united nations educational, scientific and cultural organization

international atomic energy agency

the
abdus salam
international centre for theoretical physics

SMR.1508 - 31

SUMMER SCHOOL ON PARTICLE PHYSICS

16 June - 4 July 2003

NONCOMMUTATIVE FIELD THEORY

C. CARLSON
Nuclear & Particle theory Group
College of William & Mary
Williamsburg, VA
U.S.A.

Lectures on Noncommutative Field Theory

Carl Carlson

Nuclear and Particle Theory Group

College of William and Mary

Loose outline:

- Introduction and motivation
- Fundamentals
- Serious problem
- Redemption

Egencontributions:

Bounding noncommutative QCD,
C. E. Carlson, C. D. Carone, and R. F. Lebed,
Phys. Lett. B **518**, 201 (2001) [arXiv:hep-ph/0107291].

Discerning noncommutative extra dimensions,
C. E. Carlson and C. D. Carone,
Phys. Rev. D **65**, 075007 (2002) [arXiv:hep-ph/0112143].

Noncommutative gauge theory w/o Lorentz violation,
C. E. Carlson, C. D. Carone, and N. Zobin,
Phys. Rev. D **66**, 075001 (2002) [arXiv:hep-th/0206035].

Supersymmetric noncommutative QED and Lorentz violation,
C. E. Carlson, C. D. Carone, and R. F. Lebed,
Phys. Lett. B **549**, 337 (2002) [arXiv:hep-ph/0209077].

Phenomenology of Lorentz-conserving noncommutative QED,
J. M. Conroy, H. J. Kwee, and V. Nazaryan,
arXiv:hep-ph/0305225.

Contents

I. INTRODUCTION AND MOTIVATION

Much recent research on modifications of spacetime.

e.g.,

- extra dimensions
- the subject here: noncommutative spacetime

Motivations:

- General Relativity can be taken to suggest there is a minimum sensible length scale. Position uncertainty also suggested by noncommutation of coordinate operators, so explore separately consequences of such noncommutation (Doplicher, Fredenhagen, and Roberts).
- String theory can be solved in a background field. The solution gives coordinate operators that do not commute. (Ardalan, Arfaei, and Sheikh-Jabbari; Seiberg and Witten; Connes, Douglas, and Schwarz)

  Both of these suggest/lead to the commutation relation for spacetime coordinate operators,

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} \tag{1.1}$$

although for the string theory version ("canonical version") the 4×4 array on the RHS is a set of constants and not something that transforms like a Lorentz tensor.

- An early (Snyder, AD 1947!) motivation for noncommutation of coordinates was the hope that field theory based on coordinates that could not be so sharply localized would have fewer divergence problems. This motivation no longer stands, but the commutator algebra suggested by Snyder in that era remains interesting and potentially useful.

- Final motivation: this is basic research in field theory

Elaborate on first two motivations:

A. General relativity

Measurements pack energy into the region we measure: Heisenberg.

Find: below a certain size, cannot see what we localize. It becomes a black hole.

Radius of black hole,

$$R = \frac{Gm_{eff}}{2c^2} \tag{1.2}$$

Packed energy, if size scale is a,

$$E = \frac{\hbar c}{a} \tag{1.3}$$

and of course $m_{eff} = E/c^2$.

If a is big, we are fine. At limit, $R = a$,

$$a \sim \frac{G}{c^2}\frac{\hbar}{ac} \tag{1.4}$$

$$a = \sqrt{\frac{G\hbar}{c^3}} \tag{1.5}$$

Might have guessed: the answer is the Planck length.

Point: Effectively, there is a minimum length scale. Uncertainty is position is also a consequence of non-commuting coordinate operators. Abstract the principle and try to see what follows just from idea that coordinates are operators that don't commute.

1. Note on shrinking size of Planck scale if there are extra dimensions

λ = Planck length

ϕ = gravitational potential (potential energy/unit mass)

Here let $\hbar = 1$, $c = 1$.

In 4D,

$$\phi = -\frac{\lambda_0^2 M}{r}\,, \tag{1.6}$$

where λ_0 is just what we had above with the $\hbar$ and c in place,

$$\lambda_0 = \sqrt{\frac{G\hbar}{c^3}} \approx 10^{-33}\ \mathrm{cm} \approx (1/10^{19}\ \mathrm{GeV}) \tag{1.7}$$

Suppose we have n extra dimensions. The extra dimensions have radius R. At short distance in $4+n$ dimensions, the potential falls like (radius)$^{-(n+1)}$, but at long distances, the falloff in the extra dimensions saturates and the falloff is just in the dimensions we see, and is just the $(1/r)$ that we normally see,

$$\phi = \begin{cases} -\dfrac{\lambda_n^{n+2} M}{r^{n+1}} & r << R \\[2ex] -\dfrac{\lambda_n^{n+2} M}{R^n r} & r >> R \end{cases} \tag{1.8}$$

Matching the long distance results,

$$\lambda_0^2 = \frac{\lambda_n^{n+2}}{R^n} \tag{1.9}$$

which may be manipulated into

$$\lambda_n = \lambda_0 \left(\frac{R}{\lambda_0}\right)^{n/(n+2)} \quad . \tag{1.10}$$

This says that the length scale associated with gravity could get much larger than the 10^{-19} cm we are used to, if there are extra dimensions and their length scale is not too small. For example, say there are 2 extra dimensions and that their length scale R is 1 mm = 10^{-1} cm. Then

$$\lambda_2 = 10^{-17}\ \mathrm{cm} \approx \frac{1}{10\ \mathrm{TeV}} \ , \tag{1.11}$$

getting the scale for length uncertainty into the region where we should be thinking about it now.

B. String theory

String theory solved in the presence of a background field leads to noncommutative coordinates. Not as hard to follow the manipulations as I once thought.

Standard Nambu-Goto string action, with extra term. $B^{\mu\nu}$ = background field. Will be taken constant: uniform and static.

$$S = \frac{1}{2\pi\alpha'} \int d\sigma d\tau \left\{ g^{ab}\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu + \epsilon^{ab} B_{\mu\nu}\partial_a X^\mu \partial_b X^\nu \right\} \tag{1.12}$$

For open string, the σ integral runs between endpoints at $\sigma = 0$ and $\sigma = \pi$.

To get the equations of motion, we do the standard variational principle calculation. Let $\delta S = 0$ when $X^\mu \to X^\mu + \delta X^\mu$, and integrate by parts,

$$\begin{aligned} 0 = & - \int d\sigma d\tau \left\{ g^{ab}\eta_{\mu\nu}\delta X^\mu \partial_a \partial_b X^\nu + \epsilon^{ab} B_{\mu\nu}\delta X^\mu \partial_a \partial_b X^\nu \right\} \\ & + \int d\tau \left\{ -\eta_{\mu\nu}\delta X^\mu \partial_\sigma X^\nu + B_{\mu\nu}\delta X^\mu \partial_\tau X^\nu \right\}\Big|_{\sigma=0}^{\sigma=\pi} \end{aligned} \tag{1.13}$$

(Useful to write out ($\epsilon_{01} = -\epsilon^{01} = 1$)

$$\mathcal{L} \propto \eta_{\mu\nu}\left[\partial_\tau X^\mu \partial_\tau X^\nu - \partial_\sigma X^\mu \partial_\sigma X^\nu\right] + 2B_{\mu\nu}\partial_\sigma X^\mu \partial_\tau X^\nu \ .) \tag{1.14}$$

Thus, including a background field does not change the string equation of motion,

$$(\partial_\tau^2 - \partial_\sigma^2) X^\mu = 0 \tag{1.15}$$

but it does change the boundary condition,

$$\partial_\sigma X_\mu - B_{\mu\nu} \partial_\tau X^\nu = 0 \,, \quad \sigma = 0, \pi \tag{1.16}$$

The equation of motion means that we can expand the string position X^μ in terms of functions $e^{-in\tau} \cos n\sigma$ and $e^{-in\tau} \sin n\sigma$ plus constants and linear terms. To satisfy the boundary conditions, we arrange the coefficients as,

$$X^\mu = x_0^\mu + p^\mu \tau + B^{\mu\nu} p_\nu \sigma + \sum_{n \neq 0} \frac{e^{-in\tau}}{n} \left(i a_n^\mu \cos n\sigma + B^\mu{}_\nu a_n^\nu \sin n\sigma \right) \tag{1.17}$$

[Reality of X means that $a_n^\mu = (a_{-n}^\mu)^\dagger$.] To verify the boundary conditions, write out,

$$\partial_\sigma X^\mu = B^\mu{}_\nu p^\nu + \sum_{n \neq 0} e^{-in\tau} \left(-i a_n^\mu \sin n\sigma + B^\mu{}_\nu a_n^\nu \cos n\sigma \right) \,, \tag{1.18}$$

and

$$\partial_\tau X^\nu = p^\nu + \sum_{n \neq 0} e^{-in\tau} \left(a_n^\nu \cos n\sigma - i B^\nu{}_\rho a_n^\rho \sin n\sigma \right) \,. \tag{1.19}$$

In the boundary condition equation, the momentum term and the cosine terms cancel, leaving the sine terms. But they are o.k., since the b. c. equation only has to work at $\sigma = 0$ and $\sigma = \pi$.

******* Do quantum mechanics

Find the commutation relations of the expansion coefficients of X^μ. The basic rule is

$$[X^\mu(\sigma,\tau), P^\nu(\sigma',\tau)] = i\eta^{\mu\nu}\delta(\sigma-\sigma') \tag{1.20}$$

and we shall also use

$$[P^\mu(\sigma,\tau), P^\nu(\sigma',\tau)] = 0\ . \tag{1.21}$$

The definition of the canonical momentum density is

$$P^\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\tau X_\nu)} \tag{1.22}$$

and this gives

$$2\pi\alpha' P^\nu = \partial_\tau X^\mu - B^\nu{}_\rho \partial_\sigma X^\rho \tag{1.23}$$

or

$$\begin{aligned} 2\pi\alpha' P^\nu &= p^\nu - B^\nu{}_\rho B^\rho{}_\beta p^\beta \\ &+ \sum_{n\neq 0} e^{-in\tau}\left(a_n^\nu \cos n\sigma - iB^\nu{}_\rho a_n^\rho \sin n\sigma + iB^\nu{}_\rho a_n^\rho \sin n\sigma - B^\nu{}_\rho B^\rho{}_\beta a_n^\beta \cos n\sigma\right) \end{aligned} \tag{1.24}$$

or

$$2\pi\alpha' P^\nu = \mathcal{M}^\nu{}_\beta p^\beta + \sum_{n\neq 0} e^{-in\tau} \mathcal{M}^\nu{}_\beta a_n^\beta \cos n\sigma \tag{1.25}$$

with

$$\mathcal{M}^\nu{}_\beta \equiv \left(1 - B^2\right)^\nu{}_\beta \tag{1.26}$$

Easy to show that $\mathcal{M}^{\mu\nu}$ is symmetric.

Hence from $[X^\mu, P^\nu] \ldots$,

$$\begin{aligned}&\Big[\ x_0^\mu + p^\mu\tau + B^\mu{}_\rho p^\rho\sigma + \sum_{n\neq 0} \frac{e^{-in\tau}}{n}\left(i a_n^\mu \cos n\sigma + B^\mu{}_\rho a_n^\rho \sin n\sigma\right)\ ,\\ &\mathcal{M}^\nu{}_\beta p^\beta + \sum_{n\neq 0} e^{-in\tau} \mathcal{M}^\nu{}_\beta a_n^\beta \cos n\sigma'\Big]\\ &= 2\pi i\alpha' \eta^{\mu\nu}\delta(\sigma - \sigma')\end{aligned} \tag{1.27}$$

Integrate over σ and σ' using

$$\int_0^\pi d\sigma\ \cos n\sigma = \int_0^\pi d\sigma\ \sin n\sigma = 0\ , \quad n \neq 0 \tag{1.28}$$

Get

$$\left[x_0^\mu + p^\mu\tau + \frac{1}{2}B^\mu{}_\rho p^\rho\ ,\ p^\nu\right] = 2i\alpha'(\mathcal{M}^{-1})^{\mu\nu} \tag{1.29}$$

Since this has to work for any τ,

$$[p^\mu, p^\nu] = 0 \tag{1.30}$$

and

$$[x_0^\mu, p^\nu] = 2i\alpha'(\mathcal{M}^{-1})^{\mu\nu} \tag{1.31}$$

Then use

$$\int_0^\pi d\sigma\ \cos n'\sigma\ \cos n\sigma = \frac{\pi}{2}\left(\delta_{nn'} + \delta_{n,-n'}\right) \tag{1.32}$$

Get

$$\left[\frac{1}{n'}\left(e^{-in'\tau}a^\mu_{n'} - e^{in'\tau}a^\mu_{-n'}\right)\ ,\ e^{-im'\tau}a^\nu_{m'} + e^{im'\tau}a^\nu_{-m'}\right] = 4\alpha'\left(\mathcal{M}^{-1}\right)^{\mu\nu} \tag{1.33}$$

To make the τ dependence work,

$$\left[a^\mu_n, a^\nu_{-m}\right] - \left[a^\mu_{-n}, a^\nu_m\right] = 4n\alpha'\delta_{nm}\left(\mathcal{M}^{-1}\right)^{\mu\nu} \tag{1.34}$$

Now use $[P^\mu, P^\nu] = 0$. Same manipulations lead to

$$\left[a^\mu_n, a^\nu_{-m}\right] + \left[a^\mu_{-n}, a^\nu_m\right] = 0 \tag{1.35}$$

Thus,

$$\left[a^\mu_n, a^\nu_m\right] = 2n\alpha'\delta_{n,-m}\left(\mathcal{M}^{-1}\right)^{\mu\nu} \tag{1.36}$$

Can also show

$$\left[p^\mu, a_m^\nu\right] = 0 = \left[x_0^\mu, a_m^\nu\right] \tag{1.37}$$

Now examine $[X^\mu, X^\nu]$.

$$[X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] =$$

$$\Bigg[\; x_0^\mu + p^\mu\tau + B^\mu{}_\rho p^\rho\sigma + \sum_{n\neq 0}\frac{e^{-in\tau}}{n}\left(ia_n^\mu \cos n\sigma + B^\mu{}_\rho a_n^\rho \sin n\sigma\right)\;,$$

$$x_0^\nu + p^\nu\tau + B^\nu{}_\beta p^\beta\sigma' + \sum_{m\neq 0}\frac{e^{-im\tau}}{m}\left(ia_m^\nu \cos m\sigma' + B^\nu{}_\beta a_m^\beta \sin m\sigma'\right)\Bigg]$$

$$\begin{aligned}
&= [x_0^\mu, x_0^\nu] + \left[x_0^\mu, p^\nu\tau + B^\nu{}_\beta p^\beta\sigma'\right] + \left[p^\mu\tau + B^\mu{}_\rho p^\rho\sigma, x_0^\nu\right] \\
&+\; 2\alpha'\sum_{n\neq 0}\frac{1}{n}\Big\{ -\left(\mathcal{M}^{-1}\right)^{\mu\nu}\cos n\sigma\cos n\sigma' - i\left(B\mathcal{M}^{-1}\right)^{\mu\nu}\sin n\sigma\cos n\sigma' \\
&\qquad + i\left(\mathcal{M}^{-1}B\right)^{\mu\nu}\cos n\sigma\sin n\sigma' + \left(B\mathcal{M}^{-1}B\right)^{\mu\nu}\sin n\sigma\sin n\sigma'\Big\}
\end{aligned} \tag{1.38}$$

Useful information: $B\mathcal{M}^{-1} = \mathcal{M}^{-1}B$, and $(\mathcal{M}^{-1}B)^{\mu\nu}$ is antisymmetric (both easy to prove).

Get

$$[X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] = [x_0^\mu, x_0^\nu] + 2i\alpha' \left(\mathcal{M}^{-1}B\right)^{\mu\nu} \left\{\sigma + \sigma' + \sum_{n\neq 0} \frac{1}{n}\sin n(\sigma+\sigma')\right\} \tag{1.39}$$

The function has the values

$$\sigma + \sigma' + \sum_{n\neq 0} \frac{1}{n}\sin n(\sigma+\sigma') = \begin{cases} 0 & \sigma = \sigma' = 0 \\ 2\pi & \sigma = \sigma' = \pi \\ \pi & \text{otherwise} \end{cases} \tag{1.40}$$

(Hint on proving the above: Expand functions $f(x) = x$ and $f(x) = 1$ in a sine series on the interval $x = (0, \pi)$. Examine the result ....)

Thus

$$[X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] = [x_0^\mu, x_0^\nu] + 2i\pi\alpha' \left((1-B^2)^{-1}B\right)^{\mu\nu} \begin{cases} 0 & \sigma = \sigma' = 0 \\ 2 & \sigma = \sigma' = \pi \\ 1 & \text{otherwise} \end{cases} \tag{1.41}$$

Have no information on the x_0 commutator, but no matter what its value, we have noncommutativity somewhere. Suppose zero, define the CM of the string,

$$\hat{x}^\mu = \frac{1}{\pi}\int_0^\pi d\sigma\, X^\mu \tag{1.42}$$

and find

$$[\hat{x}^\mu, \hat{x}^\nu] = 2i\pi\alpha' \left((1 - B^2)^{-1} B\right)^{\mu\nu} \equiv i\theta^{\mu\nu} \tag{1.43}$$

Summary: There is noncommutativity of coordinates if we quantize a string in the presence of a background field.

II. MULTIPLYING FIELDS: MOYAL PRODUCTS

Coordinates $\hat{x}^\mu$ are operators.

The fields $\hat{f}$ are now mappings ("functions") $\hat{f}(\hat{x})$ of arguments that do not mutually commute. Hence in general, fields $\hat{f}$ and $\hat{g}$ don't commute, not just because of canonical commutation relations between fields and canonical momentum densities, but because of the $\hat{x}$ themselves.

We don't know how to calculate with Lagrangians that depend on products of fields when the fields are not, at least classically, ordinary functions. We have to learn how. The way is to establish a mapping between fields and ordinary functions.

But ordinary functions, again not thinking of the quantum mechanics, commute if we multiply using the ordinary product. When we map products of fields to products of ordinary functions, the product will not turn out to be the ordinary product.

A. Multiplying fields

Deal with fields $\hat{f}(\hat{x})$ by relating them to ordinary functions $f(x)$ of ordinary variables.

$$\hat{f}(\hat{x}) = \int (dp) e^{-ip\hat{x}} f(p) \tag{2.1}$$

with

$$f(x) = \int (dp) e^{-ipx} f(p) \tag{2.2}$$

where $f(p)$ and $f(x)$ are an ordinary functions and $(dp) = d^4p/(2\pi)^4$.

Multiply fields,

$$\hat{f}(\hat{x})\,\hat{g}(\hat{x}) \;\equiv\; \left(\widehat{f \circ g}\right)(\hat{x}) \quad \text{or} \quad \left(\widehat{fg}\right)(\hat{x})$$

$$= \int (dp) e^{-ip\hat{x}} f(p) \int (dk) e^{-ik\hat{x}} g(k) \tag{2.3}$$

The two exponentials don't commute, so we use the Baker-Campbell-Hausdorff-… theorem,

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\ldots} \tag{2.4}$$

For $A = -ip\hat{s}$, $B = -ik\hat{x}$, and if $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$ where the $\theta^{\mu\nu}$ are just ordinary numbers, then the series terminates. One can manipulate,

$$\hat{f}\hat{g} = \;=\; \int (dp)(dk)\, e^{-i(p+k)\hat{x}} \exp\left[-\frac{i}{2} p_\mu \theta^{\mu\nu} k_\nu\right] f(p) g(k)$$

$$= \int (dp)(dk)(dx)(dy)\, e^{-i(p+k)\hat{x}} f(x) g(y) \exp\left[\frac{i}{2} \partial^x_\mu \theta^{\mu\nu} \partial^y_\nu\right] e^{ipx} e^{iky}$$

$$= \int (dp)(dk)(dx)(dy)\, e^{-i(p+k)\hat{x}} e^{ipx} e^{iky} \exp\left[\frac{i}{2}\partial_\mu^x \theta^{\mu\nu} \partial_\nu^y\right] f(x)g(y)$$

$$= \int (dp)(dq)(dx)(dy)\, e^{-iq\hat{x}} e^{ipx} e^{i(q-p)y} \exp\left[\frac{i}{2}\partial_\mu^x \theta^{\mu\nu} \partial_\nu^y\right] f(x)g(y)$$

$$= \int (dq)\, e^{-iq\hat{x}} \int (dx)\, e^{iqx} \exp\left[\frac{i}{2}\partial_\mu^x \theta^{\mu\nu} \partial_\nu^y\right] f(x)g(y) \qquad (2.5)$$

Looking at the Fourier transform in the last line, realize the correspondence

$$\widehat{f \circ g} \to \exp\left[\frac{i}{2}\partial_\mu^x \theta^{\mu\nu} \partial_\nu^y\right] f(x)g(y)\bigg|_{y=x} \equiv (f \star g)(x) \qquad (2.6)$$

Alternative writing,

$$(f \star g)(x) = f(x)\, \exp\left(\frac{i}{2}\overleftarrow{\partial}_\mu \theta^{\mu\nu} \partial_\nu\right)\, g(x) \qquad (2.7)$$

Thus, work with ordinary functions with unusual multiplication rule. Multiplication rule mimics noncommutative multiplication in the operator algebra.

Called "Moyal product" or "Weyl-Moyal product" or "star-product."

Query: What is $[x^\mu \stackrel{\star}{,} x^\nu]$? (Definition: $[x^\mu \stackrel{\star}{,} x^\nu] \equiv x^\mu \star x^\nu - x^\nu \star x^\mu$. Note the absence of the hat, so that these are just the ordinary x's.)

B. Lagrangians, using example of QED

Keep the Lagrangian the same. But: fields are operators dependent upon noncommutative coordinates. For QED,

$$\hat{\mathcal{L}} = -\frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} + \hat{\bar{\psi}}(i\ \not{\partial} - e\ \hat{\not{A}} - m)\hat{\psi} \tag{2.8}$$

and can somewhat formally define the action S from

$$S = \mathrm{Tr}\ \hat{\mathcal{L}} \tag{2.9}$$

although in practical terms we shall define the trace from the integral of the ordinary space correspondent of the Lagrangian.

The Lagrangian as a ordinary function becomes

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu} \star F^{\mu\nu} + \bar{\psi} \star (i\ \not{\partial} - e\ \not{A} - m) \star \psi \tag{2.10}$$

with

$$S = \int (dx)\ \mathcal{L} \tag{2.11}$$

C. Mini-theorems

- $f \star g \star h$ is associative (2.12)

- $$\int (dx)\; f(x) \star g(x) = \int (dx)\; g(x) \star f(x) \quad \text{(at least for bosons)} \tag{2.13}$$

- $$\int (dx)\; f(x) \star g(x) = \int (dx)\; f(x) g(x) \tag{2.14}$$

- $$\int (dx)\; f \star g \star h = \int (dx)\; f \star g\, h = \int (dx)\; f\, g \star h \tag{2.15}$$

D. Gauge transformations

If U is a unitary operator, the Lagrangian should be invariant under field redefinitions that follow from acting with U in the field ψ. Now, however, multiplications are done using star products.

$$\psi \rightarrow \psi' = U \star \psi = e^{i\alpha(x)} \star \psi \tag{2.16}$$

With ordinary multiplication, compensate for the derivative acting on $e^{i\alpha}$ by having a shift in the electromagnetic field under a gauge transformation, so that $A_\mu \to A_\mu - (1/e)\partial_\mu\alpha$. But now U and A_μ don't commute, and what works is:

$$A_\mu \to A'_\mu = U \star A_\mu \star U^\dagger + \frac{i}{e}(\partial_\mu U) \star U^\dagger \tag{2.17}$$

to make

$$(\partial_\mu - ieA'_\mu) \star \psi' = U \star (\partial_\mu - ieA_\mu) \star \psi \tag{2.18}$$

The more complicated A_μ transformation also means that the usual QED definition of $F_{\mu\nu}$, namely $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, is not gauge invariant or gauge covariant. Instead, use

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + ie\,[A_\mu \overset{\star}{,} A_\nu] \tag{2.19}$$

Then under a gauge transformation

$$F_{\mu\nu} \to F'_{\mu\nu} = U \star F_{\mu\nu} \star U^\dagger \tag{2.20}$$

Hence the QED action, Eq. (2.11), is gauge invariant.

E. Limitations of gauge transformations

We have a gauge transformation $\psi \to \psi' = U \star \psi = e^{i\alpha(x)} \star \psi$. This is for a given charge. Taking that charge as the elementary unit, it is for $Q = 1$.

Say there is second matter field with different charge, a field ϕ with charge $Q \neq 1$.

Expect

$$\phi \rightarrow \phi' = e^{iQ\alpha} \star \phi = U^Q \star \phi\ . \tag{2.21}$$

The covariant derivative for charge Q is

$$D_\mu \phi = (\partial_\mu + iQeA_\mu) \star \phi \tag{2.22}$$

and a gauge transformation is should change like

$$D_\mu \phi \rightarrow D'_\mu \phi' = U^Q \star D_\mu \phi\ . \tag{2.23}$$

The photon already has its gauge transformation behavior fixed, so we just have to try it:

$$\begin{aligned}(\partial_\mu + iQeA_\mu) \star \phi \ \rightarrow\ (\partial_\mu + iQeA'_\mu) \star \phi' =\ & U^Q \partial_\mu \phi + (\partial_\mu U^Q) \star \phi + \\ +\ & iQe\, U \star A_\mu \star U^{Q-1} \star \phi - Q(\partial_\mu U) \star U^{Q-1} \star \phi\ .\end{aligned} \tag{2.24}$$

The red would be perfect if the multiplications commuted, and the purple terms would cancel. But things don't commute, and we get the correct result only for $Q = 1$.

$Q = -1$ also possible. Let the gauge transformation be

$$\phi \rightarrow \phi' = \phi \star e^{iQ\alpha} = \phi \star U^Q \tag{2.25}$$

and let the covariant derivative be

$$D_\mu \phi = \phi \star \left(\overleftarrow{\partial}_\mu + iQeA_\mu \right) \tag{2.26}$$

Try it and succeed for $Q = -1$, fail for other Q.

Something new: for neutral particles one has to make a choice. A neutral particle can be invariant under gauge transformations and have no electromagnetic interaction, exactly as in the commutative case. But nontrivial gauge transformations and electromagnetic interactions are possible even for neutral particles.

Let the gauge transformation for a neutral particle be

$$\phi \to \phi' = e^{i\alpha} \star \phi \star e^{-i\alpha} = U \star \phi \star U^\dagger \tag{2.27}$$

and let the covariant derivative be

$$D_\mu \phi = \partial_\mu \phi + ie \left[A_\mu \overset{\star}{,} \phi \right] \tag{2.28}$$

One can work through the gauge transformation for the covariant derivative and find

$$D_\mu \phi \to D'_\mu \phi' = U \star D_\mu \phi \star U^\dagger \tag{2.29}$$

III. FEYNMAN RULES AND PHENOMENA

A. Feynman rules for QED

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} \star F^{\mu\nu} + \bar{\psi} \star (i \not{\partial} - e \not{A} - m) \star \psi \tag{3.1}$$

Perturbation theory.

Lowest order,

$$\mathcal{L}_0 = -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu) \star (\partial^\mu A^\nu - \partial^\nu A^\mu) + \bar{\psi} \star (i \not{\partial} - m) \psi \tag{3.2}$$

Quadratic in fields. Only action matters. In action, can remove star for terms that are quadratic in fields.

Hence same as commutative case.

Hence propagators same as commutative case.

Also fourier expansions of LO fields same as commutative case.

Interactions: “matter”

$$\mathcal{L} = -e \psi \star \gamma^\mu A_\mu \star \psi \tag{3.3}$$

same as

$$\mathcal{L} = -e\bar{\psi}\gamma^\mu A_\mu \star \psi = -e\bar{\psi}\gamma^\mu \left\{ A_\mu \exp\left[\frac{i}{2}\overleftarrow{\partial}_\rho \theta^{\rho\nu}\partial_\nu\right]\psi\right\} \tag{3.4}$$

Recall the field expansions,

$$\psi(x) = \sum_\lambda \int [dp]\left(a(p,\lambda)u(p,\lambda)e^{-ipx} + b^\dagger(p,\lambda)v(p,\lambda)e^{+ipx}\right) \tag{3.5}$$

(where $[dp] \equiv d^3p/((2\pi)^3 2E)$ and λ is a spin or helicity index), with similar expansions for A_μ and $\bar{\psi}$.

A derivative ∂_ν on this field becomes (ip_ν) inside the expansion, for an outgoing particle. For writing the Feynman rules, let all the momenta be outgoing.

Then with abuse of notation,

$$\mathcal{L} = -e\bar{\psi}\gamma^\mu A_\mu \star \psi = -e\bar{\psi}\gamma^\mu A_\mu \exp\left[\frac{-i}{2}q_\rho \theta^{\rho\nu} p_\nu\right]\psi \tag{3.6}$$

Hence we can read off the Feynman rule for the γee vertex (remembering the vertex goes like $i\mathcal{L}$),

$$\text{vertex} = -ie\gamma^\mu e^{ip'\wedge p} \tag{3.7}$$

which is the standard commutative vertex times a phase factor.

The notation is standard,

$$p' \wedge p \equiv \frac{1}{2}\, p'_\rho \theta^{\rho\nu} p_\nu \tag{3.8}$$

and we changed the writing by using momentum conservation, $q = -p' - p$.

Interactions: the triple photon vertex

One of the interaction terms is

$$\mathcal{L}_{3\gamma} = -\frac{1}{2} ie\,(\partial_\mu A_\nu - \partial_\nu A_\mu) \star [A^\mu \overset{\star}{,} A^\nu] \tag{3.9}$$

which, dropping the first $\star$ and again abusing the notation, can become

$$\mathcal{L}_{3\gamma} = -\frac{1}{2} ie\,(iq_{a\mu} A_\nu - iq_{a\nu} A_\mu)\, A^\mu A^\nu\, [\exp(-iq_b \wedge q_c) - \exp(+iq_b \wedge q_c)] \tag{3.10}$$

which leads to a Feynman rule (there is only a common sine function, since we can use momentum conservation to show $q_1 \wedge q_2 = q_2 \wedge q_3 = q_3 \wedge q_1$)

$$V_{3\gamma} = -2e \sin(q_1 \wedge q_2) \left\{ g_{\mu\nu}(q_{1\rho} - q_{2_\rho}) + g_{\nu\rho}(q_{2\mu} - q_{3_\mu}) + g_{\rho\mu}(q_{3\nu} - q_{1_\nu}) \right\} \tag{3.11}$$

The triple photon vertex violates charge conjugation invariance (i.e., violates Furry's theorem).

Interactions: the quadruple photon vertex

The final interaction terms is

$$\mathcal{L}_{4\gamma} = \frac{1}{4}\,[A^\mu \stackrel{\star}{,} A^\nu] \star [A_\mu \stackrel{\star}{,} A_\nu] \tag{3.12}$$

which leads to a 4 photon vertex that is,

$$\begin{aligned} V_{4\gamma} = -4ie^2 \; \Big[\; & \sin(q_1 \wedge q_2) \sin(q_3 \wedge q_4) \left(g_{\mu_1\mu_3} g_{\mu_2\mu_4} - g_{\mu_1\mu_4} g_{\mu_2\mu_3} \right) \\ + \; & \sin(q_3 \wedge q_1) \sin(q_2 \wedge q_4) \left(g_{\mu_1\mu_4} g_{\mu_2\mu_3} - g_{\mu_1\mu_2} g_{\mu_2\mu_4} \right) \\ + \; & \sin(q_1 \wedge q_4) \sin(q_2 \wedge q_3) \left(g_{\mu_1\mu_2} g_{\mu_3\mu_4} - g_{\mu_1\mu_3} g_{\mu_2\mu_4} \right) \Big] \end{aligned} \tag{3.13}$$

Clearly 0 for $\theta^{\mu\nu} \to 0$.

B. Loop integrals

We will talk about loops more below.

One early hope of NCFT was that divergences would be less severe if we "fuzzed out" the space-time points. However, it was not so. For many loop calculations in ordinary QED and other theories, the calculations are unchanged by the noncommutativity and the divergences are the same. [*Give example.*] Some loops, however, become less UV divergent with noncommutativity.

But there is a surprise: the $\theta^{\mu\nu} \to 0$ behavior is often unexpected and not continuously connected to what one would expect for $\theta^{\mu\nu} = 0$. An example comes later when we look at NCQED in a supersymmetric context, but the IR problem is neither helped nor hurt by the supersymmetry.

C. Scattering and decay phenomena

• Noncommutative field theories hot topic among formal field theorists

• There has been effort trying to see connection to experimental world.

• So far, have only discussed canonical version, which has Lorentz violation. We will soon discuss bounds which appear to make canonical NC unobservable in accelerator experiments. But the arguments for the bounds use expansions in $\mathcal{O}(\theta)$ or need cutoffs; there are those who argue that full theory may give looser bound, allowing discussion of accelerator Lorentz violating signatures of NC (and also CP violation from NC).

References (partial):

Hewett, Petriello, & Rizzo	Hinchliffe & Kersting
Liao & Grosse	Godfrey & Doncheski
Baek, Ghosh, He, & Hwang	Mahajan
Chaichian, Sheikh-Jabbari, & Tureanu	Iltan

Typically, the idea is that the 4×4 array $\theta^{\mu\nu}$ defines two fixed directions, one electric field-like, θ^{i0}, and one magnetic field-like $\epsilon^{ijk}\theta_{jk}$. As earth turns, these vectors turn in the laboratory frame, and so one may see effects that have a 24 hour [$23^h 56^m$] period.

Give two examples

Example 1: Bhabha scattering, $e^- e^+ \to e^- e^+$

(Hewett, Petriello, & Rizzo)

In the canonical version of NCQED, the only changes to the vertices are the phases. Hence, the only change will be in the interference of the two diagrams. The t-channel diagram gets a phase

$$\exp\left(i(-p_2) \wedge p_4 + ip_3 \wedge (-p_1)\right) \tag{3.14}$$

and the s-channel diagram has a phase,

$$\exp\left(ip_3 \wedge p_4 + ip_2 \wedge p_1\right) \ . \tag{3.15}$$

Keep only $\theta^{0i} = -\theta^{i0} \equiv (\vec{c}_E)^i$ and get the t-channel phase to be (in the CM)

$$\exp\left[-iE\ \vec{c}_E \cdot (\vec{p}_4 - \vec{p}_2)\right] \tag{3.16}$$

while the s-channel phase is just the opposite.

Easy setup to visualize: accelerator at equator, beam heading north, $\vec{c}_E$ perpendicular to earth rotation axis but fixed in direction, scattered particle heads east—a direction rotating with the earth and at angle ϕ to $\vec{c}_E$. The phase is

$$\exp\left[-iE|\vec{c}_E||\vec{p}_4|\cos\phi(t)\right] \tag{3.17}$$

and the cross section has a term with time dependence

$$\cos\left[\ldots\cos\phi(t)\right] \; , \tag{3.18}$$

whose effect is visible in the plot borrowed from Hewett *et al.*

IV. BOUNDING CANONICAL NC THEORIES

Best bounds: from low-energy already-done experiments that searched for Lorentz non-invariance.
Reminder: $\theta^{\mu\nu}$ is a fixed matrix in the canonical version of NCFT,
θ^{i0} and $\epsilon^{ijk}\theta_{jk}$ are fixed three-vectors that define preferred directions in a given Lorentz frame.
By loops and HO corrections, can define operators like

$$\mathcal{O}_1 \; = \; m_e \theta^{\mu\nu} \bar{\psi} \sigma_{\mu\nu} \psi$$

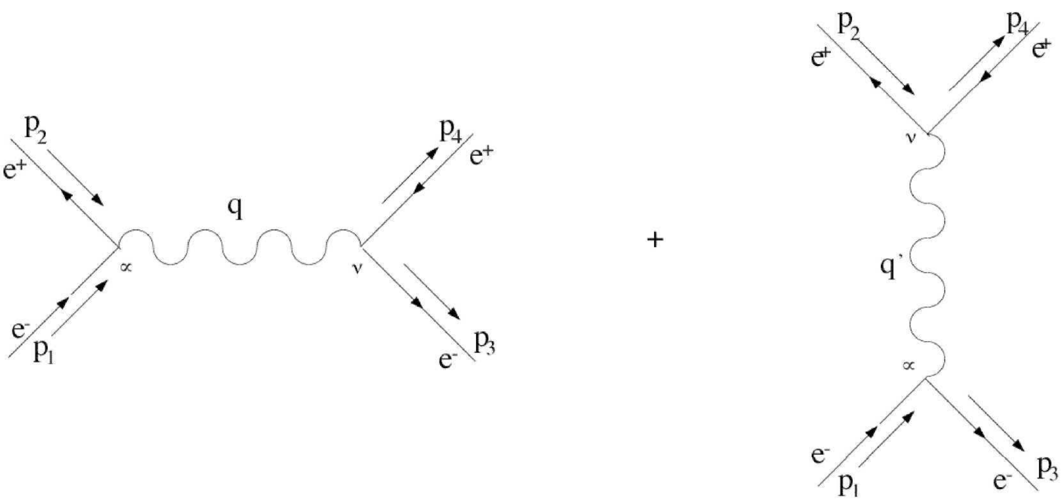


FIG. 1: Bhabha scattering

$$
\begin{aligned}
\mathcal{O}_2 &= \theta^{\mu\nu}\bar{\psi}D_\mu\gamma_\nu\psi \\
\mathcal{O}_3 &= \theta^{\mu\nu}F_{\mu\rho}F^{\rho}_{\ \nu} \\
\mathcal{O}_4 &= \theta^{\mu\nu}\theta^{\rho\sigma}F_{\mu\nu}F_{\rho\sigma}
\end{aligned} \tag{4.1}
$$

These are all Lorentz violating, if $\theta^{\mu\nu}$ is fixed.

Where do they come from. Ans.: Loops, for example Fig. 3.

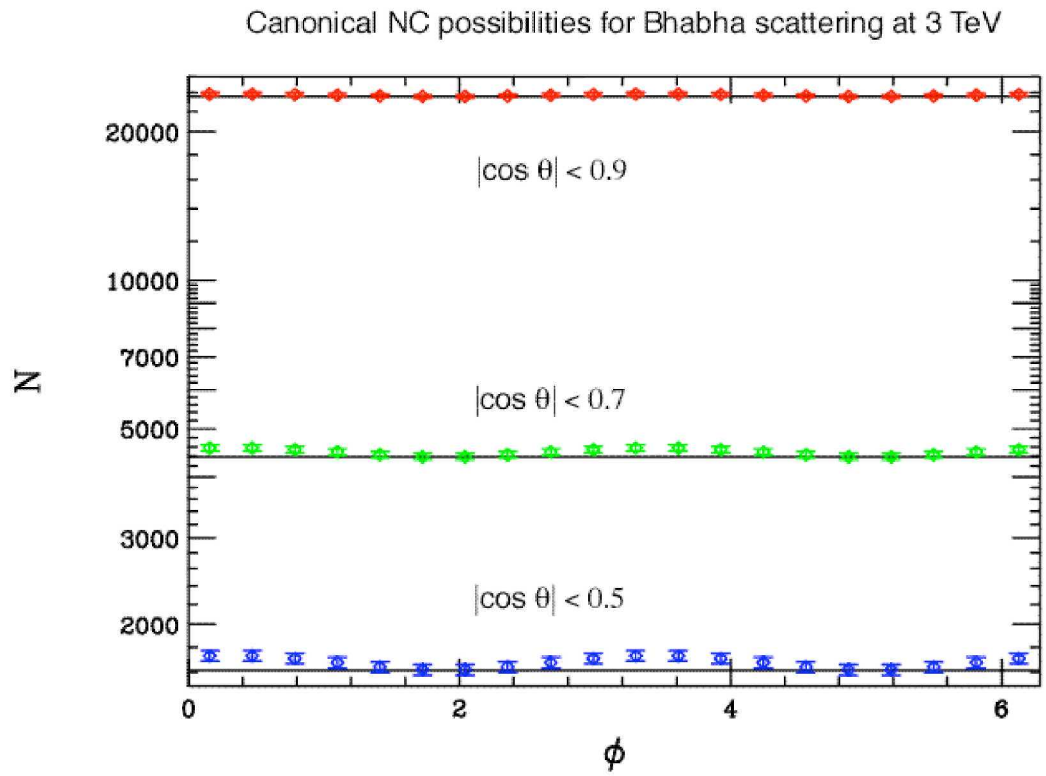


FIG. 2: Bhabha possibilities for 3 TeV and integrated luminosity 1 ab^{-1}.

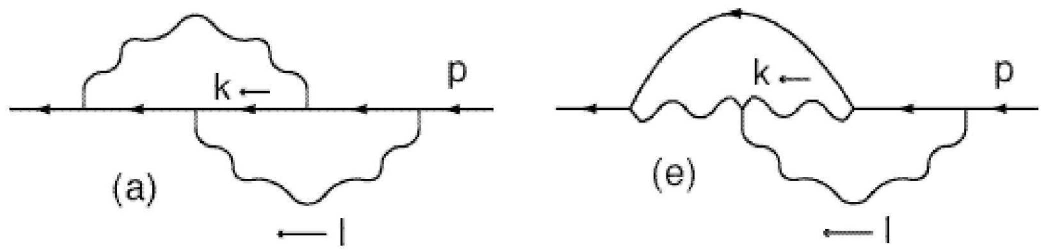


FIG. 3: Two-loop diagrams.

Consider $\mathcal{O}_1$.

Approach 1

Can do the integrals without a cutoff! Will show how. For now quote result,

$$\text{amplitude} = \frac{1}{8} m_e \alpha^2 \frac{\sigma_{\mu\nu}\theta^{\mu\nu}}{\sqrt{(-1/2)\mathrm{Tr}\,\theta^2}} \; . \tag{4.2}$$

Like $\text{coeff.} \times \vec{\sigma} \cdot \hat{n}$, where $\hat{n}$ is unit vector pointing in direction of $\epsilon^{ijk}\theta_{jk}$.

Calculated coeff. $= \frac{1}{8} m_e \alpha^2 = \frac{1}{4} R_\infty \sim 10 \text{ eV}$

Experimental bound $< O(10^{-19}) \text{ eV}$

Can't be right!

Approach 2

Cut off the integrals at some scale Λ. Motivation: belief in some "new physics" above this scale that alters and physically cuts off the interactions.

$$\text{new result} = \tfrac{3}{4} m_e \Lambda^2 \left(\tfrac{\alpha}{4\pi}\right)^2 \sigma_{\mu\nu}\theta^{\mu\nu}$$

From the experimental bound above,

$$\theta\Lambda^2 < 10^{-19} \tag{4.3}$$

where θ is some scale size for $\theta^{\mu\nu}$.

For collider experiment in process without a loop, there would be a factor

$$\theta E^2 \tag{4.4}$$

where E is some energy scale in the experiment. At most, E is about a TeV; at least Λ is about a TeV. So, if above is right, cannot expect to see much in collider experiment.

But a cutoff:

1. Violates gauge invariance

2. Has imprecise physical interpretation

Approach 3

Find a cutoff that is physical and preserves gauge invariance.

Consider ”softly” broken supersymmetric NCQED.

(”Softly” means the supersymmetry breaking is only in the mass terms.)

In the supersymmetric limit [13], the dangerous Lorentz-violating operator is forbidden:

show either by explicit calculation, loops involving the superpartners of the known electron and photon exactly cancel loops involving only already discovered particles

or by showing that there is no supersymmetric way to write down an operator like $\mathcal{O}_1 = m_e\theta^{\mu\nu}\bar{\psi}\sigma_{\mu\nu}\psi$.

Giving the superpartners a mass M different from the electron and photon mass, the dangerous operator is again generated; however, supersymmetric cancellations eliminate contributions from the ultraviolet part of the loop integrals. Thus, M serves as an effective cutoff that preserves the gauge invariance of the theory.

but too bad: the bound gets stronger.

A. The Dangerous Operator in Supersymmetric NCQED

Operational primer on supersymmetric QED

“Old” particles: electron and photon

New particles:

Photino—the partner of the photon.

- Spin 1/2 Majorana fermion (meaning self conjugate under charge conjugation)
- Count degrees of freedom: a photon has two degrees of freedom (polarizations) and so does a self conjugate spin-1/2 particle
- Represent as wavy line with straight line core on Feynman diagram (see actual diagrams).

Selectron—the partner of the electron

- Spin-0
- One scalar for left-handed electrons, another partnering right-handed electrons. Hence "left-handed scalars" and "right-handed scalars."
- Represent as dashed lines
- Charged, so put arrow on line to indicate direction of charge flow
- Count degrees of freedom: they match

There are Feynman rules for the interactions of the the supersymmetric particles. For the NC case, the rules for the new vertices are summarized in Figure 4.

Calculate the two-loop contribution to the operator in Eq. (4.1). In ordinary NCQED, there is no one-loop diagram that contributes to $\mathcal{O}_1$. Two-loop diagrams that contribute are shown in Figs. 5 and 6.

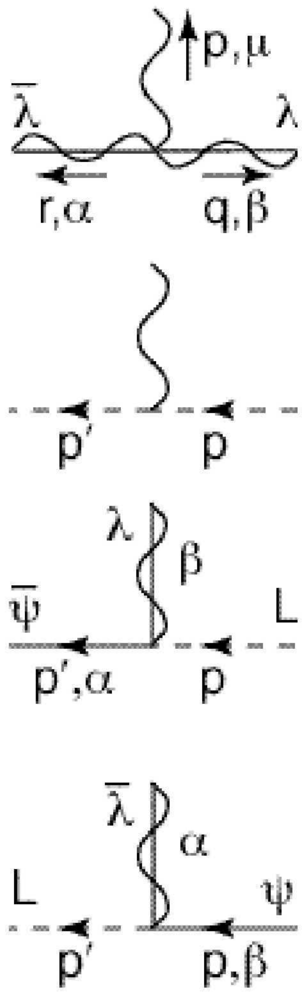
We will extract the terms proportional to $\sigma_{\mu\nu}$ and work on shell (*i.e.*, we evaluate the diagrams between spinors $\bar{u}(p)$ and $u(p)$ and use $\not{p}u(p) = m_e u(p)$.) For each of the 8 diagrams, the $\sigma_{\mu\nu}$ terms are proportional to the electron mass m_e. After extracting the overall electron mass factor, we set the electron mass and momentum p to zero in the integrals as a simplifying assumption. This leads to corrections in the final result that are wholly negligible as far as our numerical analysis is concerned.

Each of the diagrams with only electrons and photons, (a) and (e), give identical results, and the sum of the two is

$$\mathcal{M}_a + \mathcal{M}_e = 24 i m_e e^4 \int (dk)(dl) \, \frac{e^{i\, l\cdot\theta\cdot k} \sigma_{\mu\nu} k^\mu l^\nu}{k^2 l^4 (k+l)^4} \; , \tag{4.5}$$

where $(dk) \equiv d^4k/(2\pi)^4$ and $l \cdot \theta \cdot k \equiv l_\mu \theta^{\mu\nu} k_\nu$. The result (using techniques shown below) is

$$\mathcal{M}_a + \mathcal{M}_e = \frac{1}{8} m_e \alpha^2 \frac{\sigma_{\mu\nu}\theta^{\mu\nu}}{\sqrt{(-1/2)\mathrm{Tr}\, \theta^2}} \; . \tag{4.6}$$



$$2e\sin(\tfrac{1}{2}r\cdot\theta\cdot q)(\gamma_\mu)_{\alpha\beta}$$

$$-ie\exp(\tfrac{i}{2}p\cdot\theta\cdot p')(p'+p)_\mu$$

$$\frac{e}{\sqrt{2}}(1+\gamma_5)_{\alpha\beta}\exp(\tfrac{i}{2}p\cdot\theta\cdot p')$$

$$\frac{-e}{\sqrt{2}}(1-\gamma_5)_{\alpha\beta}\exp(\tfrac{i}{2}p\cdot\theta\cdot p')$$

FIG. 4: Feynman rules for superpartners in noncommutative supersymmetric QED. The rules for the right-handed scalars can be obtained from the left-handed ones shown by $\gamma_5 \to -\gamma_5$. ψ and λ represent electrons and photinos, respectively. Our sign conventions are based on those of Ref. [20].

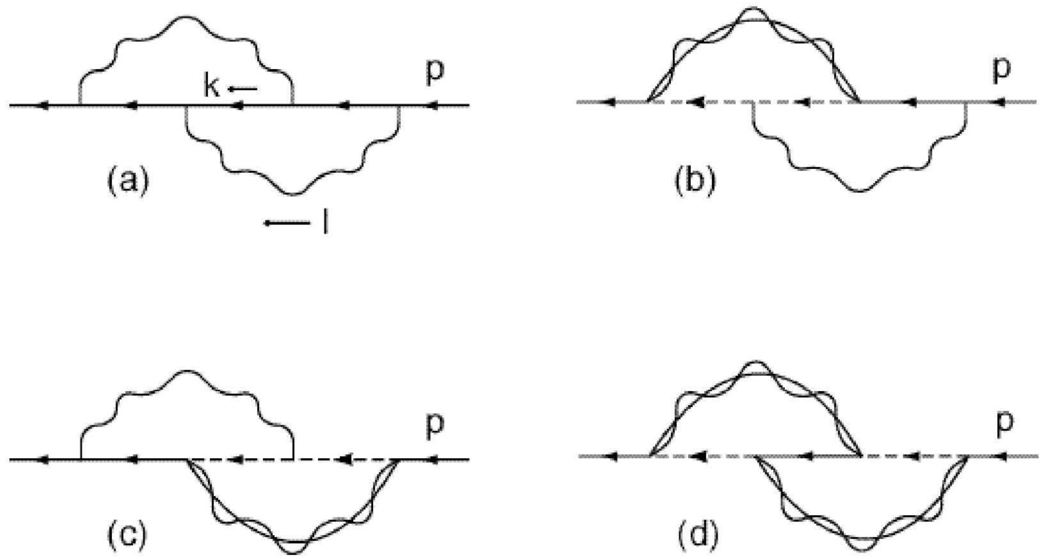


FIG. 5: Two-loop diagrams with two gauge multiplet propagators. Solid lines represent electrons, wavy lines represent photons, wavy lines with a solid core represent photinos, and dashed lines represent selectrons.

Now consider the diagrams with superpartners. The four diagrams with three superpartner propagators all give the same result, at least to the operator $\sigma_{\mu\nu}\theta^{\mu\nu}$, and similarly for the two diagrams with four superpartner propagators. We will give some detail of how the diagrams are evaluated.

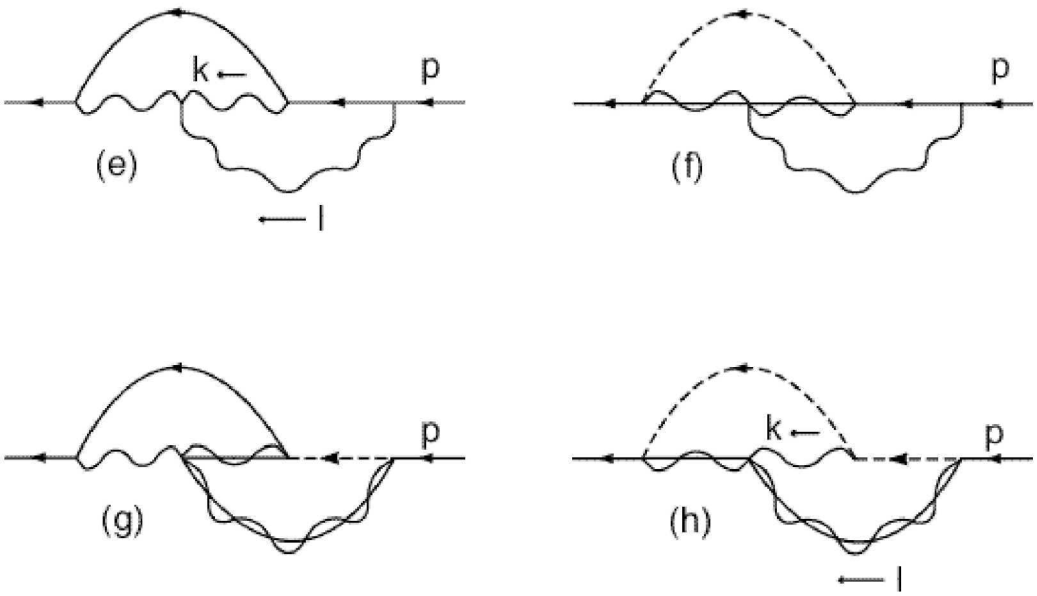


FIG. 6: Two-loop diagrams with three gauge multiplet propagators.

Using diagram (h), as an example, we have

$$\mathcal{M}_h = -4im_e e^4 \int (dk)(dl) \frac{e^{i\,l\cdot\theta\cdot k}\sigma_{\mu\nu}k^\mu l^\nu}{k^2(l^2-M^2)^2((k+l)^2-M^2)^2}\,, \tag{4.7}$$

where M is a common superpartner mass. We can combine denominators using a Feynman parameter, and shift one of the integration momenta to obtain

$$\mathcal{M}_h = -24 i m_e e^4 \int_0^1 dx\, x(1-x) \int (dk)(dl) \frac{e^{i\, l\cdot\theta\cdot k} \sigma_{\mu\nu} k^\mu l^\nu}{k^2 \left[l^2 + x(1-x)k^2 - M^2\right]^4} \,. \tag{4.8}$$

Say that only $\theta_{12} = -\theta_{21} \equiv \theta \neq 0$. Then

$$\mathcal{M}_h = 4 m_e e^4 \sigma_{12} \frac{\partial}{\partial\theta} J_h \,, \tag{4.9}$$

where after rescaling k and Euclideanizing, we have

$$J_h = 6 \int_0^1 dx \int (dk)(dl)\, \frac{e^{i(l_1 k_2 - l_2 k_1)\theta/\sqrt{x(1-x)}}}{k^2 \left[l^2 + k^2 + M^2\right]^4} \,. \tag{4.10}$$

Now the $dl_0 dl_3$ integrals can be done. After combining the remaining denominators using another Feynman parameter and rescaling the remaining components of l, we get

$$J_h = \frac{3}{8\pi^3} \int_0^1 dx \int_0^1 dy\, y \int (dk) dl_1 dl_2\, \frac{e^{i(l_1 k_2 - l_2 k_1)\theta/\sqrt{yx(1-x)}}}{\left[k^2 + l_1^2 + l_2^2 + y M^2\right]^4} \,. \tag{4.11}$$

Now do the $dk_0 dk_3$ integrals, and put the denominators into the exponential using a Schwinger parameter. After one more rescaling of the remaining momenta, we have

$$\begin{aligned} J_h &= \frac{1}{256\pi^6} \int_0^1 dx \int_0^1 dy\, y \int_0^\infty dz \int dk_1 dk_2 dl_1 dl_2 \\ &\quad\times\; e^{-yzM^2 - k_1^2 - k_2^2 - l_1^2 - l_2^2 + i(l_1 k_2 - l_2 k_1)\theta/\left(z\sqrt{yx(1-x)}\right)} \\ &= \frac{1}{256\pi^4} \int_0^1 dx \int_0^1 dy\, y \int_0^\infty dz \frac{4z^2 yx(1-x)}{4z^2 yx(1-x) + \theta^2} e^{-yzM^2} \,. \end{aligned} \tag{4.12}$$

Thus,

$$\mathcal{M}_h = -\frac{m_e \alpha^2}{2\pi^2}\sigma_{12} \int_0^1 dx \int_0^1 dy\, y \int_0^\infty dz \frac{4z^2 yx(1-x)\theta}{(4z^2 yx(1-x)+\theta^2)^2} e^{-yzM^2} \ . \tag{4.13}$$

The end result for $\mathcal{M}_f$ is a similar expression, but with the front integrals reading

$$\int_0^1 dx\, x \int_0^1 dy\, (1-y) \ldots \ . \tag{4.14}$$

Noting that the rest of the integrand is symmetric under inversion about $x = 1/2$, we can replace "x" in the line above by "$(x - 1/2) + 1/2$," and keep only the "1/2." The 6 graphs involving superpartners sum to

$$4\mathcal{M}_f + 2\mathcal{M}_h = -\frac{m_e \alpha^2}{\pi^2}\sigma_{12} \int_0^1 dx \int_0^1 dy \int_0^\infty dz \frac{4z^2 yx(1-x)\theta}{(4z^2 yx(1-x)+\theta^2)^2} e^{-yzM^2} \ . \tag{4.15}$$

In the supersymmetric limit $M \to 0$, the integrals can be done exactly. For general M, it is convenient to rescale z,

$$4\mathcal{M}_f + 2\mathcal{M}_h = -\frac{m_e \alpha^2}{\pi^2}\frac{\sigma_{12}}{(\theta M^2)^3} \int_0^1 dx \int_0^1 dy \int_0^\infty dz \frac{4z^2 x(1-x)}{(4z^2 x(1-x)(\theta M^2)^{-2}+y)^2} e^{-z} \ . \tag{4.16}$$

The y and x integrals can both be done, and the full result for the two-loop contributions to $\mathcal{O}_1$ becomes

$$\mathcal{M} = \sum_{I=a}^{g} \mathcal{M}_I = \frac{1}{8} m_e \alpha^2 \frac{\sigma_{\mu\nu}\theta^{\mu\nu}}{\sqrt{(-1/2)\mathrm{Tr}\,\theta^2}} L(\theta_M) \ , \tag{4.17}$$

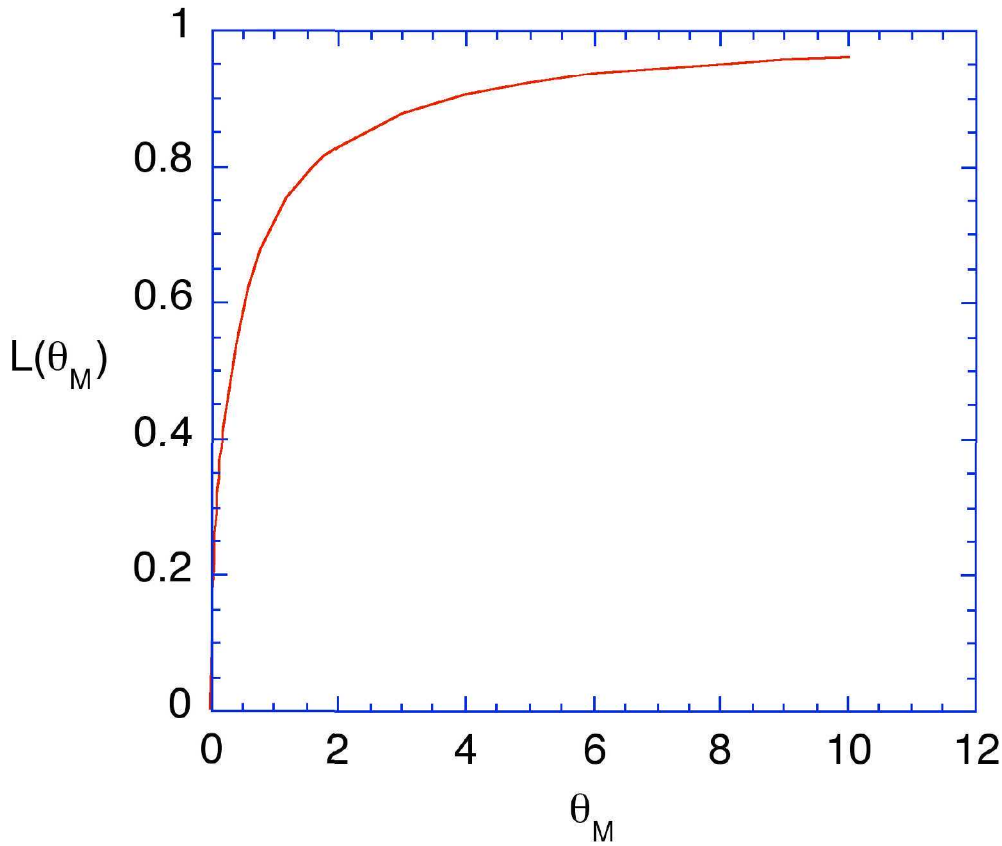


FIG. 7: The function $L(\theta_M)$.

where $\theta_M \equiv M^2\sqrt{(-1/2)\mathrm{Tr}\,\theta^2}$ and

$$L(\theta_M) = 1 - \frac{2\theta_M}{\pi^2}\int_0^\infty \frac{dz}{z\sqrt{z^2+\theta_M^2}} e^{-z} \ln\frac{\sqrt{z^2+\theta_M^2}+z}{\sqrt{z^2+\theta_M^2}-z} \, . \tag{4.18}$$

The z integral can be computed analytically, but the answer is not enlightening and we do not show it. Function L satisfies $L(0) = 0$ and $L(\infty) = 1$ and is shown in Fig. 7.

For the choice $\theta^{0i} = 0$ [14], $\theta^{\mu\nu}$ defines a 3-vector in a fixed direction $\hat{n}$ (where $\hat{n}$ is a unit vector) and the result (4.17) can be written as a effective Lagrangian

$$\mathcal{L}_{\text{eff}} = \frac{1}{2} R_\infty \, L(\theta_M)\, \vec{\sigma}\cdot\hat{n} \; , \tag{4.19}$$

where $R_\infty \equiv m_e\alpha^2/2 = 13.6$ eV. Searches for such a term in magnetic systems [15] show that matrix elements of $\mathcal{L}_{\text{eff}}$ are below 10^{-19} eV. Doing without any cutoff, $L \to 1$, is impossible. One must get a severe suppression from $L(\theta_M)$, requiring $\theta_M \equiv M^2/\Lambda_{NC}^2 \ll 1$. The slope of L near the origin is infinite, meaning that $L(\theta_M)$ has a nonanalytic behavior for $\theta_M \to 0$. Numerical evaluations suggest

$$L(\theta_M) \approx 3\,(\theta_M)^{0.78} \tag{4.20}$$

for small argument. From this we estimate $\theta_M \lesssim 10^{-26}$ or $\Lambda_{NC} \gtrsim 10^{13} M$.

B. Conclusions

- can use supersymmetry-breaking mass as a gauge-invariant regulator
- The bound that follows from searches for Lorentz violation in magnetic systems (Bluhm *et al.*) is $\theta M^2 \lesssim 10^{-26}$
- If nature uses noncommutative coordinates, it need not be done with a Lorentz-violating implementation ....

V. NCQED FOR ANY CHARGE

Problem: the group is commutative,

But: NC multiplication does not respect the commutation properties of the group.

That is, we should have

$$U_\alpha \star U_\beta \star \psi = U_\beta \star U_\alpha \star \psi \tag{5.1}$$

where U_α

$$U_\alpha = e^{i\alpha(x)} \tag{5.2}$$

but we don't.

Easier to present the problem and solution (Jurčo *et al.*) in infinitesimal form. Have

$$\delta_\alpha \psi = i\alpha \star \psi \tag{5.3}$$

and the problem that

$$\delta_\beta \delta_\alpha \psi = i\alpha \star i\beta \star \psi \neq \delta_\alpha \delta_\beta \psi = i\beta \star i\alpha \star \psi \tag{5.4}$$

What we shall do is change the transformation to

$$\delta_\alpha \psi = i\Lambda_\alpha \star \psi \tag{5.5}$$

where $\Lambda_\alpha = \alpha$ in the commutative limit, but is otherwise yet to be determined using the requirement that

$$(\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha)\, \psi = 0\ , \tag{5.6}$$

New: implement the gauge transformation with the gauge function Λ having some dependence on the gauge field A_μ, $\Lambda_\alpha = \Lambda_\alpha[A_\mu]$. Then,

$$\delta_\beta \delta_\alpha \psi = \delta_\beta \left(i\Lambda_\alpha \star \psi\right) = i\Lambda_\alpha \star \delta_\beta \psi + i\left(\delta_\beta \Lambda_\alpha\right) \star \psi = i\Lambda_\alpha \star i\Lambda_\beta \star \psi + i\left(\delta_\beta \Lambda_\alpha\right) \star \psi = (\alpha \leftrightarrow \beta) \tag{5.7}$$

or

$$-\Lambda_\alpha \star \Lambda_\beta + i\delta_\beta \Lambda_\alpha = -\Lambda_\beta \star \Lambda_\alpha + i\delta_\alpha \Lambda_\beta \tag{5.8}$$

Solve by expansion,

$$\begin{aligned}\Lambda_\alpha &= \alpha + c_1\theta^{\mu\nu}A_\mu\partial_\nu\alpha + \dots \\ \delta_\beta\Lambda_\alpha &= c_1\theta^{\mu\nu}\partial_\mu\beta\partial_\nu\alpha\end{aligned} \tag{5.9}$$

The product $\Lambda_\alpha \star \Lambda_\beta$ has a number of terms, but keeping just the ones that matter from the Moyal product gives

$$\Lambda_\alpha \star \Lambda_\beta = \alpha\beta + \frac{i}{2}\theta^{\mu\nu}\partial_\mu\alpha\partial_\nu\beta \tag{5.10}$$

and one can get a solution using $c_1 = -1/2$, so that

$$\Lambda_\alpha = \alpha - \frac{1}{2}\theta^{\mu\nu}A_\mu\partial_\nu\alpha + \dots \equiv \alpha + \Lambda_\alpha^{(1)} \tag{5.11}$$

Have to make the gauge transformation work for the matter field also. Expect,

$$\delta_\alpha\psi = i\Lambda_\alpha \star \psi \tag{5.12}$$

Need to modify and expand ψ also,

$$\psi = \psi[A_\mu] = \psi^0 + \psi^1 + \dots\ , \tag{5.13}$$

where ψ^0 is the usual field in the commutative limit, with $\delta_\alpha\psi^0 = i\alpha\psi$.

To first order in θ,

$$\delta\psi^0 + \delta\psi^1 = i(\alpha + \Lambda^1) \star (\psi^0 + \psi^1) = i\alpha\psi^0 + \frac{i^2}{2}\theta^{\mu\nu}\partial_\mu\alpha\partial_\nu\psi^0 + i\Lambda^1_\alpha\psi^0 + i\alpha\psi^1 \tag{5.14}$$

Try a solution in the form

$$\psi^1 = c_1\theta^{\mu\nu}A_\mu\partial_\nu\psi^0 \tag{5.15}$$

(other terms are possible, but turn out unneeded), and find it works for $c_1 = -1/2$. Hence the matter field has become

$$\psi = \psi^0 - \frac{1}{2}\theta^{\mu\nu}A_\mu\partial_\nu\psi^0 \tag{5.16}$$

Make sure the gauge field has the correct gauge transformation also.

$$A_\mu \to U \star A_\mu \star U^\dagger - i\left(\partial_\mu U\right) \star U^\dagger \tag{5.17}$$

becomes

$$\begin{aligned} \delta_\alpha A_\mu &= \partial_\mu\Lambda_\alpha + i\left[\Lambda_\alpha \overset{\star}{,} A_\mu\right] \\ &\overset{\rightarrow}{=} \partial_\mu\alpha + \partial_\mu\Lambda^{(1)} + i^2\theta^{\rho\nu}\partial_\rho\alpha\partial_\nu A_\mu + \ldots \end{aligned} \tag{5.18}$$

Again, we need an expansion,

$$A_\mu = A^0_\mu + A^1_\mu + \ldots = A^0_\mu + \theta^{\rho\nu}\left(c_1 A_\rho\partial_\nu A^0_\mu + c_2 A_\rho\partial_\mu A^0_\nu\right) + \ldots \tag{5.19}$$

which we discover does give a solution, with $c_1 = -1$ and $c_2 = -1/2$. The gauge field has become

$$A_\mu = A_\mu^0 + \theta^{\rho\nu}\left(\frac{1}{2}A_\rho\partial_\mu A_\nu^0 - A_\rho\partial_\nu A_\mu^0\right) + \dots \tag{5.20}$$

Get the Feynman rules, in the context of an expansion in θ of the (by now) usual

$$S = \int (dx)\left\{-\frac{1}{4}F^{\mu\nu}\star F_{\mu\nu} + \bar{\psi}\star(i\not{D} - m)\star\psi\right\} \tag{5.21}$$

Use the modified fields in the QED Lagrangian. "Free" fields are ψ^0 and A_μ^0, and the expansions give new interaction terms and additional Feynman rules.

Comments

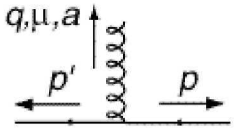
• Can show this works for a field ϕ of any charge Q, by letting $\delta_\alpha\phi^0 = iQ\alpha\phi$ and inserting $A_\mu \to QA_\mu$ wherever A_μ appears.

• Similar procedure works for $SU(N)$ non-Abelian gauge theories. Listed the Feynman rules for this case on next page.

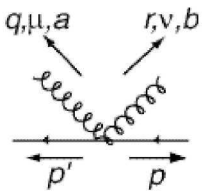
• Clear disadvantage of expansion: cannot do divergent loops, so cannot fully discuss renormalization.

• Will drop this discussion here. For QED, will discuss loops in the context of just electrons and photons, where an unexpanded NC theory does work. Regarding this expansion, we will see what use can be made of something clearly inspired by it when we try Lorentz covariant noncommutative field theory.

Revised Feynman rules (QCD version), $\mathcal{O}(\theta)$ parts only (Carlson, Carone, & Lebed)

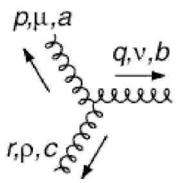


$$\frac{1}{2} g T^a \Big[(\theta p)^\mu \, (\not p' - \, m) - (\theta p')^\mu \, (\not p + \, m) - (p' \theta p) \gamma^\mu \Big]$$

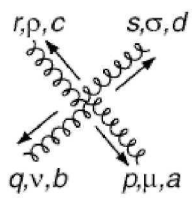


$$\frac{1}{2} g^2 \Big\{ T^a T^b [m \, \theta^{\mu\nu} + \theta^{\mu\nu\rho} (p+q)_\rho] - T^b T^a [m \, \theta^{\mu\nu} + \theta^{\mu\nu\rho} (p+r)_\rho] \Big\};$$

$$\theta^{\mu\nu\rho} \equiv \theta^{\mu\nu} \gamma^\rho + \theta^{\nu\rho} \gamma^\mu + \theta^{\rho\mu} \gamma^\nu$$



$$\frac{1}{2} g d_{abc} \Big\{ (r \theta q) \Big[(q-r)^\mu g^{\nu\rho} + \ldots \Big] + \text{long expression} \Big\}$$



$$\Big(\text{longer expression} \Big)$$

VI. LORENTZ COVARIANT NCFT

A. Lorentz covariant noncommuting coordinates

Canonical version of noncommutative spacetime

$$[\hat{x}^\mu\,,\,\hat{x}^\nu] = i\,\theta^{\mu\nu} \quad , \tag{6.1}$$

where $\theta^{\mu\nu}$ is a real, constant matrix of ordinary c-numbers. Found trouble with Lorentz noninvariance. Solution would seem simple: find a Lorentz covariant operator to replace $\theta^{\mu\nu}$. Question becomes what operator to choose, and how to realize the algebra in practice.

H. Snyder (1947) had an idea, carried through in context of quantum mechanics. (Motivation in 1947 was hope that this would lead to a divergence free field theory.) He proposed an algebra

$$\begin{aligned}
[\hat{x}^\mu, \hat{x}^\nu] &= ia^2 \hat{M}^{\mu\nu} \ , \\
\left[\hat{M}^{\mu\nu}, \hat{x}^\lambda\right] &= i\left(\hat{x}^\mu g^{\nu\lambda} - \hat{x}^\nu g^{\mu\lambda}\right) \ , \\
\left[\hat{M}^{\mu\nu}, \hat{M}^{\alpha\beta}\right] &= i\left(\hat{M}^{\mu\beta} g^{\nu\alpha} + \hat{M}^{\nu\alpha} g^{\mu\beta} - \hat{M}^{\mu\alpha} g^{\nu\beta} - \hat{M}^{\nu\beta} g^{\mu\alpha}\right) \ ,
\end{aligned} \tag{6.2}$$

where the operators $\hat{M}^{\mu\nu}$ were the Lorentz group, including angular momentum, generators. Second two commutators standard for Lorentz group, first was new; $g^{\mu\nu} = diag\,(+,-,-,-)$.

(By the way, $\hat{M}$ and $\hat{x}/a$ generate an SO(4,1) group.)

Snyder's explicit realization: use a 5-dimensional space with coordinates

$$\eta_0, \ldots, \eta_4 \qquad \text{and metric} \qquad diag\,(+,-,-,-,-) \tag{6.3}$$

Let $\eta_\mu \equiv (\eta_0, \eta_1, \eta_2, \eta_3)$, $\eta^\mu \equiv (\eta_0, -\eta_1, -\eta_2, -\eta_3)$, and

$$\begin{aligned} \hat{x}^\mu &= ia\left(\eta_4 \frac{\partial}{\partial \eta_\mu} + \eta^\mu \frac{\partial}{\partial \eta_4}\right) \,, \\ \hat{M}^{\mu\nu} &= i\left(\eta^\mu \frac{\partial}{\partial \eta_\nu} - \eta^\nu \frac{\partial}{\partial \eta_\mu}\right) \,. \end{aligned} \tag{6.4}$$

It is easy to check that this gives the CR of Eq. (6.2). Transformations that leave both η_4 and $\eta_0^2 - \eta_1^2 - \eta_2^2 - \eta_3^2 - \eta_4^2$ invariant induce ordinary Lorentz transformations on the coordinates $\hat{x}^\mu$. (Also, from Eq. (6.4), one can show that the spatial coordinate operators $\hat{x}^i$ do not have a continuous spectrum, but have eigenvalues that are integers times the length scale a. The time coordinate x^0, on the other hand, has a continuous spectrum.)

The Snyder algebra, for very unsubtle technical reasons, is hard to work with. The Baker-Campbell-Hausdorff formula that we use to deal with products of functions involves series that do not terminate. Instead work with a contracted algebra.

The contraction of an algebra is a simpler one obtained by taking the limit of some parameter. Let

$$\hat{M}^{\mu\nu} = \hat{\theta}^{\mu\nu}/b \, . \tag{6.5}$$

and let

$$b \to 0 \; , \; a \to 0 \; , \tag{6.6}$$

with

$$\frac{a^2}{b} \to 1 \; . \tag{6.7}$$

The result is the Lie algebra,

$$\begin{aligned} [\hat{x}^\mu, \hat{x}^\nu] &= i\hat{\theta}^{\mu\nu} \; , \\ \left[\hat{\theta}^{\mu\nu}, \hat{x}^\lambda\right] &= 0 \; , \\ \left[\hat{\theta}^{\mu\nu}, \hat{\theta}^{\alpha\beta}\right] &= 0 \; . \end{aligned} \tag{6.8}$$

This is the contracted Snyder Lie algebra; it is identical to the Lie algebra suggested by Doplicher, Fredenhagen, and Roberts (DFR), based on General Relativity considerations reviewed in the first lecture.

Importantly, we still have

$$\left[\hat{M}^{\mu\nu}, \hat{\theta}^{\alpha\beta}\right] = i\left(\hat{\theta}^{\mu\beta} g^{\nu\alpha} + \hat{\theta}^{\nu\alpha} g^{\mu\beta} - \hat{\theta}^{\mu\alpha} g^{\nu\beta} - \hat{\theta}^{\nu\beta} g^{\mu\alpha}\right) \; , \tag{6.9}$$

which establishes that $\hat{\theta}^{\mu\nu}$ transforms as a Lorentz tensor and that Eq. (6.8) is Lorentz covariant. (Since $a \to 0$ is part of the limit, the contracted algebra is a continuum limit of Snyder's quantized spacetime.)

B. Fields depending on new variable

Have a new fundamental operator $\hat{\theta}^{\mu\nu}$. Elements of the group defined locally by Eq. (6.8) [fields] are now $\hat{f} = \hat{f}(\hat{x}, \hat{\theta})$. Relate to ordinary c-number functions $f(x, \theta)$ with a Fourier transform, using more variables.

$$\hat{f} = \int (d\alpha)(dB)\, e^{-i(\alpha\hat{x}+B\hat{\theta})}\, \tilde{f}(\alpha, B)\,, \tag{6.10}$$

with

$$\tilde{f}(\alpha, B) = \int (dx)(d\theta)\, e^{i(\alpha x+B\theta)}\, f(x, \theta)\,. \tag{6.11}$$

Definitions of measures: $(d\alpha) \equiv (2\pi)^{-4} d^4\alpha$, $(dB) \equiv (2\pi)^{-6} d^6 B$, $(dx) \equiv d^4 x$ and $(d\theta) \equiv d^6\theta$; the $B_{\mu\nu}$ and $\theta^{\mu\nu}$ are antisymmetric parameters, and $\alpha x = \alpha_\mu x^\mu$, $B\theta \equiv B_{\mu\nu}\theta^{\mu\nu}/2$.
The measure

$$d^6 B = dB_{12} dB_{23} dB_{31} dB_{01} dB_{02} dB_{03} \tag{6.12}$$

is Lorentz invariant (if $B_{\mu\nu}$ transforms like a second-rank Lorentz tensor).

The x^μ are ordinary commuting coordinates; the $\theta^{\mu\nu}$ (no hat) are a set of new commuting parameters in ordinary function space that correspond to the $\hat{\theta}^{\mu\nu}$. While the operators $\hat{x}$ and $\hat{\theta}$ are related through commutation relations, the commuting parameters x and θ are completely independent of each other.

The mapping from the operator algebra to the space of ordinary functions allows one to define a star-product through the requirement Eq. (2.3) [$\hat{f}\hat{g} = \widehat{f \star g}$]. The derivation begins as usual,

$$\hat{f}\hat{g} = \int (d\alpha)(dB)(d\gamma)(d\Delta)\, e^{-i(\alpha\hat{x}+B\hat{\theta})}\, e^{-i(\gamma\hat{x}+\Delta\hat{\theta})}\, \tilde{f}(\alpha, B)\, \tilde{g}(\gamma, \Delta)\,, \tag{6.13}$$

and continues using the Baker-Campbell-Hausdorff … formula,

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]+\frac{1}{12}[B,[B,A]]+\dots}\,. \tag{6.14}$$

The expansion terminates after the first commutator and, after some manipulation, one obtains the same $\star$-product as in the canonical case except for the presence of the extra argument θ:

$$(f \star g)(x, \theta) = f(x, \theta)\, \exp[\frac{i}{2}\, \overleftarrow{\partial}_\mu\, \theta^{\mu\nu}\, \overrightarrow{\partial}_\nu]\, g(x, \theta)\ . \tag{6.15}$$

This star product is Lorentz covariant; the Lorentz transformation properties of θ are identical to those of $\hat{\theta}$, as one can show via the mapping defined in Eqs. (6.10) and (6.11).

C. Gauge Theory

To start, if the field ϕ transforms as some representation of a gauge group G, it is not possible to choose ϕ to be a function of x only. A θ dependence is introduced via the noncommutative generalization of the gauge transformation. Consider NCQED. The transformations are

$$\psi(x,\theta) \to \psi'(x,\theta) = U \star \psi(x,\theta) \ , \tag{6.16}$$

$$A_\mu(x,\theta) \to A'_\mu(x,\theta) = U \star A_\mu(x,\theta) \star U^{-1} + \frac{i}{e} U \star \partial_\mu U^{-1} \ , \tag{6.17}$$

where

$$U = (e^{i\Lambda})_\star \ . \tag{6.18}$$

The Lagrangian

$$\mathcal{L} = \left[-\frac{1}{4} F_{\mu\nu} \star F^{\mu\nu} + \bar{\psi} \star (i \not{D} - m) \star \psi \right] \tag{6.19}$$

is gauge invariant provided

$$D_\mu = \partial_\mu - ieA_\mu \ , \tag{6.20}$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu \stackrel{\star}{,} A_\nu] \, . \tag{6.21}$$

Superficially, Eqs. (6.19–2.20) are the same as in the case of canonical noncommutative QED [29], aside the trace average over θ. However, that the fields here are functions of both x and θ, and so are not the same as the ordinary quantum fields $\psi(x)$ and $A^\mu(x)$.

To proceed, expand the fields as a power series in the variable θ. It will look like the work in Jurčo *et al.* and Calmet *et al.* although the context is different.

Begin with the gauge parameter Λ and the gauge field A^μ

$$\Lambda_\alpha(x,\theta) = \alpha(x) + \theta^{\mu\nu}\Lambda^{(1)}_{\mu\nu}(x;\alpha) + \theta^{\mu\nu}\theta^{\eta\sigma}\Lambda^{(2)}_{\mu\nu\eta\sigma}(x;\alpha) + \cdots \quad , \tag{6.22}$$

$$A_\rho(x,\theta) = A_\rho(x) + \theta^{\mu\nu}A^{(1)}_{\mu\nu\rho}(x) + \theta^{\mu\nu}\theta^{\eta\sigma}A^{(2)}_{\mu\nu\eta\sigma\rho}(x) + \cdots \quad . \tag{6.23}$$

Identify the first term in each expansion as the ordinary gauge parameter and ordinary gauge field. In ordinary Abelian gauge theory, expect two gauge transformations parameterized by $\alpha(x)$ and $\beta(x)$ to commute

$$(\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\psi(x) = 0 \quad , \tag{6.24}$$

where ψ transforms infinitesimally as

$$\delta_\alpha\psi(x) = i\,\alpha(x)\psi(x) \quad . \tag{6.25}$$

Just so, in the noncommutative theory, expect the field $\psi(x,\theta)$ to still satisfy

$$(\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\psi(x,\theta) = 0 \ , \tag{6.26}$$

where

$$\delta_\alpha\psi(x,\theta) = i\,\Lambda_\alpha(x,\theta)\star\psi(x,\theta)\,. \tag{6.27}$$

Like manipulations we have already seen

$$i\delta_\alpha\Lambda_\beta - i\delta_\beta\Lambda_\alpha + [\Lambda_\alpha \stackrel{\star}{,} \Lambda_\beta] = 0 \ , \tag{6.28}$$

which can be solved by (going to second order this time):

$$\Lambda^{(1)}_{\mu\nu}(x;\alpha) = \frac{e}{2}\partial_\mu\alpha(x)A_\nu(x) \ , \tag{6.29}$$

$$\Lambda^{(2)}_{\mu\nu\eta\sigma}(x;\alpha) = -\frac{e^2}{2}\partial_\mu\alpha(x)A_\eta(x)\partial_\sigma A_\nu(x) \ . \tag{6.30}$$

We next need to make the gauge field transform as it should:

$$\delta_\alpha A_\sigma = \partial_\sigma\Lambda_\alpha + i[\Lambda_\alpha \stackrel{\star}{,} A_\sigma] \ , \tag{6.31}$$

which works if

$$A^{(1)}_{\mu\nu\rho}(x) = -\frac{e}{2}A_\mu(\partial_\nu A_\rho + F^0_{\nu\rho}) \ , \tag{6.32}$$

$$A^{(2)}_{\mu\nu\eta\sigma\rho}(x) = \frac{e^2}{2}(A_\mu A_\eta \partial_\sigma F^0_{\nu\rho} - \partial_\nu A_\rho \partial_\eta A_\mu A_\sigma + A_\mu F^0_{\nu\eta} F^0_{\sigma\rho}) \ , \tag{6.33}$$

where

$$F^0_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{6.34}$$

We will look at photon self interactions first, and so shall pause here before looking at the fermion expansion.

D. Generalizing the operator trace, i.e., ...

We need generalize the operator trace, since we have extra variables. A trace maps from an operator algebra to numbers and is linear, positive ($\mathrm{Tr}\,\hat{f}\hat{f}^\dagger \geq 0$), and cyclic ($\mathrm{Tr}\hat{f}\hat{g} = \mathrm{Tr}\hat{g}\hat{f}$). We propose

$$\mathrm{Tr}\hat{f} = \int d^4x\, d^6\theta\, W(\theta)\, f(x,\theta) \ . \tag{6.35}$$

The weighting function $W(\theta)$ will allow us to work with power series expansions of functions of θ. We assume that the weighting function is positive and for any large $|\theta^{\mu\nu}|$ falls to zero quickly enough so that all integrals are well defined. Also, let

$$\int d^6\theta\, W(\theta)\, \theta^{\mu\nu} = 0\ . \tag{6.36}$$

Field theory actions are like Eq. (2.9) [$S = \mathrm{Tr}\, \hat{\mathcal{L}}$],

$$S = \int d^4x\, d^6\theta\, W(\theta)\, \mathcal{L}(\phi, \partial\phi)_\star\ . \tag{6.37}$$

Also useful is $W(\theta) \equiv W(\theta_{\mu\nu}\theta^{\mu\nu})$,

$$\int d^6\theta\, W(\theta)\, \theta^{\mu\nu}\theta^{\eta\rho} = \frac{1}{12}\langle\theta^2\rangle(g^{\mu\eta}g^{\nu\rho} - g^{\mu\rho}g^{\eta\nu})\ , \tag{6.38}$$

for any Lorentz-invariant weighting function, and where

$$\langle\theta^2\rangle \equiv \int d^6\theta\, W(\theta)\, \theta_{\mu\nu}\theta^{\mu\nu}\ . \tag{6.39}$$

Of course, we normalize by

$$\int d^6\theta\, W(\theta) = 1\ . \tag{6.40}$$

Note that $\mathcal{L}(\phi, \partial\phi)_\star$ depends in general on both x and θ, but we can think of

$$\mathcal{L}(x) = \int d^6\theta\, W(\theta)\, \mathcal{L}(\phi, \partial\phi)_\star\ . \tag{6.41}$$

as playing the role of the ordinary Lagrangian.

E. Lagrangians and Feynman rules

The Lagrangian, again, is

$$\mathcal{L}(x) = \int d^6\theta\, W(\theta) \left[-\frac{1}{4} F_{\mu\nu} \star F^{\mu\nu} + \bar{\psi} \star (i\, \not{D} - m) \star \psi \right] \tag{6.42}$$

The noncommutative field strength tensor can be written out in terms of the ordinary gauge field $A^\mu(x)$. For the record,

$$\begin{aligned} F_{\mu\nu} &= F^0_{\mu\nu} + e\, \theta^{\kappa\lambda}(F^0_{\mu\kappa}F^0_{\nu\lambda} - A_\kappa \partial_\lambda F^0_{\mu\nu}) + \frac{e^2}{2}\theta^{\kappa\lambda}\theta^{\rho\eta}[F^0_{\kappa\nu}F^0_{\lambda\rho}F^0_{\eta\mu} - F^0_{\kappa\mu}F^0_{\lambda\rho}F^0_{\eta\nu} \\ &+ A_\kappa(2F^0_{\mu\rho}\partial_\lambda F^0_{\eta\nu} - 2F^0_{\nu\rho}\partial_\lambda F^0_{\eta\mu} + F^0_{\lambda\rho}\partial_\eta F^0_{\mu\nu} - \partial_\rho F^0_{\mu\nu}\partial_\lambda A_\eta) + A_\kappa A_\rho \partial_\eta \partial_\lambda F^0_{\mu\nu}]. \end{aligned} \tag{6.43}$$

Photon self-interactions may be isolated by substituting this result into the Lagrangian above and integrating over θ.

Photon self-interaction terms that are odd in θ vanish. The lowest-order nonstandard vertex is given by:

$$\mathcal{L} = \frac{\pi\alpha}{12}\langle\theta^2\rangle[F^0_{\mu\nu}F^{0\nu\eta}F^0_{\eta\rho}F^{0\rho\mu} - (F^0_{\mu\nu}F^{0\mu\nu})^2] \ . \tag{6.44}$$

This will give for example, a new contribution to photon-photon elastic scattering.

By the way, Eq. (6.44) reduces to

$$\mathcal{L} = \frac{\pi\alpha}{6}\langle\theta^2\rangle[-(\mathbf{E}^2 - \mathbf{B}^2)^2 + 2(\mathbf{E}\cdot\mathbf{B})^2] \tag{6.45}$$

in comparison to the famous Euler-Heisenberg low-energy effective Lagrangian following at the one-loop level in QED [40]

$$\mathcal{L}_{E-L} = \frac{2\alpha^2}{45 m_e^4}\left[(\mathbf{E}^2 - \mathbf{B}^2)^2 + 7(\mathbf{E}\cdot\mathbf{B})^2\right] \ . \tag{6.46}$$

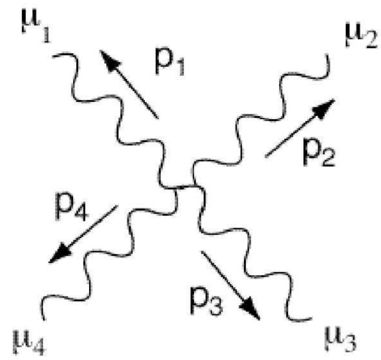


FIG. 8: Four-photon vertex.

F. Phenomenology

- Lots of new vertices.
- Seek deviations in observable scattering cross sections from expectations within standard model.
- Focus on $\gamma\gamma \to \gamma\gamma$.
- Potential window on physics beyond the standard model.

Quote the Feynman rule for the for-photon vertex Fig. 8,

$$
\begin{aligned}
V_{4\gamma} \;=\; & \frac{1}{6} i e^2 \left\langle \theta^2 \right\rangle \Big\{ -4 p_1^{\mu_2} p_2^{\mu_1} p_3^{\mu_4} p_4^{\mu_3} + p_1^{\mu_2} p_2^{\mu_3} p_3^{\mu_4} p_4^{\mu_1} + p_1^{\mu_2} p_2^{\mu_4} p_3^{\mu_1} p_4^{\mu_3} \\
& \qquad -4 p_1^{\mu_3} p_2^{\mu_4} p_3^{\mu_1} p_4^{\mu_2} + p_1^{\mu_3} p_2^{\mu_1} p_3^{\mu_4} p_4^{\mu_2} + p_1^{\mu_3} p_2^{\mu_4} p_3^{\mu_2} p_4^{\mu_1} \\
& \qquad -4 p_1^{\mu_4} p_2^{\mu_3} p_3^{\mu_2} p_4^{\mu_1} + p_1^{\mu_4} p_2^{\mu_1} p_3^{\mu_2} p_4^{\mu_3} + p_1^{\mu_4} p_2^{\mu_3} p_3^{\mu_1} p_4^{\mu_2} \\
& + \Big(g^{\mu_1 \mu_2} \Big[(p_1^{\mu_3} p_2^{\mu_4} + p_2^{\mu_3} p_1^{\mu_4})\, p_3 \cdot p_4 + 4 p_4^{\mu_3} p_3^{\mu_4} p_1 \cdot p_2 \\
& \qquad - p_4^{\mu_3} p_1^{\mu_4} p_2 \cdot p_3 - p_4^{\mu_3} p_2^{\mu_4} p_1 \cdot p_3 - p_1^{\mu_3} p_3^{\mu_4} p_2 \cdot p_4 - p_2^{\mu_3} p_3^{\mu_4} p_1 \cdot p_4 \Big] \\
& \quad + [\, (12)(34) \to (34)(12) \,] \\
& \quad + [\, (12)(34) \to (13)(42) \,] + [\, (12)(34) \to (42)(13) \,] \\
& \quad + [\, (12)(34) \to (14)(23) \,] + [\, (12)(34) \to (23)(14) \,] \Big) \\
& + \Big(g^{\mu_1 \mu_2} g^{\mu_3 \mu_4} \Big[-4 p_1 \cdot p_2 \; p_3 \cdot p_4 + p_1 \cdot p_4 \; p_2 \cdot p_3 + p_1 \cdot p_3 \; p_2 \cdot p_4 \Big] \\
& \quad + [\, (12)(34) \to (13)(42) \,] + [\, (12)(34) \to (14)(23) \,] \Big) \Big\} \; . \qquad (6.47)
\end{aligned}
$$

Calculate $d\sigma/d\Omega$ for $\gamma\gamma \to \gamma\gamma$ in the photon CM.

The NC amplitude is $90°$ out of phase to the leading log contributions to the standard model background (coming), so

$$\sigma \approx \sigma_{\rm NC} + \sigma_{\rm SM} \; . \qquad (6.48)$$

Find (no polarization)

$$\frac{d\sigma_{\rm NC}}{d\cos\Theta^*} = \frac{19\pi}{128}\left(\frac{\langle\theta^2\rangle}{12}\right)^2 \alpha^2 s^3 (3+\cos^2\Theta^*)^2 \ , \tag{6.49}$$

$\sqrt{s}$ = CM energy

Θ^* = CM scattering angle

Total cross section ($0^\circ < \Theta^* < 180^\circ$),

$$\sigma_{\rm NC} = \frac{133\pi}{80}\alpha^2 s^3 \left(\frac{\langle\theta^2\rangle}{12}\right)^2 \ . \tag{6.50}$$

For the standard model, we quote amplitudes from Gounaris *et al.* for light-by-light scattering for $s,\ |t|,\ |u| \gg m_W^2$. For reference, here are the relevant results. The differential cross section is given by

$$\left(\frac{d\sigma}{d\cos\Theta^*}\right)_{SM} = \frac{1}{128\pi s}[(\text{Im } F_{++++})^2 + (\text{Im } F_{+-+-})^2 + (\text{Im } F_{+--+})^2] \ ; \tag{6.51}$$

the dominant helicity amplitudes are mostly imaginary and

$$\text{Im } F_{++++} = -16\pi\alpha^2\left[\frac{s}{u}\ln\left|\frac{u}{m_W^2}\right| + \frac{s}{t}\ln\left|\frac{t}{m_W^2}\right|\right] \ , \tag{6.52}$$

$$\text{Im } F_{+-+-} = -12\pi\alpha^2\frac{s-t}{u} - 16\pi\alpha^2\left[\frac{u}{s}\ln\left|\frac{u}{m_W^2}\right| + \frac{u^2}{st}\ln\left|\frac{t}{m_W^2}\right|\right] \ , \tag{6.53}$$

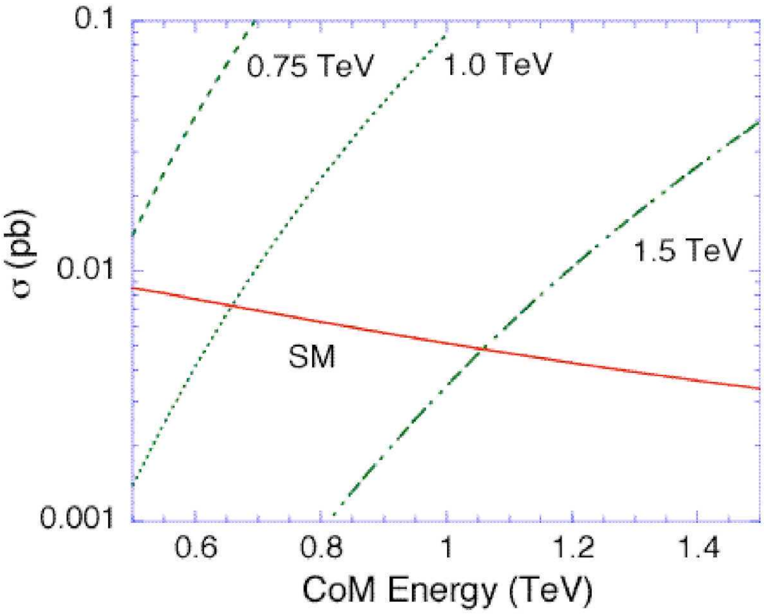


FIG. 9: Total cross sections $\sigma_{\rm NC}$ and $\sigma_{\rm SM}$ for $30° < \Theta^* < 150°$. Noncommutative results are labeled by the value of $\Lambda_{\rm NC}$, defined in the text.

and

$$\text{Im } F_{+--+}(s,t,u) = \text{Im } F_{+-+-}(s,u,t) \ . \tag{6.54}$$

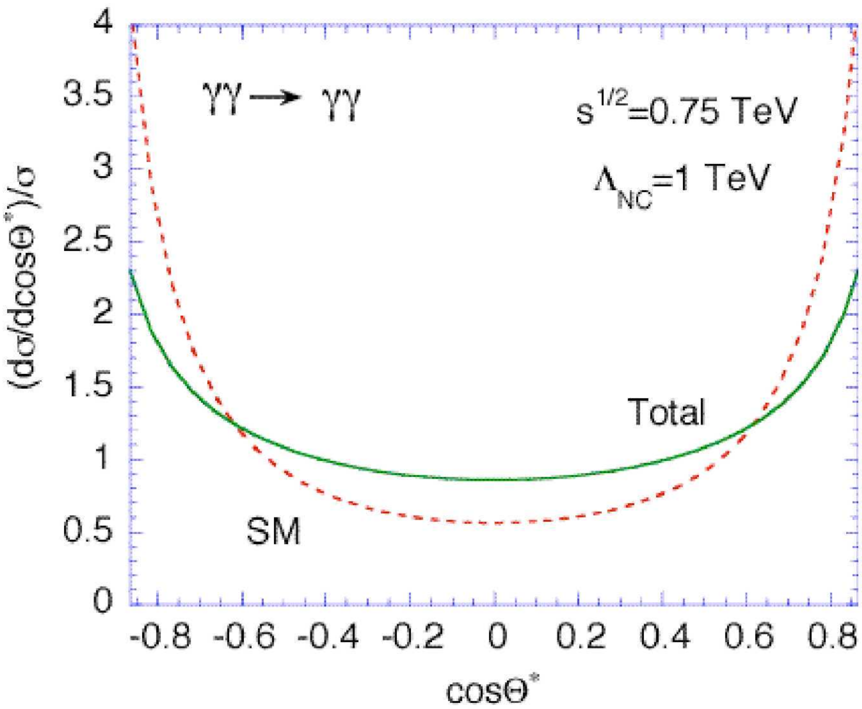


FIG. 10: Differential cross sections for $\sqrt{s} = 0.75$ TeV and $\Lambda_{\rm NC} = 1.0$ TeV, normalized to $\sigma(30° < \Theta^* < 150°)$. The dashed line indicates the standard model background and the solid line indicates the result when both the standard model and Lorentz-invariant NCQED interactions are present.

Figs. 9 and 10 show the comparison between our noncommutative result and the expectation in the standard model. Since the scale of new physics $\Lambda_{\rm NC}$ is characterized by a root-mean-square average of the components of $\theta^{\mu\nu}$, we define

$$\Lambda_{NC} = \left(\frac{12}{\langle\theta^2\rangle}\right)^{1/4} \quad , \tag{6.55}$$

We hope it is clear from the present example that our scenario may lead to potentially distinctive collider signals, and defer a complete investigation of these phenomenological issues to future work.

G. Conclusions

• Lorentz covariant:

$$[\hat{x}^{\mu}, \hat{x}^{\nu}] = i\hat{\theta}^{\mu\nu} \ . \tag{6.56}$$

• There is an explicit realization: contracted Snyder algebra, or DFR algebra.

• Tight bounds on noncommutativity parameter not present in Lorentz conserving case.

• Fields become functions of additional variable, that we deal with by expansion and integrate out.

• Looked at the Lorentz invariant version of NCQED. including two-fermion-two-photon and four-photon interactions However, no three-photon vertex is present.

• Had predictions from total and differential cross sections that were distinct from the standard model.

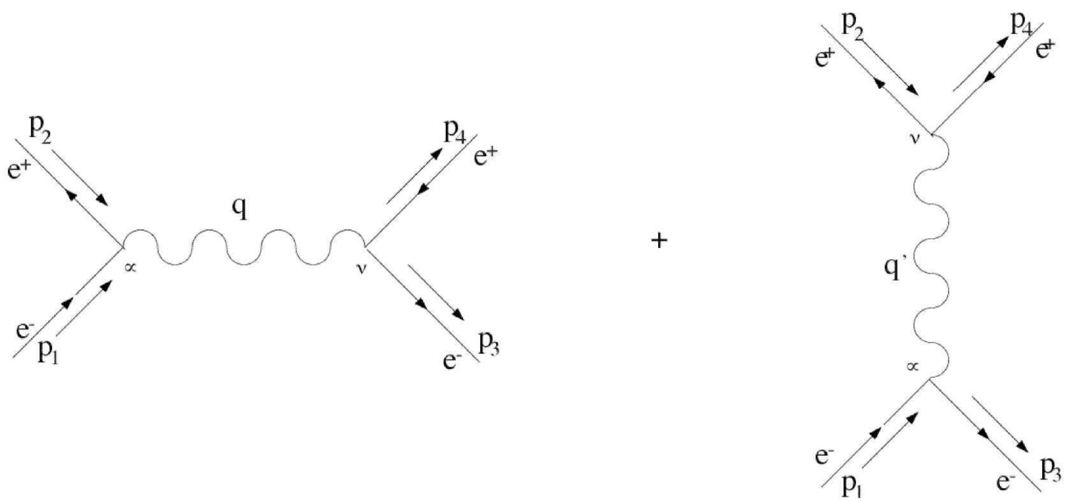


FIG. 11: Bhabha

[1] J. L. Hewett, F. J. Petriello and T. G. Rizzo, Phys. Rev. D **64**, 075012 (2001); S. Godfrey and M. A. Doncheski, Phys. Rev. D **65**, 015005 (2002); N. Mahajan, hep-ph/0110148; S. w. Baek, D. K. Ghosh, X. G. He and W. Y. Hwang, Phys. Rev. D **64**, 056001 (2001); H. Grosse and Y. Liao, Phys. Rev. D **64**, 115007 (2001); Phys. Lett. B **520**, 63 (2001).

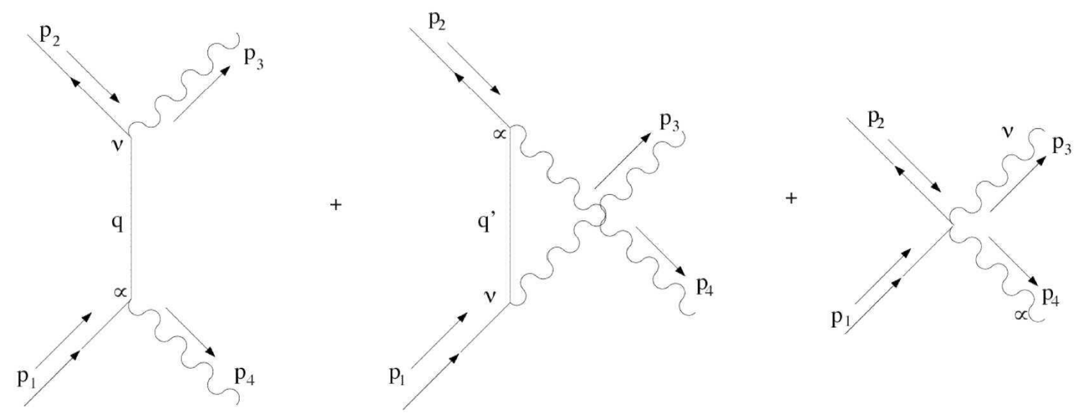


FIG. 12: pair annihilation

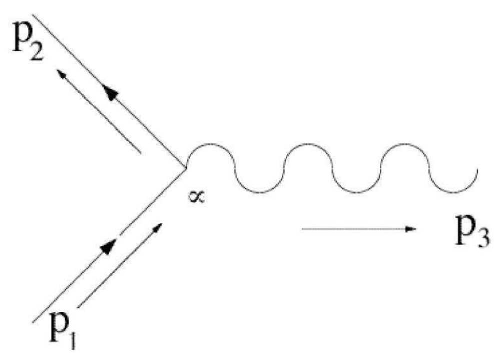


FIG. 13: vertex3

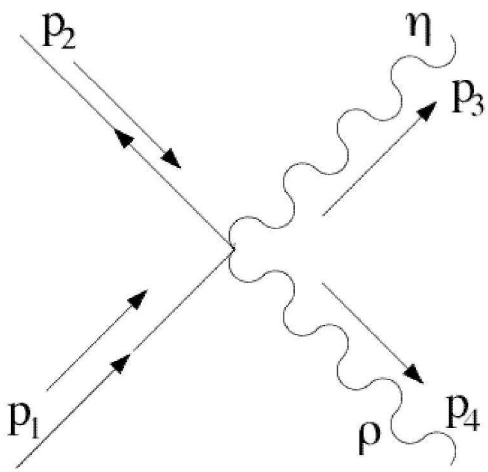


FIG. 14: vertex4

[2] I. Hinchliffe and N. Kersting, Phys. Rev. D **64**, 116007 (2001) and hep-ph/0205040; E. O. Iltan, Phys. Rev. D **66**, 034011 (2002) and New J. Phys. **4**, 54 (2002); H. Falomir *et al.*, Phys. Rev. D **66**, 045018 (2002); S. Narison, hep-ph/0208225; J. i. Kamoshita, hep-ph/0206223.

[3] S. M. Carroll *et al.*, Phys. Rev. Lett. **87**, 141601 (2001).

[4] I. Mocioiu, M. Pospelov and R. Roiban, Phys. Lett. B **489**, 390 (2000).

[5] A. Anisimov, T. Banks, M. Dine and M. Graesser, Phys. Rev. D **65**, 085032 (2002).

[6] C. E. Carlson, C. D. Carone and R. F. Lebed, Phys. Lett. B **518**, 201 (2001).

[7] C. E. Carlson and C. D. Carone, Phys. Rev. D **65**, 075007 (2002).

[8] J. Gomis, T. Mehen and M. B. Wise, JHEP **0008**, 029 (2000).

[9] I. Mocioiu, M. Pospelov and R. Roiban, Phys. Rev. D **65**, 107702 (2002).

[10] N. Seiberg and E. Witten, JHEP **9909**, 032 (1999).

[11] See, for example, M. R. Douglas and N. A. Nekrasov, Rev. Mod. Phys. **73**, 977 (2001).

[12] D. Colladay and V. A. Kostelecky, Phys. Rev. D **58**, 116002 (1998); See also, V. A. Kostelecky *et al.*, Phys. Rev. D **65**, 056006 (2002) and hep-ph/0205211.

[13] S. Terashima, Phys. Lett. B **482**, 276 (2000); C. S. Chu *et al.*, Phys. Lett. B **513**, 200 (2001).

[14] J. Gomis and T. Mehen, Nucl. Phys. B **591**, 265 (2000).

[15] R. Bluhm and V. A. Kostelecky, Phys. Rev. Lett. **84**, 1381 (2000).

[16] A. Matusis, L. Susskind and N. Toumbas, JHEP **0012**, 002 (2000).

[17] A. Armoni and E. Lopez, Nucl. Phys. B **632**, 240 (2002)

[18] A. A. Bichl *et al.*, hep-th/0203141.

[19] C. E. Carlson, C. D. Carone and N. Zobin, hep-th/0206035 (to appear in Phys. Rev. D).

[20] D. Bailin and A. Love, *Supersymmetric Gauge Field Theory And String Theory*, Institute of Physics Publishing, Bristol, UK, 1994.

[21] I. Hinchliffe and N. Kersting, "CP violation from noncommutative geometry," Phys. Rev. D **64**, 116007 (2001).

[22] J. L. Hewett, F. J. Petriello and T. G. Rizzo, "Signals for non-commutative interactions at linear colliders," Phys. Rev. D **64**, 075012 (2001).

[23] S. Godfrey and M. A. Doncheski, Phys. Rev. D **65**, 015005 (2002) [arXiv:hep-ph/0108268]; arXiv:hep-ph/0111147; N. Mahajan, "Noncommutative QED and gamma gamma scattering," arXiv:hep-ph/0110148; S. w. Baek, D. K. Ghosh, X. G. He and W. Y. Hwang, "Signatures of noncommutative QED at photon colliders," Phys. Rev. D **64**, 056001 (2001); H. Grosse and Y. Liao, "Pair production of neutral Higgs bosons through noncommutative QED interactions at linear colliders," Phys. Rev. D **64**, 115007 (2001); "Anomalous C-violating three photon decay of the neutral pion in noncommutative quantum electrodynamics," Phys. Lett. B **520**, 63 (2001); Z. Guralnik, R. Jackiw, S. Y. Pi and A. P. Polychronakos, "Testing non-commutative QED, constructing non-commutative MHD," Phys. Lett. B **517**, 450 (2001); M. Chaichian, M. M. Sheikh-Jabbari and A. Tureanu, "Hydrogen atom spectrum and the Lamb shift in noncommutative QED," Phys. Rev. Lett. **86**, 2716 (2001).

[24] S. M. Carroll, J. A. Harvey, V. A. Kostelecky, C. D. Lane and T. Okamoto, "Noncommutative field theory and Lorentz violation," Phys. Rev. Lett. **87**, 141601 (2001).

[25] E. O. Iltan, "The $Z \to \ell^+\ell^-$ and $W \to \nu_\ell \ell^+$ decays in the noncommutative standard model,"

arXiv:hep-ph/0204332; "The inclusive $b \to s\gamma$ decay in the noncommutative standard model," arXiv:hep-ph/0204129.

[26] G. Amelino-Camelia, L. Doplicher, S. k. Nam and Y. S. Seo, "Phenomenology of particle production and propagation in string-motivated canonical noncommutative spacetime," arXiv:hep-th/0109191.

[27] I. Hinchliffe and N. Kersting, "Review of the phenomenology of noncommutative geometry," arXiv:hep-ph/0205040.

[28] J. Madore, S. Schraml, P. Schupp and J. Wess, "Gauge theory on noncommutative spaces," Eur. Phys. J. C **16**, 161 (2000).

[29] M. Hayakawa, "Perturbative analysis on infrared aspects of noncommutative QED on R^4," Phys. Lett. B **478**, 394 (2000).

[30] A. Armoni, "Comments on perturbative dynamics of non-commutative Yang-Mills theory," Nucl. Phys. B **593**, 229 (2001).

[31] Y. Liao, "One loop renormalization of spontaneously broken U(2) gauge theory on noncommutative spacetime," JHEP **0111**, 067 (2001) [arXiv:hep-th/0110112].

[32] M. Chaichian, P. Presnajder, M. M. Sheikh-Jabbari and A. Tureanu, "Noncommutative standard model: Model building," arXiv:hep-th/0107055; M. Chaichian, P. Presnajder, M. M. Sheikh-Jabbari and A. Tureanu, "Noncommutative gauge field theories: A no-go theorem," Phys. Lett. B **526**, 132

(2002) [arXiv:hep-th/0107037].

[33] A. A. Bichl, J. M. Grimstrup, H. Grosse, E. Kraus, L. Popp, M. Schweda and R. Wulkenhaar, "Noncommutative Lorentz symmetry and the origin of the Seiberg-Witten map," Eur. Phys. J. C **24**, 165 (2002) [arXiv:hep-th/0108045]; J. M. Grimstrup, H. Grosse, E. Kraus, L. Popp, M. Schweda and R. Wulkenhaar, "Noncommutative spin-1/2 representations," arXiv:hep-th/0110231.

[34] B. Jurčo, L. Moller, S. Schraml, P. Schupp and J. Wess, "Construction of non-Abelian gauge theories on noncommutative spaces," Eur. Phys. J. C **21**, 383 (2001).

[35] X. Calmet, B. Jurco, P. Schupp, J. Wess and M. Wohlgenannt, "The standard model on noncommutative space-time," Eur. Phys. J. C **23**, 363 (2002) [arXiv:hep-ph/0111115].

[36] X. G. He, "Strong, electroweak interactions and their unification with noncommutative space-time," arXiv:hep-ph/0202223; W. Behr, N. G. Deshpande, G. Duplancic, P. Schupp, J. Trampetic and J. Wess, "The $Z \to \gamma\gamma$, gg decays in the noncommutative standard model," arXiv:hep-ph/0202121; N. G. Deshpande and X. G. He, "Triple neutral gauge boson couplings in noncommutative standard model," Phys. Lett. B **533**, 116 (2002) [arXiv:hep-ph/0112320]; P. Aschieri, B. Jurco, P. Schupp and J. Wess, "Non-commutative GUTs, standard model and C, P, T," arXiv:hep-th/0205214.

[37] V. A. Kostelecky and M. Mewes, "Signals for Lorentz violation in electrodynamics," arXiv:hep-ph/0205211; V. A. Kostelecky, C. D. Lane and A. G. Pickering, "One-loop renormalization of Lorentz-

violating electrodynamics," Phys. Rev. D **65**, 056006 (2002) [arXiv:hep-th/0111123]; R. Bluhm, V. A. Kostelecky, C. D. Lane and N. Russell, "Clock-comparison tests of Lorentz and CPT symmetry in space," Phys. Rev. Lett. **88**, 090801 (2002) [arXiv:hep-ph/0111141]. For additional references, see D. Colladay and V. A. Kostelecky, "Lorentz-violating extension of the standard model," Phys. Rev. D **58**, 116002 (1998) [arXiv:hep-ph/9809521].

[38] H. S. Snyder, "Quantized Space-Time," Phys. Rev. **71**, 38 (1947).

[39] See also S. Doplicher, K. Fredenhagen and J. E. Roberts, "The Quantum Structure Of Space-Time At The Planck Scale And Quantum Fields," Commun. Math. Phys. **172**, 187 (1995); "Space-time quantization induced by classical gravity," Phys. Lett. B **331**, 39 (1994).

[40] W. Heisenberg and H. Euler, "Consequences Of Dirac's Theory Of Positrons," Z. Phys. **98**, 714 (1936).

[41] G. J. Gounaris, P. I. Porfyriadis and F. M. Renard, "The $\gamma\gamma \to \gamma\gamma$ process in the standard and SUSY models at high energies," Eur. Phys. J. C **9**, 673 (1999) [arXiv:hep-ph/9902230]; "Light by light scattering at high energy: A tool to reveal new particles," Phys. Lett. B **452**, 76 (1999) [Erratum-ibid. B **513**, 431 (2001)] [arXiv:hep-ph/9812378].

[42] See, for example, K. m. Cheung, "Diphoton signals for low scale gravity in extra dimensions," Phys. Rev. D **61**, 015005 (2000) [arXiv:hep-ph/9904266].; H. Davoudiasl, "$\gamma\gamma \to \gamma\gamma$ as a test of weak scale quantum gravity at the NLC," Phys. Rev. D **60**, 084022 (1999) [arXiv:hep-ph/9904425].