Towards stability criteria for rotating superfluid stars

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Motivation

Although neutron stars are born very hot ($> 10^{11}$ K) they rapidly cool below the temperature ($\sim 10^9$ K) where several components are expected to become superfluid.

- Mature neutron stars are thought to contain: superfluid neutrons, superconducting protons, superfluid hyperons and possibly also colour superconducting quarks.
- Superfluid dynamics may be key for understanding secular instabilities (eg. the r-modes)
- Unique superfluid signatures?
- Two-stream instability when $\Omega_{\rm n} \neq \Omega_{\rm p}$.

Need to develop a framework for analysing stability of multi-fluid stars. Aim: Generalise the Lagrangian perturbation of Friedman and Schutz to two-fluid model.

The superfluid equations

The equations for a two-fluid model for a superfluid neutron star are derived from an energy functional $E(n_{\rm n}, n_{\rm p}, w^2)$ where $w_i \equiv v_i^Y - v_i^X$;

$$dE = \sum_{X={\rm n},{\rm p}} \mu_X dn_X + \alpha dw^2 \longrightarrow \tilde{\mu}_X = \frac{1}{m_{\rm B}} \frac{\partial E}{\partial n_X} \text{ and } \alpha = \frac{\partial E}{\partial w^2}$$

We get

$$\partial_t n_X + \nabla_i (n_X v_X^i) = 0$$

and

$$(\partial_t + v_X^j \nabla_j) \left(v_i^X + \varepsilon_X w_i \right) + \nabla_i \left(\Phi + \widetilde{\mu}_X \right) + \varepsilon_X w_j \nabla_i v_X^j = 0$$

where $\varepsilon_X = 2\alpha / n_X$ make manifest the entrainment effect.
To perturb these equations we use Lagrangian variation ΔQ;

$$\Delta_X Q = \delta Q + \pounds_{\xi_X} Q \longrightarrow \Delta v_X^i = \partial_t \xi_X^i$$

The perturbation equations

For simplicity, focus on the case of vanishing entrainment.

Conservation of mass for the perturbations is expressed as

$$\Delta_X n_X = -n_X \nabla_i \xi^i_X \longrightarrow \delta n_X = -\nabla_i (n_X \xi^i_X)$$

and some algebra (noting that Δ and $\partial_t + \pounds_v$ commute), leads to the perturbed Euler equations

$$\begin{aligned} \partial_t^2 \xi_i^X \quad &+ \quad 2 v^j_X \nabla_j \partial_t \xi_i^X + (v^j_X \nabla_j)^2 \xi_i^X + \nabla_i \delta\Phi + \xi^j_X \nabla_i \nabla_j (\Phi + \tilde{\mu}_X) \\ &- \quad \nabla_i \left[\left(\frac{\partial \tilde{\mu}_X}{\partial n_X} \right)_{n_Y} \underbrace{\nabla_j (n_X \xi^j_X)}_{-\delta n_X} + \left(\frac{\partial \tilde{\mu}_X}{\partial n_Y} \right)_{n_X} \underbrace{\nabla_j (n_Y \xi^j_Y)}_{-\delta n_Y} \right] = 0 \end{aligned}$$

From which the “chemical” coupling between the two fluids is clear.

A conserved quantity

Given

$$A_X \partial_t^2 \xi_X + B_X \partial_t \xi_X + C_X \xi_X + D_X \xi_Y = 0$$

and the inner product

$$\langle \eta^i, \xi_i \rangle = \int (\eta^i)^* \xi_i dV$$

one can show that

$$W_X(\eta_X, \xi_X) = \left\langle \eta_X, A_X \partial_t \xi_X + \frac{1}{2} B_X \xi_X \right\rangle - \left\langle A_X \partial_t \eta_X + \frac{1}{2} B_X \eta_X, \xi_X \right\rangle$$

leads to

$$\partial_t W_\mathrm{n} = - < \eta_\mathrm{n}, D_\mathrm{n} \xi_\mathrm{p} > + < D_\mathrm{n} \eta_\mathrm{p}, \xi_\mathrm{n} >= -\partial_t W_\mathrm{p} \neq 0$$

Hence, $W_\mathrm{n} + W_\mathrm{p}$ is conserved.

Canonical energy/angular momentum

Define

$$E_c = \frac{m_{\rm B}}{2}\left[n_{\rm n}W_{\rm n}(\dot\xi_{\rm n},\xi_{\rm n}) + n_{\rm p}W_{\rm p}(\dot\xi_{\rm p},\xi_{\rm p})\right]$$

$$J_c = -\frac{m_{\rm B}}{2}\text{Re}\ \left[n_{\rm n}W_{\rm n}(\partial_\varphi\xi_{\rm n},\xi_{\rm n}) + n_{\rm p}W_{\rm p}(\partial_\varphi\xi_{\rm p},\xi_{\rm p})\right]$$

- If the system is coupled to radiation then any initial data for which $E_c < 0$ will lead to an unstable evolution.
- Dynamical instabilities are only possible if $E_c = 0$.
- For (real frequency) normal modes one can show that

$$E_c = -\frac{\omega}{m}J_c = \sigma_p J_c$$

  where σ_p is the pattern speed of the mode.
- Can **prove** that onset of secular instability corresponds to $\sigma_p = 0$ even in the case $\Omega_{\rm n} \neq \Omega_{\rm p}$.

Executive summary

We have taken the first steps towards generalising the Lagrangian perturbation formalism to the two-fluid problem.

Essentially, we are two-thirds of the way there at this point:

- no entrainment, but $\Omega_n \neq \Omega_{\mathrm{p}}$ (this talk)
- constant entrainment, and $\Omega_{\mathrm{n}} = \Omega_{\mathrm{p}}$ (a little bit harder but OK)
- general case (God knows...)

Ultimate aim: To provide a framework for carrying out detailed stability analysis and study unique superfluid dynamics, eg. two-stream instability.