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"Active Reaction Advection Diffusion"

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Active Reaction Advection Diffusion

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thanks: A. Oberman, R. Rosner, O. Ruchayskyi, N. Vladimirova, B. Winn.

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Outline

1. Passive Reaction Advection Diffusion (PRAD)
2. Active Reaction Advection Diffusion (ARAD)

Equations

$$\begin{cases} \partial_t T + u \cdot \nabla T - \kappa \Delta T = \frac{1}{\tau} g(T) n \\ \partial_t n + u \cdot \nabla n - \frac{\kappa}{Le} \Delta n = -\frac{1}{\tau} g(T) n \end{cases}$$

κ thermal diffusivity, Le Lewis number. $Le = 1$ is special: $n + T$ conserved.

$$T_t + u \cdot \nabla\ T - \kappa \Delta T = \frac{1}{\tau} f(T)$$

$$f(T) = g(T)(1 - T)$$

The velocity

$$u = u(x, t)$$

is either given (= PRAD), or it is coupled (= ARAD) via Boussinesq:

$$\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = g \alpha e_3 T$$

- How do different flows affect bulk front speed in PRAD?
- Which flows quench PRAD?
- What is the effect of active coupling on bulk front speed in ARAD ?
- What is the effect of active coupling on stability of fronts in ARAD?

Bulk Burning Speed

The domain: $D = [0, L] \times (-\infty, \infty)$.

$$T_t + u \cdot \nabla T - \kappa \Delta T = \frac{1}{\tau} f(T)$$

T is normalized $0 \leq T(x, z, t) \leq 1$.

$$T(x, -\infty, t) = 1, \quad T(x, \infty, t) = 0.$$

T satisfies Neumann lateral boundary conditions. Reaction of KPP type: $f(T) = T(1 - T)$. Reaction of ignition type $f(T) = 0$, $T \in [0, \theta_0]$, $\quad f(1) = 0$. Advection by incompressible ambient fluid:

$$\nabla \cdot u = 0.$$

Fisher (1937) and Kolmogorov, Petrovsky and Piskunov (1937)for $u = 0$. Traveling fronts, with minimal speed v_0.

v_0 – laminar front speed,
u_0 – rms turbulent velocity,
u_T – effective turbulent front speed
A_T, A_L – turbulent and laminar front areas

$$\frac{u_T}{v_0} = \frac{A_T}{A_L}.$$

$U_T = u_T/v_0$, $U = u_0/v_0$. Physical predictions:
Shchelkin (1943): $U_T = (1 + \beta U^2)^{1/2}$
Clavin-Williams (1979): $U \ll 1$, $U_T = 1 + U^2$
Kerstein-Ashurst (1992): $U \ll 1$, random, then $U_T = 1 + U^{4/3}$
Yakhot (1988): $U_T = \exp(U^2/U_T^2)$ from G equation:

$$G_t + u \cdot \nabla G = v_0 |\nabla G|.$$

Pocheau (1994): $U_T = (1 + \beta U^\alpha)^{1/\alpha}$ for scale-invariant flows
Shy, Ronney, Buckley, Yakhot (1992), Abel, Celani, Cencini, Vergni, Vulpiani (2001):
$U_T = \exp(U/U_T)$ for an array of vortices.

Homogenization approach.

Freidlin (1979-85): $\kappa = \epsilon \to 0$. In the limit - effective Hamilton-Jacobi equation for the front. Majda, Souganidis (1994): u allowed to depend on $\kappa = \epsilon : u_\epsilon = v_1 + v_2(\frac{x}{\epsilon^\alpha}, \frac{t}{\epsilon^\alpha})$, $0 \le \alpha \le 1$. Implicit predictions: for shear flows perpendicular to the front, the added speed is roughly of the order of the speed of advection.

Traveling waves.

Traveling waves for $u \neq 0$.
Berestycki, Nirenberg (1992): traveling waves $U(x - ct, y)$ for shear flows.
Berestycki, Larrouturou, Roquejoffre (1992): study of stability.
Xin (1992-93): in periodic flows, $U(x - ct, x, y)$ periodic in the second coordinate, but for different (not KPP) reaction term.
No estimates on the velocity. Berestycki, Hamel (99): Pulsating flows.

Bulk Burning Speed

$$V(t) = \frac{1}{L}\int_D \frac{\partial T}{\partial t}(x,z,t)dxdz$$

Note, from PDE:

$$V(t) = \frac{1}{\tau L}\int_D f(T(x,z,t))dxdz$$

Note also: if $T(x,z,t) = P(x,z-ct)$ then $V(t) = c$

Theorem 1 *(CKR, 99) Arbitrary initial data, KPP nonlinearity. The bulk burning speed satisfies:*

$$V(t) \geq C v_0 \left(1 - e^{-\frac{t}{2\tau}}\right).$$

Lemma 1 *(CKR, 99) Assume* $T(x,z,t)$ *satisfies*

$$0 \leq T \leq 1,$$

$$T(x,-\infty,t) = 1, \;\; T(x,\infty,t) = 0 \;\text{ for any }\; x \in [0,L].$$

Then there exists a constant $C > 0$ *depending only on f, such that*

$$\left(\int_D f(T)\,dxdz\right)\left(\int_D |\nabla T|^2\,dxdz\right) \geq CL^2.$$

Idea of proof

$\int\limits_D f(T)dxdz < \infty, \int\limits_D |\nabla T|^2 dxdz < \infty.$
$\exists x \in [0, L]$ such that

$$\int\limits_{-\infty}^{\infty} f(T(x,z,t))dz \leq \frac{3}{L}\int\limits_D f(T)dxdz$$

and also

$$\int\limits_{\infty}^{\infty} |\nabla T(x,z,t)|^2 dz \leq \frac{3}{L}\int |\nabla T|^2 dxdz$$

hold. Let $\epsilon > 0$ and let C be a positive constant determined by the condition

$$\frac{1}{\epsilon}\inf_{[\epsilon,1-\epsilon]} f(\theta) = C$$

Because of the boundary conditions on T and continuity, $\exists z_1,\ z_2$ such that $T(x,z,t) \in [\epsilon, 1-\epsilon]$, $\forall z \in [z_1, z_2]$ and

$$|T(x,z_2,t) - T(x,z_1,t)| \geq 1 - \epsilon.$$

From the construction of C we get:

$$f(T(x,z,t)) \geq C\epsilon$$

for all $z \in [z_1, z_2]$. Integrating in z:

$$C\epsilon|z_1 - z_2| \leq \frac{3}{L}\int_D f(T)dxdz. \quad (*)$$

The gradient is bounded below:

$$\left|\int_{z_1}^{z_2} \frac{\partial T(x,z,t)}{\partial z}dz\right| \leq \sqrt{|z_1 - z_2|}\sqrt{\int_{z_1}^{z_2} \left|\frac{\partial T}{\partial z}\right|^2 dz}$$

and so

$$\frac{(1-\epsilon)^2}{|z_1 - z_2|} \leq \frac{3}{L}\int |\nabla T|^2 \, dxdz. \quad (**)$$

Multiplying $(*)$ and $(**)$ we get the lower bound.

2. Time independent shear flows

Let

$$\langle V \rangle_t = \frac{1}{t} \int\limits_0^t V(s)\, ds.$$

Theorem 2 *Let us consider a partition of the interval* $[0, L]$ *into subintervals* $I_j = [c_j - h_j, c_j + h_j]$ *on which* $u(x)$ *does not change sign. Denote* D_-, D_+ *the unions of intervals* I_j *where* $u(x) > 0$ *and* $u(x) < 0$ *respectively. Let*

$$m_+ = \frac{|D_+|}{|D_-| + |D_+|}, \;\; m_- = \frac{|D_-|}{|D_-| + |D_+|}.$$

Then there exist a constant $C_1 > 0$, *independent of the partition, of* $u(x)$, *and of the initial data* $T_0(x, y)$, *so that for any*

$$t \geq t_0 = \max\left[\frac{\kappa}{v_0^2}, \frac{L}{v_0}\right]$$

we have

$$\langle V \rangle_t \geq$$

$$C_1\left(m_+\sum_{I_j\subset D_+}\left(1+\frac{\kappa^2}{v_0^2h_j^2}\right)^{-1}\int_{c_j-\frac{h_j}{2}}^{c_j+\frac{h_j}{2}}|u(x)|\frac{dx}{L}+\right.$$

$$\left.+m_-\sum_{I_j\subset D_-}\left(1+\frac{\kappa^2}{v_0^2h_j^2}\right)^{-1}\int_{c_j-\frac{h_j}{2}}^{c_j+\frac{h_j}{2}}|u(z)|\frac{dx}{L}\right).$$

3. Time dependent shear flows.

Take a time scale τ_0. Given t, choose D_+ and D_-, unions of intervals in $[0,H]$. Set

$$J(t,\tau_0,u)=$$

$$m_+\sum_{I_j\subset D_+}\left(1+\frac{\kappa^2}{v_0^2h_j^2}\right)^{-1}\frac{1}{\tau_0}\int_t^{t+\tau_0}dt\int_{c_j-\frac{h_j}{2}}^{c_j+\frac{h_j}{2}}u(x,t)\frac{dx}{L}-$$

$$m_-\sum_{I_j\subset D_-}\left(1+\frac{\kappa^2}{v_0^2h_j^2}\right)^{-1}\frac{1}{\tau_0}\int_t^{t+\tau_0}dt\int_{c_j-\frac{h_j}{2}}^{c_j+\frac{h_j}{2}}u(x,t)\frac{dx}{L}.$$

Theorem 3 *For any t, τ_0 we have the estimate*

$$\langle V\rangle_{t+\tau_0}\geq\left(1+\frac{1}{\tau_0}\max\left[\frac{4\kappa}{v_0^2},\frac{\tilde{H}}{v_0^2}\right]\right)^{-1}\langle J(t,\tau_0,u)\rangle_t.$$

4. Percolating flows

Coordinate ρ along the streamlines, θ is the orthogonal coordinate.

$$E_1^2 d\rho^2 + E_2^2 d\theta^2 = ds^2.$$

Assumptions:

1. (Local) $\omega_1(\rho,\theta) = \frac{E_1}{E_2}(\rho,\theta), \omega_2(\rho,\theta) = \frac{E_2}{E_1}(\rho,\theta)$. Should have:

$$|\omega_i(\rho,\theta)| \leq C, \ \ |\nabla \omega_i(\rho,\theta)| \leq \frac{C}{h}.$$

2. (Global) The widths of the streams do not oscillate too much.

Theorem 4 *For any* $t \geq \tau_0 = \max\left(\frac{\kappa}{v_0^2}, \frac{L}{v_0}\right)$

$$\langle V \rangle_t \geq$$

$$C\left(m_+ \sum_{I_j \subset D_+} \left(1 + \frac{\kappa^2}{v_0^2 h_j^2}\right)^{-1} \int_{c_j - \frac{h_j}{2}}^{c_j + \frac{h_j}{2}} u(\rho,\theta) E_1(\rho,\theta) \frac{d\rho}{H} + \right.$$

$$m_- \sum_{I_j \subset D_-} \left(1 + \frac{\kappa^2}{v_0^2 h_j^2}\right)^{-1} \int_{c_j - \frac{h_j}{2}}^{c_j + \frac{h_j}{2}} |u(\rho, \theta)| E_1(\rho, \theta) \frac{d\rho}{H} \Bigg).$$

Cellular flows

Berestycki-Pomeau, heuristic argument:

$$V \sim A^{\frac{1}{4}}$$

Kiselev-Ryzhik, rigorous

$$V \geq c A^{\frac{1}{5}}.$$

Quenching

Nonlinearity of ignition type: $f(T) = 0$ if $0 \leq T \leq \theta_0$. Velocity: time independent shear $u = u(x)e_z$, $\in_0^L u(x) = 0$. Initial data: compactly supported, above ignition. Width of support (in z) of the order h. Physical domain: strip $(x, z) \in D = [0, L] \times \mathbf{R}$. BC:periodic in x.

$$\partial_t T + Au(x)T_z - \kappa \Delta T = \frac{1}{\tau} f(T)$$

Laminar front width

$$\delta = \frac{2\kappa}{v_0} = \sqrt{\kappa\tau}$$

Laminar front speed v_0

$$\tau = 4\kappa v_0^{-2}$$

Definitions

- $u \in Q$ (quenching): $\forall h$, $\exists A_0$, $\forall A \geq A_0$
$$\lim_{t\to\infty} T(x, z, t) = 0$$
uniformly.
- $u \in H(J)$ if $\forall x \in J \, \exists k$, $u^{(k)}(x) \neq 0$.

Theorem 5 *If* $u \in H([0, L])$ *then* $u \in Q$.

Theorem 6 $\exists a > 0$, *if* $J = [0, L] \setminus I$, $|I| \leq a\delta$, $u \in H(J)$, *and* $u_{|I} = c$ *constant, then* $u \in Q$.

Theorem 7 *The set of quenching profiles* u *contains an open dense subset of* $\mathcal{C}([0, L])$.

The three results above apply to systems:

$$\begin{cases} T_t + Au(x)T_z = \kappa \Delta T + \frac{1}{\tau} g(T)n \\ n_t + Au(x)n_z = \frac{\kappa}{Le} \Delta n - \frac{1}{\tau} g(T)n \end{cases}$$

Theorem 8 *If* $|I| \geq b\delta$ *and* $u_{|I} = c$ *constant then* $u \notin Q$. *More precisely,* $\exists C$, *so that if* $T_0 \geq \theta_0$ *on a region* $[0, L] \times [z_0, z_0 + h]$, *with* $h \geq C\delta$, *then* $\lim_{t\to\infty} T(x, z, t) = 1$, $\forall A \geq 0, \forall x \in [0, L], \forall z \in \mathbf{R}$, *uniformly on compacts.*

H implies quenching

Suffices $\exists t_0 > 0$, such that

$$T(x,z,t_0) \leq \theta_0, \;\; \forall (x,z) \in D.$$

Indeed, by the maximum principle,

$$T \leq \theta_0, \, \forall t \geq t_0,$$

and thus

$$T_t + Au(x)T_z = \kappa \Delta T.$$

Note:

$$R(T) \leq T$$

so

$$T(x,z,t) \leq \Phi(x,z,t) e^{\frac{t}{\tau}}$$

with

$$\Phi_t + Au(x)\Phi_z = \kappa \Delta \Phi$$

with initial datum $\Phi(x,z,0) = T_0(x,z)$, periodic in x.

$$\Phi(x,z,t) = \int_{-\infty}^{\infty} G(z-\zeta,t)\Psi(x,\zeta,t)d\zeta$$

where

$$G(z,t) = \frac{1}{4\pi\kappa t} e^{-\frac{z^2}{4\kappa t}}$$

and

$$\Psi_t + Au(x)\Psi_z = \kappa \Psi_{xx}$$

with initial datum $\Psi(x,z,0) = T_0(x,z)$ and periodic in x. (Fourier in z). Note:

$$\|\Phi(\cdot,\cdot,t)\|_{L^\infty(dxdz)} \le \|\Psi(\cdot,\cdot,t)\|_{L^\infty(dxdz)}$$

We assumed: $u \in H([0,L])$. The Lie algebra generated by the vector fields ∂_x and $\partial_t + u(x)\partial_z$ spans $\mathbf{R}^2$. This implies (Hormander, Ichihara and Kunita) that there exists a continuous transition probability density $p_A(x,\xi,z-\zeta,t)$ such that

$$\Psi(x,z,t) = \int_0^L \int_{-\infty}^{\infty} p_A(x,\xi,z-\zeta,t) T_0(\xi,\zeta) d\xi d\zeta.$$

Rescaling

$$p_A(x,\xi,z,t) = \frac{1}{A} p_1(x,\xi,\frac{z}{A},t)$$

where p_1 is the transition probability density for $A = 1$. Thus

$$\|\Psi(\cdot,\cdot,t)\|_{L^\infty(dxdz)} \le C(t)\frac{1}{A}\|T_0\|_{L^1(dxdz)}$$

where

$$C(t) = \sup_{x,z} p_1(x,z,t).$$

Obtained: If $0 \le T_0 \le 1$ is supported in $[0,L]\times[z_0 - \frac{h}{2}, z_0 + \frac{h}{2}]$, then

$$T(x,z,t) \le \Phi e^{\frac{t}{\tau}} \le e^{\frac{t}{\tau}} C(t)\frac{Lh}{A}.$$

Pick

$$A \ge \frac{e}{\theta_0} C(\tau) L h.$$

Fix $t = \tau$. Then

$$T(x,z,\tau) \le eC(\tau)\frac{Lh}{A} \le \theta_0.$$

Stability and genericity

We assume $u \in H(J)$ and $[0, L] \setminus J$ is a small interval (compared to δ). Same construction as before. We wish to show

$$\Psi(x, z, \tau) \leq \frac{\theta_0}{e}.$$

Let us split

$$T_0(x, z) \leq \chi_0(x) + \psi_0(x, z)$$

where $0 \leq \chi_0 \leq 1$ is supported in a small interval, containing the interval of constancy of u, but not more than twice its length $|I|$. We take $0 \leq \psi_0(x, z) \leq 1$ to vanish whenever x is in a neighborhood of the interval of constancy of u. It follows that

$$\Psi(x, z, t) \leq \chi(x, t) + \psi(x, z, t)$$

where $\chi(x, t)$ and $\psi(x, z, t)$ are solutions of

$$(\partial_t + Au(x)\partial_z - \kappa\partial_{xx})\,\psi = 0$$

with the indicated initial data. Because χ_0 does not depend on z, it is simply a solution of the heat equation,

$$\chi_t - \kappa\chi_{xx} = 0$$

with L periodic boundary conditions in x:

$$\chi(x,t) = \sum_{j\in\mathbf{Z}} \widehat{\chi_0}(j) e^{(\frac{2\pi}{L} ijx - \frac{4\pi^2\kappa j^2 t}{L^2})}.$$

Note that, for all $j \in \mathbf{Z}$,

$$|\widehat{\chi_0}(j)| \le \frac{1}{L}\|\chi_0\|_{L^1(dx)} \le 2\frac{|I|}{L}.$$

Then

$$|\chi(x,t) - \widehat{\chi_0}(0)| \le \frac{C}{L}\|\chi_0\|_{L^1(dx)}\frac{L}{\sqrt{\kappa t}}$$

Choosing $t = \tau$ we get

$$\|\chi(\cdot,\tau)\|_{L^\infty(dx)} \le |I|\left(\frac{C}{\delta} + \frac{2}{L}\right) \le \frac{\theta_0}{10}$$

provided

$$|I|\left(\frac{C}{\delta} + \frac{2}{L}\right) \le \frac{\theta}{10}$$

has been prearranged. Now we bound ψ. Let us pick a point $x_0 \in I$ so that $I \subset [x_0 - a\delta/2, x_0 + a\delta/2]$, and arrange that $\psi_0(x,z) = 0$ for $x \in$

$[x_0 - 2a\delta, x_0 + 2a\delta]$. If $x \in [x_0 - a\delta, x_0 + a\delta]$ then

$$\psi(x,z,t) \le \int_{-\infty}^{\infty} \int_{|\xi - x_0| \ge 2a\delta} p_A(x,\xi,z-\zeta,t) d\xi d\zeta$$

$$\le \mathcal{P}\left\{\sqrt{2\kappa}|W(t)| \ge a\delta\right\} \le \frac{\theta_0}{10}$$

where $W(t)$ is one dimensional Brownian motion and $t \le t_1 \le \tau$ is chosen small enough. Indeed, consider the SDE system

$$\begin{cases} dX(t) = \sqrt{2\kappa} dW(t),\ X(0) = x, \\ dZ(t) = Au(X(t))dt,\ z(0) = z. \end{cases}$$

Thus

$$Z(t) = z + A\int_0^t u(x + \sqrt{2\kappa}W(s))ds$$

and the solution of $(\partial_t + Au(x)\partial_z - \kappa\partial_{xx}\psi = 0$ with initial datum ψ_0 is given by

$$\psi(x,z,t) = E_{x,z}(\psi_0(X(t), Z(t)))$$

For $x \notin [x_0 - a\delta, x_0 + a\delta]$ we use the condition $u \in H(J)$. There exists a function $\tilde{u}(x)$ which coincides identically with u outside $[x_0 -$

$a\delta/2, x_0 + a\delta/2]$ such that $\tilde{u} \in H([0, L])$. We consider the processes $\tilde{X}, \tilde{Z}$ associated to $\tilde{u}$ in a similar manner. Consider the stopping time t_I, the first passage time when $X(t)$ enters $[x_0 - a\delta/2, x_0 + a\delta/2]$. Note that $X(t)$ is just one dimensional Brownian motion with diffusivity κ starting from x, so the stopping time is well understood. We have

$$\psi(x, z, t) \leq P\{(X(t), Z(t)) \in supp\psi_0\} =$$
$$P\{(X(t), Z(t)) \in supp\psi_0 \,|t_I > t\}P(t_I > t)+$$
$$P\{(X(t), Z(t)) \in supp\psi_0 \,|t_I \leq t\}P(t_I \leq t) =$$
$$P\{(\tilde{X}(t), \tilde{Z}(t)) \in supp\psi_0 \,|t_I > t\}P(t_I > t)+$$
$$P\{(X(t), Z(t)) \in supp\psi_0 \,|t_I \leq t\}P(t_I \leq t) \leq$$
$$P\{(\tilde{X}(t), \tilde{Z}(t)) \in supp\psi_0\}+$$
$$P(t_I \leq t).$$

The Brownian motion needs to travel a distance of at least $a\delta/2$ to enter, so we may choose $t \leq t_2 \leq t_1$ small enough for

$$P(t_I \leq t) \leq \frac{\theta_0}{20}.$$

On the other hand, the function

$$P\{(\tilde{X}(t), \tilde{Z}(t)) \in supp\psi_0\} = \tilde{\psi}(x, z, t)$$

satisfies the PDE with $\tilde{u}$ and initial data the characteristic function $\mathbf{1}_{supp\psi_0}$. We may take A large enough to have

$$\tilde{\psi}(x, z, t) \leq \frac{\theta_0}{20}$$

at $t = t_2$. So

$$\psi(x, z, t) \leq \frac{\theta_0}{10}$$

at $t = t_2$, and by maximum principle, for $t \geq t_2$. This implies that

$$\Psi(x, z, \tau) \leq \frac{\theta_0}{5}$$

and concludes the proof of stability.

The set of functions $u \in H([0, L])$ is dense in $\mathcal{C}([0, L])$. Moreover, if $\tilde{u} \in H([0, L])$, then there exits $v > 0$ so that if $\|u - \tilde{u}\| \leq v$, then $u \in Q$.

Indeed: for $\tilde{u} \in H([0, L])$ there exists a constant $\tilde{C}$ such that $\tilde{\Psi}(x, z, \tau) \leq \frac{\theta}{10}$ holds for all initial data supported in a box $[0, L] \times [z_0 - H, z_0 + H]$ provided $H \leq \tilde{C}A$. Take now initial data supported in a box $[0, L] \times [z_0 - h, z_0 + h]$ for the equation with Au with

$$h \leq \frac{\tilde{C}}{2} A.$$

Now

$$Z(t) = z + A \int_0^t u(x + \sqrt{2\kappa} W(s)) ds$$

and thus

$$\left| Z(t) - \tilde{Z}(t) \right| \leq A \upsilon t$$

holds almost surely. Choose υ such that

$$\upsilon \tau = \frac{\tilde{C}}{2}.$$

Then

$$\Psi(x, z, \tau) \leq P\{Z(t) \in [z_0 - h, z_0 + h]\}$$
$$\leq P\{\tilde{Z}(\tau) \in [z_0 - H, z_0 + H]\} =$$
$$\tilde{\Psi}(x, z, \tau)$$

where $\tilde{\Psi}$ corresponds to advection $A\tilde{u}$, initial data $\mathbf{1}_{[0,L]\times[z_0-H,z_0+H]}$, and

$$H = h + A\tau\upsilon \leq \tilde{C}A.$$

Thus

$$\Psi(x,z,\tau) \leq \tilde{\Psi}(x,z,\tau) \leq \frac{\theta_0}{10}$$

Reactive Boussinesq fronts

Reactive Boussinesq equations:

$$\begin{cases} \frac{\partial v}{\partial t} + v \cdot \nabla v + \nabla p - \nu \Delta v = gATe_z, \\ \nabla \cdot v = 0, \\ \frac{\partial T}{\partial t} + v \cdot \nabla T - \kappa \nabla T = \frac{1}{\tau} f(T). \end{cases}$$

The Boussinesq system has flat traveling wave solutions

$$T_{fr} = \pi(z - ct), v_{fr} = 0.$$

The momentum equation holds because the pressure can balance a temperature that depends on z and t alone. The speed c takes all values $c \geq v_0$ in the KPP case, and is unique in the bistable and ignition case. The profile $\pi(z)$ is monotonically decreasing in all three cases, and obeys

$$\kappa \pi'' + c\pi' + \frac{1}{\tau} f(\pi) = 0,$$

where $\pi' = \dfrac{d\pi}{dz}$.

Dimensional units: Space:

$$\delta = \sqrt{\kappa\tau}$$

time

$$\tau.$$

$$\pi(z) = P(z/\delta)$$

with P obeying

$$P'' + 2\bar{c}P' + f(P) = 0.$$

$$v(\mathbf{x}, t) = \frac{\delta}{\tau}\tilde{v}\left(\frac{\mathbf{x}}{\delta}, \frac{t}{\tau}\right)$$

$$\theta(\mathbf{x}, t) = \tilde{\theta}\left(\frac{\mathbf{x}}{\delta}, \frac{t}{\tau}\right).$$

Using these units, rescaling, using $\mathbf{x} = (x, z) = (x_{new}, z_{new}) = (x_{old}/\delta, z_{old}/\delta)$ and $t = t_{new} = t_{old}/\tau$, and dropping tildes, we derive the non-linear equations

$$\begin{cases} \partial_t\omega + v \cdot \nabla\omega - \sigma\Delta\omega = \sigma\rho\partial_x T \\ \partial_t T + v \cdot \nabla T - \Delta T = f(T) \end{cases}$$

where $v = (u, w)$, with

$$\Delta u = -\frac{\partial\omega}{\partial z}, \quad \Delta w = \frac{\partial\omega}{\partial x}$$

Nondimensional parameters: Prandtl number

$$\sigma = \frac{\nu}{\kappa}$$

Rayleigh number (across a laminar front width)

$$\rho = \frac{gA\delta^3}{\kappa\nu}.$$

Boundary conditions

$$\begin{cases} T(x,z,t) \to 1 \text{ as } z \to -\infty, \\ T(x,z,t) \to 0 \text{ as } z \to +\infty, \\ v(x,z,t) \to 0, \text{ as } |z| \to \infty. \\ \omega(x,z,t) \to 0 \text{ as } |z| \to \infty. \end{cases}$$

The boundary conditions in x are periodic

$$T(x+\lambda,z,t) = T(x,z,t),$$
$$v(x+\lambda,z,t) = v(x,z,t),$$
$$\omega(x+\lambda,z,t) = \omega(x,z,t)$$

with period

$$\lambda = \frac{L}{\delta}.$$

General bounds

$D = [0, \lambda] \times \mathbf{R}$.

$$\|g\|_{L^2}^2 = \frac{1}{\lambda} \int_D |g(x,z)|^2 dxdz$$

Bulk burning speed:

$$V(t) = \frac{1}{\lambda} \int_D \frac{\partial T(x,z,t)}{\partial t} dxdz$$

From PDE:

$$V(t) = \frac{1}{\lambda} \int_D f(T(x,z,t)) dxdz.$$

$$\overline{V}(t) = \frac{1}{t} \int_0^t V(s) ds.$$

Consider the average quantities

$$W(t) = \frac{1}{t} \int_0^t \|w(\cdot, s)\|_{L^\infty} ds$$

and

$$N(t) = \frac{1}{t} \int_0^t \|\nabla T(\cdot, s)\|_{L^2}^2 ds$$

Theorem 9 *(CKR 2002) Solutions with front-like initial data obey*

$$N(t) \leq C_1 \rho^2 \lambda^5 + C_2 + (\frac{K_1}{\sqrt{t}} + \frac{K_2}{t}),$$

$$W(t) \leq C_3 \rho^2 \lambda^5 + C_4 \rho \lambda^{5/2} + \frac{K_3}{t^{\frac{1}{4}}} + \frac{K_4}{t^{\frac{1}{2}}}$$

and

$$\limsup_{t\to\infty} \overline{V}(t) \leq 2 + C_5 \rho^2 \lambda^5 + C_6 \rho \lambda^{5/2}$$

with C_j *depending only on the nonlinearity* f*,* C *and with* K_1, K_2, K_3, K_4 *depending on the initial data .*

Lemma 2 *Assume that there exists a constant* $\alpha \in \mathbf{R}$ *so that the front-like initial data* $T_0(x, z)$ *obeys*

$$T_0(x, z) \leq \exp(\alpha - z)$$

and

$$(1 - T_0(x, z)) \leq \exp(\alpha + z).$$

Then the solution obeys the bounds

$$T(x, z, t) \leq \exp\left[\alpha - z + 2t + \int_0^t \|w(\cdot, s)\|_{L^\infty} ds\right]$$

and

$$(1 - T(x, z, t)) \leq \exp\left[\alpha + z + t - \int_0^t \|w(\cdot, s)\|_{L^\infty} ds\right]$$

for all $t \geq 0$.

Lemma 3 *Consider front-like initial data. Then the solutions obey*

$$\overline{V}(t) \leq W(t) + 2 + \frac{\gamma}{t}$$

for all $t \geq 0$ *with* γ *depending on the initial data.*

$W(t)$ bounded below by $N(t)$:

Lemma 4 *Consider front-like initial data. Then the solutions obey*

$$N(t) \le C_1 W(t) + C_2 + \frac{\Gamma}{t}$$

with C_j *depending only on* f *and with* Γ *depending on the initial data.*

The next step consists of bounding the quantity $W(t)$ in terms of $N(t)$, using the vorticity equation.

Lemma 5 *There exists an absolute constant* C *depending on the nonlinearity* f *only, so that for all* $t > 0$ *one has*

$$W(t) \le C\lambda^{3/2} \left\{ \rho\lambda\sqrt{N(t)} + \frac{1}{\sqrt{\sigma t}} \|\omega_0\|_{L^2} \right\}$$

where $\omega_0(x, z)$ *is the initial data for* $\omega(x, z, t)$.

Ideas for proofs

For the bound of T we seek a supersolution of the form:

$$\theta_+(z,t) = \exp\left[-az + \int_0^t \|w(\cdot,s)\|_{L^\infty} ds + 2t + \alpha\right].$$

$$\frac{\partial \theta_+}{\partial t} + v \cdot \nabla \theta_+ - \Delta \theta_+ - f(\theta_+) \geq 0.$$

For the bound of $1 - T$ we seek a subsolution for T of the form

$$\theta_-(z,t) = 1 - \exp\left[z - \int_0^t \|w(\cdot,s)\|_{L^\infty} ds + t + \alpha\right]$$

and, using the fact that $f \geq 0$ on $[0,1]$, the condition

$$\frac{\partial \theta_-}{\partial t} + v \cdot \nabla \theta_- - \Delta \theta_- - f(\theta_-) \leq 0$$

follows. For the bound

$$\overline{V}(t) \leq W(t) + 2 + \frac{\gamma}{t}$$

we write

$$\overline{V}(t) = \frac{1}{\lambda t} \int_D \left(T(x,z,t) - T_0(x,z)\right) dxdz.$$

which we bound as

$$\overline{V}(t) \le \frac{1}{\lambda t}\int_0^\lambda dx \left[\int_{-\infty}^0 (1 - T_0(x,z))\, dz + \int_0^\infty T(x,z,t)dz\right],$$

using the fact that $T(t,x,z) \le 1$. Now, denoting

$$B_1(t) = \alpha + 2t + \int_0^t \|w(\cdot,s)\|_{L^\infty} ds$$

we have from the exponential upper bound ahead of the front:

$$\int_{B_1(t)}^\infty T(x,z,t)dz \le 1,$$

while, because $T \le 1$, we have

$$\int_0^{B_1(t)} T(x,z,t)dz \le B_1(t)$$

and this finishes the proof of the lemma. For the lower bound

$$N(t) \le C_1 W(t) + C_2 + \frac{\Gamma}{t}$$

we start by computing

$$\frac{d}{dt}\frac{1}{\lambda}\int_D T(1-T)dxdz$$

$$-\frac{1}{\lambda}\int_D (1-2T(x,z,t))f(T(x,z,t))dxdz =$$
$$-\frac{2}{\lambda}\int_D |\nabla T(x,z,t)|^2 dxdz.$$

Taking a time average we get

$$\frac{1}{\lambda t}\int_D \left(T(x,z,t)(1-T(x,z,t)) - T_0(x,z)(1-T_0(x,z))\right) dx$$
$$+\overline{V}(t) \geq 2N(t).$$

We observe that

$$\frac{1}{\lambda}\int_D T(x,z,t)(1-T(x,z,t))dxdz \leq$$
$$\int_0^\lambda \frac{dx}{\lambda}\int_{-\infty}^{-B_2(t)}(1-T(x,z,t))dz$$
$$+\int_0^\lambda \frac{dx}{\lambda}\int_{-B_2(t)}^{B_1(t)} 1dz$$
$$+\int_0^\lambda \frac{dx}{\lambda}\int_{B_1(t)}^{\infty} T(x,z,t)dz,$$

where $B_1(t)$ is given above and

$$B_2(t) = \alpha + t + \int_0^t \|w(\cdot,s)\|_{L^\infty} ds.$$

We use

$$\int_0^\lambda \frac{dx}{\lambda}\int_{-\infty}^{-B_2(t)}(1-T(x,z,t))dz \leq 1,$$

as follows from the bound at minus infinity. Similarly,

$$\int_0^\lambda \frac{dx}{\lambda} \int_{B_1(t)}^\infty T(x,z,t)dz \le 1$$

The second term $B_1(t)+B_2(t)$. Thus, returning we have

$$2N(t) \le \overline{V}(t) + 3 + 2W(t) + \frac{c}{t}.$$

This finishes the lower bound proof. The upper bound of W in terms of $\sqrt{N}$ is done using energy estimates. We introduce

$$\overline{T}(z,t) := \int_0^\lambda T(x,z,t)\frac{dx}{\lambda}$$

and obtain

$$\frac{1}{2}\frac{d}{dt}\int_D |\omega(x,z,t)|^2 \frac{dxdz}{\lambda} + \sigma \int_D |\nabla\omega(x,z,t)|^2 \frac{dxdz}{\lambda} = -\sigma\rho \int_D \frac{\partial\omega(x,z,t)}{\partial x}\left(T(x,z,t) - \overline{T}(z,t)\right)\frac{dxdz}{\lambda}.$$

Using Young's inequality together with the inequality

$$\int_D \left|T(x,z,t) - \overline{T}(z,t)\right|^2 \frac{dxdz}{\lambda} \le \lambda^2 \int_D \left|\nabla T(x,z,t)\right|^2 \frac{dxdz}{\lambda}$$

we deduce

$$\frac{1}{2}\frac{d}{dt}\int_D |\omega(x,z,t)|^2 \frac{dxdz}{\lambda} + \sigma \int_D |\nabla\omega(x,z,t)|^2 \frac{dxdz}{\lambda} \leq \frac{\sigma}{2}\int_D \left|\frac{\partial\omega(x,z,t)}{\partial x}\right|^2 \frac{dxdz}{\lambda} + \frac{\sigma\rho^2\lambda^2}{2}\int_D |\nabla T(x,z,t)|^2 \frac{dxdz}{\lambda}.$$

Integrating in time we deduce

$$\frac{1}{t}\int_0^t ds \int_D |\nabla\omega(x,z,s)|^2 \frac{dxdz}{\lambda} \leq \rho^2\lambda^2 N(t) + \frac{1}{\sigma t}\|\omega_0\|_{L^2}^2.$$

Let us represent the function w in terms of its Fourier series

$$w(x,z,t) = \sum_{k\in\frac{2\pi}{\lambda}\mathbf{Z}} w_k(z,t)e^{ikx}$$

and note that, in view of incompressibility, $w_0(z,t)$ is independent of z, and hence the boundary conditions at $z \pm \infty$ imply that

$$w_0(z,t) = 0.$$

In view of the embedding inequality

$$\|w(\cdot,t)\|_{L^\infty} \leq C\lambda^{3/2}\|\nabla\omega(\cdot,t)\|_{L^2}$$

the last lemma follows.

Nonlinear stability of planar fronts in narrow domains

Narrow domain: small aspect ratio λ. The nonlinearity f is of either one of the three types: KPP, ignition or bistable.

Theorem 10 *(CKR 2002). There exist constants $C_1 > 0$ and $C_2 > 0$ such that if $\lambda < C_1$, and $\rho < C_2/\lambda^3$, then the only solutions of traveling front type $T = T(x, z - ct), v = v(x, z - ct)$, are planar fronts of the form $T = P(z - ct), v = 0$.*

The second result in this section is about arbitrary solutions. We show that all solutions of the Boussinesq system in a narrow domain eventually become planar:

Theorem 11 *(CKR 2002). There exist constants $C_1 > 0$ and $C_2 > 0$ so that if $\lambda < C_1$ and $\rho < C_2/\lambda^3$, then*

$$\|\omega(\cdot, t)\|_{L^2} + \|T_x(\cdot, t)\|_{L^2} \to 0 \quad \text{as } t \to +\infty.$$

Moreover, the front speed is uniformly bounded:

$$\limsup_{t\to+\infty} \bar{V}(t) \le 2.$$

Linear instability

Linear instability of planar fronts with respect to large wavelength perturbations. Galilean transformation: $z \mapsto z - v_0 t$ following the flat front. We write $T(x,z,t) = \pi(z - v_0 t) + \theta(x, z - v_0 t, t)$, and $v(x,z,t) = v(x, z - v_0 t, t)$. We linearize:

$$\frac{\partial \theta}{\partial t} - 2\frac{\partial \theta}{\partial z} - \Delta\theta - f'(P(z))\theta = -wP'(z)$$

and

$$\frac{\partial \omega}{\partial t} - 2\frac{\partial \omega}{\partial z} - \sigma\Delta\omega = \sigma\rho\frac{\partial \theta}{\partial x}$$

Infinite Prandtl number:

$$-\Delta\omega = \rho\frac{\partial \theta}{\partial x},$$

which implies

$$w = -\rho(\partial_x)^2(-\Delta)^{-2}\theta$$

We express $\theta(x,z,t)$ in terms of its Fourier series:

$$\theta(x,z,t) = \sum_{k\in\frac{2\pi}{\lambda}\mathbf{Z}} g_k(z,t)e^{ikx}.$$

The linearized temperature equation transforms into

$$\frac{\partial g_k}{\partial t} - 2\frac{\partial g_k}{\partial z} + \left(k^2 - \partial_{zz}\right) g_k - f'(P)g_k = \rho Q K g_k$$

with $k = \pm\frac{2\pi}{\lambda}, \pm 2\frac{2\pi}{\lambda}, \ldots$, the operator K defined by the Fourier transform

$$Kg = k^2 \left(k^2 - \partial_{zz}\right)^{-2} g$$

and

$$Q(z) = -P'(z) > 0.$$

We take a positive wave number

$$k = \frac{2\pi n}{\lambda}$$

The operator K is given explicitly by a convolution with a positive function

$$(Kg)(z) = \frac{1}{4k}\int_{-\infty}^{\infty}(1+k|z-\zeta|)e^{-k|z-\zeta|}g(\zeta)d\zeta.$$

It is well known that the profile P is decreasing in the case of KPP, bistable and ignition nonlinearities so that the function Q is positive. Moreover

$$Q(z) \geq a e^{-b|z|}$$

holds for all z, with $a > 0$ and $b > 0$ absolute numbers that depend only on the nonlinearity $f(T)$. Let us consider a function $\phi(z)$ which has the properties

$$e^{-k|z|} \leq \phi(z) \leq C e^{-k|z|}$$

with $C > 1$ and

$$|\phi'(z)| \leq C k e^{-k|z|}, \quad |\phi''(z)| \leq C k^2 e^{-k|z|}.$$

We obtain the ordinary differential inequality

$$\frac{d}{dt} \int \phi(z) g_k(z) dz \geq \left(\frac{\rho}{4k} \frac{a}{2C(b+2k)} - \nu_k\right) \int \phi(z) g_k(z) dz,$$

with

$$\nu_k \leq 2C(1 + k + k^2),$$

and thus $\|g_k\|_{L^1(\mathbf{R})}$ grows exponentially in time. Therefore we have the following theorem for the infinite Prandtl number case:

Theorem 12 *(CKR 2002). Let* $P(z - 2t)$*,* $u = 0$ *be a planar,* x*-independent traveling front solution of the infinite Prandtl number Boussinesq system*

$$\frac{\partial T}{\partial t} + v \cdot \nabla T - \Delta T = f(T)$$

$$-\Delta v + \nabla p = \rho T e_z, \quad \nabla \cdot u = 0,$$

with front boundary conditions for T *at* $z = \pm\infty$*, vanishing velocity at* $z = \pm\infty$ *and periodic boundary conditions in* x *of period* λ*. There exists a positive constant* $\beta > 0$ *such that, if*

$$\rho\lambda > \beta,$$

then the solution P *is linearly unstable. This means that there exist infinitesimal perturbations which grow exponentially, when viewed in a Galilean frame of reference moving with the traveling front. Their exponential growth rate is proportional to* $\rho\lambda$*.*

Similar in porous media (Darcy's law).

Finite Prandtl number Take a function $\phi(z)$ with properties above, multiply by ϕ and integrate.

$$Y(t) = \int \phi(z) g_k(z,t) dz.$$

Consider also

$$Z(t) = e^{2k(\sigma k+2)t} Y(t).$$

We can prove

$$\frac{dZ}{dt} + \beta Z \geq \alpha \int_0^t Z(s) ds$$

with

$$\beta = C_3(1 + k + k^2) - 2k(\sigma k + 2)$$

and

$$\alpha = C_4 \sigma \rho k.$$

Equations always have at least one exponentially growing solution because $\alpha > 0$ (irrespective of the sign of β). We get exponential growth for $Z(t)$. This will imply exponential growth for $Y(t)$ if

$$\rho > C \left(\frac{2}{\sigma} + k \right) \left(1 + k + k^2 \right).$$

Theorem 13 *(CKR 2002). Planar reactive Boussinesq fronts are linearly unstable to large wavelength perturbations whenever the local Rayleigh number* ρ *based on the laminar front thickness is large compared to the inverse of the Prandtl number,*

$$\rho > \frac{2C}{\sigma}.$$

Perturbations with wave numbers k *satisfying the constraint above grow exponentially in a frame of reference moving with the planar front. The growth rate is proportional to* $\sqrt{\sigma\rho k}$.

Remark. The exponential growth rate proportional to the square root of the wave number is a signature of the Rayleigh-Taylor instability, operating here only at large scales. When the wavelength of the initial perturbation is decreased to a length comparable to the thickness of the planar front, the perturbation decays in time.

Summary

Passive advection reaction systems have enhanced bulk burning speed. The enhancement is linear in the amplitude of the fluid's velocity, if the flow is percolating. Lower bounds of a lower power exist is the flow is cellular.

Passive advection reaction systems can be quenched by flows that are nondegenerate in the sense of condition H, and small perturbations thereof. Quenching is a robust property, unlike hypoellipticity.

Active reactive Boussinesq system in a strip have bounded bulk burning speed. Asymptotic front acceleration does not occur in this system. For small aspect ratios and for small Rayleigh numbers, the only traveling modes are planar, and all front-like solutions become planar. For large enough Rayleigh numbers, if the aspect ratio is large, then the planar fronts lose stability to longwave perturbations. The instability is of Rayleigh-Taylor type.