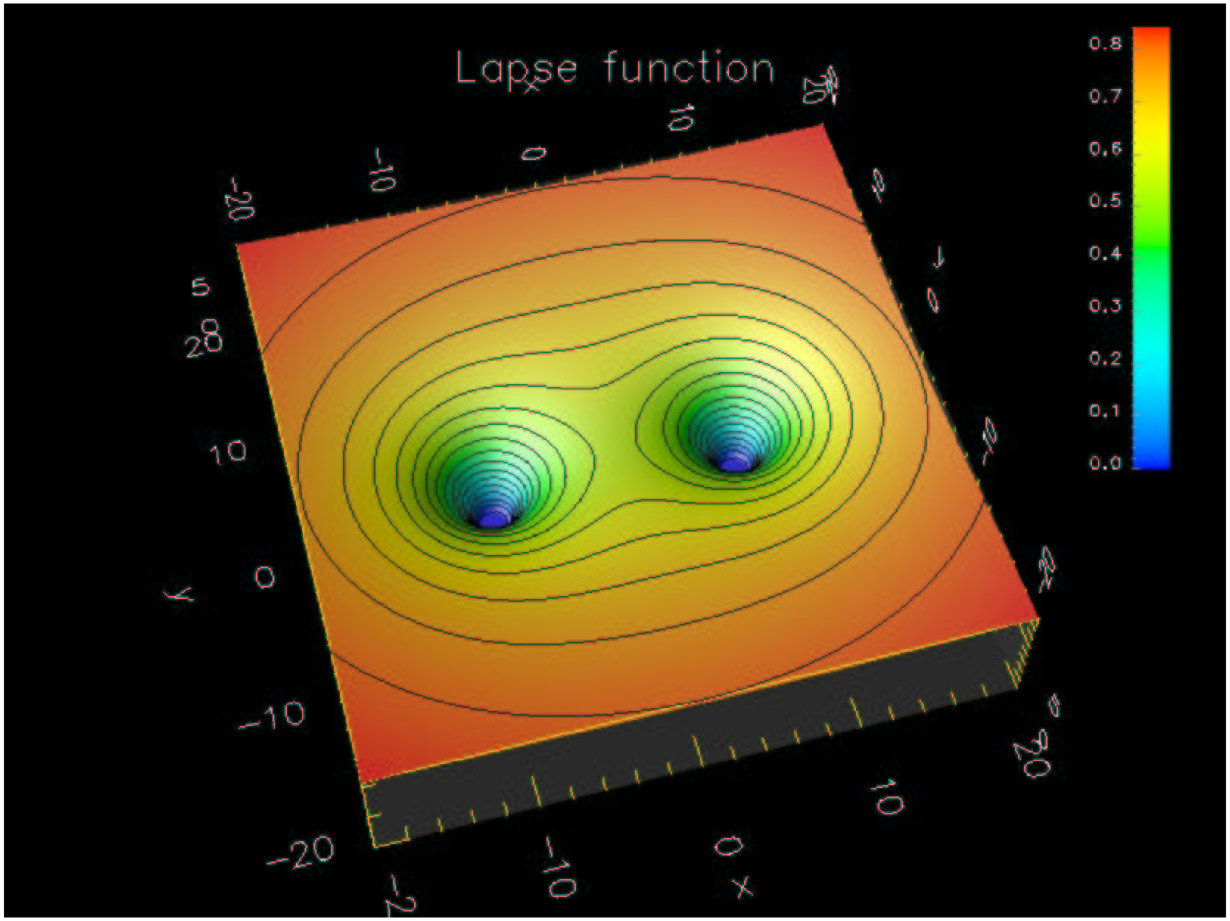
FAMILIES OF INITIAL DATA FOR BINARY BLACK HOLES



Philippe Grandclément

Trieste 2003

PART I : INTRODUCTION AND TOPOLOGIES

3+1 DECOMPOSITION

Decomposition of space-time into space **AND** time.

Metric is given by :

$$\mathrm{d}s^2 = -\left(N^2 - N^i N_j\right)\mathrm{d}t^2 + 2N_i \mathrm{d}t\mathrm{d}x^i + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

- Lapse N : choice of time coordinate.
- Shift $\vec{N}$: choice of spatial coordinates.
- Spatial projection of the metric γ_{ij}.

Second fundamental form **extrinsic curvature tensor :**

$$K_{ij} = -\frac{1}{2}\mathcal{L}_{\mathbf{n}}\gamma_{ij}.$$

EINSTEIN'S EQUATIONS IN 3+1 (vacuum case)

Type	Einstein	Maxwell
Constraints	Hamiltonian $R + K^2 - K_{ij}K^{ij} = 0$ Momentum : $D_j K^{ij} - D^i K = 0$	$\nabla \cdot \vec{E} = 0$ $\nabla \cdot \vec{B} = 0$
Evolution	$\frac{\partial \gamma_{ij}}{\partial t} - \mathcal{L}_{\vec{N}} \gamma_{ij} = -2NK_{ij}$ $\frac{\partial K_{ij}}{\partial t} - \mathcal{L}_{\vec{N}} K_{ij} = -D_i D_j N) +$ $N\left(R_{ij} - 2K_{ik}K^k_j + KK_{ij}\right)$	$\frac{\partial \vec{E}}{\partial t} = \frac{1}{\varepsilon_0 \mu_0}\left(\vec{\nabla} \times \vec{B}\right)$ $\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E}$

R_{ij} Ricci tensor associated with γ_{ij}.

D_i is covariant derivative with respect to γ_{ij}.

INITIAL DATA PROBLEM

Solving the initial value problem is as hard as doing the evolution.

One has to give $\gamma_{ij}\left(t=0\right)$ and $K_{ij}\left(t=0\right)$ verifying :

$$R - K^{ij}K_{ij} + K^2 = 0$$

$$D_j\left[K^{ij} - K\gamma^{ij}\right] = 0$$

But ...

- There exist a lot of solutions.
- How can we precise the physical content of such solutions ?

CONFORMALLY FLAT SOLUTIONS

$$\gamma_{ij} = \Psi^4 f_{ij}$$

Properties :

- exact at 1 PN.
- small deviation for binary neutron stars (2%).
- false for extreme Kerr.

With **maximum slicing** ($K = 0$) :

$$\Delta\Psi = -\frac{1}{8}\Psi^{-7}\hat{K}_{ij}\hat{K}^{ij}$$

$$\bar{D}_i\hat{K}^{ij} = 0$$

where $K_{ij} = \Psi^{-2}\hat{K}_{ij}$ and $K^{ij} = \Psi^{10}\hat{K}^{ij}$.

INITIALLY STATIC SOLUTIONS

Solutions with $\hat{K}_{ij} = 0$

Only has to solve for the Hamiltonian constraint : $\Delta\Psi = 0$

Additional conditions :

- Signature of $g_{\mu\nu}$ implies that $\Psi > 0$.
- Asymptotical flatness : $\Psi \rightarrow 1$ when $r \rightarrow \infty$

Several solutions, even for two black holes...

BRILL-LINDQUIST TOPOLOGY

D.R. Brill and R.W. Lindquist, *Phys. Rev.* **131**, 471 (1963).

Ψ singular function at two points :

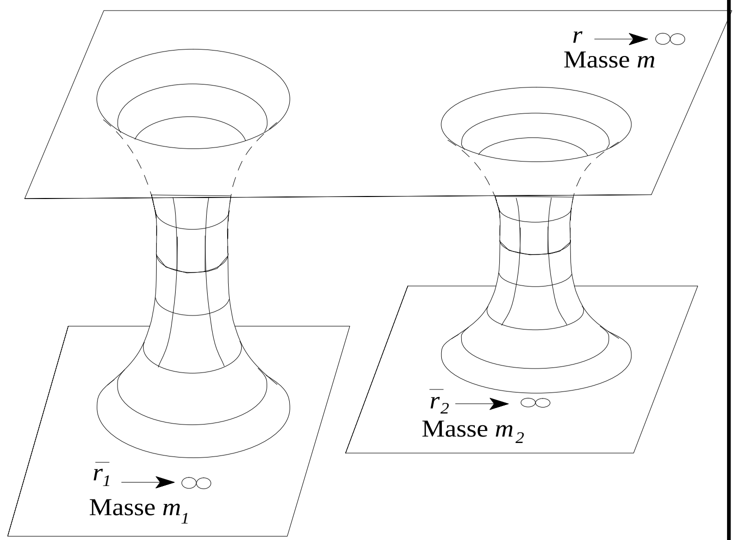
$$\Psi = 1 + \frac{\alpha_1}{||\vec{r} - \vec{c}_1||} + \frac{\alpha_2}{||\vec{r} - \vec{c}_2||}$$

Three asymptotically flat regions :

- mass $m = 2\,(\alpha_1 + \alpha_2)$ when $r \to \infty$.
- mass $m_1 = 2\alpha_1 \left(1 + \frac{\alpha_2}{c_{12}}\right)$ when $\vec{r} \to \vec{c}_1$.
- mass $m_2 = 2\alpha_1 \left(1 + \frac{\alpha_1}{c_{12}}\right)$ when $\vec{r} \to \vec{c}_2$.

with $c_{12} = ||\vec{c}_1 - \vec{c}_2||$.

The masses are defined by $\Psi \to 1 + \frac{m}{2r}$.

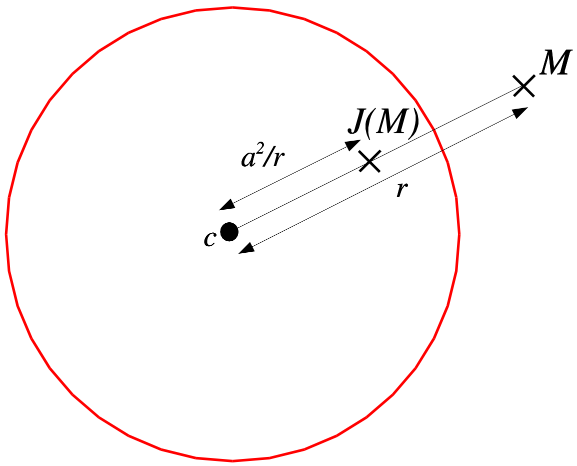


INVERSION-SYMMETRIC SOLUTION

Define two spheres centered on $\vec{c}_i$ of radii a_i.

Inversion with respect to a sphere is define by :

$$M\left(r_i, \theta_i, \varphi_i\right) \rightarrow J\left(M\right)\left(\frac{a_i^2}{r_i}, \theta_i, \varphi_i\right)$$



Choice of topology : the three metric is isometric with respect to both J_i.

It is equivalent to imposing the following boundary condition on the spheres :

$$\left.\frac{\partial \Psi}{\partial r_i} + \frac{1}{2a_i}\Psi\right|_{r_i=a_1} = 0$$

MISNER-LINDQUIST SOLUTION

C.W. Misner, *Ann. Phys.* **24**, 102 (1963).

R.W. Lindquist, *J. Math. Phys.* **4**, 938 (1963).

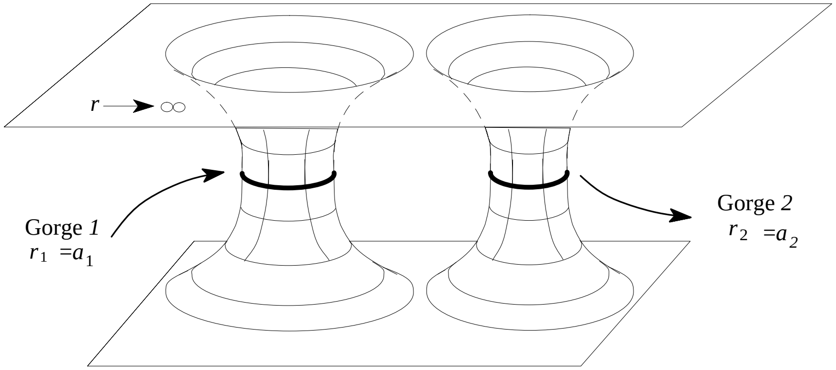
The isometric Ψ is given by :

$$\Psi = 1 + \sum_{n=1}^{\infty} c_n \left(\frac{1}{\left\| \vec{r} - \vec{d}_n \right\|} + \frac{1}{\left\| \vec{r} + \vec{d}_n \right\|} \right)$$

where c_n and d_n depends explicitly on $\vec{c}_i$ and a_i.

Two asymptotically flat regions :

- mass $m = 4\sum_{n=1}^{\infty} c_n$ when $r \to \infty$.
- regions $\vec{r} \to \vec{c}_1$ and $\vec{r} \to \vec{c}_2$ coincides (because of the inversion) and have also the mass m



PART II : CONFORMAL TT DECOMPOSITIONS

"FREELY" SPECIFIABLE VARIABLES

The 3-metric and the extrinsic curvature tensor are written as :

$$\gamma_{ij} = \Psi^4 \tilde{\gamma}_{ij}$$

$$K^{ij} = \Psi^{10} \hat{K}^{ij} = \Psi^{-10} \left[\tilde{A}^{ij}_{\mathrm{TT}} + (LX)^{ij} \right] + \frac{1}{3} \gamma^{ij} K$$

where $(LX)^{ij} = D^i X^j + D^j X^i - \frac{2}{3} \gamma^{ij} D_k X^k$

One can pick any choice for :

- the conformal metric $\tilde{\gamma}_{ij}$.
- the trace K.
- the transverse traceless part $\tilde{A}^{ij}_{\mathrm{TT}}$

The constraints give elliptic equations for Ψ and $\vec{X}$.

BOWEN-YORK EXTRINSIC CURVATURE TENSOR

Assumptions :

- conformal flatness : $\tilde{\gamma}_{ij} = f_{ij}$.
- maximum slicing : $K = 0$.
- purely longitudinal K_{ij} : $\tilde{A}^{ij}_{\rm TT} = 0$.

Existence of analytical solution for K_{ij} :

$$\hat{K}_{ij}\left(\vec{P}\right) = \frac{3}{2r^2}\left[P_i n_j + P_j n_i - (f_{ij} - n_i n_j)\, P^k n_k\right] \Longrightarrow \text{global impulsion } \vec{P}$$

and

$$\hat{K}_{ij}\left(\vec{S}\right) = \frac{3}{r^3}\left[\epsilon_{kil} S^l n^k n_j + \epsilon_{kjl} S^l n^k n_i\right] \Longrightarrow \text{global momentum } \vec{S}$$

$\vec{n}$ is a radial unit vector field.

J.M. Bowen, *Gen. Relativ. Gravit.* **11**, 227 (1979) ; J.M. Bowen and J.W. York, *Phys. Rev. D* **21**, 2047 (1980) ; A.D. Kulkarni, L.C. Shepley and J.M. York, *Phys. Lett. A* **96**, 228 (1983).

TWO BLACK HOLES

For two black holes, one can use a sum of the form :

$$\hat{K}_{ij} = \hat{K}_{1ij}\left(\vec{P}_1\right) + \hat{K}_{2ij}\left(\vec{P}_2\right) + \hat{K}_{1ij}\left(\vec{S}_1\right) + \hat{K}_{2ij}\left(\vec{S}_2\right)$$

recall that $\bar{D}_i \hat{K}^{ij} = 0$ is linear.

As we are interested in circular orbits, we will choose $\vec{P}_1 + \vec{P}_2 = \vec{0}$.

THE PUNCTURE METHOD

Extension of **Brill-Lindquist** topology :

- Use directly the previous sum centered on two points.
- Impose singularities at those two points by imposing :

$$\Psi = \left[\frac{\alpha_1}{||\vec{r} - \vec{c}_1||} + \frac{\alpha_2}{||\vec{r} - \vec{c}_2||} \right]^{-1} + u$$

- then solve the Hamiltonian constraint for u, which is regular everywhere.

T.W. Baumgarte, *Phys. Rev. D* **62**, 024018 (2000).

S. Brandt and B. Brügmann, *Phys. Rev. Lett.* **78**, 3606 (1997).

CONFORMAL IMAGING APPROACH

Extension of the **Misner-Lindquist** topology :

- Use a symmetric version of the extrinsic curvature tensor.
- Solve the Hamiltonian constrain by imposing boundary conditions on the two spheres.
- In this case the spheres are apparent horizons.

G.B. Cook, *Phys. Rev. D* **50**, 5025 (1994).

H.P. Pfeiffer, S.A. Teukolsky and G.B. Cook, *Phys. Rev. D* **62**, 104018 (2000).

EFFECTIVE POTENTIAL METHOD

Remaining question : *What value of $\vec{P}$ should be used to get circular orbits ?*

Define a potential (binding energy) $V = \frac{M_{\rm ADM} - m}{\mu}$.

Bare mass m is **NOT** well define in GR.

Utilization of Christodoulou formula :

$$m_i = M_{\rm ir} + \frac{S_i}{4M_{\rm ir}}$$

where $M_{\rm ir}$ is the irreducible mass :

$$M_{\rm ir} = \sqrt{\frac{A}{16\pi}}$$

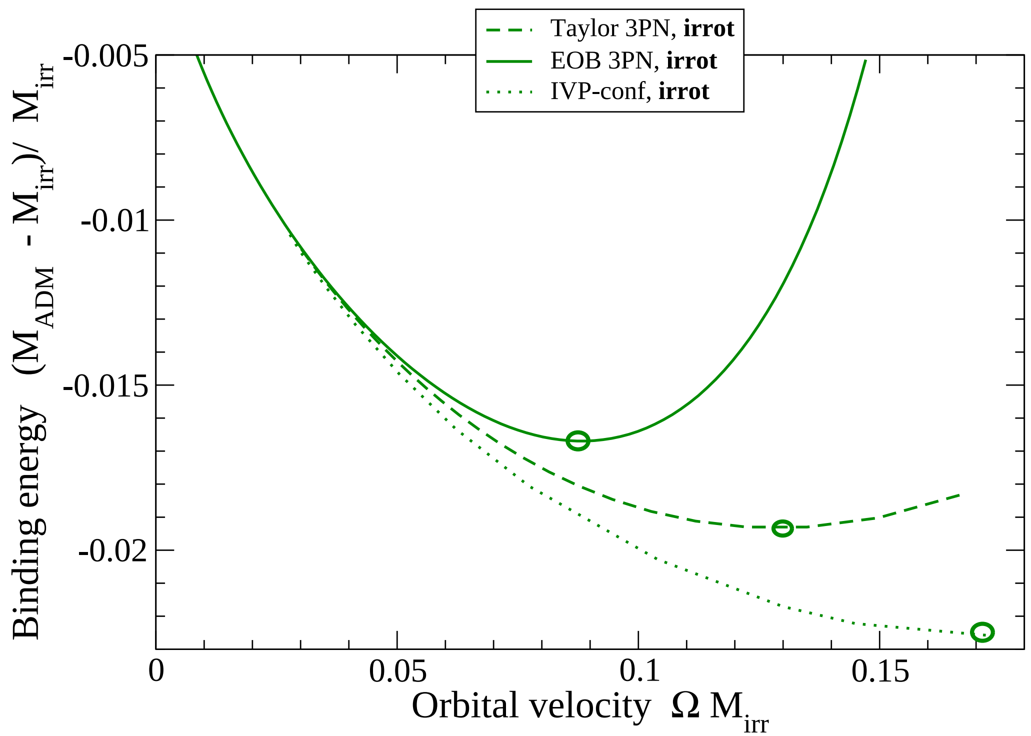
A being the area of the *apparent* horizon of the holes.

$\vec{P}$ is chosen so that $\frac{\partial V}{\partial P} = 0$.

First law of BH thermodynamics, along a sequence

$$\delta M = \Omega \delta J$$

PROBLEMS



Discrepancy with post-Newtonian results and no circular orbits found when doing the evolution...

THE SUSPECTS

Several questionable steps :

- **conformal flatness**
  - certainly an issue.
  - has long been held responsible.
- **choice of the extrinsic curvature tensor** : Bowen-York ansatz.
  - likely to be the most important difference.
- **Use of effective potential method**
  - ambiguous definition of the individual masses.
  - use of Christodoulou formula
  - unlikely because also used by PN.
  - validity tested on some simple analytic cases.

KERR-SCHILD INITIAL DATA

In Kerr-Schild coordinates it is easy to apply a Lorentz boost to the BH.

Idea : use a superposition of two boosted Kerr-Schild black holes for the freely specifiable variables.

For example :

$$\tilde{\gamma_{ij}} = f_{ij} + 2B_1 H_1 l_{1i} l_{1j} + 2B_2 H_2 l_{12i} l_{2j}$$

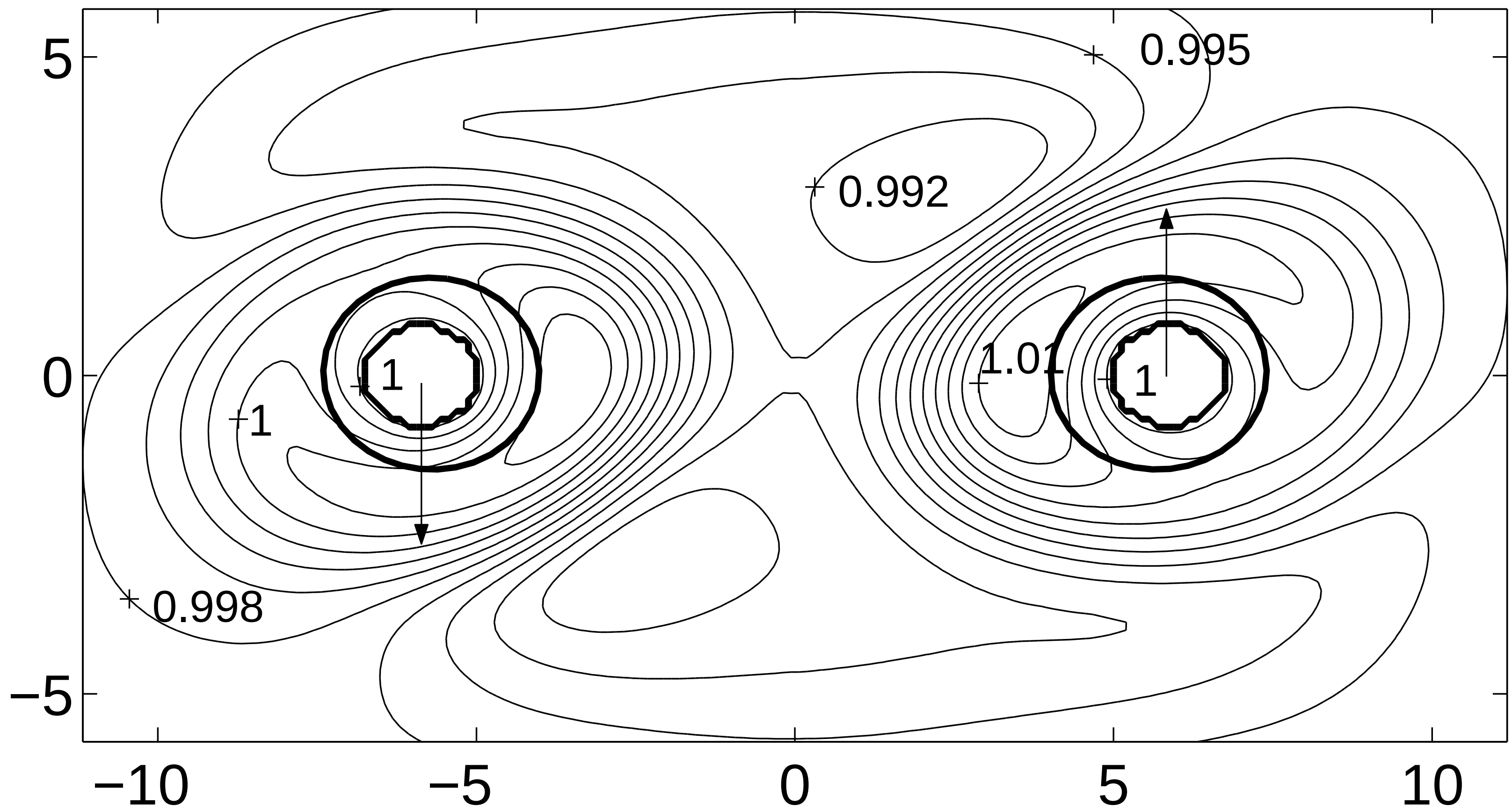
where H_i and $\vec{l_i}$ are given by the Kerr-Schild coordinates, for one BH.

The functions B_i are attenuation functions, that ensures that, near the holes the geometry is identical to a single Kerr BH.

Solve for Ψ and $\vec{X}$ (conformal TT decomposition) to obtain a solution of the constraints.

P. Marronetti, M.F. Huq, P. Laguna, L. Lehnerm, R.A. Matzner and D. Shoemaker, *Phys. Rev. D* **62**, 024017 (2000) ; P. Marronetti and R.A. Matzner, *Phys. Rev. Lett.* **85**, 5500 (2000) ; R.A. Matzner, M.F. Huq and D. Shoemaker, *Phys. Rev. D* **59**, 024015 (1999).

TYPICAL CONFORMAL FACTOR

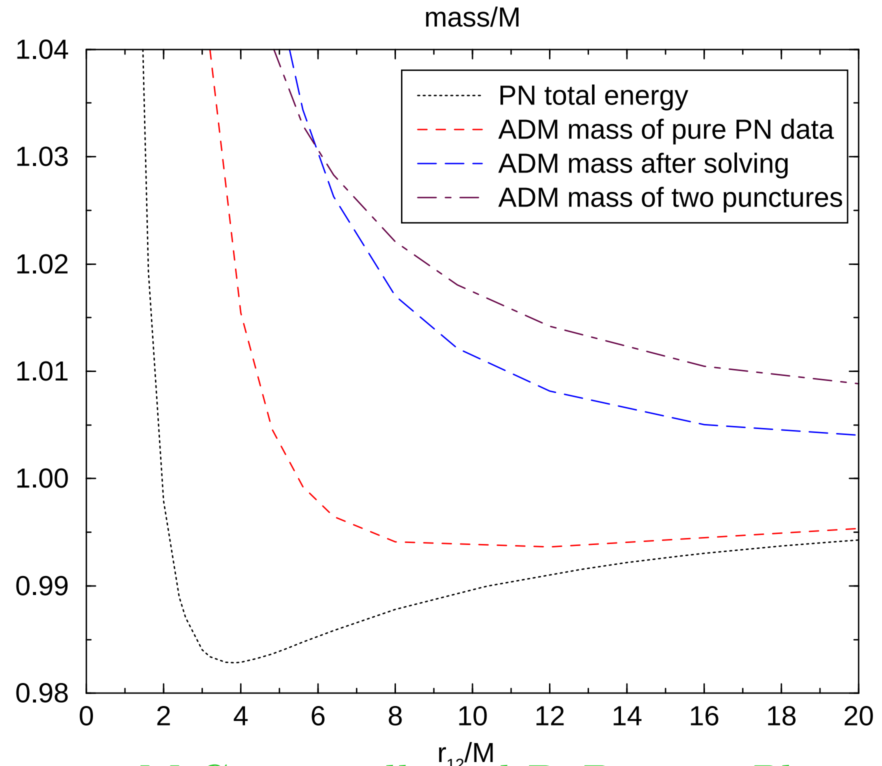


Not conformally flat and used in some evolutionary codes but no sequences published and no circular orbits found

"PN-BASED" INITIAL DATA

- Choose the "freely" specifiable variables by using their post-Newtonian expressions (not conformally flat).
- Use a version of the conformal TT decomposition to solve the constraints.

In general, the agreement with PN, at the end of the procedure is bad.



W. Tichy, B. Brügmann, M Campanelli and P. Diener, *Phys. Rev. D* **67**, 064008 (2003).

PART III:
AN HELICAL KILLING VECTOR APPROACH
a.k.a.
THE THIN-SANDWICH DECOMPOSITION
a.k.a
THE MEUDON INITIAL DATA

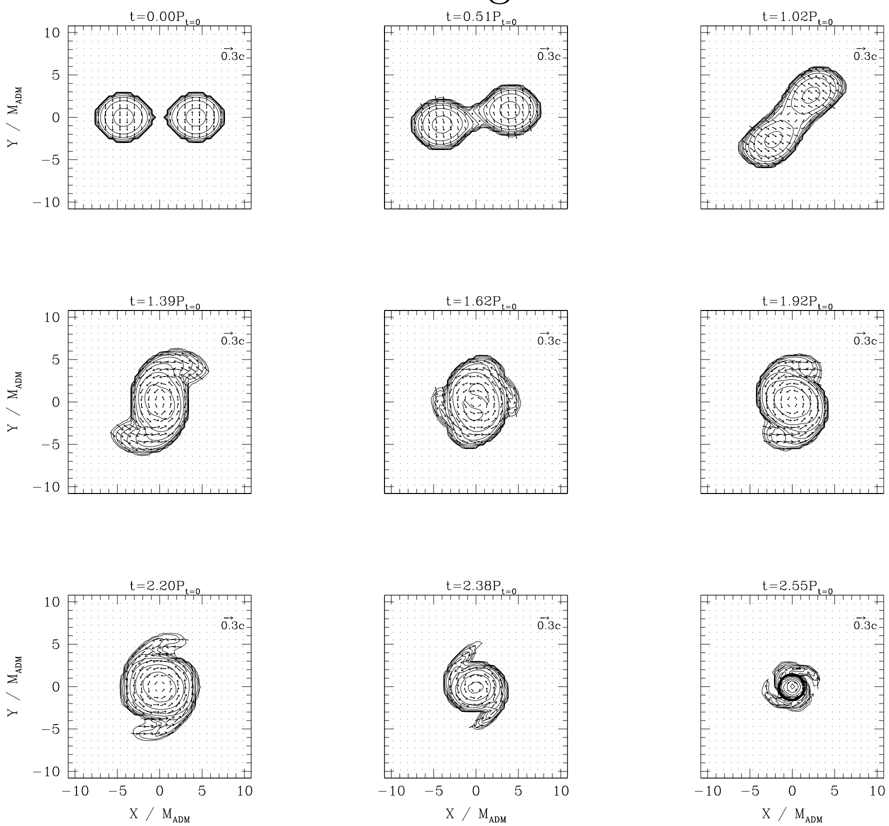
E. Gourgoulhon, P. Grandclément and S. Bonazolla, *Phys. Rev. D* **65**, 044020 (2002).

P. Grandclément, E. Gourgoulhon and S. Bonazolla, *Phys. Rev. D* **65**, 044021 (2002).

AND THE NEUTRON STARS ?

Quasi-equilibrium sequences have been computed by several groups, in the conformal flatness approximation.

The evolution of those configurations **HAVE** exhibited circular orbits.



M. Shibata and K. Uryu, *Prog. Theor. Phys.* **107**, 265 (2002).

Why not apply similar techniques for binary black holes ?

HELICAL KILLING VECTOR

Circular orbits $\Longrightarrow$ Helical Killing vector $\vec{l}$.

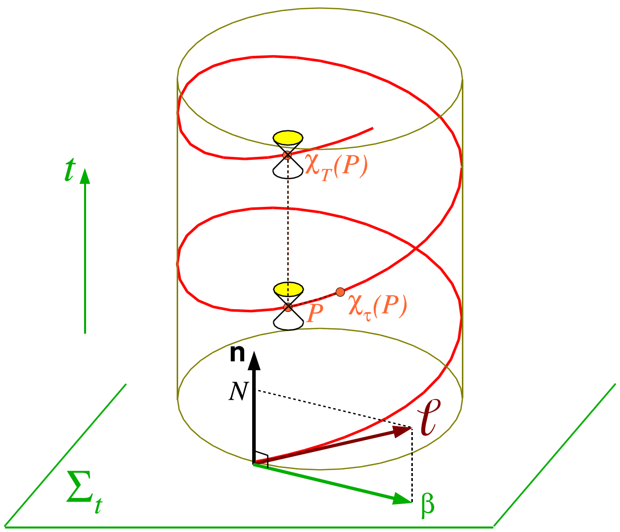
Advance δt in time $\Longleftrightarrow$ Rotation of $\delta\varphi = \Omega\delta t$.

Inertial coordinates :

$$l^\alpha = \left(\frac{\partial}{\partial t}\right)^\alpha + \Omega\left(\frac{\partial}{\partial \varphi}\right)^\alpha$$

Corotating coordinates :

- such as $l^\alpha = \left(\frac{\partial}{\partial t}\right)^\alpha$.
- coordinate t is ignorable.
- corotating shift $\beta^i = N^i + \Omega\left(\frac{\partial}{\partial \varphi}\right)^i$.
- functions N and γ_{ij} are the same.



ELLIPTIC EQUATIONS

Additional hypothesis : $K = 0$ and $\tilde{\gamma}_{ij} = f_{ij}$.

We solve 5 of the 10 Einstein's equations :

- Hamiltonian constraint : $$\Delta\Psi = -\frac{\Psi^5}{8}\hat{A}_{ij}\hat{A}^{ij}$$
- Momentum constraints : $$\Delta\beta^i + \frac{1}{3}\bar{D}^i\bar{D}_j\beta^j = 2\hat{A}^{ij}\left(\bar{D}_jN - 6N\bar{D}_j\ln\Psi\right)$$
- Trace of $\dfrac{\partial K_{ij}}{\partial t}$: $$\Delta N = N\Psi^4\hat{A}_{ij}\hat{A}^{ij} - 2\bar{D}_j\ln\Psi\,\bar{D}^jN$$

with $\hat{A}_{ij} = \Psi^{-4}K_{ij}$ and $\hat{A}^{ij} = \Psi^4K^{ij}$.

Definition of $\mathbf{K} \Longrightarrow \hat{A}^{ij} = \dfrac{1}{2N}\left(L\beta\right)^{ij}$

$\left(L\beta\right)^{ij}$ is the conformal Killing operator : $\left(L\beta\right)^{ij} = \bar{D}^i\beta^j + \bar{D}^j\beta^i - \frac{2}{3}\bar{D}_k\beta^kf^{ij}$

Set of 5 non-linear, highly-coupled, elliptic equations.

THE THIN-SANDWICH FORMULATION

Decomposition **different** from the conformal TT.

$$\gamma_{ij} = \Psi^4 \tilde{\gamma}_{ij}$$

$$K^{ij} = \Psi^{-4} \left[\frac{(L\beta)^{ij} - \tilde{u}^{ij}}{2N} \right] - \frac{1}{3}\gamma^{ij} K$$

"Freely" specifiable variables :K, $\dot{K}$, $\tilde{u}^{ij}$ and $\tilde{\gamma}^{ij}$.

- Maximum slicing $K = 0$.
- If we evolve the initial data with lapse N and shift $\vec{\beta}$, then :
  - $\partial_t \widetilde{\gamma^{ij}} = \tilde{u}^{ij} \Longrightarrow \tilde{u}^{ij} = 0$.
  - $\partial_t K = 0 \Longrightarrow$ same equation for N.
- Conformal flatness : $\tilde{\gamma}_{ij} = f_{ij}$.

The solve for Ψ and $\vec{\beta}$ to verify the constraints $\Longrightarrow$ same equations than before.

COMPARING THE TWO DECOMPOSITIONS

- **Conformal TT** : choose the momentum via $\tilde{A}^{ij}_{\mathrm{TT}} \Longrightarrow$ Hamiltonian representation.
- **Thin sandwich** : choose velocity $\partial_t \tilde{\gamma}_{ij} \Longrightarrow$ Lagrangian representation.

The thin-sandwich formulation is better suited for quasi-equilibrium data.

H.P. Pfeiffer and J.W. York, *Phys. Rev. D* **67**, 044022 (2003).

CHOICE OF TOPOLOGY

Extension of Misner-Lindquist.

Boundary condition on the throats (isometry conditions) :

- **Lapse** : antisymmetric choice

$$N|_S = 0$$

- **Conformal factor**

$$\left.\left(\frac{\partial\Psi}{\partial r} + \frac{\Psi}{2r}\right)\right|_S = 0$$

- **shift vector** : COROTATION (rigidity theorem)

$$\left.\vec{\beta}\right|_S = 0$$

The throats are Killing **AND** apparent horizons.

ASYMPTOTIC FLATNESS

At infinity we recover Minkowski space-time :

$$N \to 1 \quad \text{when} \quad r \to \infty$$

$$\Psi \to 1 \quad \text{when} \quad r \to \infty$$

$$\vec{\beta} \to \Omega \frac{\partial}{\partial \varphi} \quad \text{when} \quad r \to \infty$$

ISOMETRY AND REGULARITY

To have a regular **K** :

$$\left.\begin{array}{rcl} \hat{A}^{ij} & = & \dfrac{(L\beta)^{ij}}{2N} \\ N|_{S_i} & = & 0 \end{array}\right\} \Longrightarrow (L\beta)^{ij}\Big|_{S_i} = 0.$$

One can show that to have **RIGIDITY**, **REGULARITY** and **ISOMETRY** one must have :

$$\begin{array}{rcl} \vec{\beta}\Big|_{S_i} & = & 0 \\ \partial_r \vec{\beta}\Big|_{S_i} & = & 0 \end{array}$$

REGULARIZATION OF THE SHIFT

One solves for $\vec{\beta}$, using Dirichlet-type boundary condition :

$$\left.\vec{\beta}\right|_{S_i} = 0$$

At each iteration one modifies the shift vector by :

$$\vec{\beta}_{\rm new} = \vec{\beta} + \vec{\beta}_{\rm cor}$$

$\vec{\beta}_{\rm cor}$ is chosen so that :

$$\begin{aligned} \left.\vec{\beta}_{\rm new}\right|_{S_i} &= 0 \\ \left.\partial_r \vec{\beta}_{\rm new}\right|_{S_i} &= 0. \end{aligned}$$

At the end of a calculation :

- if $\vec{\beta}_{\rm cor} \rightarrow 0$: exact solution.
- if $\vec{\beta}_{\rm cor}$ is small : approximate solution.
- else not a solution !

DETERMINATION OF Ω

Ω only present in the boundary condition for $\vec{\beta}$.

One can solve for ANY value of Ω (example : $\Omega = 0 \Longrightarrow$ Misner-Lindquist).

SUPPLEMENTARY CONDITION : the $O\left(r^{-1}\right)$ part of the metric when $(r \to \infty)$ is identical to **Schwarzschild**.

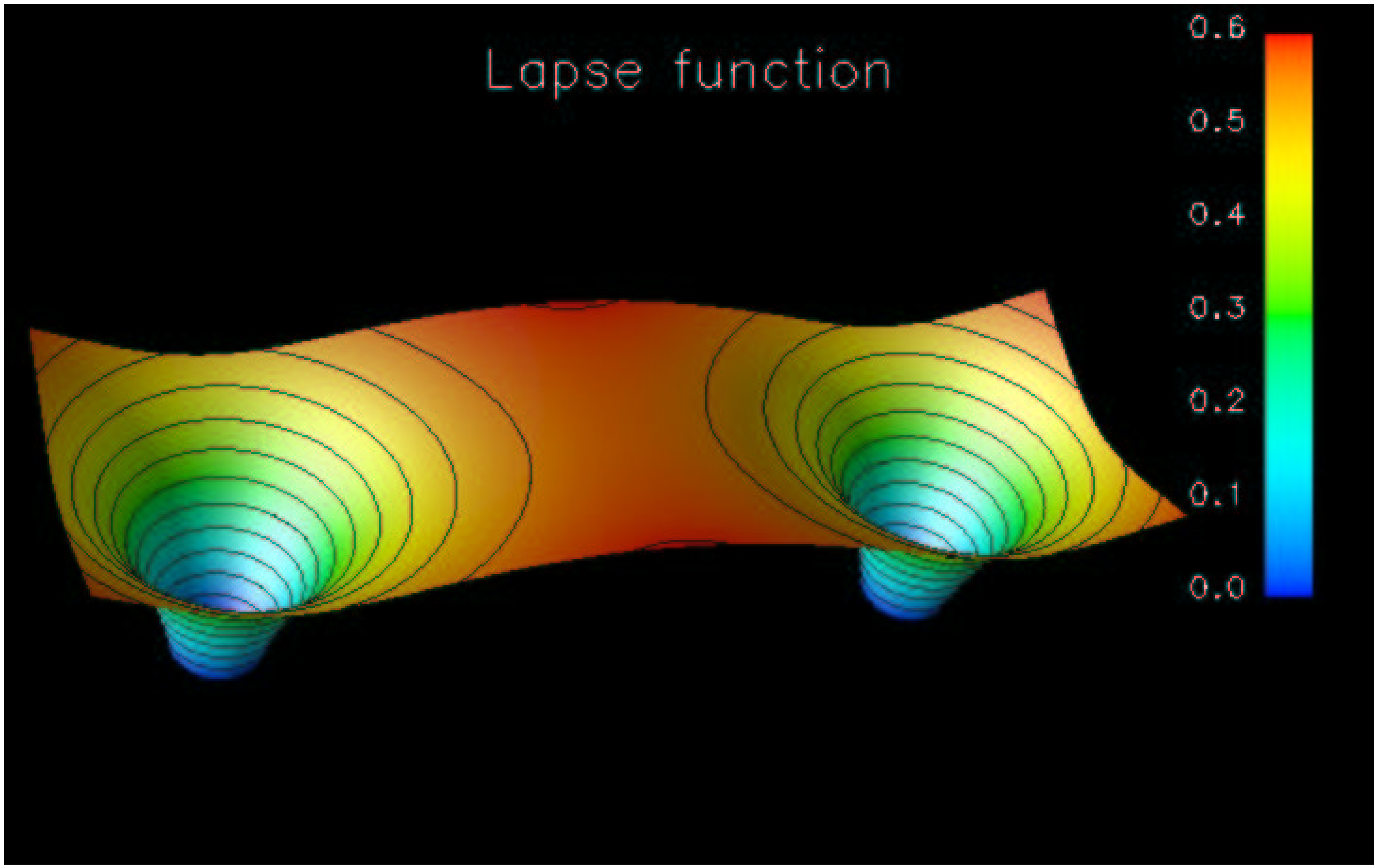
A priori : $\Psi \sim 1 + \dfrac{M_{\mathrm{ADM}}}{2r}$ and $N \sim 1 - \dfrac{M_{\mathrm{K}}}{r}$

One chooses the **ONLY** Ω such that : $M_{\mathrm{K}} = M_{\mathrm{ADM}} \Longleftrightarrow \Psi^2 N \sim 1 + \dfrac{\alpha}{r^2}$

Justifications :

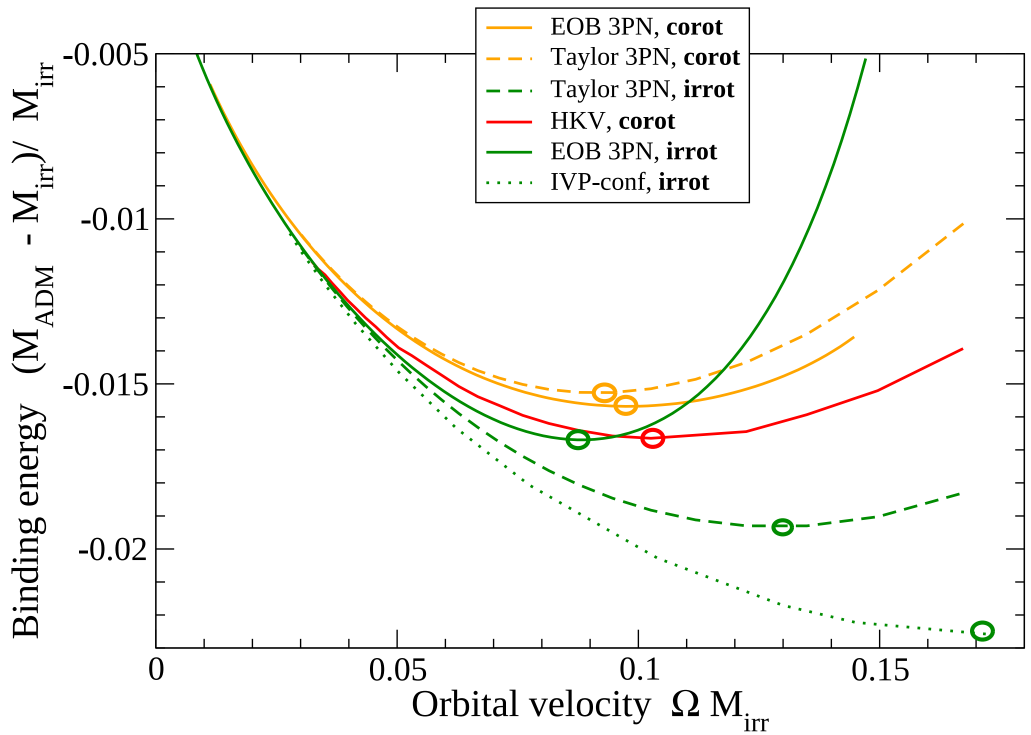
- exact stationary asymptotical space-times.
- Newtonian limit $\Longrightarrow$ virial theorem.
- True for binary neutron stars.

LAPSE IN THE ORBITAL PLANE



ISCO configuration

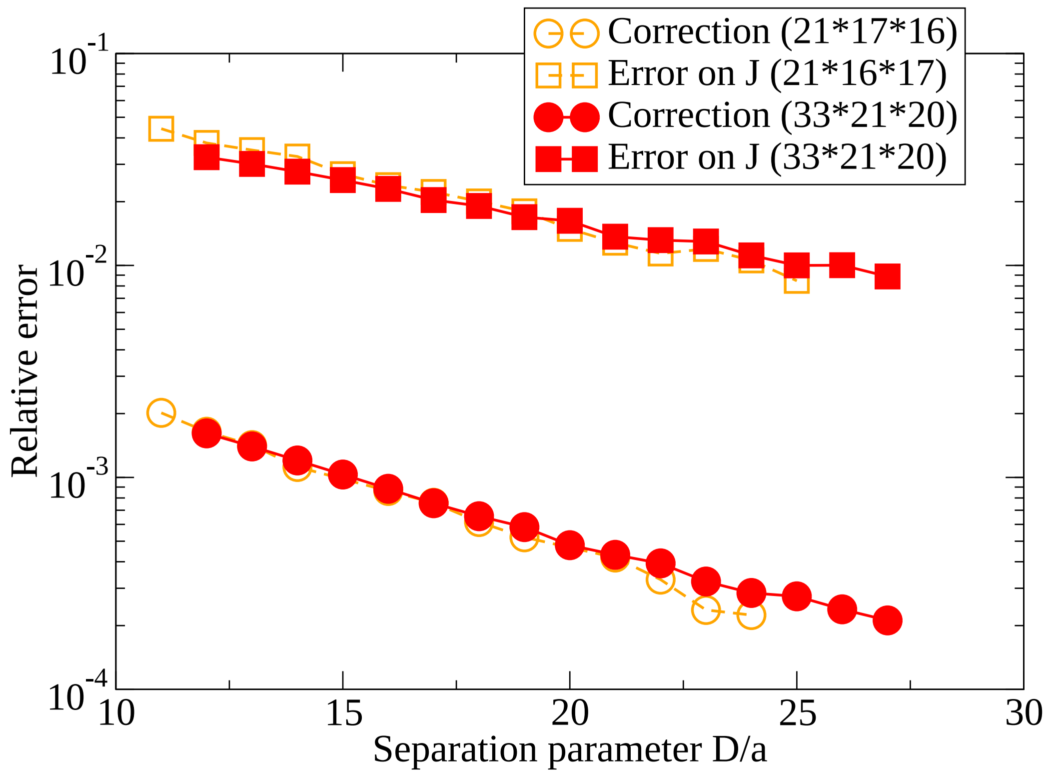
ENCOURAGING RESULTS



T. Damour, E. Gourgoulhon and P. Grandclément, *Phys. Rev. D* **66**, 024007 (2002).

PART IV: WHAT IS NEXT ?

THE REGULARIZATION FUNCTION



$$J_\infty = J_S \Longleftrightarrow \vec{\beta}_{\text{cor}} = 0$$

NEW BOUNDARY CONDITIONS ON THE HOLES

Cook proposed to relax the symmetry boundary conditions and to replace them by demanding that :

- the throats are apparent horizons $\Longrightarrow$ $\tilde{s}^k \tilde{\nabla}_k \ln \Psi \Big|_S = -\frac{1}{4} \left(\tilde{h}^{ij} \tilde{\nabla}_i \tilde{s}_j - \Psi^2 J \right) \Big|_S$
- the remain apparent horizons (Killing vector tangent to the surfaces) $\Longrightarrow$ $\beta^i \big|_S = N \Psi^{-2} \tilde{s}^i \big|_S$ (corotation).

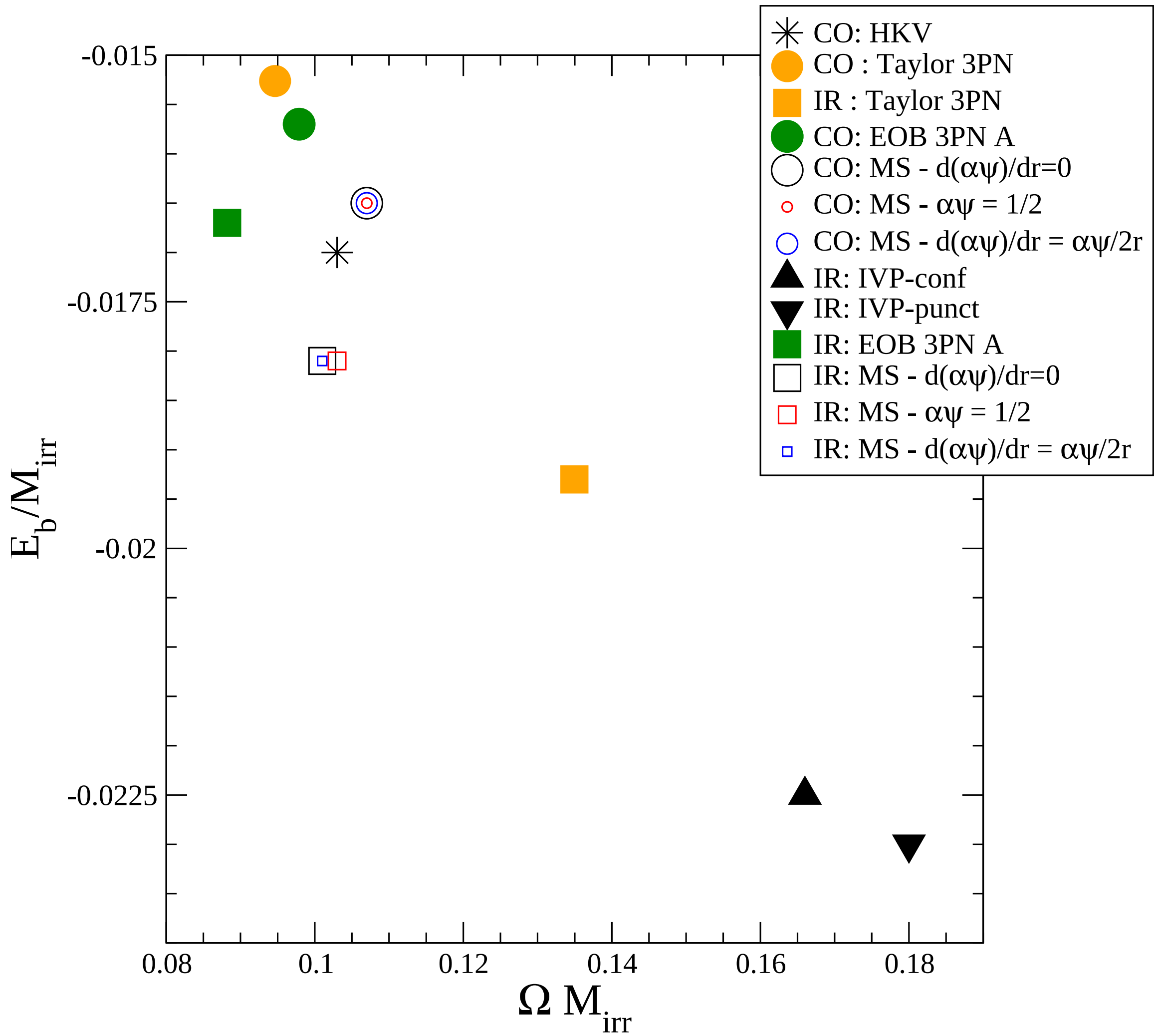
No condition for the lapse.

If we choose spheres and $N|_S = 0$ then we obtained the same equations than the one used by the Meudon group.

The "new" boundary conditions are just more general and would admit $N|_S \neq 0$ $\Longrightarrow$ no regularity problem.

G.B. Cook, *Phys. Rev. D* **65**, 084003 (2002)

RESULTS



G.B. Cook and H.P. Pfeiffer, in prep.

RELAXING THE EFFECTIVE POTENTIAL METHOD

Proposition :

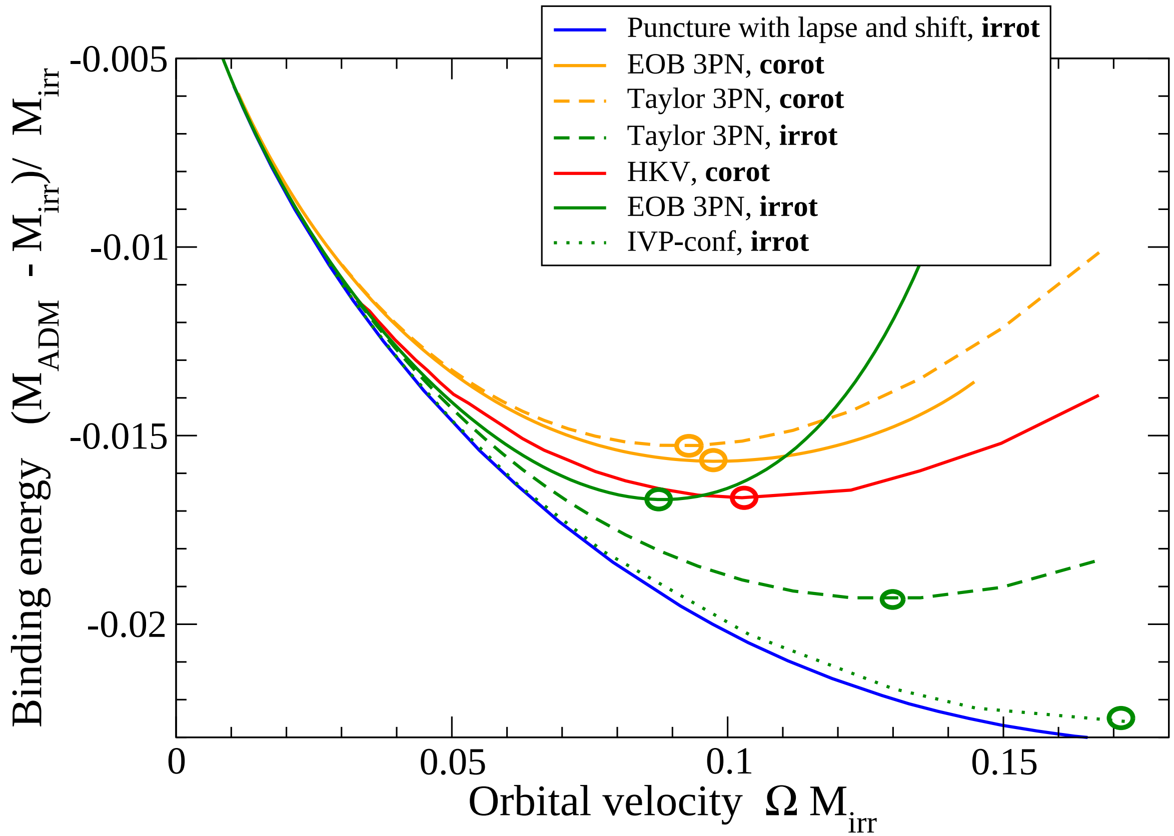
- Apply the standard puncture method with the conformal TT decomposition of K_{ij}.
- Determine a lapse by solving $\partial_t K = 0$.
- Determine a shift by solving $\partial_t \gamma_{ij} = 0$.
- Determination of Ω and puncture parameters via criterium similar to $M_{\rm ADM} = M_{\rm K}$.

Good test of the validity of the effective potential method to determine Ω.

W. Tichy, B. Brügmann and P. Laguna, gr-qc/0306020.

W. Tichy and B. Brügmann, gr-qc/0307627.

VERY SMALL DIFFERENCE



Effective potential method $\Longleftrightarrow$ $M_{\mathrm{ADM}} = M_{\mathrm{K}}$

REVISITING THE PUNCTURE

In order to combine the puncture method and the thin-sandwich formulation one would need :

- quasi-stationary coordinates (stationary slicing).
- asymptotically flat regions near the puncture.
- Positive lapse everywhere $N > 0$.

Hannam *et al.* have shown that those conditions can **NOT** be verified, even for a single Schwarzschild black hole.

The puncture method seems to lose its appeal in the thin-sandwich frame-work.

M.D. Hannam, C.R. Evans, G.B. Cook and T.W. Baumgarte, gr-qc/0306028.

CONCLUDING REMARKS

Conclusions :

- The main difference between initial data is the choice of extrinsic curvature tensor.
- The thin-sandwich formulation is better suited for quasi-equilibrium sequences.
- First good agreement between PN and numerical results.

Still to be done :

- Use apparent horizon boundary conditions.
- Remove conformal flatness, possibly making use of the other Einstein's equations.
- Assess the quality of the initial data by evolving them.