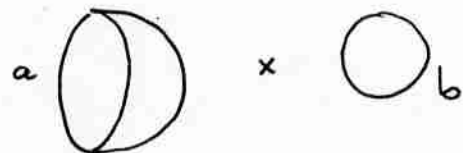
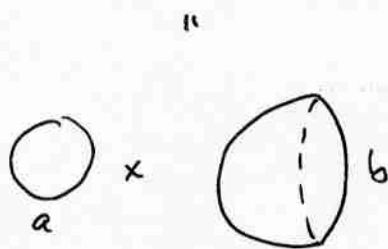
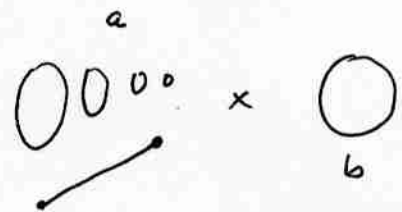
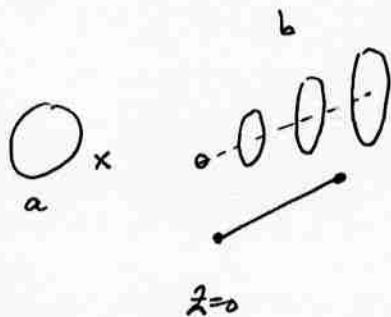
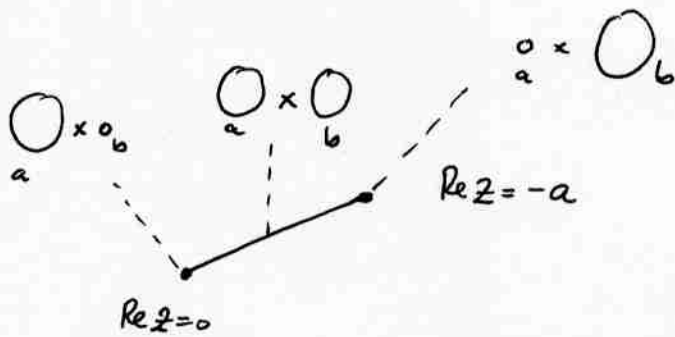
fixed points of  $\pi^2$  action:

•  $x = y = 0$   $U(1)_a$  degenerates at  $z = -a$

•  $u = v = 0$   $U(1)_b$  " at  $z = 0$



where is the 3-sphere here?



$S_a' \times D_b$

solid torus

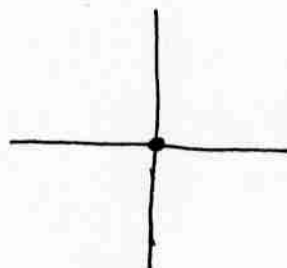


$D_a \times S_b'$

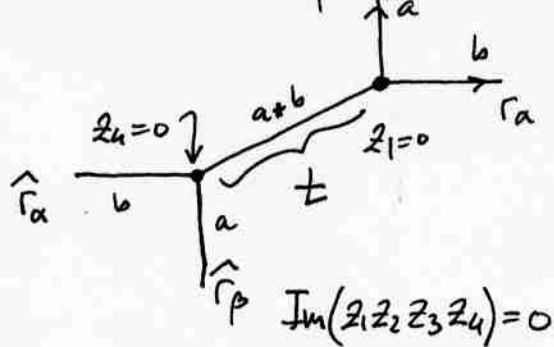
solid torus

this is our  $S^3$ !

singular manifold



resolved conifold



$$r_\alpha = |z_2|^2 - |z_1|^2$$

$$r_\beta = |z_3|^2 - |z_1|^2$$

$$\hat{r}_\alpha = |z_4|^2 - |z_2|^2 - t$$

$$\hat{r}_\beta = |z_4|^2 - |z_3|^2 - t$$

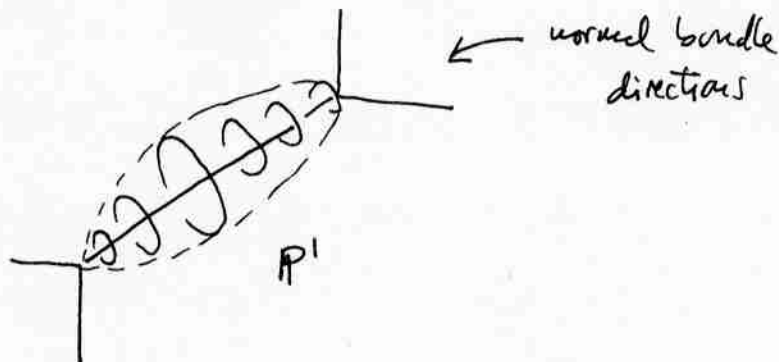
$$\Pi^2 = U(1)_a \times U(1)_b$$

$$z_1 \rightarrow e^{i\alpha} z_1$$

$$z_3 \rightarrow e^{-i(\alpha+\beta)} z_3$$

$$z_2 \rightarrow e^{i\beta} z_2$$

notice that on the diagonal there is a nontrivial cycle  $S^1$  (a-b) which degenerates at the endpoints:



Summarizing: there is a natural geometry, the resolved conifold, which comes from the deformed conifold through geometric transition

# CONJECTURE (Gopakumar - Vafa)

$CS/\mathbb{S}^3$  = open top string on  $T^*\mathbb{S}^3$   $\neq$  closed strings on the resolved manifold.

in the sense of 't Hooft

check: we have to sum over holes

remember  $F = \log Z(\mathbb{S}^3) = F^{np} + F^p$

$$\log \frac{(2\pi g_s)^{\frac{1}{2}N^2}}{\text{vol}(U(N))} + \underbrace{\sum_{g,h} F_{g,h} t^h x^{2g}}_{\text{do the sum here first}}$$

$$F_g^{\text{pert}}(t) = \sum_{h=1}^{\infty} F_{g,h} t^h \quad \text{where} \quad F_{g,h} = \frac{23(2g-2+h)}{(2\pi)^{2g-2+h}} \frac{B_{2g}}{2g(2g-1)} \binom{2g-3+h}{h} \quad g \geq 2$$

hints for this sum:

\* write  $3(2g-2+h) = \sum_{n=1}^{\infty} \frac{1}{n^{2g-2+h}}$  and  $\underline{=0}$  for  $\underline{h \text{ odd}}$

so that the sum for  $h$  reduces to

$$\sum_{h \text{ even}} \frac{1}{(2\pi n)^{2g-2+h}} \binom{2g-3+h}{h} t^h = \frac{1}{(2\pi n)^{2g-2}} \left\{ \frac{1}{2\pi n} \left( \frac{1-t}{1+t/2\pi n} \right)^{2g-2} + \frac{1}{(1+t/2\pi n)^{2g-2}} \right\}$$

where we used the binomial sum

$$\frac{1}{(1-x)^g} = \sum_{n=0}^{\infty} \binom{g+n-1}{n}$$

so we obtain

$$F_g^{\text{pert}}(t) = \frac{(-1)^g B_{2g} B_{2g-2}}{2g(2g-2)(2g-2)!} +$$

$$+ \frac{B_{2g}}{2g(2g-2)} \underbrace{\sum_{n \in \mathbb{Z}} \frac{1}{(t+2\pi n)^{2g-2}}}_{\text{except for } n=0} \quad \checkmark$$

$$F_g^{\text{np}}(t) = \frac{B_{2g}}{2g(2g-2)} \frac{1}{t^{2g-2}} \quad g \geq 2$$

Since  $F^{\text{np}} = \log \frac{(2\pi g_s)^{\frac{1}{2}} N^2}{\text{vol}(U(N))} =$

$$= \frac{N^2}{2} \left( \log(Ng_s) - \frac{3}{2} \right) - \frac{1}{2} \log N + 3'(-1)$$

$$+ \underbrace{\sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g-2)} N^{2-2g}}_{\sum_{g=2}^{\infty} F_g^{\text{np}}(t) \times t^{2g-2}}$$

$$F = \sum_{g=0}^{\infty} F_g(t) g_s^{2g-2} = \sum_{d=1}^{\infty} \frac{e^{-d \text{tKähler}}}{\left(2 \sin \frac{dg_s}{2}\right)^2}$$

$\Rightarrow$  one single Gopakumar-Vafa  
(BPS) invariant

$$Z(\mathbb{B}^3; N, g_s) = \int dM e^{-\frac{1}{g_s} (\log M)^2}$$

$\star$  comment on CS  
matrix model  
and summing over  $h$   
 $V(M) = \frac{1}{g_s} (\log M)^2$

and 
$$\frac{B_{2g}}{2g(2g-2)} \sum_{n \in \mathbb{Z}} \frac{1}{(t+2\pi n)^{2g-2}} = \frac{|B_{2g}|}{2g(2g-2)} \text{Li}_{3-2g}(e^{-it})$$

$= F_g(t)$  for the resolved manifold!

$-it \equiv t_{\text{kähler}}^{\text{op}}$

of course, for  $F_0(t) \sim \frac{t^2}{2} (\log t - 3/2) \sim VY$  for small  $t$

Notice that we have resummed a series expanded around

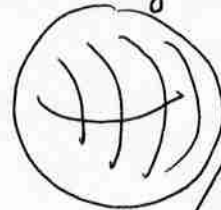
$t=0$  to obtain a series around  $\boxed{e^{-t} \rightarrow 0 \atop t \rightarrow \infty}$  large volume limit

CS perturbation theory

large volume limit  
 $\sim$  GW theory

$t=0$   $\oplus$

$t=\infty$



So we checked the conjecture at the level of partition function / free energy

## 5. Generalizations

### 5.1. Knots in $\mathbb{S}^3$

i.e. what are the implications of this for Wilson loop operators?

$\mathcal{R} \rightarrow W_R^{\mathcal{R}}(A)$  operator

gives an open string sector in the closed theory?



in order to specify a b.c. in top strings we need a Lagrangian submanifold. So we obtain a geometric map:

$$\underset{\mathbb{S}^3}{\overset{\wedge}{\mathbb{R}}} \longrightarrow \mathcal{L}_K \subset \mathcal{O}(-1)^2 \longrightarrow \mathbb{P}^1$$

Ooguri-Vafa

→ <sup>very</sup> important implications  
for knot invariants!

5.2. The above is an example of string theory  $\leftrightarrow$  gauge theory duality.

Now, on the string theory side we don't learn much — a single background  $\mathcal{O}(-1)^2 \rightarrow \mathbb{P}^1$ , with rather trivial invariant.

Can we understand topological strings on other CY's  $X$ ?  
using CS

two approaches:

$$1) \quad CS/M \longleftrightarrow \begin{array}{c} \text{open top string} \\ \text{on } T^*M \end{array} \longleftrightarrow \begin{array}{c} \text{closed strings} \\ \text{on } X_M \end{array}$$

$$X_{\mathbb{S}^3} = \mathcal{O}(-1)^2 \rightarrow \mathbb{P}^1 \quad \text{but also}$$

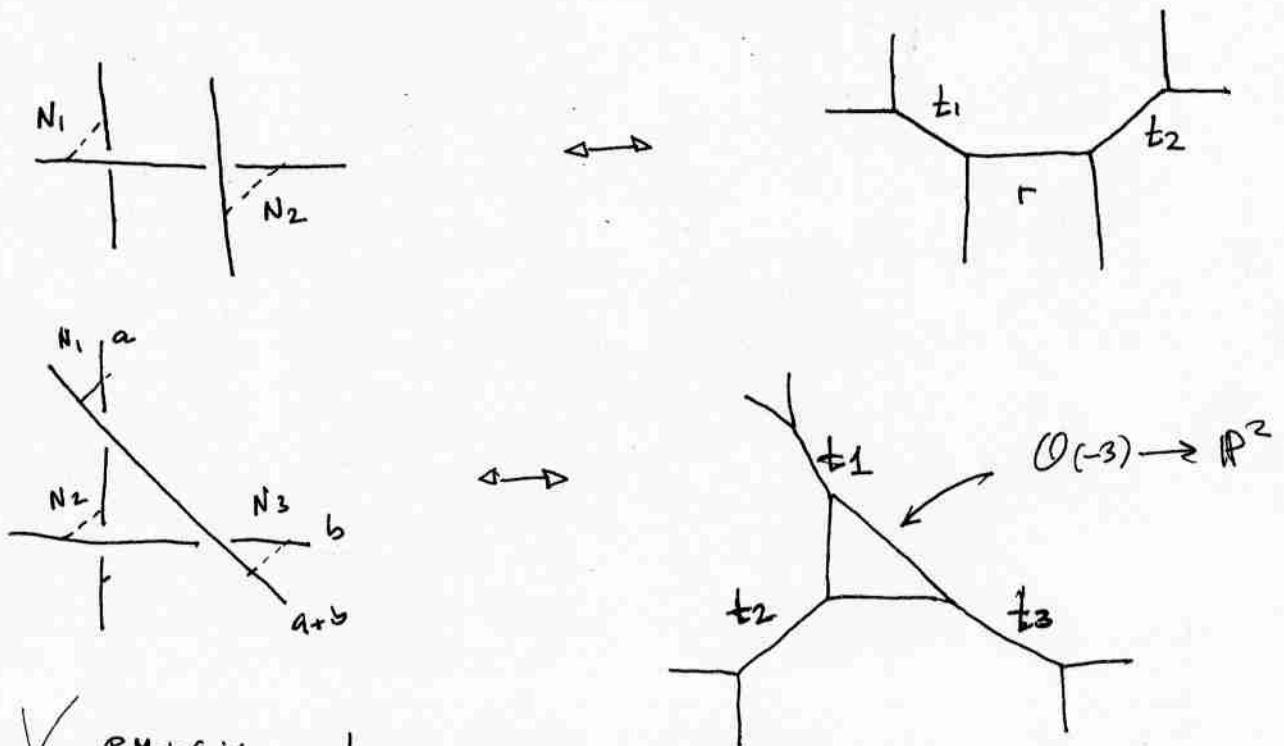
$$X_{\mathbb{S}^3/2p} = A_{p-1} \text{ fibration over } \mathbb{P}^1 \quad \left[ \begin{array}{c} \text{geometric realization} \\ \text{of } SU(p), N=2 \\ \text{SYM} \end{array} \right]$$

Aganagic, Klemm, M.M., Vafa

2) do large  $N$  / geometric transitions locally  
remember the basic geometric statement

$$\text{---} \left| \begin{array}{c} \text{---} \\ \text{---} \end{array} \right. \longleftrightarrow \text{---} \begin{array}{c} \text{---} \\ \text{---} \end{array}$$

Now, we can use these diagrams to encode other geometries!

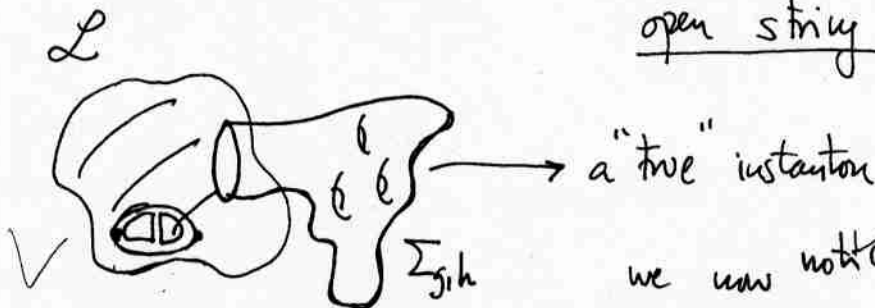


✓  $py + qx = \text{const}$

How do we compute the open topological string on the left?

Notice that, although locally these manifolds contain  $T^*M$ , they are not of that form. What corrections do we expect?

General picture by Witten: there are now nontrivial open string instantons



↓  
some strings with both ends on  $L$  or

we now notice that the boundaries of  $\Sigma_{g,h}$  are in fact knots in  $L$

in general, the effective action will be

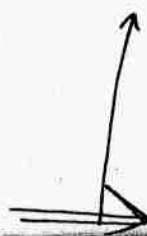
if we have more  $L_i$ 's we get

$$\sum_i S_{CS}^{L_i}(A_i) + \sum_{\substack{\Sigma_{gh} \\ \text{instantons}}} e^{-A_{\Sigma_{gh}}(t)} \prod_j \text{Tr} P_{exp} \oint_{R_j} A$$

$A_i$  are  $U(N_i)$  gauge fields

$$\partial \Sigma_{gh} = \cup R_j$$

$R_j$  ends on  $L_i$



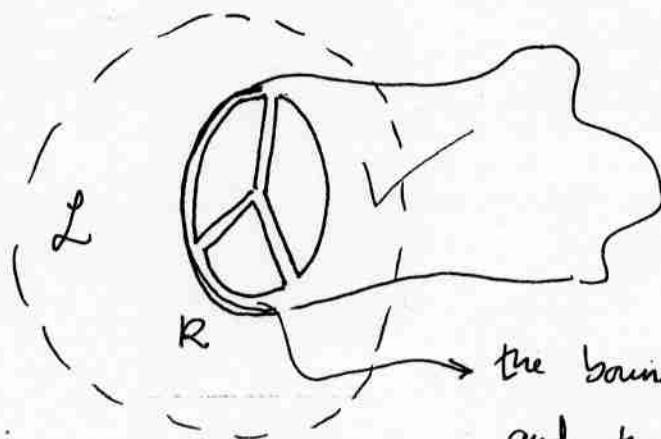
same coupling constant

$$S = S_{CS}^L(A) + \sum_{\substack{\Sigma_{g,h} \\ \text{instantons}}} e^{-A \Sigma_{g,h}(t)} \prod_i \text{Tr} P_{\text{exp}} \oint_{K_i} A$$

$\checkmark$   $2\Sigma_{g,h} = U K_i$

coupling to background fields

notice that in doing the path integral we insert fatgraphs inside the  $K_i$ , as it happens when computing vevs of Wilson loops in the  $1/N$  double line expansion

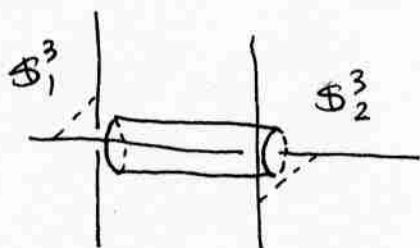


so the resulting object is a partially degenerated instanton

↑

the boundary of  $\Sigma_{g,h}$  is "cleared" and the integral leads to gluon functional insertions

Can we make the sum over instantons more concrete in examples?



in this example we have a single annulus going from one sphere to the other! The boundaries of the annulus are unknots in  $S^3$

$$\text{Sum over instantons} = \sum_{d=1}^{\infty} \frac{e^{-dr}}{d} \text{Tr} U_1^d \text{Tr} U_2^{-d}$$

$r$  = Kähler paramet for the tube

$U_{1,2}$  holonomies of  $U(N_1), U(N_2)$  in  $S^3_1, S^3_2$

localization  
Diakonov-Florez-Grassi a "primitive" annulus of area  $r$   
Aganagic-M.M.-Vafa + multicoverings

annuli always along the fixed point loci  
 simple, important result:

$$\exp \left( \sum_{d=1}^{\infty} \frac{e^{-dr}}{d} \text{Tr} U_1^d \text{Tr} U_2^{-d} \right) = \sum_R e^{-l_R \cdot r} \text{Tr}_R U_1 \text{Tr}_R U_2^{-1}$$

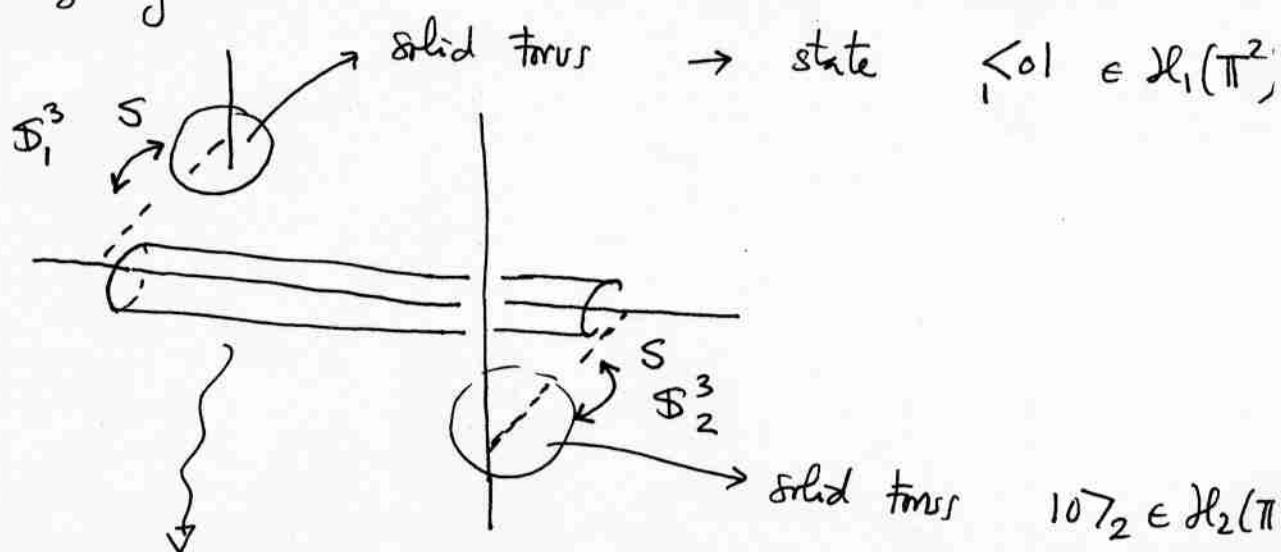
⇒ exercise

We have to evaluate now:

$$\mathcal{Z}(N_1, N_2, g_s, r) = \int DA_1 DA_2 e^{i(S_{CS}(A_1) + S_{CS}(A_2))} \times \sum_R e^{-l_R \cdot r} \text{Tr}_R U_1 \text{Tr}_R U_2^{-1}$$

we have to know the topology of the boundaries associated to  $U_1, U_2$  (what knots are they?)

instead of looking at the geometry, we use operator formalism and string



this is a state in  $\mathcal{H}_1(\pi^2) \otimes \mathcal{H}_2^*(\pi^2)$

obviously  $\sum_R |R\rangle_1 e^{-l_R \cdot r} \langle R|_2$

Since we are inserting Wilson loops in reps  $R_1, R_2$   
on the NONCONTRACTIBLE cycle of the torus

\* we then evaluate:

$$\begin{aligned} Z(N_1, N_2, g_s, r) &= \sum_R \langle 0 | S | R \rangle_1 e^{-k_R \cdot r} \langle R | S | 0 \rangle_2 \\ &= \langle 0 | S | 0 \rangle_1 \langle 0 | S | 0 \rangle_2 \sum_R \frac{S_{R0}^{(1)}}{S_{00}} e^{-k_R \cdot r} \frac{S_{0R}^{(1)}}{S_{00}} \end{aligned}$$

$$F(N_1, N_2, g_s, r) = F(\mathbb{S}_1^3) + F(\mathbb{S}_2^3)$$

$$+ \log \left\{ \sum_R (\dim_{\mathfrak{g}} R)_{U(N_1)} e^{-k_R \cdot r} (\dim_{\mathfrak{g}} R)_{U(N_2)} \right\}$$

let's write the first terms of the above:

$$1 + \left( \frac{e^{t_1/2} - e^{-t_1/2}}{g^{1/2} - g^{-1/2}} \right) e^{-r} \left( \frac{e^{t_2/2} - e^{-t_2/2}}{g^{1/2} - g^{-1/2}} \right)$$

+...

redefine  $r \Rightarrow r + \frac{t_1 + t_2}{2}$

$$1 - e^{-t_1} - e^{-t_2} + e^{-t_1 - t_2}$$

so we get

$$\frac{(1 - e^{-t_1})(1 - e^{-t_2})}{(g^{1/2} - g^{-1/2})^2} e^{-r} \quad \text{at first order.}$$

$$\rightarrow -\left(2 \sin^2 \frac{g_s}{2}\right)$$

so we read  $n_{g=0, r, t_1, 0} = 1, \quad n_{g=0, r, t_1, 0} = +1, \dots$

we can read off invariant