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NON-PERTURBATIVE METHODS IN QUANTUM FIELD THEORY

Lecture IV

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Semiclassical Methods for the Description of Tunneling in High-Energy Collisions

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Research done in collaboration with F. Bezrukov, D. Levkov, V. Rubakov, P. Tinyakov

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(Based on a talk given at BNL and The University of Florida.)

The problem

Processes with weak coupling: g is small.

Non-perturbative: e.g. tunneling $P \propto e^{-\frac{C}{g^2}}$.

Exclusive initial state.

Baryon number violation

in high-energy electroweak processes

The standard model of electroweak interactions is formulated in terms of bosonic fields describing $\gamma, W^{\pm}, Z$ and the Higgs particle, and fields describing fermionic particles.

The configurations of the bosonic fields can be classified into inequivalent **topological sectors**.

Generally in electroweak processes, as in all known particle interactions, net baryon number $N_B = n_B - n_{\bar{B}}$ is conserved. By a subtle quantum-mechanical effect (renormalization anomaly) N_B changes if the process changes the topology of the fields.

SU(2)-Higgs theory

$$S = \int dx^4 \left\{ -\frac{1}{2}\,\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} \right.$$
$$\left. + (D_\mu \Phi)^\dagger D^\mu \Phi - \lambda(\Phi^\dagger \Phi - 1)^2 \right\}$$

We consider spherically symmetric configurations described by two complex fields and one real field: $\chi(r,t), \bar{\chi}(r,t), \phi(r,t), \bar{\phi}(r,t), a(r,t)$.

($\chi(r,t), \bar{\chi}(r,t)$ parameterize the transverse components of A_μ, $\phi(r,t), \bar{\phi}(r,t)$ parameterize Φ, $a(r,t)$ the radial component of A_μ in the $A_0 = 0$ gauge.)

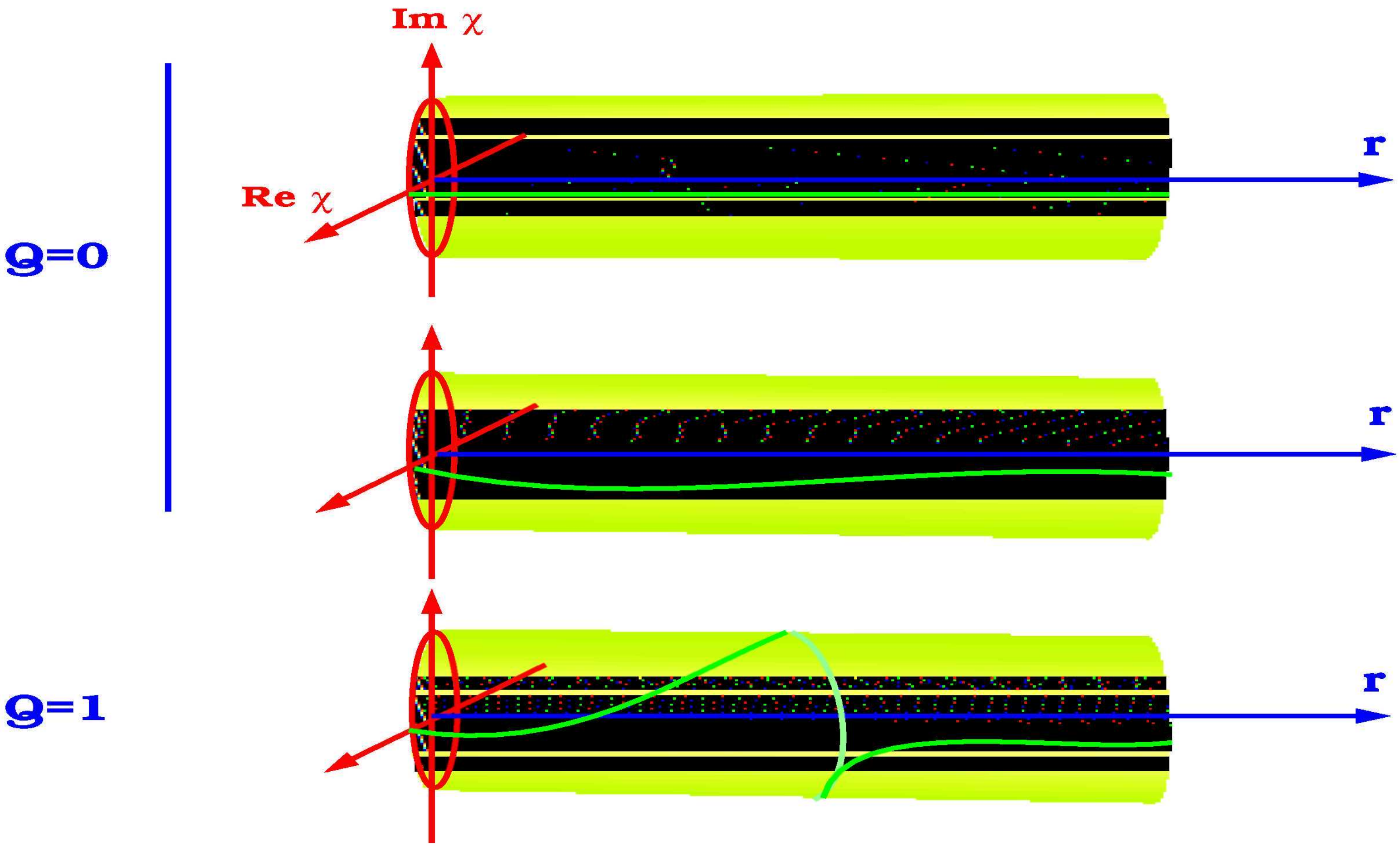
Action and Field topology

$$S = 4\pi \int dt \int_0^\infty dr \Big[-\frac{1}{4} r^2 f_{\mu\nu} f^{\mu\nu} + (D_\mu \chi)^* D^\mu \chi + r^2 (D_\mu \phi)^* D^\mu \phi$$
$$- \frac{1}{2r^2} (|\chi|^2 - 1)^2 - \frac{1}{2} (|\chi|^2 + 1) |\phi|^2$$
$$- \mathrm{Re}(i \chi^* \phi^2) - \lambda r^2 (|\phi|^2 - 1)^2 \Big]$$

Non-trivial topology occurs because in the state of lowest energy (vacuum) $|\chi| = \chi_0 > 0, \quad |\phi| = \phi_0 > 0$.

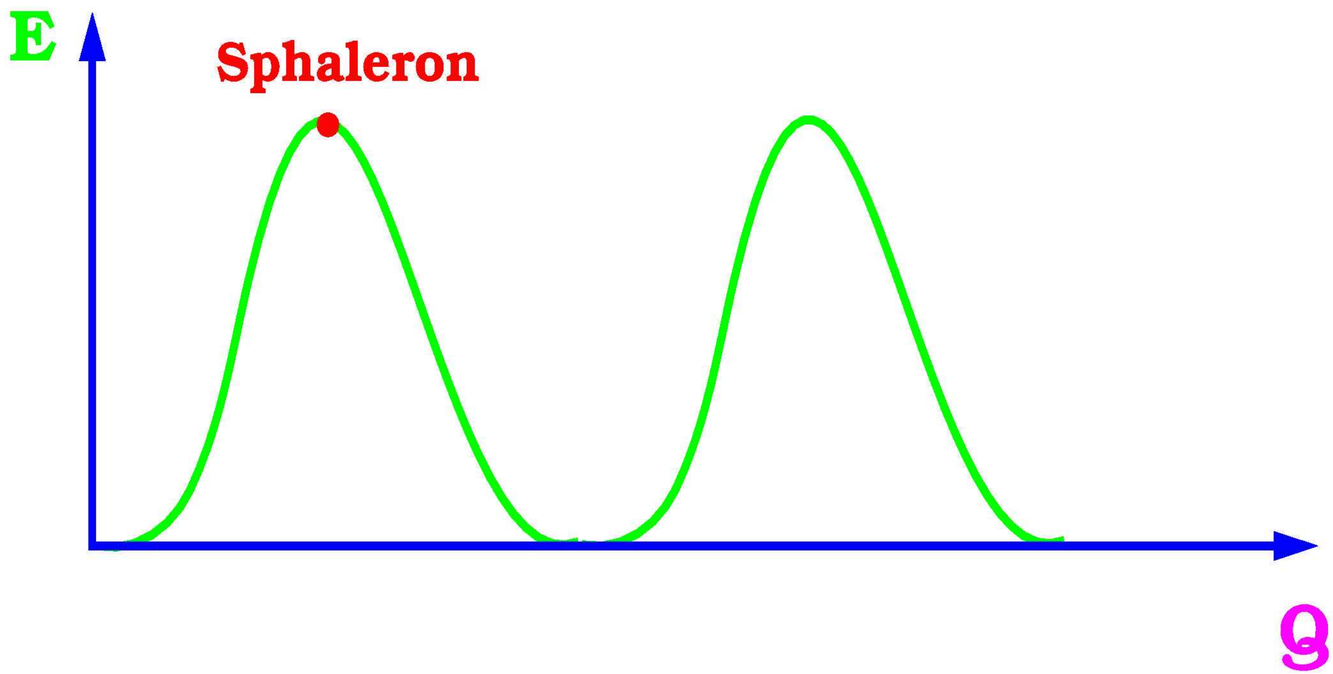
Regularity demands $\chi(r = 0, t) = \chi(r = \infty, t) = \mathrm{const}$: vacuum states are characterized by their topological winding number Q.

$\chi(r,t)$:



In a transition where $\Delta Q \neq 0$ also $\Delta N_B \neq 0$ ($\Delta N_B = 3\Delta Q$)

Sectors with different Q are separated by an energy barrier:

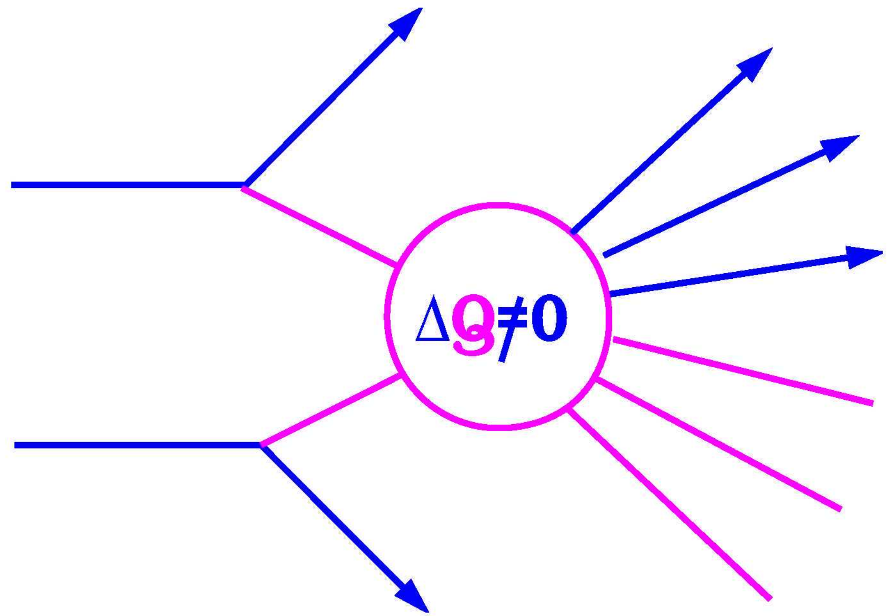


The sphaleron is an unstable solution lying on top of the barrier.

For $E > E_{sph}$ Q can change in a classical evolution.

For $E < E_{sph}$ Q can change only through quantum mechanical tunneling.

Can a two particle collision induce $\Delta Q \neq 0$ with appreciable probability, and, if so, at what energy?



Use semiclassical methods (g is small), but the initial state is not an inclusive state.

Semiclassical technique

Consider the inclusive multiparticle probability of tunneling from a "microcanonical" state with energy $E = \tilde{E}/g^2$ and number of particles $N = \tilde{N}/g^2$:

$$\sigma(E, N) = \sum_{i,f} |\langle f|\hat{S}\hat{P}_E\hat{P}_N|i\rangle|^2,$$

where sum is performed over all states $|i\rangle$ and $|f\rangle$ in different topological sectors, and $\hat{P}_E$, $\hat{P}_N$ are projection operators onto subspaces of fixed energy and number of particles. Take then $\tilde{N} \to 0$. (Rubakov, Tinyakov, 92, Rubakov, Son, Tinyakov, 92)

Writing a functional integral representation and making use of a saddle point approximation one can arrive to the following prescription:

The fields must solve the classical equations of motion.

For $t \ll 0$ or $t \gg 0$ expand in normal modes $a(k)$:

$E = 1/g^2 \int dk \omega(k) |a(k)|^2, \quad N = 1/g^2 \int dk |a(k)|^2$:

$a(k) = e^{-\theta} \bar{a}(k)^*$ for $t \ll 0$

$a(k) = \bar{a}(k)^*$ for $t \gg 0$

Tunneling probability

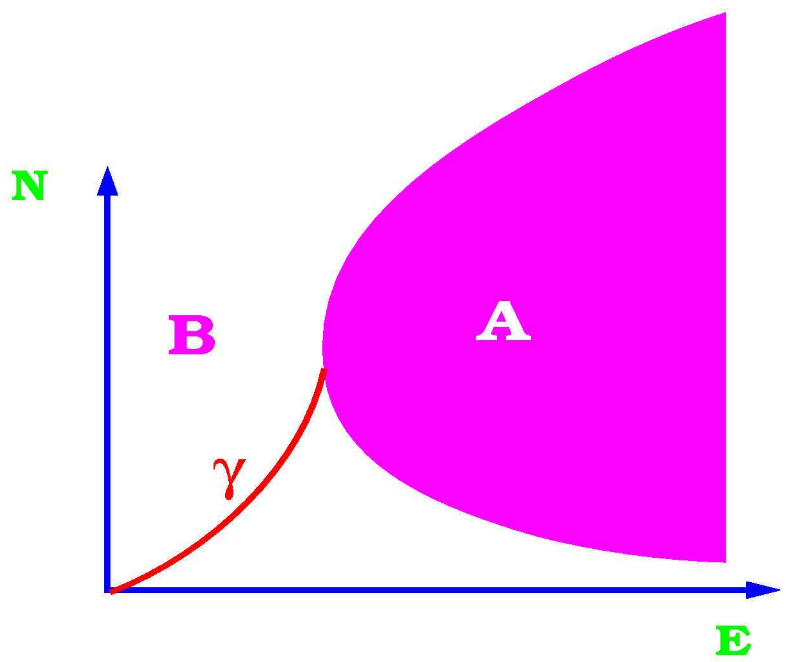
$$\sigma(E,N) \sim \exp\left\{-\frac{4\pi}{g^2}F(\tilde{E},\tilde{N})\right\},$$

$$-F(\tilde{E},\tilde{N}) = \tilde{N}\theta + \tilde{E}T + 2\,\mathrm{Re}[iS] + \mathrm{Re}\,\mathcal{B}_i$$

where:

$$\mathcal{B}_i = \frac{1}{2}\int dk[a(k)a(-k)\mathrm{e}^{-2i\omega_k t_i} - \bar{a}(k)\bar{a}(-k)\mathrm{e}^{2i\omega_k t_i}]$$

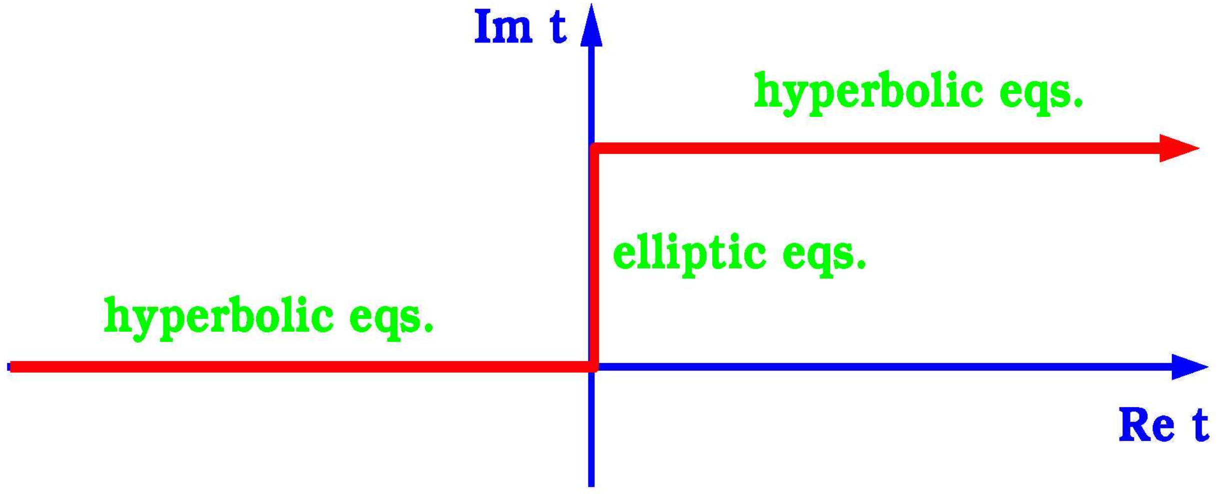
Energy vs. incoming particle number



In A topology changing processes are classically allowed, in B they can only occur through tunneling.

Classically forbidden processes

Saddle point approximation leads to solving the evolution equations along a complex time contour:



Computational method

Discretize the Lagrangian keeping exact gauge invariance.

Solve the equations of motion globally (elliptic equations cannot be evolved) at all space-time nodes, i.e. solve $N_r \times N_t$ ($\times 5$ complex) non-linear equations:

-start from an approximate solution;

-use Newton-Raphson method

$e_i(\phi_j) = 0 \qquad i = (r,t)$

$\rightarrow \sum_i \frac{\partial e_i}{\partial \phi_j}\delta\phi_j + e(\phi_{j,0}) = 0$

this is a set of $N_r \times N_t$ ($\times 5$ complex) linear equations.

ϕ_t is coupled only to ϕ_{t+1} and ϕ_{t-1}: eliminate alternate time slices, which requires full $5N_r \times 5N_r$ matrix manipulations, and reduce to boundary conditions for fields at initial and final times.

Computational challenges

Nature of PDEs change from hyperbolic to elliptic to hyperbolic again: solve equations globally.

Field equations are non-linear and boundary conditions demand continuation complex phase-space ($a(k) \neq \bar{a}(k)^*$): solve by deformation procedure.

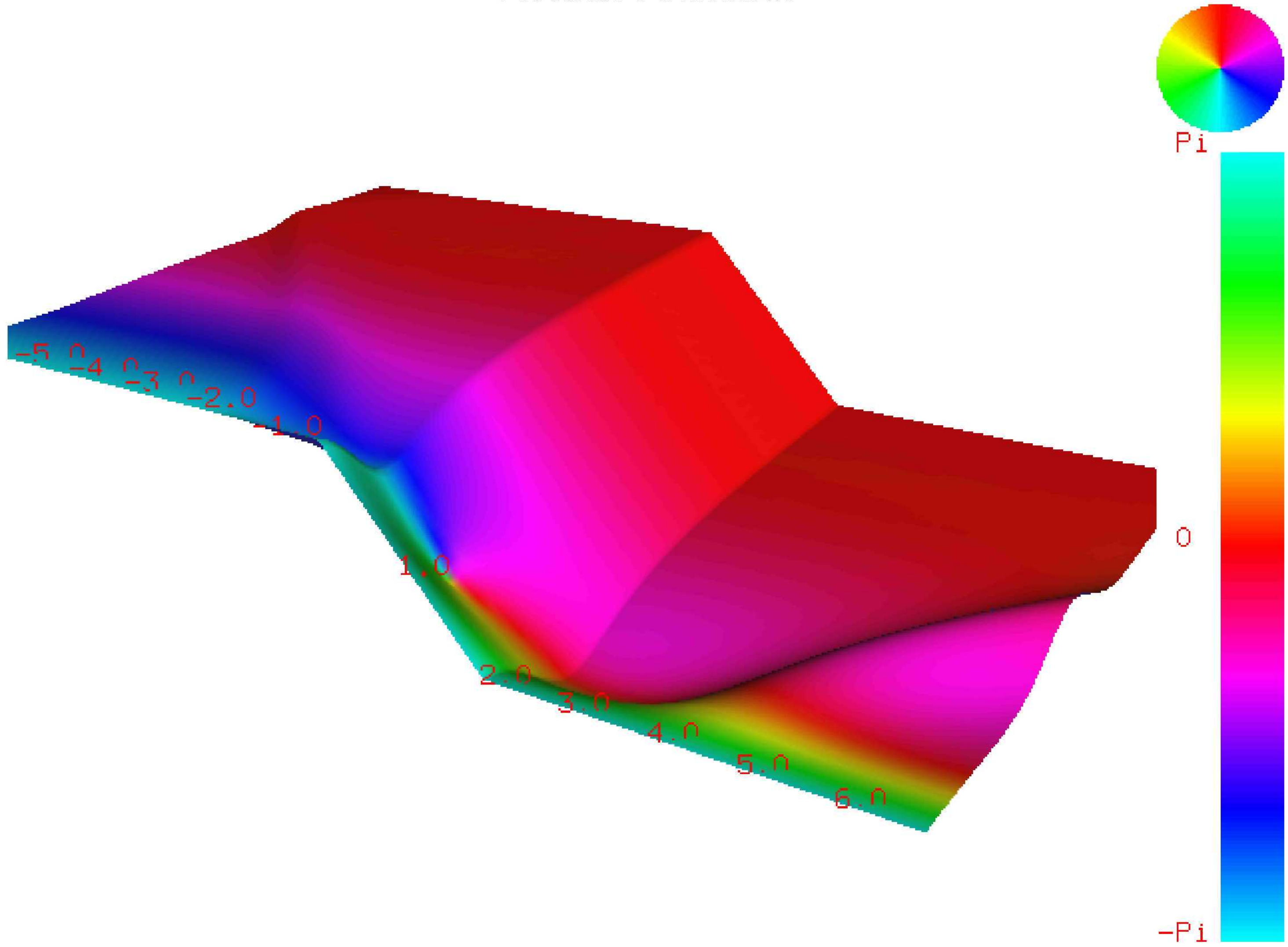
Very large number of degrees of freedom: eliminate alternate time slices.

Gauge invariance implies redundant degrees of freedom: use gauge covariant discretization and $A_0 = 0$ gauge.

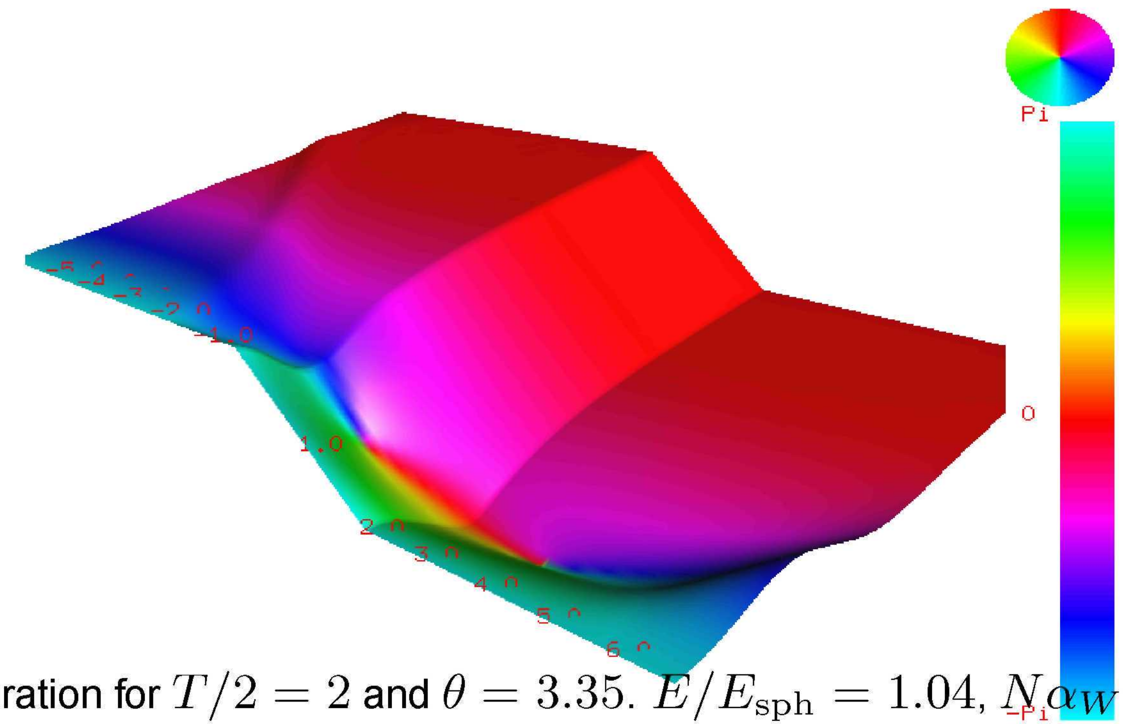
Time translation invariance implies presence of zero-mode in the deformation procedure: avoid by replacing $a(k) = \exp(-\theta)\bar{a}(k)^*$ with $a(k) = \exp(-\theta)\bar{a}(k)$ for just one of the modes.

Configuration for $T/2 = 2$ and $\theta = 3$. $E/E_{\rm sph} = 0.86$, $N\alpha_W = 0.74$.

Rotate. Animate.



One further problem: solutions apparently cannot be continued beyond the sphaleron's energy - the fields evolve back into the original topological sector.



Configuration for $T/2 = 2$ and $\theta = 3.35$. $E/E_{\rm sph} = 1.04$, $N\alpha_W = 0.94$.

Rotate. Animate.

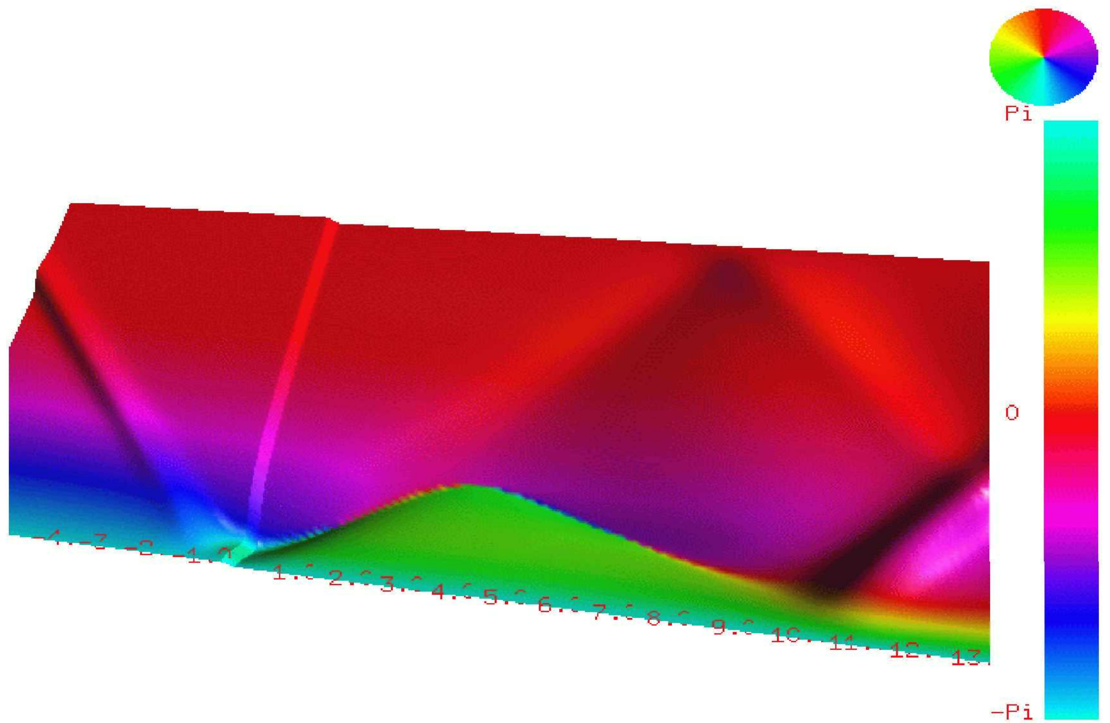
Origin of the problem and Solution

As one crosses the barrier energy, the solutions stay on top of the barrier for progressively longer times: clue found in the analysis of a quantum-mechanical model with a single oscillator hitting a barrier. (Bonini, Cohen, Rebbi and Rubakov, 1999).

The final boundary condition should then be truly applied at asymptotic times (numerically impossible).

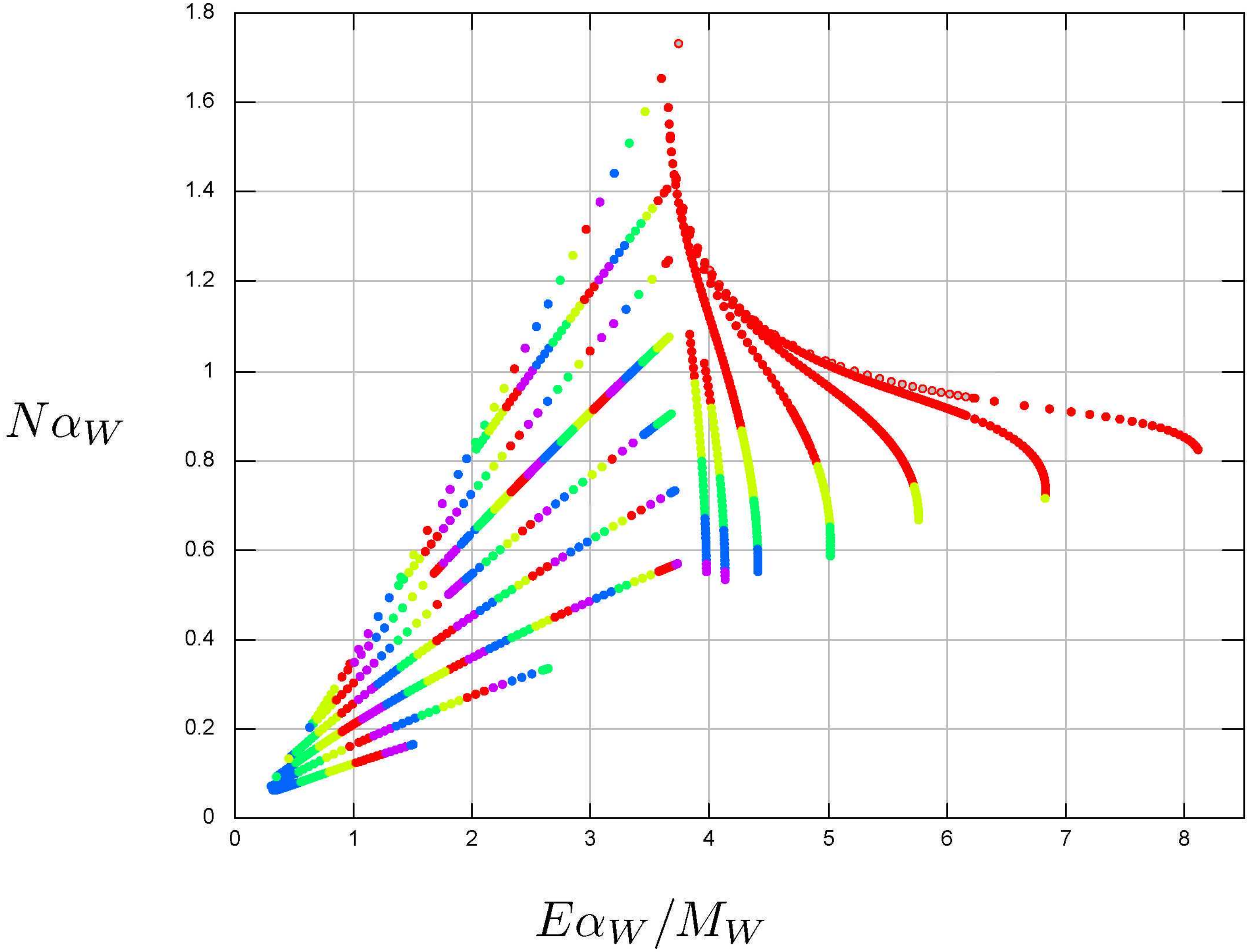
A suitable regularization procedure allows one to go around the singularity structure and recover a saddle-point solution. (Bezrukov and Levkov, quant-ph/0301022).

Tunneling solution at an energy larger than the sphaleron's energy:

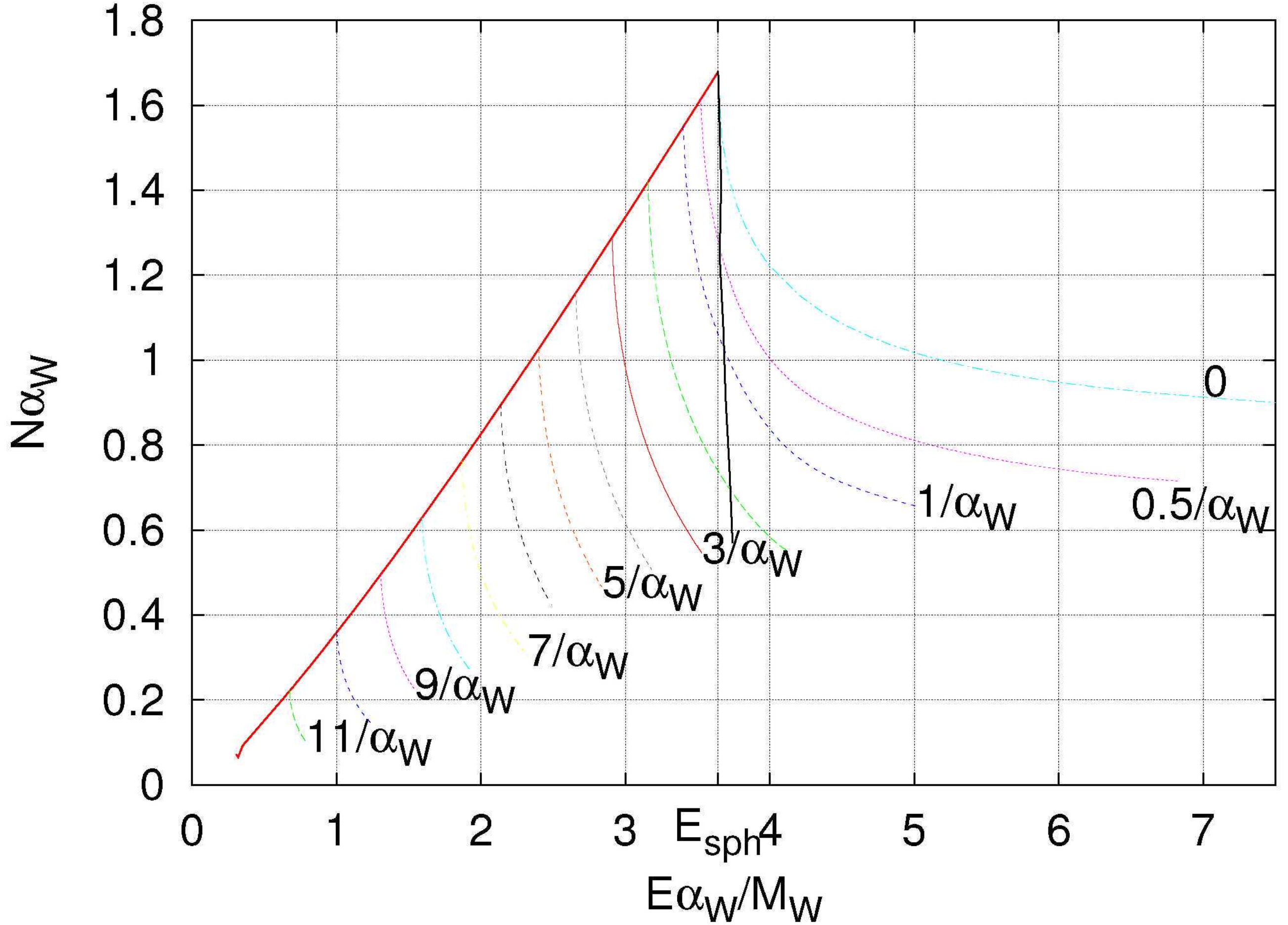


Configuration for $T/2 = 0.2$ and $\theta = 1.8$. $E/E_{\rm sph} = 1.24$, $N\alpha_W = 0.98$.

Rotate. Animate.



Search for solutions: lines of constant T, supression factor is color coded.



Lines of $F(E,N)/\alpha_W = \log(\text{suppression factor}) = \text{const.}$

Conclusions

The research shows that the semiclassical procedure of Rubakov, Son and Tinyakov, coupled to suitable computational techniques, can produce quantitative results for semi-inclusive, weakly coupled non-perturbative processes.

The study opens new dimensions for the application of computational methods to quantum field theory.

It illustrates the power of coupling analytical methods with advanced computational techniques.