Gauge conditions for binary black hole puncture data based on an approximate helical Killing vector

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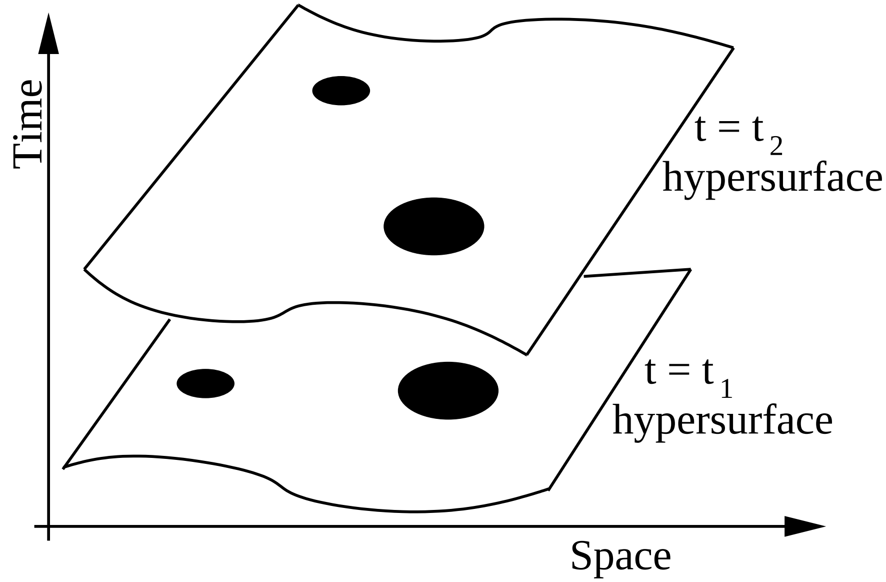
Plan of the talk:

- Initial data, constraints and the 3+1 Split of the spacetime metric
- Overview about black hole puncture data
- Helical Killing vectors and quasi-equilibrium
- How to choose lapse and shift, such that *standard* puncture data allow for an approximate helical Killing vector
- Summary

Initial data, the 3+1 Split of spacetime, and constraints



- Spacetime is foliated by $t = const$ slices, which are described by the 12 quantities g_{ij} (intrinsic spatial metric) and K_{ij} (extrinsic curvature)

- The initial data g_{ij}, K_{ij} are subject to 4 constraints:

$$R[g] + K^2 - K_{ij}K^{ij} = 0$$

$$\nabla_j(K^{ij} - Kg^{ij}) = 0$$

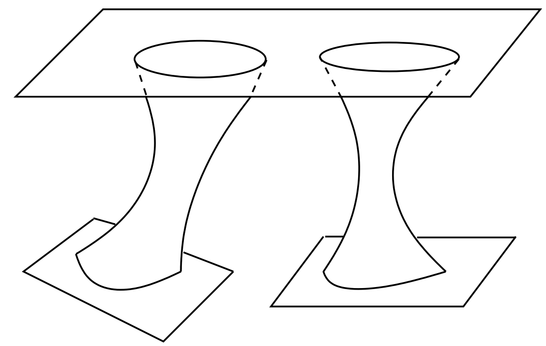
[analog to $\partial^i E_i = 0$ in E&M]

$\Rightarrow$ There are 8 freely specifiable quantities

$\Rightarrow$ The 4 constraints alone do not tell us what we should choose as initial data

- In order to get black hole (BH) initial data we need more information

Binary black hole puncture data



- conformal flatness: $g_{ij} = \phi^4 \delta_{ij}$
- conformal factor: $\phi = 1 + \frac{m_1}{2r_1} + \frac{m_2}{2r_2} + u$
- maximal slicing: $K = K_i{}^i = 0$
- Bowen-York curvature: $K^{ij} = \phi^{-10} \bar{L}W^{ij}$, $\quad W^i = \sum_{A=1}^{2} \left[-\frac{1}{4r_A} \left(7 P_A^i + s_A^i s_{Aj} P_A^j \right) \right]$

$\Rightarrow$
- Momentum constraint is already satisfied
- Hamiltonian constraint becomes: $\bar{\nabla}^2 u + \frac{1}{8}\phi^{-7} \bar{L}W^{ij} \bar{L}W_{ij} = 0$ (H)

- we solve the elliptic Eq. (H) for u, subject to the boundary condition that $u \to 0$ for $r \to \infty$

$\Rightarrow$ we obtain binary black hole initial data that fulfill Einstein's equations

Questions:

- How should we pick the puncture parameters m_A, P_A and D
- How realistic are these data?

Toward realistic initial data for numerical relativity

In principle, we want initial data which represent a black hole (BH) binary, that has slowly been inspiraling already for a long time, due to the emission of gravitational waves.

Post-Newtonian (PN) calculations predict that the BHs are moving on quasi-circular orbits with slowly shrinking radius, i.e. there are the two timescales

$$T_{orbit} \ll T_{inspiral}$$

and in corotating coordinates the system is in quasi-equilibrium, i.e.

$$\partial_t g_{ij} \approx \partial_t K_{ij} \approx 0.$$

- First approximation:
  assume that $T_{inspiral} \to \infty$ and that the two BHs are in a circular orbit

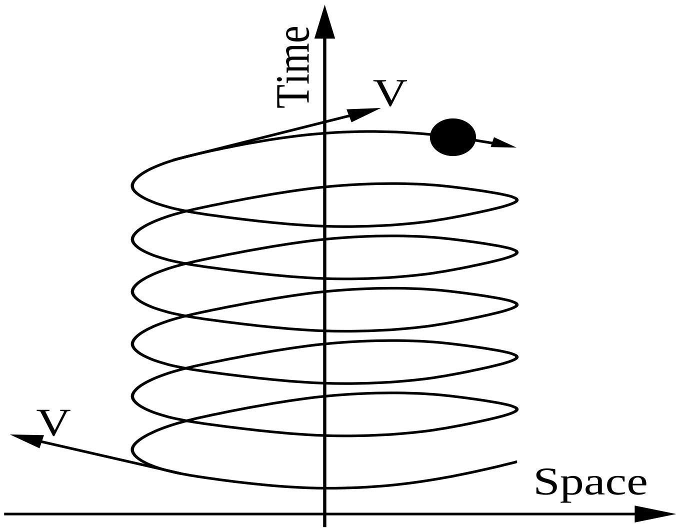
$\Rightarrow$ a (corotating) coordinate system exists in which

$$\partial_t g_{ij} = \partial_t K_{ij} = 0$$

- there exists a Killing vector V^a
- in corotating coordinates

$$V^a = \left(\frac{\partial}{\partial t}\right)^a = \alpha n^a + \beta^a$$

- V^a is a helical Killing vector:
  $n^a = T^a$ and $\beta^a = \Omega\Phi^a$ at spatial infinity



- lapse α and shift β^i are pure gauge, and determine the coordinates

$\Rightarrow$ choose α and β^i, s.t. we have manifest time independence, i.e. $V^a = \left(\frac{\partial}{\partial t}\right)^a$

Are puncture data compatible with an approximate helical Killing vector, i.e. with quasi-equilibrium?

- The momenta P_A^i must be compatible with quasi-circular orbits: We choose the center of mass to be at rest and let P_1^i and P_2^i be perpendicular to the line connecting the two BHs.

  $\Rightarrow$ The momenta are characterized by 1 parameter $P = |P_1^i| = |P_2^i|$.

- If there is an exact helical Killing vector V^a then the Komar integral

$$I_K(V,S) = -\frac{1}{8\pi}\oint_S \epsilon_{abcd}\nabla^c V^d$$

  is related to the ADM mass as follows:

$$I_K(\alpha n, S_\infty) = M_\infty^{ADM} \qquad I_K(\alpha n, S_1) = c_1 M_1^{ADM} \qquad I_K(\alpha n, S_2) = c_2 M_2^{ADM} \quad (*)$$

  where α is chosen such that $\lim_{r\to\infty}\alpha = 1$ and $\lim_{r_A\to 0}\alpha = -c_A$.

  $\Rightarrow$ $(*)$ constitutes 3 equations for the 3 parameters P, c_1, c_2.

Idea:

Use *standard puncture data*, but try to find a lapse α and a momentum parameter P such that the necessary conditions $(*)$ for a helical Killing vector are satisfied. Otherwise the puncture data cannot be in quasi-equilibrium!

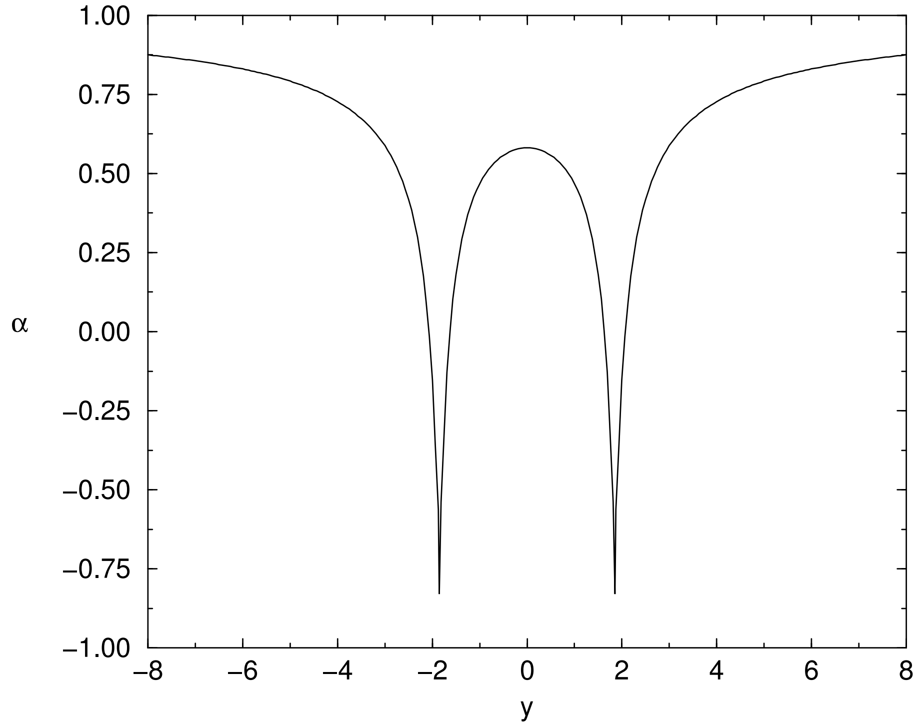
Choice of lapse

- We make the ansatz $\alpha\phi = 1 - \left(\frac{c_1 m_1}{2r_1} + \frac{c_2 m_2}{2r_2}\right) + v,$

  so that $\lim_{r\to\infty}\alpha = 1$ and $\lim_{r_A\to 0}\alpha = -c_A$.

- Since we are interested in quasi-equilibrium situations we choose a maximal slicing lapse α such that
$$\partial_t K = 0 \quad \Leftrightarrow \quad \bar{\nabla}^2 v = \frac{7}{8}(\alpha\phi)\phi^{-8}\bar{L}W_{ij}\bar{L}W^{ij}.$$

- If P, c_1, c_2 are chosen such that the ADM and Komar integrals agree at infinity and at both punctures (i.e. if $(*)$ is satisfied) the lapse is

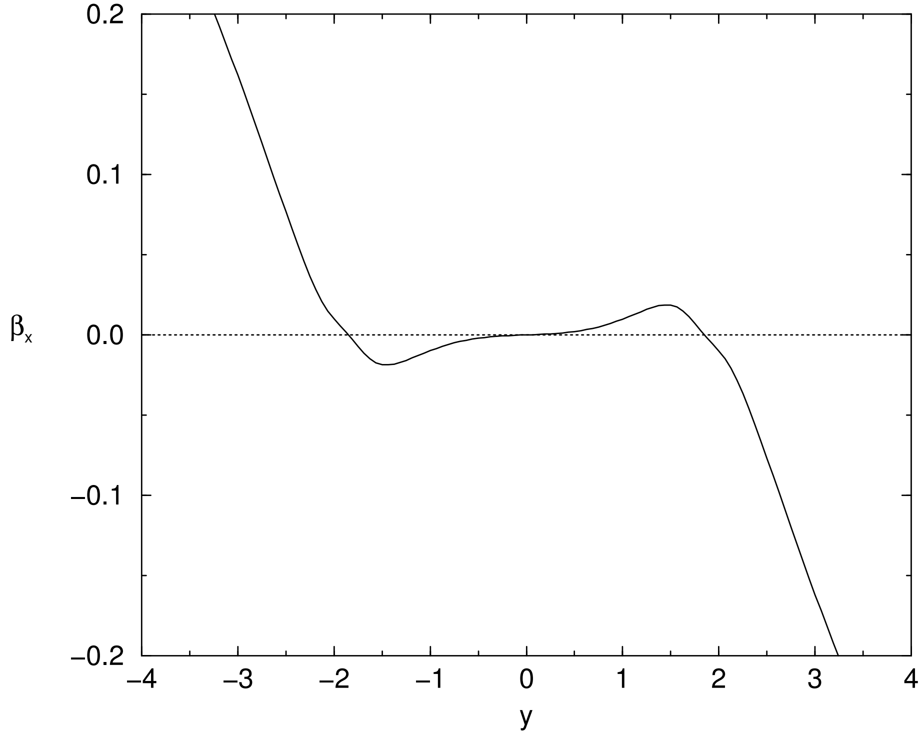


  – lapse for an equal mass binary
  – the BHs are at $y = \pm 1.840M$
  – the lapse is $c_1 = c_2 = -0.83$ at both punctures
  – the lapse approaches 1 in the far zone

- If the same procedure is applied to a single puncture we obtain the Schwarzschild lapse of isotropic coordinates, which goes from -1 to 1.

Choice of shift

- We choose the shift with the aim to minimize the time evolution of the conformal metric $\bar{g}_{ij}$
- We would like $\partial_t \bar{g}_{ij} \overset{?}{=} 0$. This however cannot be achieved by adjusting the shift. Instead we use $\bar{\bar{\nabla}}^j \partial_t \bar{g}_{ij} = 0$ to obtain
$$\bar{\nabla}_j \bar{L}\beta^{ij} = \bar{\nabla}_j \left(2\alpha\phi^{-6}\bar{L}W^{ij}\right),$$
- This is an elliptic equation for the shift, which we solve subject to the boundary condition
$$\lim_{r\to\infty} \beta^i = \Omega\Phi^i$$
- A puncture doesn't have momentum when viewed from the asymptotically flat region near it. Hence we choose Ω such that $\beta^i_A = 0$.
- Then the shift is



- β^x for an equal mass binary
- the BHs are at $y = \pm 1.840M$
- β^i vanishes at each puncture
- Ω can be determined: $\Omega = 0.1/M$
- with this shift $\partial_t \tilde{\Gamma}^i = 0$ and $\partial_t g_{ij}$ is reduced

Advantages of our choice of lapse and shift

- The ADM and Komar masses agree at infinity and at both punctures, as they should if a helical Killing vector exists.
- The shift is zero at each puncture, which is a good choice as punctures do not move, when viewed from the other asymptotic end.
- The angular velocity satisfies $2J_\infty^{ADM}\Omega = M_\infty^{ADM} - I_K(V, S_1 \cup S_2)$, which is another necessary condition for a helical Killing vector.
- The inferred Ω agrees well with results obtained using the effective potential method.
- Since P and Ω can be determined for each separation D, we can easily construct quasi-equilibrium sequences.

Timescales on which metric components evolve

for $\Omega = 0.1/M$:

orbital timescale $T_{orbit} \sim 2\pi/\Omega \sim 60M$

If the data are in quasi-equilibrium and if lapse and shift are chosen such that $(\frac{\partial}{\partial t})^a$ is an approximate helical Killing vector, ϕ, g_{ij}, K and K_{ij} should evolve on a timescale $T_{inspiral} \gg T_{orbit}$

- $T_\phi \sim 2\pi \max \left| \frac{\phi}{\partial_t \phi} \right| \sim 1200M$
- $T_{g_{ij}} \sim 2\pi \left(\frac{1}{\max |\partial_t g_{ij}|} \right) \sim 300M$
- $T_K \sim \infty, \quad \text{since } \partial_t K = 0$
- $T_{K_{ij}} \sim 2\pi \left(\frac{\max |K_{ij}|}{\max |\partial_t K_{ij}|} \right) \sim 95M$

Summary

- We have found a gauge choice for BH puncture data, which fulfills several necessary conditions for the existence of a helical Killing vector, such as agreement between ADM and Komar masses.
- With our gauge choice the metric, conformal factor and trace of the extrinsic curvature evolve on a timescale longer than the orbital timescale.
- The tracefree part of the extrinsic curvature still evolves on the orbital timescale.
- Using the conditions for the existence of a helical Killing vector, we can easily construct quasi-equilibrium sequences, for puncture data.