Improved dynamics and gravitational waveforms from relativistic core collapse simulations

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Conference on Sources of Graviational Waves.
2003 - ICTP. Trieste (Italy).

Introduction

Gravitational Waves from Core Collapse Supernovae:

- **Müller, 1982 : 2D Newtonian finite-differences simulations**
  $\longrightarrow$ low efficiencies of core collapse generating GW.
- **Finn & Evans, 1990 : Confirmed Müller's results and improved quadrupole formula.**
- **Bonazzola & Marck, 1993 : 3D simulations using pseudospectral methods $\longrightarrow$ still low efficiencies.**
- **Zwerger & Müller, 1997**
- **Dimmelmeier, Font & Müller, 2001 : First relativistic attempt.**
  2D axisymmetric simulations with CFC metric (Isenberg, Wilson & Mathews)
- **Shibata & Sekiguchi (in preparation) : Full General Relativistic simulations in 2D using cartoon method**

In all cases initial models were rotating polytropes.

CFC $\longrightarrow$ CFC+

- **CFC**
  - **5 Elliptic equations $\longrightarrow$ stability.**
  - **Exact Full GR in spherical symmetry .**
  - **Without spherical symmetry has the same precision as first Post-Newtonian order (Kley & Schäfer, 1999).**
  - **Satisfy ADM constraint equations in core collapse scenarios (Dimmelmeier, Font & Müller, 2001).**
- **CFC+**
  Extension of CFC by Second Post-Newtonian Terms (G.Faye):
  - **Appear non-diagonal terms in Spatial part of metric .**
  - **We still don't have radiation reaction (appears at 2.5 PN order).**

Motivation

- **CFC approximation vs Full General Relativity.**
  - **Elliptic equations is a way to get stable equations.**
    (Full General Relativistic equations are unstable and complicated).
  - **In Core Collapse SNe and pulsating NS, CFC seems to be enough.**
- **Assessment of CFC approximation in core collapse supernovae.**
- **Study other scenarios such as collapse to BH or extreme rotating configurations of NS.**
  - **Failed supernova (collapsar model of GRB).**
  - **Late stage of coalescing neutron stars.**
  - **Self gravitating tori around black hole (runaway instability, magneto-rotational instability).**

Formulation - {3+1}

{3+1} metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

Extrinsic curvature

$$K_{ij} = -\frac{1}{2}\mathcal{L}_{n^\mu}\gamma_{ij}$$

Evolution equations

$$\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i \\
\partial_t K_{ij} &= -\nabla_i \nabla_j \alpha + \alpha(R_{ij} + K K_{ij} - 2K_{ik}K^k_j) + \beta^k \nabla_k K_{ij} \\
&\quad + K_{ik}\nabla_j \beta^k + K_{jk}\nabla_i \beta^k - 8\pi\alpha\left(S_{ij} - \frac{\gamma_{ij}}{2}(S - \rho_H)\right)
\end{aligned}$$

Constraint equations

$$\begin{aligned}
R + K^2 - K_{ij}K^{ij} - 16\pi\rho_H &= 0 \\
\nabla_i(K^{ij} - \gamma^{ij}K) - 8\pi S^j &= 0
\end{aligned}$$

Formulation - CFC

Conformal flat metric:

$$\gamma_{ij}^{CFC} = \phi^4 \hat{\gamma}_{ij} \qquad \longrightarrow \qquad \gamma_{ij}^{CFC} = 0 \quad \textbf{if} \quad i \neq j$$

Maximal slicing condition:

$$K_i^i = 0$$

Equations for α, β^i, and ϕ:

$$\hat{\Delta}\phi = -2\pi\phi^5\left(\rho h W^2 - P + \frac{K_{ij}K^{ij}}{16\pi}\right)$$

$$\hat{\Delta}(\alpha\phi) = 2\pi\alpha\phi^5\left(\rho h(3W^2 - 2) + 5P + \frac{7K_{ij}K^{ij}}{16\pi}\right)$$

$$\hat{\Delta}\beta^i = 16\pi\alpha\phi^5 S^i + 2K^{ij}\hat{\nabla}_j\left(\frac{\alpha}{\phi^6}\right) - \frac{1}{3}\hat{\nabla}^i\hat{\nabla}_k\beta^k$$

$\hat{\Delta}$ **: Laplacian with respect to the flat metric** $\hat{\gamma}_{ij}$

Formulation - CFC+

CFC+ metric:

$$\gamma_{ij}^{CFC+} = \gamma_{ij}^{CFC} + h_{ij}^{TT} \qquad \longrightarrow \qquad Tr h^{TT} = 0$$

The Second Post-Newtonian deviation from isotropy is the solution of

$$\Delta h_{ij}^{TT} = \frac{1}{c^4} \delta^{TT}{}^{kl}_{ij} (-16\pi \rho v_k v_l - 4\partial_k U \, \partial_l U) + \mathcal{O}\left(\frac{1}{c^6}\right)$$

Modified equations for α, β^i, and ϕ:

$$\hat{\Delta}\phi = -2\pi\phi^5 \left(\rho h W^2 - P + \frac{K_{ij}K^{ij}}{16\pi}\right)$$

$$\hat{\Delta}(\alpha\phi) = 2\pi\alpha\phi^5 \left(\rho h(3W^2 - 2) + 5P + \frac{7K_{ij}K^{ij}}{16\pi}\right) - \frac{1}{c^2} h_{ij}^{TT} \partial_{ij} U$$

$$\hat{\Delta}\beta^i = 16\pi\alpha\phi^5 S^i + 2K^{ij}\hat{\nabla}_j \left(\frac{\alpha}{\phi^6}\right) - \frac{1}{3}\hat{\nabla}^i \hat{\nabla}_k \beta^k$$

Formulation - CFC+

We can solve h_{ij}^{TT} equations introducing some intermediate potentials

$$\left.\begin{aligned}(\Delta\mathcal{S})_i &= (-4\pi\rho v_i v_j - \partial_i U \partial_j U)\, x^j \\ (\Delta\mathcal{S})_{ij} &= -4\pi\rho v_i v_j - \partial_i U \partial_j U \\ \Delta\mathcal{S} &= -4\pi\rho v_i v_j\, x^i x^j \\ (\Delta\mathcal{T})_i &= (-4\pi\rho v_j v^j - \partial_j U \partial^j U)\, x_i \\ (\Delta\mathcal{R})_i &= \partial_i(\partial_k U \partial_l U)\, x^k x^l.\end{aligned}\right\} \text{16 elliptic equations}$$

such that,

$$\begin{aligned} h_{ij}^{TT} = &\ \frac{1}{2}\mathcal{S}_{ij} - \frac{7}{4}\delta_{ij}\mathcal{S}_k^k - 3x^k\partial_{(i}\mathcal{S}_{j)k} + 3\partial_{(i}\mathcal{S}_{j)} + \frac{1}{4}x^j\partial_i\mathcal{S}_k^k \\ & -\frac{1}{4}\partial_i\mathcal{T}_j - \frac{1}{2}x^k\partial_{ij}\mathcal{S}_k + \frac{1}{4}x^k x^l\partial_{ij}\mathcal{S}_{kl} + \frac{1}{4}\partial_{ij}\mathcal{S} - \frac{1}{4}\partial_i\mathcal{R}_j \end{aligned}$$

Formulation - Hydrodynamics

Conserved quantities:

$$\left.\begin{aligned} D &= -J^\mu n_\mu = \rho W \\ S^i &= -\perp^i_\nu T^{\mu\nu} n_\mu = \rho h W^2 v^i \\ \tau &= T^{\mu\nu} n_\mu n_\nu - J^\mu n_\mu = \rho h W^2 - P - D \end{aligned}\right\} \quad F^0 = (D, S_j, \tau)$$

General Relativistic Hydrodynamic Equations written as flux-conservative hyperbolic system of conservation laws:

$$\frac{1}{\sqrt{-g}}\left[\frac{\partial\sqrt{\gamma}F^0}{\partial x^0} + \frac{\partial\sqrt{-g}F^i}{\partial x^i}\right] = Q$$

F^0 **: state vector**

F^i **: flux vector**

Q **: source vector**

Sources:

$$Q = \left(0, T^{\mu\nu}\left(\frac{\partial g_{\mu j}}{\partial x^\mu} - \Gamma^\lambda_{\mu\nu} g_{\lambda j}\right), \alpha\left(T^{\mu 0}\frac{\partial \ln\alpha}{\partial x^\mu} - T^{\mu\nu}\Gamma^0_{\mu\nu}\right)\right)$$

Numerical Methods - Grid setup

- **2D Axisymetric code with $v_\phi \neq 0$ + Equatorial symmetry (Dimmelmeier et al, 2001)**
- **Spherical polar coordinates (r, θ)**
- **Equidistant $\Delta\theta$**
- **logarithmic Δr_i (for Core Collapse) or equidistant Δr_i (for NS)**
- **Regions**
  - **Star**
  - **Atmosphere**
  - **Extended outer domain for CFC+ metric calculations**

Numerical Methods - Hydrodynamics

We use High Resolution Shock Capturing (HRSC) Methods.

- **Reconstruction of hydro quantities (ρ, v_i, ϵ) at cell interfaces $\longrightarrow$ PPM (third order reconstruction)**
- **Solve approximate Riemann problem at each cell, calculate fluxes through interfaces $\longrightarrow$ Marquina flux formula**
  - **Eigenvectors in CFC : diagonal γ_{ij} (Banyuls et al. 1997).**
  - **Eigenvectors in CFC+ : general metric (Font et al. 2000).**
- **Time evolution of conserved quantities (D, S_i, τ) $\longrightarrow$ third order Runge-Kutta method**
- **Recovery of Hydro quantities (ρ, v_i, ϵ)**

Numerical Methods - Metric Solver

- **Intermediate potentials:** $\mathcal{S}_i$, $\mathcal{S}_{ij}$, $\mathcal{S}$, $\mathcal{T}_i$, $\mathcal{R}_i$
  - **Linear solver (LU decomposition using standard LAPACK routines)**
  - **Spectral methods using LORENE (Work in progress...)**
- **CFC quantities:** ϕ, α, β^i
  - **Newton-Raphson with sparse linear solver**
  - **Integral formulation with Green's functions**
  - **Spectral methods LORENE**

Numerical Methods - Metric Solver

Boundary conditions
Multipole development in compact supported integrals:

$$
\begin{aligned}
\mathcal{S}^i &= \frac{1}{r}\int d^3x\rho(v^iv_kx^k + x^i(U + x^k\partial_kU)) + \frac{M^2}{2r}n^i + \mathcal{O}(\frac{1}{r^2}) \\
\mathcal{S}^{ij} &= \frac{1}{r}\int d^3x\rho(v^iv^j + \frac{\delta^{ij}}{2} + x^j\partial_iU) + \mathcal{O}(\frac{1}{r^2}) \\
\mathcal{S} &= \frac{1}{r}\int d^3x\rho v_kv_lx^kx^l + \mathcal{O}(\frac{1}{r^2}) \\
\mathcal{T}^i &= \frac{1}{r}\int d^3x\rho(v_kv^kx^i + x^iU) + \frac{M^2}{2r}n^i + \mathcal{O}(\frac{1}{r^2}) \\
\mathcal{R}^i &= \frac{M^2n^i}{r} + \mathcal{O}(\frac{1}{r^2})
\end{aligned}
$$

Numerical Methods - Gravitational Wave Extraction

Quadrupole Formula

$$h_{\theta\theta}^{TT} = \frac{1}{8}\sqrt{\frac{15}{\pi}} \sin\theta^2 \frac{A_{20}^{E2}}{r}$$

$$A_{20}^{E2} = \frac{d^2 M_{20}^{E2}}{dt^2} = \frac{d^2}{dt^2}\left(\frac{32\pi^{3/2}}{\sqrt{15}} \int \rho\left(\frac{3}{2}\cos\theta^2 - \frac{1}{2}\right) r^4 \sin\theta dr d\theta\right)$$

Stress formula (Blanchet, Damour & Schäfer 1990)

$$\begin{aligned} A_{20}^{E2} = & \frac{32\pi^{3/2}}{\sqrt{15}} \int \rho\Big(v_r v_r (3\cos\theta^2 - 1) + v_\theta v_\theta (2 - 3\cos\theta^2) - v_\phi v_\phi \\ & - 6 v_r v_\theta \sin\theta\cos\theta - r\frac{\partial\Phi}{\partial r}(2\cos\theta^2 - 1) + 3\frac{\partial\Phi}{\theta}\sin\theta\cos\theta\Big) r^2 \sin\theta dr d\theta \end{aligned}$$

Results - Rotating Neutron Star

We have generated initial models for rotating NS in equilibrium using Hachisu's self-consistent field method (Komatsu, Eriguchi & Hachisu, 1998).

Free parameters:

- ρ_c **: central density**
- r_p/r_e **: ratio of equatorial to polar radius.**
- A **: rotational law**

$$\Omega = \frac{\Omega_C}{1 + d^2/A^2}$$

$A >> R \rightarrow$ **rigid rotation**
$A < R \rightarrow$ **differential rotation**

For the spherical models we use TOV solution.

Results - Rotating Neutron Star

RNS models used:

- **Non Rotating (Spherical) Neutron Stars**
  $\rho_c = 7.91 \times 10^{14} g\, cm^{-3}$
  Polytropic equation of state : $P = K\rho^\gamma$
  $K = 100$ **(**$G = c = M_\odot = 1$**) and** $\gamma = 2$

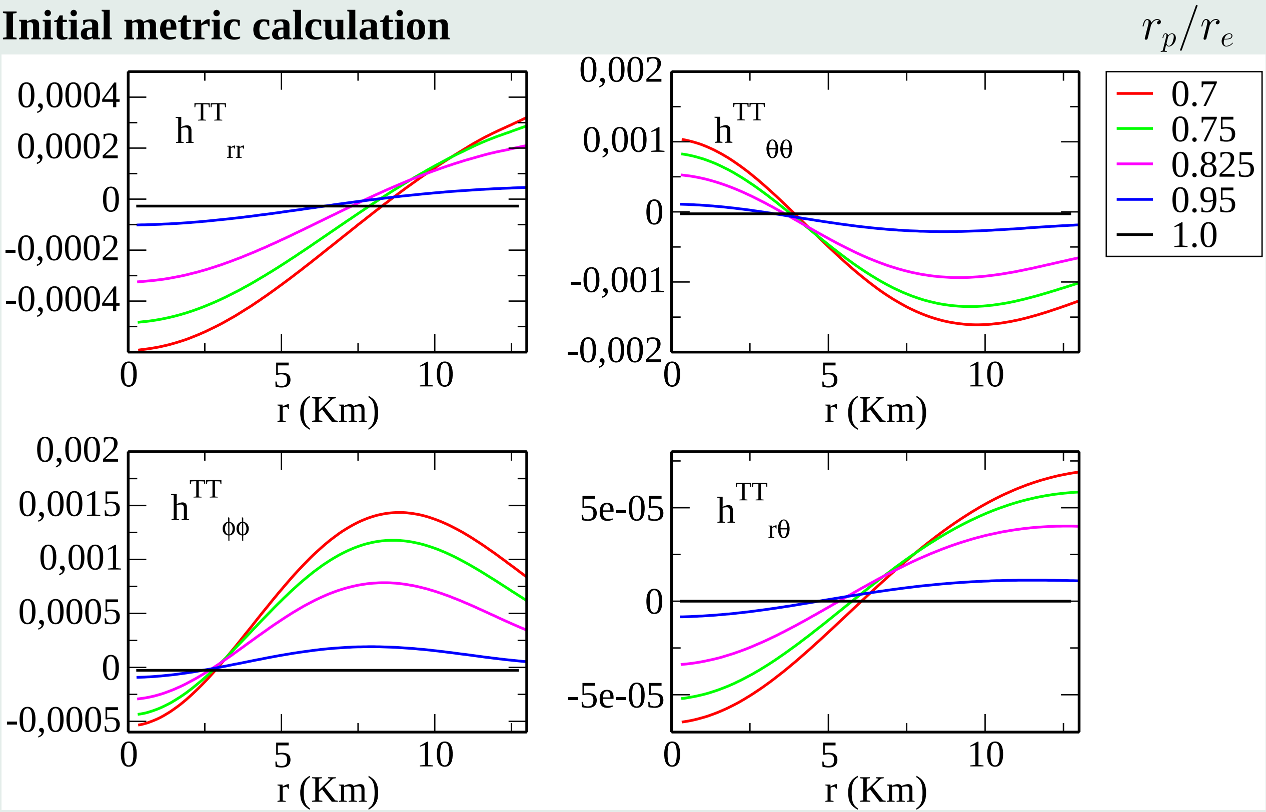
  $M = 1.4 M_\odot$ **and** $R = 14.15$ **km**

- **Rotating Neutron Stars**
  $\rho_c = 7.91 \times 10^{14} g\, cm^{-3}$
  rigid rotation : $A \rightarrow \infty$
  r_p/r_e **= 0.7**
  Polytropic equation of state : $P = K\rho^\gamma$
  $K = 100$ **(**$G = c = M_\odot = 1$**) and** $\gamma = 2$

  $M = 1.63 M_\odot$
  $r_e = 19.6$ **km**
  $\Omega = 4997 s^{-1} \rightarrow$ **93 % mass-shedding limit.**

Results - Rotating Neutron Star

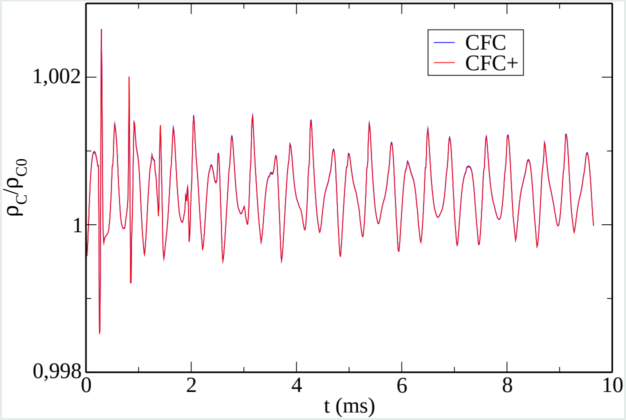
Initial metric calculation



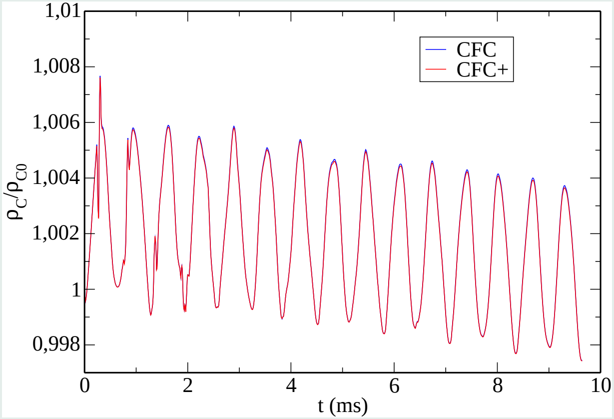
$$n_r = 150, n_\theta = 20$$

Results - Radial Modes of spherical NS

Fixed Space-Time



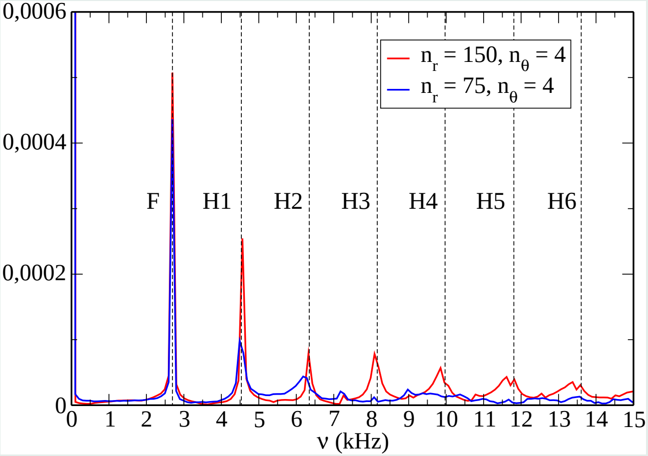
Coupled evolution



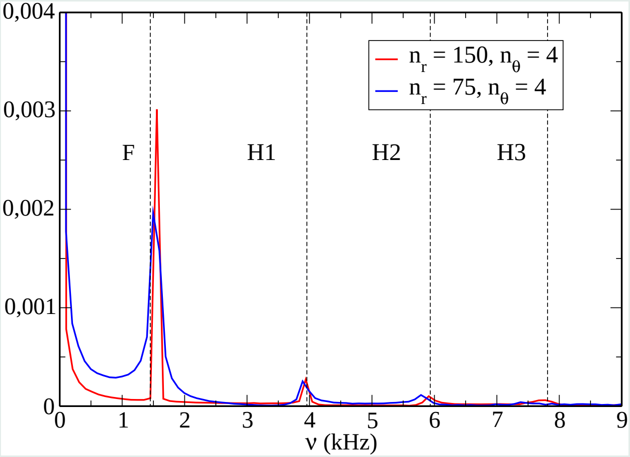
$$n_r = 150, n_\theta = 20$$

Results - Radial Modes of spherical NS

Fixed Space-Time



Coupled evolution



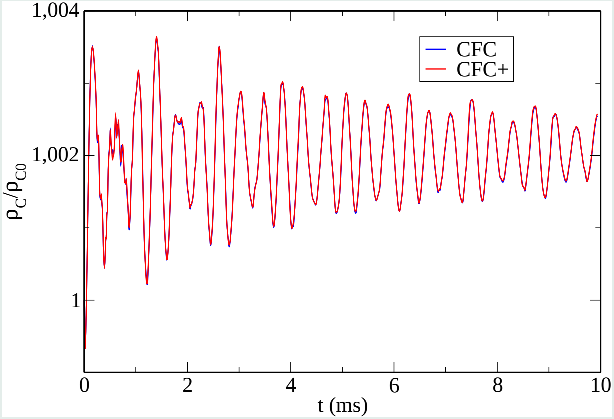
Mode	CFC+ (kHz)	GR Code ⋆ (kHz)	Rel.Diff. (%)
F	2.74	2.696	1.6
H1	4.58	4.534	1.0
H2	6.35	6.346	0.06
H3	8.12	8.161	0.5
H4	9.84	9.971	1.3
H5	11.88	11.806	0.6
H6	13.47	13.605	1.0

Mode	CFC+ (kHz)	GR Code ⋆ (kHz)	Rel.Diff. (%)
F	1.57	1.450	8
H1	3.94	3.958	0.5
H2	5.91	5.935	0.4
H3	7.74	7.812	0.9

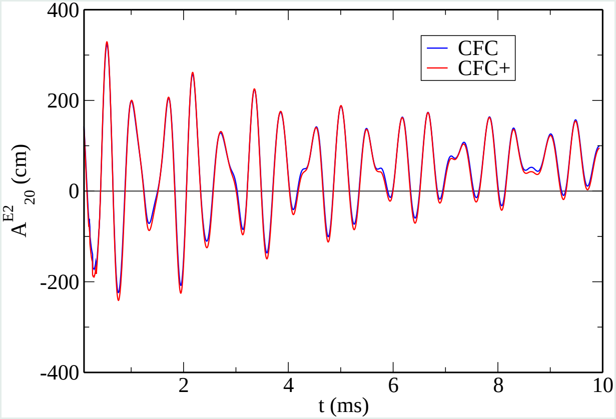
⋆ Font et al. 2002

Results - Quasi-Radial Modes of RNS - Fixed Space-Time

Central density

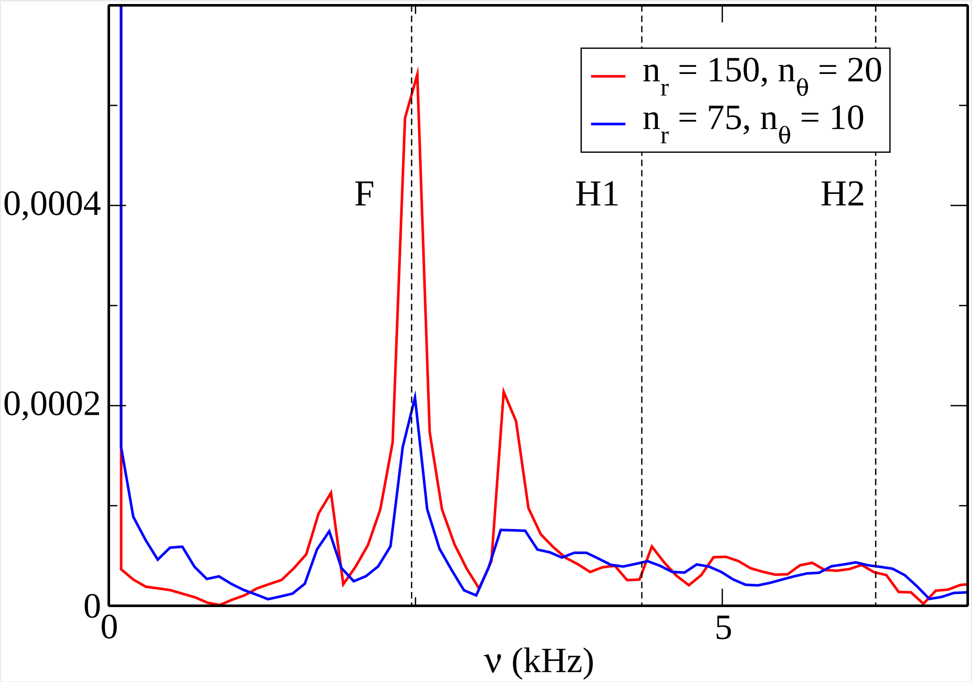


GW signal



$$n_r = 150, n_\theta = 20$$

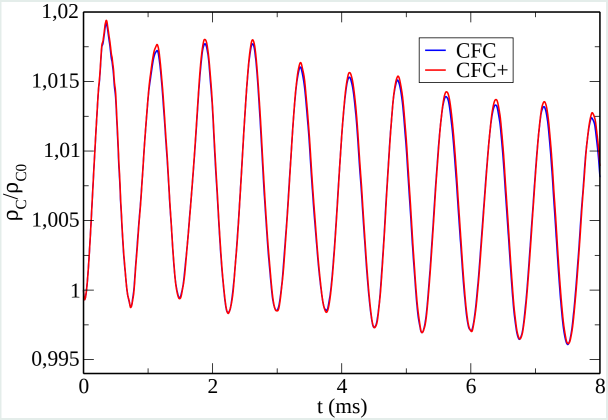
Results - Quasi-Radial Modes of RNS - Fixed Space-Time



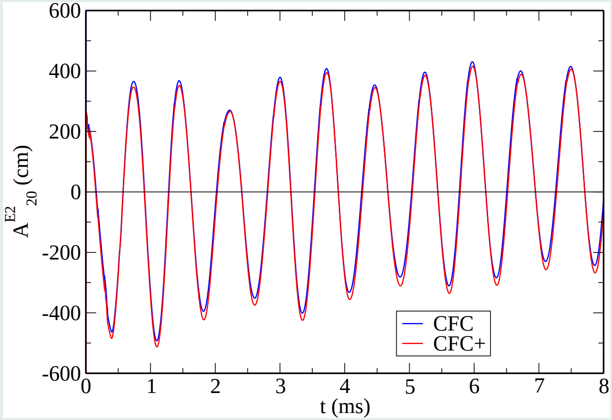
Mode	CFC+ (kHz)	GR Code * (kHz)	Rel.Diff. (%)
F	2.51	2.468	1.7
H1	4.44	4.344	2

Results - Quasi-Radial Modes of RNS - Coupled evolution

Central density

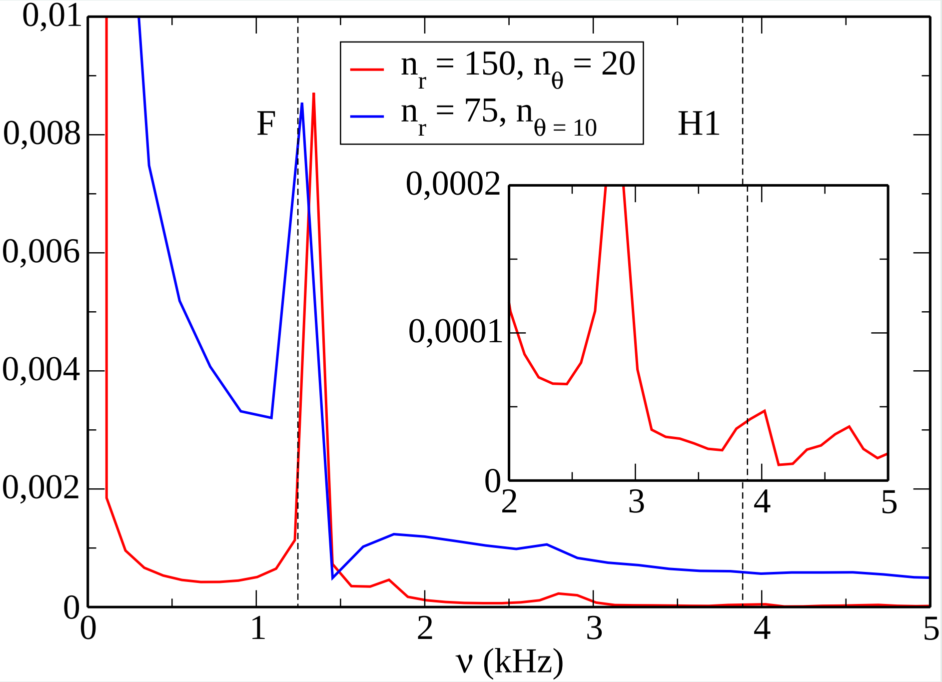


GW signal



$$n_r = 150, n_\theta = 20$$

Results - Quasi-Radial Modes of RNS - Coupled evolution



Mode	CFC+ (kHz)	GR Code ⋆ (kHz)	Rel.Diff. (%)
F	1.35	1.247	8
H1	4.03	3.887	4

Results - Core Collapse Supernova

Iron Core $\longrightarrow$ Rotating 4/3 polytope

$\rho_c = 10^{10} g cm^{-3}$

A1B3G5	**A4B5G5**
$A = 50 \times 10^8 cm$	$A = 0.1 \times 10^8 cm$
(rigid rotation)	**(differential rotation)**
$r_p / r_e = 0.665$	$r_p / r_e = 0.63$
$\beta \equiv \frac{\|T\|}{\|W\|} = 0.894\%$	$\beta \equiv \frac{\|T\|}{\|W\|} = 4\%$

Polytropic equation of state : $P = K\rho^\gamma$ **with** $\gamma = 4/3$

Collapse begins reducing γ**:** $4/3 \longrightarrow 1.28$

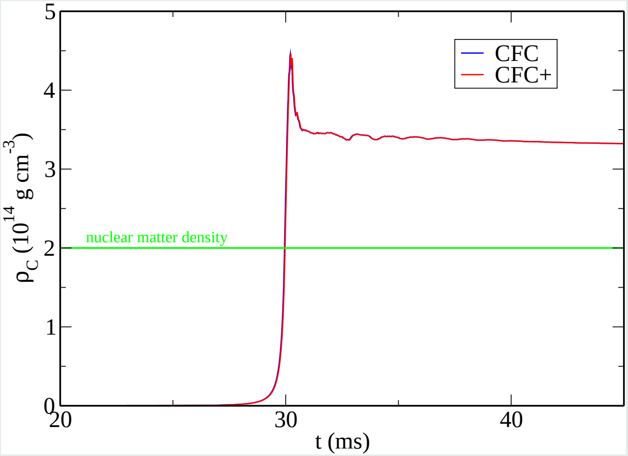
During collapse we use hybrid equation of state:

$$
\begin{aligned}
P &= P_{th} + P_p \\
P_{th} &= (\gamma_{th} - 1)\rho\epsilon_{th} \qquad P_p = K\rho^\gamma
\end{aligned}
$$

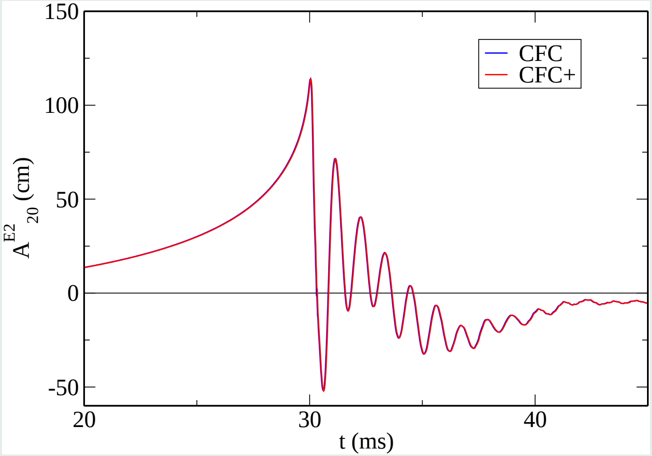
with $\quad \gamma_{th} = 1.5 \quad$ **and** $\quad \gamma \left\{ \begin{array}{ll} 1.28 & \textbf{if } \rho < \rho_{nuc} \\ 2.5 & \textbf{if } \rho \geq \rho_{nuc} \end{array} \right.$

Results - Core Collapse Supernova

Central density



GW signal

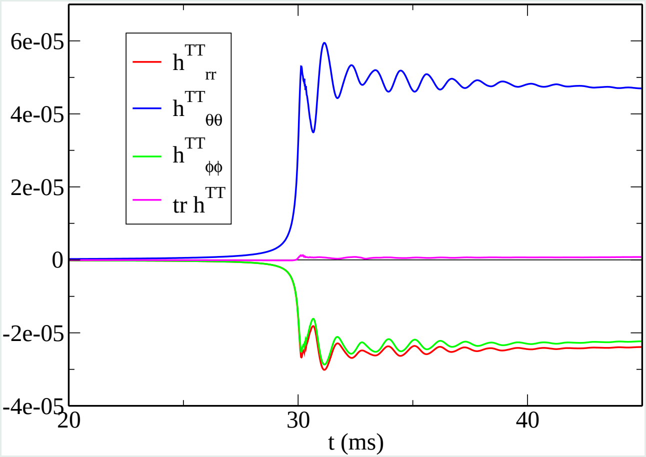


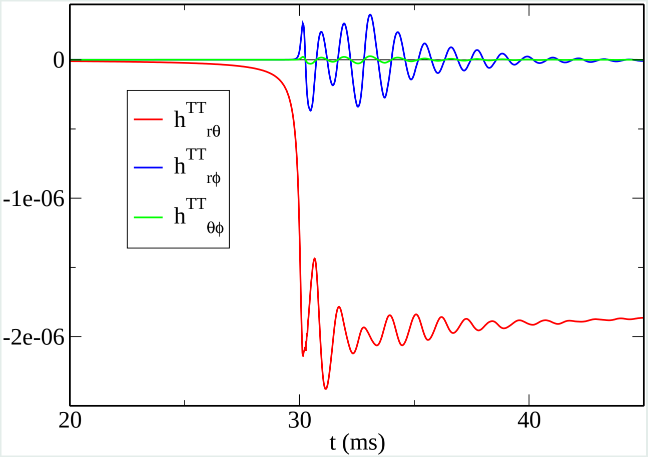
A1B3G5 model

$$n_r = 300, n_\theta = 30$$

Results - Core Collapse Supernova

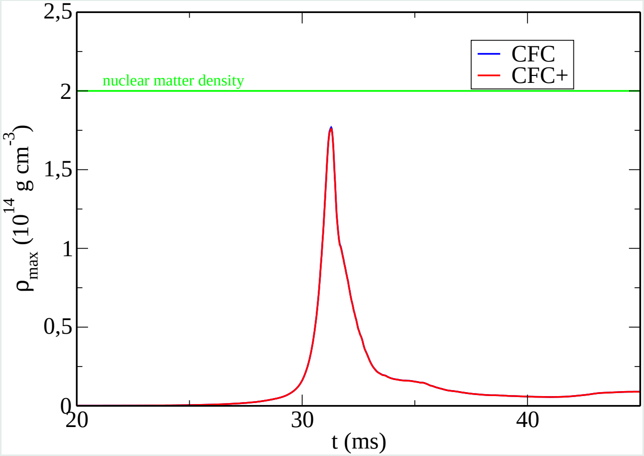
h_{ij}^{TT} correction at center



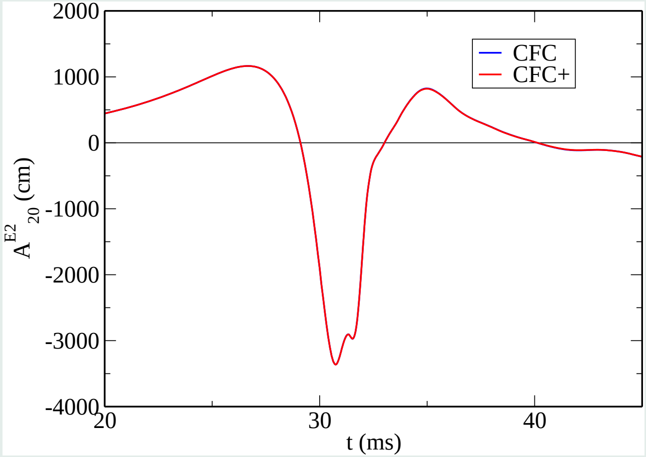


Results - Core Collapse Supernova

Central density



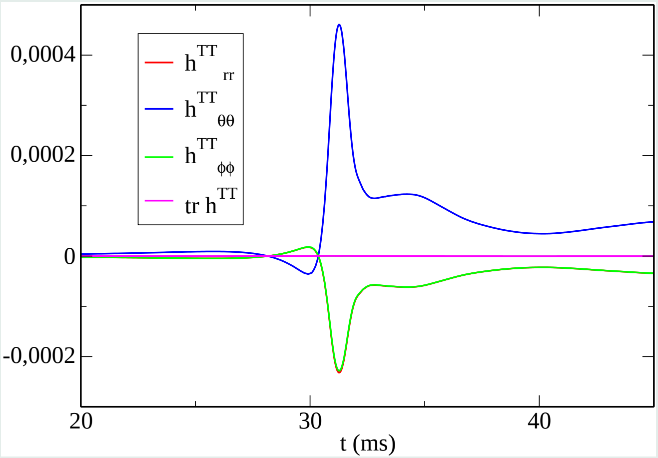
GW signal

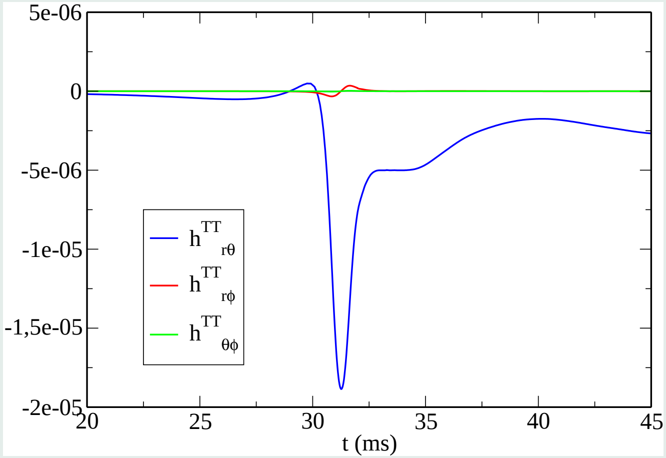


A4B5G5 model $n_r = 300, n_\theta = 30$

Results - Core Collapse Supernova

h_{ij}^{TT} correction at center





Conclusions

- **CFC and CFC+ are stable methods for calculating the metric in axisymmetry (without the stability problems of actual Full General Relativistic codes).**
- **CFC and CFC+ codes have lower computational cost than actual Full Relativistic ones.**
- **In the pulsating NS and Core Collapse SNe scenario CFC seems to be enough to describe the dynamics of the system:**
  - **Second Post-Newtonian deviations from isotropy are very small.**
  - **good agreement with Full General Relativistic codes.**
- **We need to study scenarios were CFC+ approximation is better than CFC:**
  - **Collapse to Black Hole.**
  - **Coalescing Neutron Stars (initial models).**
  - **Self gravitating tori around black hole.**

In the future..

- **Improve metric calculation using LORENE (Work in progress in collaboration with J. Novak).**
- **Explore the full parameter-space in the Core Collapse SNe scenario.**
- **Improve Physics**
  - **More realistic equation of state (Work in progress in collaboration with J. Pons).**
  - **Simplified neutrino transport.**
  - **Magnetic fields.**